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Asymptotic Stability and Strict Boundedness for Non-autonomous Nonlinear Difference Equations with Time-varying Delay

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Abstract In this paper, we derive some new results on the asymptotic stability and strict boundedness for a new class of non-autonomous nonlinear difference equations with time-varying delay. We employ fixed point theory and compute some difference inequalities to derive the new results. We apply these results to determine the extinction condition and the persistence condition for some discrete population models.

Keywords Fixed point · Asymptotic stability · Boundedness · Extinction · Persistence · Time-varying delay

Mathematics Subject Classification (2010) 39A10 · 39A12

1 Introduction

In this paper, we consider the following new class of non-autonomous nonlinear difference equations with time-varying delay

$$x_{n+1} = \lambda_n x_n + \alpha_n F(n, x_{n-\omega_n}), \quad n \in \mathbb{N}_{n_0}, \tag{1}$$

where $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}_{n_0} = \{n_0+1, n_0+2, \dots\}$, $n_0 \in \mathbb{N}$, (λ_n) and (α_n) are sequences of positive real numbers, (ω_n) is a sequence of positive integer numbers, the map F maps $\mathbb{N} \times [0, \infty)$ to $[0, \infty)$. Let $\psi : D_{n_0} \longrightarrow [0, \infty)$ be the bounded initial

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value function, where D_{n_0} is defined as follows: for each $n_0 \in \mathbb{N}$, if (ω_n) is bounded from above by ω, then D_{n_0} is a set of integers belonging to the interval $[n_0 - \omega, n_0]$. Conversely, if (ω_n) is unbounded from above then, D_{n_0} is a set of integers belonging to $(-\infty, n_0]$.

By a solution of (1), we mean a sequence $(x_{n,n_0,\psi})$ such that $x_n = \psi_n$ on D_{n_0} and $(x_{n,n_0,\psi})$ satisfies (1) for $n \in \mathbb{N}_{n_0}$. Clearly, (1) has a unique non-negative solution $(x_{n,n_0,\psi})$ with the given initial condition ψ.

Note that when $\lambda_n \equiv \lambda$, $\alpha_n \equiv \alpha$ are positive constants, $\omega_n \equiv \omega$ is a positive integer and $F(n, x) = f(x)$, (1) is reduced to the following form

$$x_{n+1} = \lambda x_n + \alpha f(x_{n-\omega}). \tag{2}$$

The above equation includes several discrete models derived from mathematical biology such as the Nicholson's blowflies model and the bobwhite quail population model. The qualitative properties of solutions of the autonomous nonlinear difference equation (2) and their applications have attracted a lot of attention from many authors (see, [1, 3–9, 11, 15] and the references therein).

However, in the real world, the parameters in discrete population models are not fixed constants, they are in fact time-varying. Due to the time dependence of the parameters, these models need to be non-autonomous. Beside their theoretical interest, non-autonomous nonlinear difference equations with time delay are important in mathematical biology. In the context of biology, the asymptotic stability of the zero solution and the strict boundedness of the solutions of difference equations describe the extinction and persistence, respectively, in discrete population models. In fact, both properties are very important in biology. It is vital to avoid extinction and hence it is necessary to obtain extinctive conditions. On the other hand, persistence characterizes the long-term survival of all populations in an ecosystem. From a biological point of view, persistence of a system means the survival of all populations of the system in the future time.

In the literature, some qualitative properties of solutions of non-autonomous nonlinear difference equations of the form (1) have been investigated (see, [2, 12, 13, 16, 17] and the references therein). In most cases, the analysis has been restricted to only constant time delays, i.e., $\omega_n \equiv \omega$. Even for the case where the time delay is constant, strict boundedness of the equation (1) has still not been studied elsewhere. This paper studies the strict boundedness of the equation (1). We also investigate the asymptotic stability of the zero solution of the equation (1) with time-varying delay.

In the next section, we derive a new sufficient condition on the asymptotic stability of the zero solution of the equation (1) by using the fixed point theory (see, [10, 14, 18]). We also derive a new sufficient condition on the strict boundedness of the solutions of the equation (1) by using an analysis method that involves computation of some difference inequalities. Some applications with numerical examples and simulations are presented in Section 3. A conclusion is given in Section 4.

2 Main Results

Definition 1 The zero solution of (1) is Lyapunov stable if for any $\varepsilon > 0$ and any integer $n_0 \geq 0$ there exists a $\delta = \delta(n_0, \epsilon) > 0$ such that $\psi_n \leq \delta$ on D_{n_0} implies $x_{n,n_0,\psi} \leq \varepsilon$ for $n \in \mathbb{N}_{n_0}$.

The zero solution of (1) is asymptotically stable if it is Lyapunov stable and if for any integer $n_0 \geq 0$ there exists a positive constant $c = c(n_0)$ such that $\psi_n \leq c$ on D_{n_0} implies $x_{n,n_0,\psi} \to 0$ as $n \to \infty$.

