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Local Property of a Class of m-Subharmonic Functions

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Abstract In the paper, we introduce a new class of m-subharmonic functions with finite weighted complex m-Hessian. We prove that this class has local property.

Keywords m-Subharmonic functions · Weighted energy classes of m-subharmonic functions · Complex m-Hessian · Local property

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1 Introduction

Let Ω be a hyperconvex domain in $\mathbb{C}^n$. By $PSH(\Omega)$ (resp. $PSH^-(\Omega)$), we denote the cone of plurisubharmonic functions (resp. negative plurisubharmonic functions) on Ω. In [15], the authors introduced and investigated the notion of local class as follows. A class $\mathcal{J}(\Omega) \subset PSH^-(\Omega)$ is said to be a local class if $\varphi \in \mathcal{J}(\Omega)$ then $\varphi \in \mathcal{J}(D)$ for all hyperconvex domains $D \Subset \Omega$ and if $\varphi \in PSH^-(\Omega), \varphi|_{\Omega_i} \in \mathcal{J}(\Omega_i)\ \forall i \in I$ with $\Omega = \bigcup_{i\in I} \Omega_i$ then $\varphi \in \mathcal{J}(\Omega)$. As is well known, Błocki (see [8]) proved the class $\mathcal{E}(\Omega)$ introduced and investigated by Cegrell in [10], is a local class. Moreover, in [10], Cegrell has proved this class is the biggest on which the complex Monge–Ampère operator $(dd^c.)^n$ is well defined as a Radon measure, and it is continuous under decreasing sequences. On the other hand, another weighted energy class $\mathcal{E}_\chi(\Omega)$ which extends the classes $\mathcal{E}_p(\Omega)$ and $\mathcal{F}(\Omega)$

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in [9] and [10] introduced and investigated recently by Benelkourchi et al. [4] is as follows. Let $\chi : \mathbb{R}^- \longrightarrow \mathbb{R}^+$ be a decreasing function. Then, as in [4], we define

$$\mathcal{E}_\chi(\Omega) = \left\{ \varphi \in PSH^-(\Omega) : \exists \mathcal{E}_0(\Omega) \ni \varphi_j \searrow \varphi, \ \sup_{j \geq 1} \int_\Omega \chi(\varphi_j)(dd^c\varphi_j)^n < +\infty \right\},$$

where $\mathcal{E}_0(\Omega)$ is the cone of bounded plurisubharmonic functions φ defined on Ω with finite total Monge–Ampère mass and $\lim_{z\to\xi} \varphi(z) = 0$ for all $\xi \in \partial\Omega$. Note that from Corollary 4.4 in [3], it follows that if $\varphi \in \mathcal{E}_\chi(\Omega)$ then $\lim_{z\to\xi} \varphi(z) = 0$ for all $\xi \in \partial\Omega$. Hence, if $\varphi \in \mathcal{E}_\chi(\Omega)$, then $\varphi \notin \mathcal{E}_\chi(D)$ with D a relatively compact hyperconvex domain in Ω. Thus, the class $\mathcal{E}_\chi(\Omega)$ is not a "local" one. In this paper, by relying on ideas from the paper of Benelkourchi et al. [4] and on Cegrell classes of m-subharmonic functions introduced and studied recently in [12], we introduce weighted energy classes of m-subharmonic functions $\mathcal{F}_{m,\chi}(\Omega)$ and $\mathcal{E}_{m,\chi}(\Omega)$. Under slight hypotheses for weights χ, we achieve that the class $\mathcal{F}_{m,\chi}(\Omega)$ is a convex cone (see Proposition 2 below). We also show that the complex Hessian operator $H_m(u) = (dd^c u)^m \wedge \beta^{n-m}$ is well defined on the class $\mathcal{E}_{m,\chi}(\Omega)$ where $\beta = dd^c\|z\|^2$ denotes the canonical Kähler form of $\mathbb{C}^n$. Furthermore, we prove that the class $\mathcal{E}_{m,\chi}(\Omega)$ is a local class (see Theorem 1 in Section 4 below). In this article, we prove the following main result.

Theorem 1 *Let Ω be a hyperconvex domain in $\mathbb{C}^n$ and m be an integer with $1 \leq m \leq n$. Assume that $u \in SH_m^-(\Omega)$ and $\chi \in \mathcal{K}$ such that $\chi''(t) \geq 0$ $\forall t < 0$. Then the following statements are equivalent.*

a) $u \in \mathcal{E}_{m,\chi}(\Omega)$.

b) *For all $K \Subset \Omega$, there exists a sequence $\{u_j\} \subset \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\Omega)$, $u_j \searrow u$ on K such that*

$$\sup_j \int_K \chi(u_j)|u_j|^p (dd^c u_j)^{m-p} \wedge \beta^{n-m+p} < \infty$$

for every $p = 0, \ldots, m$.

c) *For every $W \Subset \Omega$ such that W is a hyperconvex domain, we have $u|_W \in \mathcal{E}_{m,\chi}(W)$.*

d) *For every $z \in \Omega$, there exists a hyperconvex domain $V_z \Subset \Omega$ such that $z \in V_z$ and $u|_{V_z} \in \mathcal{E}_{m,\chi}(V_z)$.*

Finally, using the main results above, we prove an interesting corollary. Namely, we have

Corollary 1 *Assume that Ω is a bounded hyperconvex domain, and $\chi \in \mathcal{K}$ satisfies all hypotheses of Theorem 1. Then $\mathcal{E}_{m,\chi}(\Omega) \subset \mathcal{E}_{m-1,\chi}(\Omega)$.*

The paper is organized as follows. Beside the introduction, the paper has three sections. In Section 2, we recall the definitions and results concerning to m-subharmonic functions which were introduced and investigated intensively in recent years by many authors, see [5, 13, 21]. We also recall the Cegrell classes of m-subharmonic functions $\mathcal{F}_m(\Omega)$ and $\mathcal{E}_m(\Omega)$ introduced and studied in [12]. In Section 3, we introduce two new weighted energy classes of m-subharmonic functions $\mathcal{F}_{m,\chi}(\Omega)$ and $\mathcal{E}_{m,\chi}(\Omega)$. Section 4 is devoted to the proof of the local property of the class $\mathcal{E}_{m,\chi}(\Omega)$ under some extra assumptions on weights χ. To show this property of the class $\mathcal{E}_{m,\chi}(\Omega)$, we need a result about subextension for the class $\mathcal{F}_{m,\chi}(\Omega)$ (see Lemma 5 below) which is of independent interest. Finally, by relying on the local property of the class $\mathcal{E}_{m,\chi}(\Omega)$, we prove a corollary for this class.

2 Preliminaries

Some elements of pluripotential theory that will be used throughout the paper can be found in [1, 17, 18, 20], while elements of the theory of m-subharmonic functions and the complex Hessian operator can be found in [5, 13, 21]. Now, we recall the definition of some Cegrell classes of plurisubharmonic functions (see [9] and [10]), as well as the class of m-subharmonic functions introduced by Błocki in [5] and the classes $\mathcal{E}_m^0(\Omega)$ and $\mathcal{F}_m(\Omega)$ introduced and investigated by Chinh in [12] recently. Let Ω be an open subset in $\mathbb{C}^n$. By $\beta = dd^c\|z\|^2$, we denote the canonical Kähler form of $\mathbb{C}^n$ with the volume element $dV_n = \frac{1}{n!}\beta^n$ where $d = \partial + \overline{\partial}$ and $d^c = \frac{\partial - \overline{\partial}}{4i}$, hence $dd^c = \frac{i}{2}\partial\overline{\partial}$.

2.1 The Cegrell Classes

As in [9, 10], we define the classes $\mathcal{E}_0(\Omega)$ and $\mathcal{F}(\Omega)$ as follows. Let Ω be a bounded hyperconvex domain. That means that Ω is a connected, bounded open subset, and there exists a negative plurisubharmonic function ϱ such that for all $c < 0$ the set $\Omega_c = \{z \in \Omega : \varrho(z) < c\} \Subset \Omega$. Set

$$\mathcal{E}_0 = \mathcal{E}_0(\Omega) = \left\{\varphi \in PSH^-(\Omega) \cap L^\infty(\Omega) : \lim_{z\to\xi}\varphi(z) = 0\ \forall\xi \in \partial\Omega,\ \int_\Omega (dd^c\varphi)^n < \infty\right\}$$

and

$$\mathcal{F} = \mathcal{F}(\Omega) = \left\{\varphi \in PSH^-(\Omega) : \exists\mathcal{E}_0 \ni \varphi_j \searrow \varphi,\ \sup_j \int_\Omega (dd^c\varphi_j)^n < \infty\right\}.$$

2.2 m-Subharmonic Functions

We recall the class of m-subharmonic functions introduced and investigated in [5] recently. For $1 \le m \le n$, we define

$$\widehat{\Gamma}_m = \left\{\eta \in \mathbb{C}_{(1,1)} : \eta \wedge \beta^{n-1} \ge 0, \ldots, \eta^m \wedge \beta^{n-m} \ge 0\right\},$$

where $\mathbb{C}_{(1,1)}$ denotes the space of $(1, 1)$-forms with constant coefficients.