Definition 2 A solution $(x_{n,n_0,\psi})$ of (1) is said to be strictly bounded if

$$0 < \liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n < \infty.$$

The following theorem provides a sufficient condition for the zero solution of (1) to be asymptotically stable. For the sake of convenience, we adopt the notation $\sum_{n=a}^{b} x_n = 0$, $\prod_{n=a}^{b} x_n = 1$ for any $a > b$.

Theorem 1 *Assume that the following conditions are satisfied:*

i) *For each $n \in \mathbb{N}$, $F(n, 0) = 0$ and $F(n, x)$ is L-locally Lipschitz in x. That is, there is a $K > 0$ such that if $0 \le x \le K$, $0 \le y \le K$, then*

$$|F(n, x) - F(n, y)| \le L_n|x - y|$$

for a positive constant L_n.

ii) *There exist $\sigma \in (0, 1)$, $\beta \in (0, 1)$ and $n_1 \in \mathbb{N}_{n_0}$ such that $\lambda_n \in (0, \sigma)$ $\forall n \in \mathbb{N}$ and*

$$\sum_{t=n_0}^{n-1} L_t\alpha_t \prod_{s=t+1}^{n-1} \lambda_s \le \beta, \quad n \ge n_1.$$

iii) *$n - \omega_n \to \infty$ as $n \to \infty$.*

Then the zero solution of (1) is asymptotically stable.

Proof We first prove that (x_n) is a solution of (1) if and only if

$$x_n = x_{n_0} \prod_{s=n_0}^{n-1} \lambda_s + \sum_{t=n_0}^{n-1} \alpha_t F(t, x_{t-\omega_t}) \prod_{s=t+1}^{n-1} \lambda_s \qquad \forall n_0 \in \mathbb{N}.$$

Indeed, it is easy to see that (1) is equivalent to the equation

$$\Delta\left(x_n \prod_{s=n_0}^{n-1} \lambda_s^{-1}\right) = \alpha_n F(n, x_{n-\omega_n}) \prod_{s=n_0}^{n} \lambda_s^{-1}, \tag{3}$$

where $\Delta x_n = x_{n+1} - x_n$. Summing (3) from n_0 to $n-1$ gives

$$\sum_{t=n_0}^{n-1} \Delta\left(x_t \prod_{s=n_0}^{t-1} \lambda_s^{-1}\right) = \sum_{t=n_0}^{n-1} \alpha_t F(t, x_{t-\omega_t}) \prod_{s=n_0}^{t} \lambda_s^{-1},$$

$$x_n = x_{n_0} \prod_{s=n_0}^{n-1} \lambda_s + \sum_{t=n_0}^{n-1} \alpha_t F(t, x_{t-\omega_t}) \prod_{s=t+1}^{n-1} \lambda_s.$$

Clearly, for any $\varepsilon > 0$ and any integer $n_0 \ge 0$, there exists a $\delta = \delta(n_0, \epsilon)$ such that $0 < \delta \le \varepsilon(1-\beta)$. Let ψ be an initial value function satisfying $\psi_n \le \delta$ on D_{n_0}. Define

$$S = \{\varphi : D_{n_0} \cup \mathbb{N}_{n_0} \longrightarrow [0, \infty) \mid \varphi_n = \psi_n \text{ on } D_{n_0} \text{ and } \|\varphi\| \le \epsilon\},$$

where $\|\varphi\| = \max_{n\in D_{n_0}\cup\mathbb{N}_{n_0}} |\varphi_n|$. We will prove that $(S, \|\cdot\|)$ is a complete metric space. Indeed, $\forall\varphi, \eta, \psi \in S$, we have the following:

i) $\|\varphi - \eta\| = \max_{n\in D_{n_0}\cup\mathbb{N}_{n_0}} |(\varphi - \eta)_n| \ge 0$ and $\|\varphi - \eta\| = 0$ if and only if $\varphi = \eta$.

ii)

$$\begin{aligned}\|\varphi-\eta\| &= \max_{n\in D_{n_0}\cup\mathbb{N}_{n_0}} |(\varphi-\eta)_n| = \max_{n\in D_{n_0}\cup\mathbb{N}_{n_0}} |\varphi_n-\eta_n| \\ &= \max_{n\in D_{n_0}\cup\mathbb{N}_{n_0}} |\eta_n-\varphi_n| = \max_{n\in D_{n_0}\cup\mathbb{N}_{n_0}} |(\eta-\varphi)_n| = \|\eta-\varphi\|.\end{aligned}$$

iii)

$$\begin{aligned}\|\varphi-\psi\| &= \max_{n\in D_{n_0}\cup\mathbb{N}_{n_0}} |(\varphi-\psi)_n| = \max_{n\in D_{n_0}\cup\mathbb{N}_{n_0}} |\varphi_n-\psi_n| \\ &= \max_{n\in D_{n_0}\cup\mathbb{N}_{n_0}} |\varphi_n-\eta_n+\eta_n-\psi_n| \le \max_{n\in D_{n_0}\cup\mathbb{N}_{n_0}} (|\varphi_n-\eta_n|+|\eta_n-\psi_n|) \\ &\le \max_{n\in D_{n_0}\cup\mathbb{N}_{n_0}} |\varphi_n-\eta_n| + \max_{n\in D_{n_0}\cup\mathbb{N}_{n_0}} |\eta_n-\psi_n| = \|\varphi-\eta\|+\|\eta-\psi\|.\end{aligned}$$