Definition 1 Let u be a subharmonic function on an open subset $\Omega \subset \mathbb{C}^n$. u is said to be a m-subharmonic function on Ω if for every $\eta_1, \ldots, \eta_{m-1}$ in $\widehat{\Gamma}_m$ the inequality

$$dd^c u \wedge \eta_1 \wedge \cdots \wedge \eta_{m-1} \wedge \beta^{n-m} \ge 0$$

holds in the sense of currents.

By $SH_m(\Omega)$ (resp. $SH_m^-(\Omega)$), we denote the cone of m-subharmonic functions (resp. negative m-subharmonic functions) on Ω. Before formulating the basic properties of m-subharmonic functions, we recall the following (see [5]).

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $1 \le m \le n$, define

$$S_m(\lambda) = \sum_{1\le j_1<\cdots<j_m\le n} \lambda_{j_1}\cdots\lambda_{j_m}.$$

Set

$$\Gamma_m = \{S_1 \ge 0\} \cap \{S_2 \ge 0\} \cap \cdots \cap \{S_m \ge 0\}.$$

By $\mathcal{H}$, we denote the vector space of complex hermitian $n \times n$ matrices over $\mathbb{R}$. For $A \in \mathcal{H}$, let $\lambda(A) = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ be the eigenvalues of A. Set

$$\widetilde{S}_m(A) = S_m(\lambda(A)).$$

As in [14], we define

$$\widetilde{\Gamma}_m = \{A \in \mathcal{H} : \lambda(A) \in \Gamma_m\} = \left\{\widetilde{S}_1 \geq 0\right\} \cap \cdots \cap \left\{\widetilde{S}_m \geq 0\right\}.$$

Now, we list the basic properties of m-subharmonic functions whose proofs repeat analogous reasonings for plurisubharmonic functions, hence we omit them.

Proposition 1 *Let Ω be an open set in $\mathbb{C}^n$. Then we have*

a) $PSH(\Omega) = SH_n(\Omega) \subset SH_{n-1}(\Omega) \subset \cdots \subset SH_1(\Omega) = SH(\Omega)$. *Hence,* $u \in SH_m(\Omega)$, $1 \leq m \leq n$, *then* $u \in SH_r(\Omega)$ *for every* $1 \leq r \leq m$.
b) *If u is C^2 smooth then it is m-subharmonic if and only if the form $dd^c u$ is pointwise in $\widehat{\Gamma}_m$.*
c) *If* $u, v \in SH_m(\Omega)$ *and* $\alpha, \beta > 0$ *then* $\alpha u + \beta v \in SH_m(\Omega)$.
d) *If* $u, v \in SH_m(\Omega)$ *then so is* $\max\{u, v\}$.
e) *If* $\{u_j\}_{j=1}^{\infty}$ *is a family of m-subharmonic functions,* $u = \sup_j u_j < +\infty$ *and u is upper semicontinuous then u is a m-subharmonic function.*
f) *If* $\{u_j\}_{j=1}^{\infty}$ *is a decreasing sequence of m-subharmonic functions then so is* $u = \lim_{j \to +\infty} u_j$.
g) *Let* $\rho \geq 0$ *be a smooth radial function in* $\mathbb{C}^n$ *vanishing outside the unit ball and satisfying* $\int_{\mathbb{C}^n} \rho dV_n = 1$, *where* dV_n *denotes the Lebesgue measure of* $\mathbb{C}^n$. *For* $u \in SH_m(\Omega)$, *we define*

$$u_\varepsilon(z) := (u * \rho_\varepsilon)(z) = \int_{\mathbb{B}(0,\varepsilon)} u(z - \xi)\rho_\varepsilon(\xi) dV_n(\xi) \quad \forall z \in \Omega_\varepsilon,$$

where $\rho_\varepsilon(z) := \frac{1}{\varepsilon^{2n}}\rho(z/\varepsilon)$ *and* $\Omega_\varepsilon = \{z \in \Omega : d(z, \partial\Omega) > \varepsilon\}$. *Then* $u_\varepsilon \in SH_m(\Omega_\varepsilon) \cap \mathcal{C}^\infty(\Omega_\varepsilon)$ *and* $u_\varepsilon \downarrow u$ *as* $\varepsilon \downarrow 0$.
h) *Let* $u_1, \ldots, u_p \in SH_m(\Omega)$ *and* $\chi : \mathbb{R}^p \to \mathbb{R}$ *be a convex function which is non decreasing in each variable. If χ is extended by continuity to a function* $[-\infty, +\infty)^p \to [-\infty, \infty)$, *then* $\chi(u_1, \ldots, u_p) \in SH_m(\Omega)$.

Example 1 Let $u(z_1, z_2, z_3) = 5|z_1|^2 + 4|z_2|^2 - |z_3|^2$. By using (b) of Proposition 1, it is easy to see that $u \in SH_2(\mathbb{C}^3)$. However, u is not a plurisubharmonic function in $\mathbb{C}^3$ because the restriction of u on the line $(0, 0, z_3)$ is not subharmonic.

Now, as in [5, 13], we define the complex Hessian operator of locally bounded m-subharmonic functions as follows.

Definition 2 Assume that $u_1, \ldots, u_p \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega)$. Then the complex Hessian operator $H_m(u_1, \ldots, u_p)$ is defined inductively by

$$dd^c u_p \wedge \cdots \wedge dd^c u_1 \wedge \beta^{n-m} = dd^c \left(u_p dd^c u_{p-1} \wedge \cdots \wedge dd^c u_1 \wedge \beta^{n-m}\right).$$

From the definition of m-subharmonic functions and using arguments as in the proof of Theorem 2.1 in [1], we note that $H_m(u_1, \ldots, u_p)$ is a closed positive current of bidegree

$(n-m+p, n-m+p)$, and this operator is continuous under decreasing sequences of locally bounded m-subharmonic functions. Hence, for $p=m$, $dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge \beta^{n-m}$ is a nonnegative Borel measure. In particular, when $u=u_1=\cdots=u_m \in SH_m(\Omega)\cap L^\infty_{loc}(\Omega)$, the Borel measure

$$H_m(u) = (dd^c u)^m \wedge \beta^{n-m}$$

is well defined and is called the complex Hessian of u.

2.3 m-Maximal Functions

Similarly in pluripotential theory now we recall a class of m-maximal functions introduced and investigated in [5] recently.

Definition 3 A m-subharmonic function $u \in SH_m(\Omega)$ is called m-maximal if every $v \in SH_m(\Omega)$, $v \le u$ outside a compact subset of Ω implies that $v \le u$ on Ω.

By $MSH_m(\Omega)$ we denote the set of m-maximal functions on Ω. Theorem 3.6 in [5] claims that a locally bounded m-subharmonic function u on a bounded domain $\Omega \subset \mathbb{C}^n$ belongs to $MSH_m(\Omega)$ if and only if it solves the homogeneous Hessian equation $H_m(u) = (dd^c u)^m \wedge \beta^{n-m} = 0$.

2.4 The $\mathcal{E}^0_m(\Omega)$ and $\mathcal{F}_m(\Omega)$ Classes

Next, we recall the classes $\mathcal{E}^0_m(\Omega)$ and $\mathcal{F}_m(\Omega)$ introduced and investigated in [12]. First, we give the following.

Let Ω be a bounded domain in $\mathbb{C}^n$. Ω is said to be m-hyperconvex if there exists a continuous m-subharmonic function $u : \Omega \longrightarrow \mathbb{R}^-$ such that $\Omega_c = \{u < c\} \Subset \Omega$ for every $c < 0$. As above, every plurisubharmonic function is m-subharmonic with $m \ge 1$ then every hyperconvex domain in $\mathbb{C}^n$ is m-hyperconvex. Let $\Omega \subset \mathbb{C}^n$ be a m-hyperconvex domain. Set

$$\mathcal{E}^0_m = \mathcal{E}^0_m(\Omega) = \left\{ u \in SH^-_m(\Omega) \cap L^\infty(\Omega) : \lim_{z\to\partial\Omega} u(z) = 0, \ \int_\Omega H_m(u) < \infty \right\},$$

$$\mathcal{F}_m = \mathcal{F}_m(\Omega) = \left\{ u \in SH^-_m(\Omega) : \exists \mathcal{E}^0_m \ni u_j \searrow u, \ \sup_j \int_\Omega H_m(u_j) < \infty \right\},$$

and

$$\mathcal{E}_m = \mathcal{E}_m(\Omega) = \Big\{ u \in SH^-_m(\Omega) : \forall z_0 \in \Omega, \exists \text{ a neighborhood } \omega \ni z_0, \text{ and}$$

$$\mathcal{E}^0_m \ni u_j \searrow u \ \text{ on } \ \omega, \sup_j \int_\Omega H_m(u_j) < \infty \Big\},$$

where $H_m(u) = (dd^c u)^m \wedge \beta^{n-m}$ denotes the Hessian measure of $u \in SH^-_m(\Omega) \cap L^\infty(\Omega)$. From Theorem 3.14 in [12], it follows that if $u \in \mathcal{E}_m(\Omega)$, the complex Hessian $H_m(u) = (dd^c u)^m \wedge \beta^{n-m}$ is well defined and is a Radon measure on Ω. On the other hand, by Remark 3.6 in [12], we may give the following description of the class $\mathcal{E}_m(\Omega)$:

$$\mathcal{E}_m = \mathcal{E}_m(\Omega) = \left\{ u \in SH^-_m(\Omega) : \forall \, U \Subset \Omega, \exists v \in \mathcal{F}_m(\Omega), \ v = u \ \text{ on } \ U \right\}.$$

2.5 *m*-Capacity

We recall the notion of m-capacity introduced in [12].