It follows from i), ii), and iii) that $\|\cdot\|$ is a metric in S.

Next, we suppose that (φ^ℓ) is a Cauchy sequence in S. We have

$$\forall\varepsilon>0, \exists\ell_0\in\mathbb{N}: \forall k,\ell\ge\ell_0: \quad \|\varphi^\ell-\varphi^k\|<\varepsilon$$

or

$$\forall\varepsilon>0, \exists\ell_0\in\mathbb{N}: \forall k,\ell\ge\ell_0: \quad \max_{n\in D_{n_0}\cup\mathbb{N}_{n_0}} |(\varphi^\ell-\varphi^k)_n|<\varepsilon$$

or

$$\forall\varepsilon>0, \exists\ell_0\in\mathbb{N}: \forall k,\ell\ge\ell_0: \quad |(\varphi^\ell-\varphi^k)_n|<\varepsilon \quad \forall n\in D_{n_0}\cup\mathbb{N}_{n_0}.$$

Fix n, (φ_n^ℓ) is a Cauchy sequence in $[0,\infty)\subset\mathbb{R}$. In view of $\mathbb{R}$ is a complete metric space,

$$\exists\varphi_n\in[0,\infty): \quad \varphi_n=\lim_{\ell\to\infty}\varphi_n^\ell.$$

We prove $\varphi\in S$. Indeed, since $\varphi^\ell\in S$, $\varphi_n^\ell=\psi_n$ on D_{n_0}. It implies $\varphi_n=\lim_{\ell\to\infty}\varphi_n^\ell=\psi_n$ on D_{n_0}. Moreover, since $\|\varphi^\ell\|\le\epsilon$, we have $\|\varphi\|\le\epsilon$. Therefore, $(S,\|\cdot\|)$ is a complete metric space.

Define a mapping $P:S\longrightarrow S$ by $(P\varphi)_n=\psi_n$ on D_{n_0} and

$$(P\varphi)_n=\psi_{n_0}\prod_{s=n_0}^{n-1}\lambda_s+\sum_{t=n_0}^{n-1}\alpha_t F(t,\varphi_{t-\omega_t})\prod_{s=t+1}^{n-1}\lambda_s,\ n\in\mathbb{N}_{n_0}. \tag{4}$$

Since $\|\varphi\|\le\epsilon$, $\varphi_{t-\omega_t}\le\epsilon$. Thus, $F(t,\varphi_{t-\omega_t})\le L_t\varphi_{t-\omega_t}\le L_t\epsilon$. This and (4) imply that for $n\ge n_1$, $(P\varphi)_n\le\delta+\beta\epsilon\le\epsilon$. Hence, P maps from S to itself. Moreover, let $\varphi,\eta\in S$, we get for $n\ge n_1$,

$$\begin{aligned}|(P\varphi)_n-(P\eta)_n| &= \left|\sum_{t=n_0}^{n-1}\alpha_t F(t,\varphi_{t-\omega_t})\prod_{s=t+1}^{n-1}\lambda_s-\sum_{t=n_0}^{n-1}\alpha_t F(t,\eta_{t-\omega_t})\prod_{s=t+1}^{n-1}\lambda_s\right| \\ &\le \sum_{t=n_0}^{n-1}L_t\alpha_t\prod_{s=t+1}^{n-1}\lambda_s\|\varphi-\eta\| \\ &\le \beta\|\varphi-\eta\|.\end{aligned}$$

Therefore, P is a contraction map. By the contraction mapping principle, P has a unique fixed point $\varphi^*\in S$, which satisfies $\varphi_n^*=\psi_n$ for $n\in D_{n_0}$ and

$$\varphi_n^*=\varphi_{n_0}^*\prod_{s=n_0}^{n-1}\lambda_s+\sum_{t=n_0}^{n-1}\alpha_t F(t,\varphi_{t-\omega_t}^*)\prod_{s=t+1}^{n-1}\lambda_s \quad \forall n\in\mathbb{N}_{n_0},$$

i.e., (φ_n^*) is a solution of (1). Hence, the zero solution of (1) is Lyapunov stable.

Next, for the initial value function ψ as above, we define

$$T = \{\varphi : D_{n_0} \cup \mathbb{N}_{n_0} \longrightarrow [0, \infty) \mid \varphi_n = \psi_n \text{ on } D_{n_0}, \|\varphi\| \leq \varepsilon, \varphi_n \to 0 \text{ as } n \to \infty\}.$$

One can check that $(T, \|\cdot\|)$ is a complete metric space. Define $P : T \longrightarrow T$ by (4). Then, for any $\varphi \in T$, we have $(P\varphi)_n \leq \epsilon$ for $n \geq n_1$. On the other hand, since $0 < \lambda_n < \sigma < 1$ $\forall n \in \mathbb{N}$, it follows that

$$\prod_{s=n_0}^{n-1} \lambda_s \to 0 \quad \text{as } n \to \infty. \tag{5}$$

This implies $\psi_{n_0} \prod_{s=n_0}^{n-1} \lambda_s$ goes to zero as n goes to infinity. Now, let $\varphi \in T$ then $\varphi_{n-\omega_n} \leq \varepsilon$. Also, since $\varphi_{n-\omega_n} \to 0$ as $n - \omega_n \to \infty$, there exists an integer $n_2 > n_1$ such that for $n \geq n_2$, $\varphi_{n-\omega_n} < \varepsilon_1$ for $\varepsilon_1 > 0$. Due to (5), there exists an integer $n_3 > n_2$ such that

$$\prod_{s=n_2}^{n-1} \lambda_s < \frac{\varepsilon_1}{\beta\varepsilon} \qquad \forall n \geq n_3.$$

Therefore, for all $n \geq n_3$, we get

$$\begin{aligned}
\sum_{t=n_0}^{n-1} \alpha_t F(t, \varphi_{t-\omega_t}) \prod_{s=t+1}^{n-1} \lambda_s &\leq \varepsilon \sum_{t=n_0}^{n_2-1} L_t\alpha_t \prod_{s=t+1}^{n-1} \lambda_s + \varepsilon_1 \sum_{t=n_2}^{n-1} L_t\alpha_t \prod_{s=t+1}^{n-1} \lambda_s \\
&\leq \varepsilon \prod_{s=n_2}^{n-1} \lambda_s \sum_{t=n_0}^{n_2-1} L_t\alpha_t \prod_{s=t+1}^{n_2-1} \lambda_s + \varepsilon_1 \sum_{t=n_2}^{n-1} L_t\alpha_t \prod_{s=t+1}^{n-1} \lambda_s \\
&\leq \varepsilon\beta \prod_{s=n_2}^{n-1} \lambda_s + \varepsilon_1\beta \leq \varepsilon\beta \cdot \frac{\varepsilon_1}{\varepsilon\beta} + \varepsilon_1\beta = \varepsilon_1 + \varepsilon_1\beta,
\end{aligned}$$

which indicates that $\sum_{t=n_0}^{n-1} \alpha_t F(t, \varphi_{t-\omega_t}) \prod_{s=t+1}^{n-1} \lambda_s$ goes to zero as n goes to infinity. Hence, P maps from T to itself. Moreover, P is a contraction map under the supremum norm. By the contraction mapping principle, P has a unique fixed point that solves (1) and goes to zero as n goes to infinity. The proof is complete. □

Corollary 1 *If conditions* i) *and* ii) *of Theorem 1 hold, then* (1) *has a unique positive solution for every initial value function* ψ *satisfying* $\psi_n > 0$ *on* D_{n_0}.

Proof Let ψ be an initial value function satisfying $\psi_n > 0$ on D_{n_0}. Define

$$W = \{\varphi : D_{n_0} \cup \mathbb{N}_{n_0} \longrightarrow (0, \infty) \mid \varphi_n = \psi_n \text{ on } D_{n_0}\},$$

where $\|\varphi\| = \max_{n \in D_{n_0} \cup \mathbb{N}_{n_0}} |\varphi_n|$. It is not hard to check that $(W, \|\cdot\|)$ is a complete metric space.

Define a mapping $P : W \longrightarrow W$ by (4). Since $\psi_n > 0$ on D_{n_0}, (λ_n) and (α_n) are sequences of positive real numbers and the map F maps $\mathbb{N} \times [0, \infty)$ to $[0, \infty)$, we have $(P\varphi)_n > 0$ for all $n \in D_{n_0} \cup \mathbb{N}_{n_0}$. Hence, P maps from W to itself. Moreover, from the proof of Theorem 1, we see that P is a contraction map under the supremum norm. By the contraction mapping principle, P has a unique fixed point $\varphi^* \in W$, which satisfies $\varphi^*_n = \psi_n$ for $n \in D_{n_0}$ and

$$\varphi^*_n = \varphi^*_{n_0} \prod_{s=n_0}^{n-1} \lambda_s + \sum_{t=n_0}^{n-1} \alpha_t F(t, \varphi^*_{t-\omega_t}) \prod_{s=t+1}^{n-1} \lambda_s \qquad \forall n \in \mathbb{N}_{n_0},$$

i.e., (φ^*_n) is a positive solution of (1). The proof is complete. □

Remark 1 For the special case where $\lambda_n \equiv \lambda \in (0, 1)$, $\alpha_n \equiv \alpha = 1$, $\omega_n \equiv \omega$ is a positive integer, $F(n, x) = f(x) < Lx < (1 - \lambda)x \ \forall x > 0$ and $f(0) = 0$, where L is the Lipschitz coefficient of f, then Theorem 1 is reduced to the same sufficient condition of Theorem 2 given in [6].