Definition 4 Let $E \subset \Omega$ be a Borel subset. The m-capacity of E with respect to Ω is defined by

$$C_m(E) = C_m(E, \Omega) = \sup\left\{\int_E (dd^c u)^m \wedge \beta^{n-m} : u \in SH_m(\Omega), -1 \leq u \leq 0\right\}.$$

Proposition 2.10 in [12] gives some elementary properties of the m-capacity similar as the capacity presented in [1]. Namely, we have

a) $C_m\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} C_m(E_j)$.
b) If $E_j \nearrow E$ then $C_m(E_j) \nearrow C_m(E)$.

We need the following lemma which is used in the proof for the convexity of the class $\mathcal{E}_{m,\chi}(\Omega)$.

Lemma 1 *Assume that* $\varphi \in \mathcal{E}_m^0(\Omega)$. *Then*

$$(dd^c\varphi)^m \wedge \beta^{n-m}(\{\varphi < -t\}) \leq t^m C_m(\{\varphi < -t\})$$

and

$$t^m C_m(\{\varphi < -2t\}) \leq (dd^c\varphi) \wedge \beta^{n-m}(\{\varphi < -t\}).$$

Proof Let $v \in SH_m(\Omega)$, $-1 < v < 0$. For all $t > 0$, we have the following inclusion:

$$\{\varphi < -2t\} \subset \left\{\frac{\varphi}{t} < v - 1\right\} \subset \{\varphi < -t\}.$$

By the comparison principle (Theorem 1.4 in [13]), we get

$$\begin{aligned}
\int_{\{\varphi<-2t\}} (dd^c v)^m \wedge \beta^{n-m} &\leq \int_{\{\frac{\varphi}{t}<v-1\}} (dd^c v)^m \wedge \beta^{n-m} \\
&\leq \int_{\{\frac{\varphi}{t}<v-1\}} \frac{1}{t^m}(dd^c\varphi)^m \wedge \beta^{n-m} \\
&\leq \frac{1}{t^m}\int_{\{\varphi<-t\}} (dd^c\varphi)^m \wedge \beta^{n-m}.
\end{aligned}$$

Hence, taking the supremum over all v, we obtain

$$t^m C_m(\{\varphi < -2t\}) \leq (dd^c\varphi)^m \wedge \beta^{n-m}(\{\varphi < -t\}).$$

By similar arguments as in the proof of Proposition 3.4 in [11], it follows that

$$(dd^c\varphi)^m \wedge \beta^{n-m}(\{\varphi < -t\}) = \int_{\{\varphi<-t\}} (dd^c\varphi)^m \wedge \beta^{n-m} \leq t^m C_m(\{\varphi < -t\}).$$

The proof is complete. □

3 The Classes $\mathcal{F}_{m,\chi}(\Omega)$, $\mathcal{E}_{m,\chi}(\Omega)$

In what follows, we assume that Ω is a bounded hyperconvex domain in $\mathbb{C}^n$. Now, we introduce two weighted pluricomplex energy classes of m-subharmonic functions defined as follows.

Definition 5 Let $\chi : \mathbb{R}^- \lg \mathbb{R}^+$ be a decreasing function and $1 \leq m \leq n$. We define

$$\mathcal{F}_{m,\chi}(\Omega) = \Bigg\{ u \in SH_m^-(\Omega) : \exists \{u_j\} \subset \mathcal{E}_m^0(\Omega),\ u_j \searrow u \text{ on } \Omega \\ \sup_j \int_\Omega \chi(u_j)(dd^c u_j)^m \wedge \beta^{n-m} < +\infty \Bigg\}$$

and $\mathcal{E}_{m,\chi}(\Omega) = \{u \in SH_m^-(\Omega) : \forall K \Subset \Omega, \exists v \in \mathcal{F}_{m,\chi}(\Omega), v = u \text{ on } K\}$.

Remark 1 (a) From the above definitions of the two classes $\mathcal{F}_{m,\chi}(\Omega)$ and $\mathcal{E}_{m,\chi}(\Omega)$, we note that in the case $\chi(t) \equiv 1$ for all $t < 0$ we get the pluricomplex energy classes $\mathcal{F}_m(\Omega)$ and $\mathcal{E}_m(\Omega)$ introduced and investigated in [12].

(b) In the case $m = n$, the class $\mathcal{F}_{n,\chi}(\Omega)$ coincides with the class of plurisubharmonic functions with weak singularities $\mathcal{E}_{-\chi}(\Omega)$ erase early introduced and investigated in [4].

(c) In the case $m = n$ and $\chi(t) \equiv 1$ for all $t < 0$, the classes $\mathcal{F}_{n,\chi}(\Omega)$ and $\mathcal{E}_{n,\chi}(\Omega)$ coincide with the classes $\mathcal{F}(\Omega)$ and $\mathcal{E}(\Omega)$ in [10].

We need the following lemma.

Lemma 2 *Let $\chi : \mathbb{R}^- \to \mathbb{R}^+$ be a decreasing function such that $\chi(2t) \leq a\chi(t)$ with some $a > 1$. Assume that $1 \leq m \leq n$ and $u, v \in \mathcal{E}_m^0(\Omega)$. Then the following hold:*

(a) *If $u \leq v$, then*

$$\int_\Omega \chi(v)(dd^c v)^m \wedge \beta^{n-m} \leq 2^m \max(a, 2) \int_\Omega \chi(u)(dd^c u)^m \wedge \beta^{n-m}.$$

(b) *For every $0 \leq \lambda \leq 1$, we have*

$$\int_\Omega \chi(\lambda u + (1-\lambda)v)(dd^c(\lambda u + (1-\lambda)v))^m \wedge \beta^{n-m} \\ \leq 2^m \max(a, 2) \left(\int_\Omega \chi(u)(dd^c u)^m \wedge \beta^{n-m} + \int_\Omega \chi(v)(dd^c v)^m \wedge \beta^{n-m} \right).$$

Proof (a) First, we assume that $\chi(0) = 0$. Set

$$\chi_j(t) := \chi(t) + \frac{(1-e^t)}{j}, \quad t < 0.$$

Then χ_j is a strictly decreasing function, $\chi < \chi_j < \chi + \frac{1}{j}$ and $\chi_j(2t) \le \max(a, 2)\cdot\chi_j(t)$ for every $t < 0$. Moreover, since $\{v < -t\} \subset \{u < -t\}$ for every $t > 0$ so by Lemma 1, we have

$$\begin{aligned}\int_\Omega \chi_j(v)(dd^c v)^m \wedge \beta^{n-m} &= -\int_0^{+\infty} \chi_j'(-t)(dd^c v)^m \wedge \beta^{n-m}(\{v < -t\})dt\\ &\le -\int_0^{+\infty} t^m \chi_j'(-t)C_m(\{v < -t\})dt\\ &\le -\int_0^{+\infty} t^m \chi_j'(-t)C_m(\{u < -t\})dt\\ &\le -2^m \int_0^{+\infty} \chi_j'(-t)(dd^c u)^m \wedge \beta^{n-m}(\{u < -t/2\})dt\\ &= \int_\Omega \chi_j(2u)(dd^c(2u))^m \wedge \beta^{n-m}\\ &\le 2^m \max(a, 2)\int_\Omega \chi_j(u)(dd^c u)^m \wedge \beta^{n-m}\\ &\le 2^m \max(a, 2)\left(\int_\Omega \left(\chi(u) + \frac{1}{j}\right)(dd^c u)^m \wedge \beta^{n-m}\right).\end{aligned}$$

Letting $j \to \infty$, we get

$$\int_\Omega \chi(v)(dd^c v)^m \wedge \beta^{n-m} \le 2^m \max(a, 2)\int_\Omega \chi(u)(dd^c u)^m \wedge \beta^{n-m}.$$

In the general case, we set $\Phi_j(t) = \min(\chi(t); -jt)$. Then Φ_j are decreasing functions such that $\Phi_j(0) = 0$ and $\Phi_j \nearrow \chi$ on $(-\infty, 0)$. By the first case, we have

$$\int_\Omega \Phi_j(v)(dd^c v)^m \wedge \beta^{n-m} \le 2^m \max(a, 2)\int_\Omega \Phi_j(u)(dd^c u)^m \wedge \beta^{n-m}.$$

Letting $j \to \infty$, we obtain

$$\int_\Omega \chi(v)(dd^c v)^m \wedge \beta^{n-m} \le 2^m \max(a, 2)\int_\Omega \chi(u)(dd^c u)^m \wedge \beta^{n-m}.$$