From now on, we define $\limsup_{n\to\infty, x\to a} F(n, x)$ and $\liminf_{n\to\infty, x\to a} F(n, x)$ (here, a is zero or ∞) as follows:

$$\limsup_{n\to\infty, x\to a} F(n, x) = \sup_{x_n \to a} \left\{ \limsup_{n\to\infty} \{F(n, x_n) : x_n \to a\} \right\},$$

where $F(n, x)$ is assumed to be bounded from above.

$$\liminf_{n\to\infty, x\to a} F(n, x) = \inf_{x_n \to a} \left\{ \liminf_{n\to\infty} \{F(n, x_n) : x_n \to a\} \right\},$$

where $F(n, x)$ is assumed to be bounded from below.

The following theorem presents a sufficient condition for the strict boundedness of (1).

Theorem 2 *Assume that* $\omega_n \equiv \omega$ *where* ω *is a positive integer,* $0 < \lambda_* \le \lambda_n \le \lambda^* < 1$, $0 < \alpha_* \le \alpha_n \le \alpha^* < \infty \ \forall n \in \mathbb{N}$, $\sup_{n\in\mathbb{N}, x\ge 0} F(n, x) \in (0, \infty)$ *and*

$$\limsup_{n\to\infty, x\to\infty} \frac{F(n, x)}{x} < \frac{1 - \lambda^*}{\alpha^*}, \tag{6}$$

$$\liminf_{n\to\infty, x\to 0} \frac{F(n, x)}{x} > \frac{1 - \lambda_*}{\alpha_*}. \tag{7}$$

Then every solution of (1) *is strictly bounded.*

Proof Let us denote $B = \sup_{n\in\mathbb{N}, x\ge 0} F(n, x)$. We first prove that (x_n) is bounded from above. Assume on the contrary that $\limsup_{n\to\infty} x_n = \infty$. For each integer $n \ge 0$, we define

$$k_n := \max \left\{ \rho : 0 \le \rho \le n, x_\rho = \max_{0\le i\le n} x_i \right\}.$$

Observe that $k_0 \le k_1 \le \cdots \le k_n \to \infty$ and that

$$\lim_{n\to\infty} x_{k_n} = \infty.$$

But

$$\begin{aligned} x_{k_n} &= \lambda_{k_n-1} x_{k_n-1} + \alpha_{k_n-1} F(k_n-1, x_{k_n-1-\omega}) \\ &\le \lambda^* x_{k_n-1} + \alpha^* F(k_n-1, x_{k_n-1-\omega}) \\ &= \cdots = (\lambda^*)^{\omega+1} x_{k_n-1-\omega} + \alpha^* \sum_{j=0}^{\omega} (\lambda^*)^j F(k_n-1, x_{k_n-1-\omega-j}) \\ &\le (\lambda^*)^{\omega+1} x_{k_n-1-\omega} + \alpha^* B\big(1+\lambda^*+\cdots+(\lambda^*)^{\omega}\big), \end{aligned}$$

so

$$\lim_{n\to\infty} x_{k_n-1-\omega} = \infty.$$

Let $n_0 > 0$ be such that $k_{n_0} > 0$. We have for $n > n_0$,

$$\begin{aligned} \lambda^* x_{k_n-1} + \alpha^* F(k_n-1, x_{k_n-1-\omega}) &\ge \lambda_{k_n-1} x_{k_n-1} + \alpha_{k_n-1} F(k_n-1, x_{k_n-1-\omega}) \\ &= x_{k_n} \ge x_{k_n-1} \end{aligned}$$

and therefore,

$$\frac{F(k_n-1, x_{k_n-1-\omega})}{x_{k_n-1-\omega}} \ge \frac{F(k_n-1, x_{k_n-1-\omega})}{x_{k_n-1}} \ge \frac{1-\lambda^*}{\alpha^*}.$$

This implies that

$$\limsup_{n\to\infty, x\to\infty} \frac{F(n,x)}{x} \ge \frac{F(k_n-1, x_{k_n-1-\omega})}{x_{k_n-1-\omega}} \ge \frac{1-\lambda^*}{\alpha^*},$$

which contradicts (6). Thus, (x_n) is bounded from above.