(b) As in the proof of (a), we can assume that $\chi(0) = 0$. Since $\{\lambda u + (1-\lambda)v < -t\} \subset \{u < -t\} \cup \{v < -t\}$, so we have

$$\begin{aligned}&\int_\Omega \chi(\lambda u + (1-\lambda)v)(dd^c(\lambda u + (1-\lambda)v))^m \wedge \beta^{n-m}\\ &\le \int_\Omega \chi_j(\lambda u + (1-\lambda)v)(dd^c(\lambda u + (1-\lambda)v))^m \wedge \beta^{n-m}\\ &\le -\int_0^{+\infty} t^m \chi_j'(-t)C_m(\{u < -t\})dt - \int_0^{+\infty} t^m \chi_j'(-t)C_m(\{v < -t\})dt\\ &\le 2^m \max(a, 2)\left(\int_\Omega \left(\chi(u) + \frac{1}{j}\right)(dd^c u)^m \wedge \beta^{n-m} + \int_\Omega \left(\chi(v) + \frac{1}{j}\right)(dd^c v)^m \wedge \beta^{n-m}\right).\end{aligned}$$

Letting $j \to \infty$, we get

$$\begin{aligned}&\int_\Omega \chi(\lambda u + (1-\lambda)v)(dd^c(\lambda u + (1-\lambda)v))^m \wedge \beta^{n-m}\\ &\quad \le 2^m \max(a, 2)\left(\int_\Omega \chi(u)(dd^c u)^m \wedge \beta^{n-m} + \int_\Omega \chi(v)(dd^c v)^m \wedge \beta^{n-m}\right).\end{aligned}$$

□

Proposition 2 *Let $\chi : \mathbb{R}^- \longrightarrow \mathbb{R}^+$ be a decreasing function such that $\chi(2t) \leq a\chi(t)$ with some $a > 1$. Then the following hold:*

(a) *If $u \in \mathcal{F}_{m,\chi}(\Omega)$ (resp. $\mathcal{E}_{m,\chi}(\Omega)$) and $v \in SH_m^-(\Omega)$ with $u \leq v$ then $v \in \mathcal{F}_{m,\chi}(\Omega)$ (resp. $\mathcal{E}_{m,\chi}(\Omega)$).*
(b) *If $u, v \in \mathcal{F}_{m,\chi}(\Omega)$ (resp. $\mathcal{E}_{m,\chi}(\Omega)$) and $\alpha, \gamma \geq 0$ then $\alpha u + \gamma v \in \mathcal{F}_{m,\chi}(\Omega)$ (resp. $\mathcal{E}_{m,\chi}(\Omega)$).*

Proof (a) It suffices to prove that the conclusion holds for the class $\mathcal{F}_{m,\chi}(\Omega)$. Assume that $u \in \mathcal{F}_{m,\chi}(\Omega)$ and $u \leq v$, $v \in SH_m^-(\Omega)$. From Definition 5, there exists a sequence $\{u_j\} \subset \mathcal{E}_m^0(\Omega)$, $u_j \searrow u$ on Ω with

$$\sup_j \int_\Omega \chi(u_j)(dd^c u_j)^m \wedge \beta^{n-m} < \infty.$$

Set $v_j = \max(u_j, v) \in \mathcal{E}_m^0(\Omega)$, $v_j \searrow v$ on Ω and $u_j \leq v_j$. By Lemma 2, we have

$$\sup_j \int_\Omega \chi(v_j)(dd^c v_j)^m \wedge \beta^{n-m} \leq 2^m \max(a, 2) \sup_j \int_\Omega \chi(u_j)(dd^c u_j)^m \wedge \beta^{n-m} < +\infty.$$

Hence, $v \in \mathcal{F}_{m,\chi}(\Omega)$.

(b) First, we prove that if $u \in \mathcal{F}_{m,\chi}(\Omega)$ then $\alpha u \in \mathcal{F}_{m,\chi}(\Omega)$. Indeed, let $k \in \mathbb{N}^*$ with $2^k > \alpha$ and let $\{u_j\} \subset \mathcal{E}_m^0(\Omega)$, $u_j \searrow u$ on Ω with

$$\sup_j \int_\Omega \chi(u_j)(dd^c u_j)^m \wedge \beta^{n-m} < \infty.$$

It is clear that $\{\alpha u_j\} \subset \mathcal{E}_m^0(\Omega)$, $\alpha u_j \searrow \alpha u$ on Ω. Moreover, since $\chi(\alpha u_j) \leq \chi(2^k u_j) \leq a^k \chi(u_j)$ so

$$\sup_j \int_\Omega \chi(\alpha u_j)(dd^c \alpha u_j)^m \wedge \beta^{n-m} \leq a^k \alpha^m \sup_j \int_\Omega \chi(u_j)(dd^c u_j)^m \wedge \beta^{n-m} < \infty.$$

Hence, $\alpha u \in \mathcal{F}_{m,\chi}(\Omega)$. By the above proof, we can assume that $\alpha + \gamma = 1$. Let $\{u_j\}$, $\{v_j\} \subset \mathcal{E}_m^0(\Omega)$, $u_j \searrow u$ on Ω, $v_j \searrow u$ on Ω, $\sup_j \int_\Omega \chi(u_j)(dd^c u_j)^m \wedge \beta^{n-m} < \infty$ and $\sup_j \int_\Omega \chi(v_j)(dd^c u_j)^m \wedge \beta^{n-m} < \infty$. By Lemma 2, we have

$$\begin{aligned}
&\sup_j \int_\Omega \chi(\alpha u_j + \gamma v_j)(dd^c(\alpha u_j + \gamma v_j))^m \wedge \beta^{n-m} \\
&\quad \leq 2^m \max(a, 2) \left(\sup_j \int_\Omega \chi(u_j)(dd^c u_j)^m \wedge \beta^{n-m} + \sup_j \int_\Omega \chi(v_j)(dd^c u_j)^m \wedge \beta^{n-m} \right) \\
&\quad < \infty.
\end{aligned}$$

Hence, the desired conclusion follows.

□

Proposition 3 *Let $\chi : \mathbb{R}^- \longrightarrow \mathbb{R}^+$ be a decreasing function such that $\chi(2t) \leq a\chi(t)$ for all $t < 0$ with some $a > 1$. Then for every $u \in \mathcal{F}_{m,\chi}(\Omega)$, there exists a sequence $\{u_j\} \subset \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\Omega)$ such that $u_j \searrow u$ and*

$$\sup_j \int_\Omega \chi(u_j)(dd^c u_j)^m \wedge \beta^{n-m} < \infty.$$

Proof Let $\Omega_j \Subset \Omega_{j+1} \Subset \Omega$ be such that $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$ and let $\{v_j\} \subset \mathcal{E}_m^0(\Omega)$ be such that $v_j \searrow u$ and

$$\sup_j \int_\Omega \chi(v_j)(dd^c u_j)^m \wedge \beta^{n-m} < \infty.$$

Theorem 3.1 in [12] implies that there exists a sequence $\{w_j\} \subset \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\Omega)$ such that $w_j \searrow u$. Set

$$u_j = \sup\left\{\varphi \in SH_m^-(\Omega) : \varphi \le \frac{j-1}{j} w_j \text{ on } \Omega_j\right\}.$$

It is easy to see that $u_j \searrow u$ on Ω. By Theorem 1.2.7 in [6] and Proposition 3.2 in [5], we get $u_j \in \mathcal{C}(\Omega)$. Moreover, since $w_j \le u_j$ so $u_j \in \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\Omega)$. Now, since $v_j \searrow u$ as $j \to \infty$ and $u \le w_k$ so there exists j_0 such that $v_{j_0} \le \frac{k-1}{k} w_k$ on Ω_k. Therefore, $v_{j_0} \le u_k$ on Ω. Lemma 2 implies that

$$\begin{aligned}\int_\Omega \chi(u_k)(dd^c u_k)^m \wedge \beta^{n-m} &\le 2^m \max(a, 2) \int_\Omega \chi(v_{j_0})(dd^c v_{j_0})^m \wedge \beta^{n-m} \\ &\le 2^m \max(a, 2) \sup_j \int_\Omega \chi(v_j)(dd^c v_j)^m \wedge \beta^{n-m}.\end{aligned}$$

Thus,

$$\sup_k \int_\Omega \chi(u_k) dd^c u_k)^m \wedge \beta^{n-m} \le 2^m \max(a, 2) \sup_j \int_\Omega \chi(v_j)(dd^c v_j)^m \wedge \beta^{n-m} < \infty.$$

□

The following proposition shows that the Hessian operator is well defined on the class $\mathcal{E}_{m,\chi}(\Omega)$.

Proposition 4 *Let $\chi : \mathbb{R}^- \longrightarrow \mathbb{R}^+$ be a decreasing function such that $\chi \not\equiv 0$ and $\chi(2t) \le a\chi(t)$ for all $t < 0$ with some $a > 1$. Then $\mathcal{E}_{m,\chi}(\Omega) \subset \mathcal{E}_m(\Omega)$, and hence, the Hessian $H_m(u) = (dd^c u)^m \wedge \beta^{n-m}$ is well defined as a positive Radon measure on Ω.*

Proof Without loss of generality, we can assume that $\chi(t) > 0$ for every $t < 0$. Let $u \in \mathcal{E}_{m,\chi}(\Omega)$ and $z_0 \in \Omega$. Take a neighborhood $\omega \Subset \Omega$ of z_0 and a sequence $\{u_j\} \subset \mathcal{E}_m^0(\Omega)$ such that $\sup_{\overline{\omega}} u_1 < 0$, $u_j \searrow u$ on ω and

$$\sup_j \int_\Omega \chi(u_j) H_m(u_j) < \infty.$$

For each $j \ge 1$, set

$$\widetilde{u}_j = \sup\{u \in SH_m^-(\Omega) : u|_\omega \le u_j|_\omega\}.$$

Then $u_j \le \widetilde{u}_j$ on Ω and $u_j = \widetilde{u}_j$ on ω and, by using arguments as in [7], we arrive at $\widetilde{u}_j \in MSH_m(\Omega \setminus \overline{\omega})$. This yields that $\widetilde{u}_j \in \mathcal{E}_m^0(\Omega)$ and $H_m(\widetilde{u}_j) = 0$ on $\Omega \setminus \overline{\omega}$. Moreover, it is easy to see that $\widetilde{u}_j \searrow \widetilde{u}$ on Ω. On the other hand, as in the proof of Lemma 2, we have

$$\sup_j \int_\Omega \chi(\widetilde{u}_j) H_m(\widetilde{u}_j) < \infty.$$

Moreover, we may assume that $\inf_{\overline{\omega}} \chi(\widetilde{u}_1) = c_1 > 0$. Then

$$c_1 \sup_j \int_\Omega H_m(\widetilde{u}_j) = c_1 \sup_j \int_{\overline{\omega}} H_m(\widetilde{u}_j) \leq \sup_j \int_{\overline{\omega}} \chi(\widetilde{u}_1) H_m(\widetilde{u}_j) \leq \sup_j \int_\Omega \chi(\widetilde{u}_j) H_m(\widetilde{u}_j) < \infty.$$

Hence,

$$\sup_j \int_\Omega H_m(\widetilde{u}_j) < \infty$$

and it follows that $\widetilde{u} \in \mathcal{F}_m(\Omega)$. It is easy to see that $\widetilde{u} = u$ on ω, and this yields that $u \in \mathcal{E}_m(\Omega)$. Theorem 3.14 in [12] implies that $H_m(u)$ is a positive Radon measure on Ω. The proof is complete. □

Now we prove our main result about the local property of the class $\mathcal{E}_{m,\chi}(\Omega)$.

4 The Local Property of the Class $\mathcal{E}_{m,\chi}(\Omega)$

First, we give the following definition which is similar as in [15] for plurisubharmonic functions.

Definition 6 A class $\mathcal{J}(\Omega) \subset SH_m^-(\Omega)$ is said to be a local class if $\varphi \in \mathcal{J}(\Omega)$ then $\varphi \in \mathcal{J}(D)$ for all hyperconvex domains $D \Subset \Omega$ and if $\varphi \in SH_m^-(\Omega), \varphi|_{\Omega_j} \in \mathcal{J}(\Omega_j)\ \forall j \in I$ with $\Omega = \bigcup_{j \in I} \Omega_j$, then $\varphi \in \mathcal{J}(\Omega)$.

In [15], the authors introduced the class $\mathcal{E}_{\chi,loc}(\Omega)$ and established the local property for this class. This section is devoted to study the local property of the class $\mathcal{E}_{m,\chi}(\Omega)$.

In the sequel of the paper, we will use the following notation. We will write "$A \lesssim B$" if there exists a constant C such that $A \leq CB$.

Proposition 5 *Set*

$$\mathcal{K} = \{\chi : \mathbb{R}^- \longrightarrow \mathbb{R}^+, \chi \textit{ is decreasing and } -t^2\chi''(t) \lesssim t\chi'(t) \lesssim \chi(t)\ \forall t < 0\}.$$

Then the class $\mathcal{K}$ has the following properties.

(a) *If $\chi_1, \chi_2 \in \mathcal{K}$ and $a_1, a_2 \geq 0$ then $a_1\chi_1 + a_2\chi_2 \in \mathcal{K}$.*
(b) *If $\chi_1, \chi_2 \in \mathcal{K}$ then $\chi_1 \cdot \chi_2 \in \mathcal{K}$.*
(c) *If $\chi \in \mathcal{K}$ then $\chi^p \in \mathcal{K}$ for all $p > 0$.*
(d) *If $\chi \in \mathcal{K}$, then $(-t)\chi(t) \in \mathcal{K}$. More generally $|t^k|\chi(t) \in \mathcal{K}$ for all $k = 0, 1, 2, \ldots$.*

Proof The proof is standard hence we omit it. □

Remark 2 If $\chi \in \mathcal{K}$, then $\chi(2t) \leq a\chi(t)\ \forall t < 0$ with some $a > 1$. Indeed, by hypothesis $t\chi'(t) \leq C\chi(t), C =$ constant > 0. We set $s(t) = \frac{\chi(t)}{(-t)^C}$. Then $s'(t) \geq 0\ \forall t < 0$, hence $s(t)$ is an increasing function. This implies that $s(2t) \leq s(t)$, and we have $\chi(2t) \leq 2^C\chi(t)$.

The following result is necessary for the proof of the local property of the class $\mathcal{E}_{m,\chi}(\Omega)$.

Lemma 3 *Let $u, v \in SH_m^-(\Omega) \cap L^\infty(\Omega)$ with $u \le v$ on Ω, $\chi \in \mathcal{K}$ and $T = dd^c\varphi_1 \wedge \cdots \wedge dd^c\varphi_{m-1} \wedge \beta^{n-m}$ with $\varphi_j \in SH_m^-(\Omega) \cap L^\infty(\Omega)$, $j = 1 \ldots, m-1$. Then for every $p \ge 0$, we have*

$$\int_{\Omega'} \chi(u) dd^c v \wedge T \le c \int_{\Omega''} \chi(u)(dd^c u + |u|\beta) \wedge T,$$

where $\Omega' \Subset \Omega'' \Subset \Omega$ and c is a constant only depending on Ω', Ω'', Ω and χ.

Proof Choose $\Phi \in \mathcal{C}_0^\infty(\Omega)$, $0 \le \Phi \le 1$ and $\Phi|_{\Omega'} = 1$, supp $\Phi \Subset \Omega''' \Subset \Omega''$. Then, by integration by parts

$$\int_{\Omega'} \chi(u) dd^c v \wedge T = \int_{\Omega'} \Phi\chi(u) dd^c v \wedge T \le \int_{\Omega} \Phi\chi(u) dd^c v \wedge T = \int_{\Omega} v dd^c(\Phi\chi(u)) \wedge T.$$

On the other hand,

$$\begin{aligned} dd^c(\Phi\chi(u)) &= d(d^c(\Phi\chi(u))) \\ &= \chi(u) dd^c\Phi + \Phi(\chi'(u) dd^c u + \chi''(u) du \wedge d^c u) + \chi'(u)(d\Phi \wedge d^c u + du \wedge d^c\Phi). \end{aligned}$$

Since $\forall t$, $d(u + t\Phi) \wedge d^c(u + t\Phi) \wedge T \ge 0$, we have

$$\pm u(du \wedge d^c\Phi + d\Phi \wedge d^c u) \wedge T \le (du \wedge d^c u + u^2 d\Phi \wedge d^c\Phi) \wedge T$$

and

$$\chi'(u)(d\Phi \wedge d^c u + du \wedge \Phi) \wedge T \ge -\chi'(u)\left(u d\Phi \wedge d^c\Phi + \frac{1}{u} du \wedge d^c u\right) \wedge T.$$

Now, we can choose $A > 0$ sufficiently large such that $dd^c\Phi \ge -A dd^c\|z\|^2$, $d\Phi \wedge d^c\Phi \le A dd^c\|z\|^2$. Thus, we have the following estimates

$$\begin{aligned} dd^c(\Phi\chi(u)) \wedge T \ \ge\ & -A\chi(u) dd^c\|z\|^2 \wedge T + \Phi\chi'(u) dd^c u \wedge T + \Phi\chi''(u) du \wedge d^c u \wedge T \\ & -\chi'(u)(u d\Phi \wedge d^c\Phi + \frac{1}{u} du \wedge d^c u) \wedge T. \end{aligned} \quad (1)$$

In the case $\chi''(u) \le 0$, we have the following

$$\begin{aligned} v dd^c(\Phi\chi(u)) \wedge T \ \le\ & -Au\chi(u) dd^c\|z\|^2 \wedge T + u\chi'(u) dd^c u \wedge T \\ & + u \min\{\chi''(u), 0\} du \wedge d^c u \wedge T - u^2\chi'(u) d\Phi \wedge d^c\Phi \wedge T \\ & -\chi'(u) du \wedge d^c u \wedge T. \end{aligned}$$

In the case $\chi''(u) \ge 0$, from (1), we note that $\Phi v\chi''(u) du \wedge d^c u \wedge T \le 0$, and it is easy to obtain the above estimates. Now, we have the following estimates

$$\begin{aligned} \int_{\Omega'} \chi(u) dd^c v \wedge T \ \le\ & A \int_{\Omega'''} -u\chi(u) dd^c\|z\|^2 \wedge T + \int_{\Omega'''} u\chi'(u) dd^c u \wedge T \\ & + \int_{\Omega'''} u \min\{\chi''(u), 0\} du \wedge d^c u \wedge T + \int_{\Omega'''} -u^2\chi'(u) d\Phi \wedge d^c\Phi \wedge T \\ & + \int_{\Omega'''} -\chi'(u) du \wedge d^c u \wedge T. \end{aligned}$$