Next, we prove that $\liminf_{n\to\infty} x_n > 0$. Assume on the contrary, that

$$\liminf_{n\to\infty} x_n = 0.$$

For each integer $n \ge 0$, we define

$$s_n := \max\left\{\rho : 0 \le \rho \le n, x_\rho = \min_{0\le i\le n} x_i\right\}.$$

Clearly, $s_0 \le s_1 \le \cdots \le s_n \to \infty$ and that

$$\lim_{n\to\infty} x_{s_n} = 0.$$

But

$$\begin{aligned} x_{s_n} &= \lambda_{s_n-1} x_{s_n-1} + \alpha_{s_n-1} F(s_n-1, x_{s_n-1-\omega}) \\ &\ge \lambda_* x_{s_n-1} + \alpha_* F(s_n-1, x_{s_n-1-\omega}) \\ &= \cdots = (\lambda_*)^{\omega+1} x_{s_n-1-\omega} + \alpha_* \sum_{j=0}^{\omega} (\lambda_*)^j F(s_n-1, x_{s_n-1-\omega-j}) \\ &\ge (\lambda_*)^{\omega+1} x_{s_n-1-\omega}, \end{aligned}$$

so

$$\lim_{n\to\infty} x_{s_n-1-\omega} = 0.$$

On the other hand, we have for $n > n_0$,

$$\begin{aligned} \lambda_* x_{s_n-1} + \alpha_* F(s_n-1, x_{s_n-1-\omega}) &\le \lambda_{s_n-1} x_{s_n-1} + \alpha_{s_n-1} F(s_n-1, x_{s_n-1-\omega}) \\ &= x_{s_n} \le x_{s_n-1} \end{aligned}$$

and therefore,

$$\liminf_{n\to\infty,x\to 0}\frac{F(n,x)}{x}\le\frac{F(s_n-1,x_{s_n-1-\omega})}{x_{s_n-1-\omega}}\le\frac{F(s_n-1,x_{s_n-1-\omega})}{x_{s_n-1}}\le\frac{1-\lambda_*}{\alpha_*},$$

which contradicts (7). The proof is complete. □

Remark 2 For the special case when $\lambda_n\equiv\lambda\in(0,1)$, $\alpha_n\equiv\alpha=1$, $\omega_n\equiv\omega$ is a positive integer, $F(n,x)=f(x)$, $\sup_{x\ge 0}f(x)\in(0,\infty)$, $\limsup_{x\to\infty}\frac{f(x)}{x}<1-\lambda$ and $\liminf_{x\to 0}\frac{f(x)}{x}>1-\lambda$ then Theorem 2 provides a new sufficient condition for the persistence of (2). Note also that this sufficient condition requires less restrictive assumptions than the one derived in [6].

3 Applications

In this section, we apply the results obtained in Section 2 to the Nicholson's blowflies model

$$x_{n+1}=\lambda_n x_n+\alpha_n x_{n-\omega_n}e^{-q_n x_{n-\omega_n}}\tag{8}$$

and the bobwhite quail population model

$$x_{n+1}=\lambda_n x_n+\alpha_n\frac{x_{n-\omega_n}}{1+x^k_{n-\omega_n}},\tag{9}$$

where (λ_n), (α_n), (q_n) are sequences of positive real numbers, $k\in(0,\infty)$ and (ω_n) is a sequence of positive integer numbers.

Definition 3 A positive solution $(x_{n,n_0,\psi})$ of (8) or (9) is called extinctive if $\lim_{n\to\infty}x_n=0$. The model (8) or (9) is called extinctive if all solutions of its are extinctive.

Definition 4 A positive solution $(x_{n,n_0,\psi})$ of (8) or (9) is called persistent if

$$0<\liminf_{n\to\infty}x_n\le\limsup_{n\to\infty}x_n<\infty.$$

Clearly, the model (8) is in the form of (1) with $F(n,x)=xe^{-q_n x}$. We have $F'_x=\frac{dF}{dx}=(1-xq_n)e^{-q_n x}$. Thus, for each $n\in\mathbb{N}$,

$$|F'_x(n,x)|\le 1+q_n\quad\forall x\in(0,1),$$

which implies that $F(n,x)$ is L-locally Lipschitz in x with $L_n=1+q_n$. Hence, we have the following conclusion from Theorem 1.

Corollary 2 *Assume that $n-\omega_n\to\infty$ as $n\to\infty$, there exist $\sigma\in(0,1)$, $\beta\in(0,1)$ and $n_1\in\mathbb{N}_{n_0}$ such that $\lambda_n\in(0,\sigma)$ $\forall n\in\mathbb{N}$ and*

$$\sum_{t=n_0}^{n-1}(1+q_t)\alpha_t\prod_{s=t+1}^{n-1}\lambda_s\le\beta,\quad n\ge n_1.$$

Then the model (8) is extinctive.

On the other hand, for $0<q_n\le q^*<\infty$ $\forall n\in\mathbb{N}$, we have

$$\limsup_{n\to\infty,x\to\infty}\frac{F(n,x)}{x}=\limsup_{n\to\infty,x\to\infty}\frac{1}{e^{q_n x}}=0$$

and

$$\liminf_{n\to\infty,x\to 0}\frac{F(n,x)}{x}=\liminf_{n\to\infty,x\to 0}\frac{1}{e^{q_n x}}=1.$$

Hence, we have the following conclusion from Theorem 2.

Corollary 3 *If* $\omega_n\equiv\omega$ *where* ω *is a positive integer,* $0<\lambda_*\le\lambda_n\le\lambda^*<1, 0<\alpha_*\le\alpha_n\le\alpha^*<\infty$, $q_n>0\ \forall n\in\mathbb{N}$ *and* $\lambda_*+\alpha_*>1$, *then every positive solution of* (8) *is persistent.*

Next, we consider the model (9). It is in the form of (1) with

$$F(n,x)=f(x)=\frac{x}{1+x^k}.$$

After some simple manipulations, we obtain

$$f'(x)=\frac{(1-k)x^k+1}{(1+x^k)^2}$$

and

$$f''(x)=\frac{-kx^{k-1}(k+1)+(1-k)x^k}{(1+x^k)^3}.$$

Thus,

$$|f'(x)|\le|f'(0)|=1\quad\text{when } k=1$$

and

$$\max_{x>0}|f'(x)|=\left|f'\left(\left(\frac{k+1}{k-1}\right)^{\frac{1}{k}}\right)\right|=\frac{(k-1)^2}{4k}\quad\text{when } k>1.$$

Therefore, $F(n,x)$ is L-locally Lipschitz in x with $L=1$ when $k=1$ and with $L=\frac{(k-1)^2}{4k}$ when $k>1$. Hence, we have the following conclusion from Theorem 1.

Corollary 4 *Assume that* $n-\omega_n\to\infty$ *as* $n\to\infty$, $\lambda_n\in(0,1)\ \forall n\in\mathbb{N}$ *and there exist* $\beta\in(0,1)$ *and* $n_1\in\mathbb{N}_{n_0}$ *such that either* $k=1$ *and*

$$\sum_{t=n_0}^{n-1}\alpha_t\prod_{s=t+1}^{n-1}\lambda_s\le\beta,\quad n\ge n_1,$$

or $k>1$ *and*

$$\sum_{t=n_0}^{n-1}\alpha_t\prod_{s=t+1}^{n-1}\lambda_s\le\beta\frac{4k}{(k-1)^2},\quad n\ge n_1.$$

Then the model (9) *is extinctive.*

On the other hand, we have for $k>0$,

$$\limsup_{x\to\infty}\frac{f(x)}{x}=\limsup_{n\to\infty,x\to\infty}\frac{1}{1+x^k}=0$$

and

$$\liminf_{x\to 0}\frac{f(x)}{x}=\liminf_{n\to\infty,x\to 0}\frac{1}{1+x^k}=1.$$

Hence, we have the following conclusion from Theorem 2.

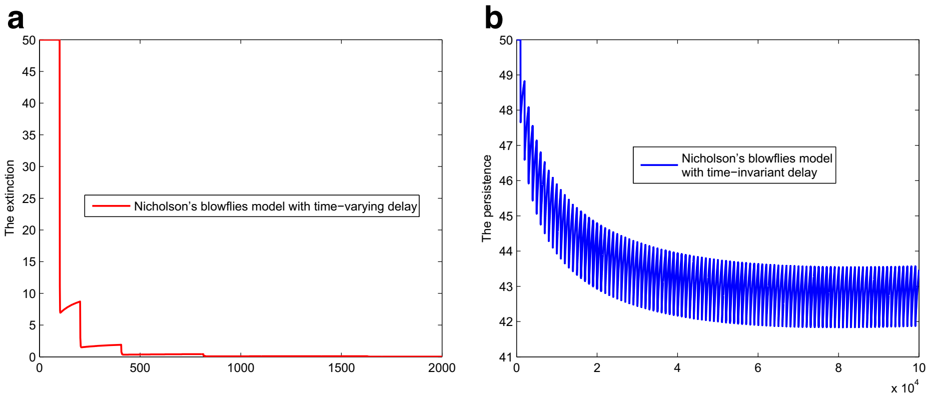

Fig. 1 Responses of the extinction and the persistence of Nicholson's blowflies model

Corollary 5 *If* $\omega_n \equiv \omega$ *where* ω *is a positive integer,* $0 < \lambda_* \leq \lambda_n \leq \lambda^* < 1, 0 < \alpha_* \leq \alpha_n \leq \alpha^* < \infty$ $\forall n \in \mathbb{N}$, $k > 0$ *and* $\lambda_* + \alpha_* > 1$, *then every positive solution of* (9) *is persistent.*