On the other hand, by hypothesis about the class $\mathcal{K}$, we have $u\chi'(u) \leq c_1\chi(u)$ and $(-u^2)\chi'(u) \leq c_1(-u)\chi(u)$, $u\chi''(u) \leq c_2(-\chi'(u))$. Therefore,

$$\begin{aligned}\int_{\Omega'} \chi(u)dd^c v \wedge T \leq\ & A\int_{\Omega'''} -u\chi(u)dd^c\|z\|^2 \wedge T + c_1\int_{\Omega'''} \chi(u)dd^c u \wedge T \\ & -(c_2+1)\int_{\Omega'''} \chi'(u)du \wedge d^c u \wedge T + Ac_1\int_{\Omega'''} \chi(u)d\Phi \wedge dd^c\|z\|^2 \wedge T \\ =\ & A(c_1+1)\int_{\Omega'''} |u|\chi(u)dd^c\|z\|^2 \wedge T + c_1\int_{\Omega'''} \chi(u)dd^c u \wedge T \\ & -(c_2+1)\int_{\Omega'''} \chi'(u)du \wedge d^c u \wedge T.\end{aligned}$$

Set $\chi_1(t) = -\int_0^t \chi(x)dx$ then

$$\chi_1'(t) = -\chi(t); \quad \chi_1''(t) = -\chi'(t); \quad \chi(t)|t| \geq \chi_1(t) \geq \chi\left(\frac{t}{2}\right)\frac{|t|}{2}.$$

Now, we choose $\psi \in \mathcal{C}_0^\infty$, $\psi|_{\Omega'''} = 1$, $\operatorname{supp}\psi \Subset \Omega''$, then we have

$$\begin{aligned}-\int_{\Omega'''} \chi'(u)du \wedge d^c u \wedge T =\ & -\int_{\Omega'''} d\chi(u) \wedge du \wedge d^c u \wedge T \leq \int_{\Omega} \psi d\chi(u) \wedge d^c u \wedge T \\ =\ & \int_{\Omega} \chi(u)d\psi \wedge d^c u \wedge T + \int_{\Omega} \psi\chi(u)dd^c u \wedge T \\ =\ & \int_{\Omega} \chi(u)d\psi \wedge d^c u \wedge T + \int_{\Omega''} \psi\chi(u)dd^c u \wedge T \\ =\ & -\int_{\Omega} d\psi d^c \chi_1(u) \wedge T + \int_{\Omega''} \psi\chi(u)dd^c u \wedge T \\ =\ & \int_{\Omega} \chi_1(u)dd^c\psi \wedge T + \int_{\Omega''} \psi\chi(u)dd^c u \wedge T \\ \leq\ & B\int_{\Omega''} \chi(u)|u|dd^c\|z\|^2 \wedge T + \int_{\Omega''} \psi\chi(u)dd^c u \wedge T\end{aligned}$$

with $B > 0$ sufficiently large.

Finally, we have

$$\begin{aligned}\int_{\Omega'} \chi(u)dd^c v \wedge T \leq\ & A(c_1+1)\int_{\Omega'''} |u|\chi(u)dd^c\|z\|^2 \wedge T + c_1\int_{\Omega'''} \chi(u)dd^c u \wedge T \\ & +(c_2+1)B\int_{\Omega'''} \chi(u)|u|dd^c\|z\|^2 \wedge T + (c_2+1)\int_{\Omega'''} \chi(u)dd^c u \wedge T \\ \leq\ & c\left[\int_{\Omega'''} \chi(u)dd^c u \wedge T + \int_{\Omega'''} \chi(u)|u|dd^c\|z\|^2 \wedge T\right].\end{aligned}$$

□

The next lemma is a crucial tool for the proof of the local property of the class $\mathcal{E}_{m,\chi}(\Omega)$.

Lemma 4 *Let Ω be a hyperconvex domain in $\mathbb{C}^n$ and $1 \leq m \leq n$. Assume that $u \in \mathcal{E}_m^0(\Omega)$ and $\chi \in \mathcal{K}$ such that $\chi''(t) \geq 0\ \forall t < 0$. Then for $\Omega' \Subset \Omega$, there exists a constant $C = C(\Omega')$ such that the following holds:*

$$\int_{\Omega'} \chi(u)|u|^p(dd^c u)^{m-p} \wedge \beta^{n-m+p} \leq C\int_{\Omega} \chi(u)(dd^c u)^m \wedge \beta^{n-m} < +\infty. \qquad (2)$$

Furthermore, if $u \in \mathcal{F}_{m,\chi}(\Omega)$ *then*

$$\int_{\Omega'} \chi(u)|u|^p (dd^c u)^{m-p} \wedge \beta^{n-m+p} < +\infty$$

for all $p = 1, \ldots, m$.

Proof Set $\chi_0(t) = \chi(t)$ and for each $k \geq 1$, let $\chi_k(t) = -\int_0^t \chi_{k-1}(x)dx$. From the hypothesis $\chi \in \mathcal{K}$, then $\chi(2t) \leq a\chi(t)$ and it is easy to check that $\chi_k \in \mathcal{K}$ and $\chi(t)(-t)^k \lesssim \chi_k(t) \lesssim \chi(t)(-t)^k$.

Now, choose $R > 0$ large enough such that $\|z\|^2 \leq R^2$ on Ω. Let $\varphi \in \mathcal{E}_m^0(\Omega)$ and $A > 0$ such that $\|z\|^2 - R^2 \geq A\varphi$ on Ω'. Set $h = \max(\|z\|^2 - R^2; A\varphi)$ then $h \in \mathcal{E}_m^0(\Omega)$ and $dd^c h = dd^c\|z\|^2 = \beta$ on Ω'. First, we claim that (2) holds for $u \in \mathcal{E}_m^0(\Omega)$. Indeed, we have

$$\begin{aligned}\int_{\Omega'} \chi(u)|u|^p (dd^c u)^{m-p} \wedge (dd^c h)^p \wedge \beta^{n-m} &\lesssim \int_{\Omega} \chi(u)|u|^p (dd^c u)^{m-p} \wedge (dd^c h)^p \wedge \beta^{n-m} \\ &\approx \int_{\Omega} \chi_p(u)(dd^c u)^{m-p} \wedge (dd^c h)^p \wedge \beta^{n-m}.\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}\int_{\Omega} \chi_p(u)(dd^c u)^{m-p} \wedge (dd^c h)^p \wedge \beta^{n-m} &= \int_{\Omega} h(dd^c u)^{m-p} dd^c \chi_p(u) \wedge (dd^c h)^{p-1} \wedge \beta^{n-m} \\ &= \int_{\Omega} h(dd^c u)^{m-p} \left[\chi_p''(u) du \wedge d^c u + \chi_p'(u) dd^c u\right] \\ &\quad \wedge (dd^c h)^{p-1} \wedge \beta^{n-m} \\ &\leq \int_{\Omega} h\chi_p'(u)(dd^c u)^{m-p+1} \wedge (dd^c h)^{p-1} \wedge \beta^{n-m} \\ &\leq \|h\|_{L^\infty(\Omega)} \int_{\Omega} \chi_{p-1}(dd^c u)^{m-p+1} \wedge (dd^c h)^{p-1} \wedge \beta^{n-m} \\ &\leq \cdots \\ &\leq \|h\|^p_{L^\infty(\Omega)} \int_{\Omega} \chi(u)(dd^c u)^m \wedge \beta^{n-m} < +\infty.\end{aligned}$$

Hence, if we set $C = C(\Omega') = p!\|h\|^p_{L^\infty(\Omega)}$ then

$$\begin{aligned}+\infty > C\int_{\Omega} \chi(u)(dd^c u)^m \wedge \beta^{n-m} &\geq \int_{\Omega} \chi(u)|u|^p (dd^c u)^{m-p} \wedge (dd^c h)^p \wedge \beta^{n-m} \\ &\geq \int_{\Omega'} \chi(u)|u|^p (dd^c u)^{m-p} \wedge (dd^c h)^p \wedge \beta^{n-m} \\ &= \int_{\Omega'} \chi(u)|u|^p (dd^c u)^{m-p} \wedge (dd^c\|z\|^2)^p \wedge \beta^{n-m}.\end{aligned}$$

Finally, we prove (2) holds for $u \in \mathcal{F}_{m,\chi}(\Omega)$. Indeed, we take $u_j \in \mathcal{E}_m^0(\Omega)$, $u_j \searrow u$ on Ω such that

$$\sup_{j \geq 1} \int_{\Omega} \chi(u_j)(dd^c u_j)^m \wedge \beta^{n-m} < +\infty.$$

By dominated convergence theorem and $(dd^c u_j)^{m-p} \wedge (dd^c\|z\|^2)^{n-m+p}$ is weakly convergent to $(dd^c u)^{m-p} \wedge (dd^c\|z\|^2)^{n-m+p}$ in the sense of currents

$$\begin{aligned}
&\int_{\Omega'} \chi(u)|u|^p \left(dd^c u\right)^{m-p} \wedge \left(dd^c\|z\|^2\right)^{n-m+p} \\
&\quad \le \liminf_j \int_{\Omega'} \chi(u_j)|u_j|^p \left(dd^c u_j\right)^{m-p} \wedge \left(dd^c\|z\|^2\right)^{n-m+p} \\
&\quad \le \liminf_j \int_{\Omega} \chi(u_j)|u_j|^p \left(dd^c u_j\right)^{m-p} \wedge \left(dd^c h\right)^p \wedge \left(dd^c\|z\|^2\right)^{n-m} \\
&\quad \le C \sup_j \int_{\Omega} \chi(u_j) \left(dd^c u_j\right)^m \wedge \left(dd^c\|z\|^2\right)^{n-m} < +\infty.
\end{aligned}$$

□

We also need the following result on subextension for the class $\mathcal{F}_{m,\chi}(\Omega)$.

Lemma 5 *Assume that $\Omega \Subset \widetilde{\Omega}$ and $u \in \mathcal{F}_{m,\chi}(\Omega)$. Then there exists $\widetilde{u} \in \mathcal{F}_{m,\chi}(\widetilde{\Omega})$ such that $\widetilde{u} \le u$ on Ω.*

Proof We split the proof into three steps.

Step 1. We prove that if $v \in \mathcal{C}(\widetilde{\Omega})$, $v \le 0$, supp $v \Subset \widetilde{\Omega}$ then $\widetilde{v} := \sup\{w \in SH_m^-(\widetilde{\Omega}) : w \le v \text{ on } \widetilde{\Omega}\} \in \mathcal{E}_m^0(\widetilde{\Omega}) \cap \mathcal{C}(\widetilde{\Omega})$ and $(dd^c\widetilde{v})^m \wedge \beta^{n-m} = 0$ on $\{\widetilde{v} < v\}$. Indeed, let $\varphi \in \mathcal{E}_m^0(\widetilde{\Omega}) \cap \mathcal{C}(\widetilde{\Omega})$ be such that $\varphi \le \inf_{\widetilde{\Omega}} v$ on supp v. Since $\varphi \le \widetilde{v}$ so $\widetilde{v} \in \mathcal{E}_m^0(\widetilde{\Omega})$. Moreover, by Proposition 3.2 in [5], we have $\widetilde{v} \in \mathcal{C}(\widetilde{\Omega})$. Let $w \in SH_m(\{\widetilde{v} < v\})$ be such that $w \le \widetilde{v}$ outside a compact subset K of $\{\widetilde{v} < v\}$. Set

$$w_1 = \begin{cases} \max(w, \widetilde{v}) & \text{on } \{\widetilde{v} < v\}, \\ \widetilde{v} & \text{on } \widetilde{\Omega}\backslash(\{\widetilde{v} < v\}). \end{cases}$$

Since $\widetilde{v}$ and v are continuous so $\varepsilon = -\sup_K(\widetilde{v} - v) > 0$. Choose $\delta \in (0, 1)$ such that $-\delta \inf_{\widetilde{\Omega}} \widetilde{v} < \varepsilon$. We have $(1-\delta)\widetilde{v} \le \widetilde{v} + \varepsilon \le v$ on K. Hence, $(1-\delta)\widetilde{v} + \delta w_1 \le v$ on $\widetilde{\Omega}$ and we get $(1-\delta)\widetilde{v} + \delta w_1 = \widetilde{v}$. Thus, $w \le \widetilde{v}$ on $\{\widetilde{v} < v\}$. Hence, $\widetilde{v}$ is m-maximal in $\{\widetilde{v} < v\}$. By [5], we get $(dd^c\widetilde{v})^m \wedge \beta^{n-m} = 0$ on $\{\widetilde{v} < v\}$.

Step 2. Next, we prove that if $u \in \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\Omega)$ then there exists $\widetilde{u} \in \mathcal{E}_m^0(\widetilde{\Omega})$, $(dd^c\widetilde{u})^m \wedge \beta^{n-m} = 0$ on $(\widetilde{\Omega}\backslash\Omega) \cup (\{\widetilde{u} < u\} \cap \Omega)$ and $(dd^c\widetilde{u})^m \wedge \beta^{n-m} \le (dd^c u)^m \wedge \beta^{n-m}$ on $\{\widetilde{u} = u\} \cap \Omega$. Indeed, set

$$v = \begin{cases} u & \text{on } \Omega, \\ 0 & \text{on } \widetilde{\Omega}\backslash\Omega. \end{cases}$$

It is easy to see that $v \in \mathcal{C}(\widetilde{\Omega})$ and supp $v \subset \Omega \Subset \widetilde{\Omega}$. Hence, we have $\widetilde{u} = \widetilde{v} \in \mathcal{E}_m^0(\widetilde{\Omega}) \cap \mathcal{C}(\widetilde{\Omega})$ and $(dd^c\widetilde{u})^m \wedge \beta^{n-m} = 0$ on $\{\widetilde{v} < v\} \cap \widetilde{\Omega} = (\widetilde{\Omega}\backslash\Omega) \cup (\{\widetilde{u} < u\} \cap \Omega)$. Let K be a compact set in $\{\widetilde{u} = u\} \cap \Omega$. Then for $\varepsilon > 0$, we have $K \Subset \{\widetilde{u} + \varepsilon > u\} \cap \Omega$ so we have

$$\begin{aligned}
\int_K (dd^c\widetilde{u})^m \wedge \beta^{n-m} &= \int_K 1_{\{\widetilde{u}+\varepsilon > u\}} (dd^c\widetilde{u})^m \wedge \beta^{n-m} \\
&= \int_K 1_{\{\widetilde{u}+\varepsilon > u\}} (dd^c \max(\widetilde{u} + \varepsilon, u))^m \wedge \beta^{n-m} \\
&\le \int_K (dd^c \max(\widetilde{u} + \varepsilon, u))^m \wedge \beta^{n-m},
\end{aligned}$$

where the equality in the second line follows by using the same arguments as in [2] (also see the proof of Theorem 3.23 in [12]). However, $\max(\widetilde{u}+\varepsilon, u) \searrow u$ on Ω as $\varepsilon \to 0$ so by [21] it follows that $(dd^c \max(\widetilde{u}+\varepsilon, u))^m \wedge \beta^{n-m}$ is weakly convergent to $(dd^c u)^m \wedge \beta^{n-m}$ as $\varepsilon \to 0$. On the other hand, 1_K is upper semicontinuous on Ω so we can approximate 1_K with a decreasing sequence of continuous functions φ_j. Hence, we infer that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \int_\Omega 1_K (dd^c \max(\widetilde{u}+\varepsilon, u))^m \wedge \beta^{n-m} \\
&\quad = \limsup_{\varepsilon\to 0} \left[\lim_j \int_\Omega \varphi_j (dd^c \max(\widetilde{u}+\varepsilon, u))^m \wedge \beta^{n-m}\right] \\
&\quad \le \limsup_{\varepsilon\to 0} \left(\int_\Omega \varphi_j (dd^c \max(\widetilde{u}+\varepsilon, u))^m \wedge \beta^{n-m}\right) \\
&\quad \le \int_\Omega \varphi_j (dd^c u)^m \wedge \beta^{n-m} \searrow \int_K (dd^c u)^m \wedge \beta^{n-m}.
\end{aligned}$$

as $j \to \infty$. This yields that $(dd^c\widetilde{u})^m \wedge \beta^{n-m} \le (dd^c u)^m \wedge \beta^{n-m}$ on $\{\widetilde{u} = u\} \cap \Omega$.

Step 3. Now, let $u_j \in \mathcal{E}^0_m(\Omega) \cap \mathcal{C}(\Omega)$ be such that $u_j \searrow u$ and

$$\sup_j \int_\Omega \chi(u_j) \left(dd^c u_j\right)^m \wedge \beta^{n-m} < \infty.$$

By Step 2, we have

$$\begin{aligned}
\int_{\widetilde{\Omega}} \chi(\widetilde{u}_j) \left(dd^c\widetilde{u}_j\right)^m \wedge \beta^{n-m} &= \int_{\{\widetilde{u}_j=u_j\}\cap\Omega} \chi(\widetilde{u}_j)(dd^c\widetilde{u}_j)^m \wedge \beta^{n-m} \\
&\le \int_{\{\widetilde{u}_j=u_j\}\cap\Omega} \chi(u_j)(dd^c u_j)^m \wedge \beta^{n-m} \\
&\le \int_\Omega \chi(u_j)(dd^c u_j)^m \wedge \beta^{n-m}.
\end{aligned}$$

Hence,

$$\sup_j \int\limits_{\widetilde{\Omega}} \chi(\widetilde{u}_j)(dd^c\widetilde{u}_j)^m \wedge \beta^{n-m} \le \sup_j \int\limits_\Omega \chi(u_j)(dd^c u_j)^m \wedge \beta^{n-m} < \infty.$$

Thus, $\widetilde{u} := \lim_{j\to\infty} \widetilde{u}_j \in \mathcal{F}_{m,\chi}(\widetilde{\Omega})$ and $\widetilde{u} \le u$ on Ω.