4 Numerical Examples and Simulations

Consider the model (8) with $\lambda_n = \frac{5n+6}{50(n+2)}$, $\alpha_n = \frac{n+1}{5n+2}$, $q_n = \frac{1}{n+1}$ $\forall n \in \mathbb{N}$, $\omega_n = \left[\frac{n}{2}\right]$, where $[\cdot]$ is the integer function. Clearly, $\lambda_n \in (0, 1)$ $\forall n \in \mathbb{N}$ and $n - \left[\frac{n}{2}\right] \to \infty$ as $n \to \infty$. Moreover, we have for $n_0 = 0$,

$$\sum_{t=0}^{n-1}(1+q_t)\alpha_t \prod_{s=t+1}^{n-1} \lambda_s = \sum_{t=0}^{n-1}\left(\frac{t+2}{5t+2}\right) \prod_{s=t+1}^{n-1}\left(\frac{5s+6}{50(s+2)}\right) \leq \left(\frac{1}{10}\right)^{n-t-1} n, \quad n \in \mathbb{N}.$$

From this inequality and due to $\lim_{n\to\infty}(\frac{1}{10})^{n-t-1}n = 0$, there exist a constant $\beta \in (0, 1)$ and $n_1 \in \mathbb{N}_{n_0}$ such that

$$\sum_{t=0}^{n-1}(1+q_t)\alpha_t \prod_{s=t+1}^{n-1} \lambda_s \leq \beta, \quad n \geq n_1.$$

Hence, according to Corollary 2, the model (8) is extinctive (see, Fig. 1a).

Next, we consider the model (8) with $\lambda_n = \frac{5n+10}{10^4(n+5)}$, $\alpha_n = \frac{5n+9}{5n+2}$, $q_n = \frac{1}{n+1}$ $\forall n \in \mathbb{N}$, $\omega_n \equiv 1000$. We have $\lambda_* = \frac{1}{5\times10^3}$, $\lambda^* = \frac{1}{2\times10^3}$, $\alpha_* = 1$, $\alpha^* = \frac{9}{2}$, $\lambda_* + \alpha_* > 1$. According to Corollary 3, every positive solution of the model (8) is persistent (see, Fig. 1b).

Similarly, by letting $\lambda_n = \frac{5n+6}{5.05(n+2)}$, $\alpha_n = \frac{n+2}{10^5 n+1}$ $\forall n \in \mathbb{N}$, $k = 9$, $\omega_n = \left[\frac{n}{2}\right]$, where $[\cdot]$ is the integer function and using Corollary 4, we can show that the model (9) is extinctive (see, Fig. 2a). By letting $\lambda_n = \frac{5n+6}{9(n+2)}$, $\alpha_n = \frac{5n+9}{5n+2}$ $\forall n \in \mathbb{N}$, $k = \frac{1}{10}$, $\omega_n \equiv 1000$ and using Corollary 5, we can see that every positive solution of the model (9) is persistent (see, Fig. 2b).

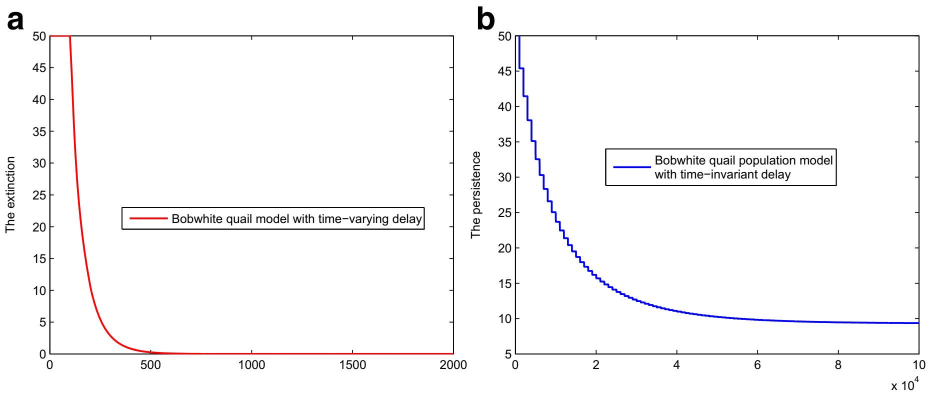

Fig. 2 Responses of the extinction and the persistence of Bobwhite quail model

5 Conclusion

In this paper, we have considered a new class of non-autonomous nonlinear difference equations with time-varying delay. By using the fixed point theory together with some analytical techniques, we have derived new sufficient conditions for the system to be asymptotically stable and strictly bounded. The results have great implication for biological systems in the sense that these conditions are related to the extinction and survival properties of population models. We have applied the new results to analyze the extinction and persistence of the Nicholson's blowflies model and the bobwhite quail population model. Simulations have also been presented. A possible future research topic is to prove the global attractivity of periodic solutions of the equation (1) when the nonlinear term $F(n, x)$ is unimodal.

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