□

The following result deals with the local property of the class $\mathcal{E}_{m,\chi}(\Omega)$. Namely, we have the following.

Theorem 1 *Let Ω be a hyperconvex domain in $\mathbb{C}^n$ and m be an integer with $1 \le m \le n$. Assume that $u \in SH^-_m(\Omega)$ and $\chi \in \mathcal{K}$ such that $\chi''(t) \ge 0\ \forall t < 0$. Then the following statements are equivalent.*

a) $u \in \mathcal{E}_{m,\chi}(\Omega)$.

b) *For all $K \Subset \Omega$, there exists a sequence $\{u_j\} \subset \mathcal{E}^0_m(\Omega) \cap \mathcal{C}(\Omega)$, $u_j \searrow u$ on K such that*

$$\sup_j \int_K \chi(u_j)|u_j|^p (dd^c u_j)^{m-p} \wedge \beta^{n-m+p} < \infty$$

for every $p = 0, \ldots, m$.

c) *For every $W \Subset \Omega$ such that W is a hyperconvex domain, we have $u|_W \in \mathcal{E}_{m,\chi}(W)$.*
d) *For every $z \in \Omega$, there exists a hyperconvex domain $V_z \Subset \Omega$ such that $z \in V_z$ and $u|_{V_z} \in \mathcal{E}_{m,\chi}(V_z)$.*

Proof Let χ_k be as in Lemma 4.

"a) $\Longrightarrow$ b)" Let $K \Subset \Omega$ be given. Since $u \in \mathcal{E}_{m,\chi}(\Omega)$, then there exists $v \in \mathcal{F}_{m,\chi}(\Omega)$ with $v = u$ on K. By the definition of the class $\mathcal{F}_{m,\chi}(\Omega)$, there exists a sequence $\{u_j\} \subset \mathcal{E}^0_m(\Omega) \cap \mathcal{C}(\Omega)$, $u_j \searrow v$ on Ω with

$$\sup_j \int_\Omega \chi(u_j)(dd^c u_j)^m \wedge \beta^{n-m} < \infty. \tag{3}$$

Then $u_j \searrow u$ on K. We have to prove

$$\sup_j \int_K \chi(u_j)|u_j|^p (dd^c u_j)^{m-p} \wedge \beta^{n-m+p} < \infty$$

for $p = 0, 1, \ldots, m$. It is obvious that the conclusion holds for $p = 0$. Assume that $1 \leq p \leq m$. Then, by Lemma 4, we get that

$$\sup_j \int_K \chi(u_j)|u_j|^p (dd^c u_j)^{m-p} \wedge \beta^{n-m+p} \leq C \sup_j \int_\Omega \chi(u_j)(dd^c u_j)^m \wedge \beta^{n-m} < \infty$$

and the desired conclusion follows.

"b) $\Longrightarrow$ c)" Let $W \Subset \Omega$ be a hyperconvex domain. Take $U \Subset W \Subset \Omega$ and a sequence $\mathcal{E}^0_m(\Omega) \ni u_j \searrow u$ on W such that

$$\sup_j \int_W \chi(u_j)|u_j|^p (dd^c u_j)^{m-p} \wedge \beta^{n-m+p} < \infty$$

for $p = 0, 1, \ldots, m$. Set $\widetilde{u}_j = \sup\{\varphi \in SH^-_m(W) : \varphi \leq u_j \text{ on } U\} \in \mathcal{E}^0_m(W)$. Next, choose $U \Subset \Omega_1 \Subset \ldots \Subset \Omega_m \Subset W$. Since $u_j \leq \widetilde{u}_j$ on W and $(dd^c\widetilde{u}_j)^m \wedge \beta^{n-m} = 0$ on $W \backslash \overline{U}$ so by applying Lemma 3 many times, we arrive at

$$\begin{aligned}
&\int_W \chi(\widetilde{u}_j)\left(dd^c\widetilde{u}_j\right)^m \wedge \beta^{n-m} = \int_{\overline{U}} \chi(\widetilde{u}_j)\left(dd^c\widetilde{u}_j\right)^m \wedge \beta^{n-m} \\
&\quad\lesssim \int_{\Omega_1} \chi(u_j)\left(dd^c u_j + |u_j|\beta\right) \wedge \left(dd^c\widetilde{u}_j\right)^{m-1} \wedge \beta^{n-m} \\
&\quad\lesssim \int_{\Omega_1} \chi(u_j) dd^c\widetilde{u}_j \wedge \left(dd^c\widetilde{u}_j\right)^{m-2} \wedge dd^c u_j \wedge \beta^{n-m} \\
&\qquad + \int_{\Omega_1} \chi_1(u_j)|u_j| dd^c\widetilde{u}_j \wedge \left(dd^c\widetilde{u}_j\right)^{m-2} \wedge \beta^{n-m+1} \\
&\quad\lesssim \int_{\Omega_2} \chi(u_j)\left(dd^c u_j + |u_j|\beta\right) \wedge \left(dd^c\widetilde{u}_j\right)^{m-2} \wedge dd^c u_j \wedge \beta^{n-m} \\
&\qquad + \int_{\Omega_2} \chi_1(u_j)|u_j|\left(dd^c u_j + |u_j|\beta\right) \wedge \left(dd^c\widetilde{u}_j\right)^{m-2} \wedge \beta^{n-m+1} \\
&\quad\lesssim \int_{\Omega_2} \chi(u_j)\left[|u_j|^2\beta^2 + |u_j|\beta \wedge dd^c u_j + \left(dd^c u_j\right)^2\right] \wedge \left(dd^c\widetilde{u}_j\right)^{m-2} \wedge \beta^{n-m} \\
&\quad\lesssim \cdots \\
&\quad\lesssim \int_{\Omega_m} \chi(u_j)\left[|u_j|^m\beta^m + |u_j|^{m-1} dd^c u_j \wedge \beta^{m-1} + \cdots + \left(dd^c u_j\right)^m\right] \wedge \beta^{n-m}.
\end{aligned}$$

Hence,

$$\sup_j \int_W \chi(u_j)\left(dd^c\widetilde{u}_j\right)^m \wedge \beta^{n-m}$$
$$\lesssim \sup_j \chi(u_j)\int_{\Omega_m}\left[|u_j|^m\beta^m + |u_j|^{m-1}dd^c u_j\wedge\beta^{m-1} + \cdots + \left(dd^c u_j\right)^m\right]\wedge\beta^{n-m}$$
$$\lesssim \sup_j \int_W \chi(u_j)\left[|u_j|^m\beta^m + |u_j|^{m-1}dd^c u_j\wedge\beta^{m-1} + \cdots + \left(dd^c u_j\right)^m\right]\wedge\beta^{n-m} < \infty.$$

Thus, $u_{U,W} := \lim \widetilde{u}_j \in \mathcal{F}_{m,\chi}(W)$. Since $U \Subset W$ is arbitrary and $u_{U,W} = u$ on U so $u \in \mathcal{E}_m(W)$.

"c) $\Longrightarrow$ d)" It is obvious.

"d) $\Longrightarrow$ a)" Assume that $\Omega' \Subset \Omega$. Choose $z_j \in \Omega$, $j = 1, 2, \ldots, s$ such that $\Omega' \Subset \bigcup_{j=1}^s V_{z_j}$, where V_{z_j} are hyperconvex domains. Let $W_{z_j} \Subset V_{z_j}$ be such that $\Omega' \Subset \bigcup_{j=1}^s W_{z_j}$. Since $u|_{V_{z_j}} \in \mathcal{E}_{m,\chi}(V_{z_j})$ so there exists $v_j \in \mathcal{F}_{m,\chi}(V_{z_j})$ such that $v_j = u$ on W_{z_j}. By Lemma 5, there exists $\widetilde{v}_j \in \mathcal{F}_{m,\chi}(\Omega)$ such that $\widetilde{v}_j \le v_j$ on V_{z_j}. Then by Proposition 2, we have $\widetilde{v} := \widetilde{v}_1 + \cdots + \widetilde{v}_s \in \mathcal{F}_{m,\chi}(\Omega)$ and, hence, $\max(\widetilde{v}, u) \in \mathcal{F}_{m,\chi}(\Omega)$. However, $\max(\widetilde{v}, u) = u$ on Ω', then $u \in \mathcal{E}_{m,\chi}(\Omega)$. The proof is complete. □

From the above theorem, we get the following property of the class $\mathcal{E}_{m,\chi}(\Omega)$.

Corollary 1 *Assume that Ω is a bounded hyperconvex domain, and $\chi \in \mathcal{K}$ satisfies all hypotheses of Theorem 1. Then $\mathcal{E}_{m,\chi}(\Omega) \subset \mathcal{E}_{m-1,\chi}(\Omega)$.*

Proof Assume that $u \in \mathcal{E}_{m,\chi}(\Omega)$. Let $K \Subset \Omega$. Take a domain Ω' with $\Omega' \Subset \Omega$. By Theorem 1, there exists a sequence $\{u_j\} \subset \mathcal{E}_m^0(\Omega) \cap \mathcal{C}(\Omega)$ such that $u_j \searrow u$ on Ω' and

$$\sup_j \int_{\Omega'} \chi(u_j)\left[|u_j|^m\beta^m + |u_j|^{m-1}dd^c u_j\wedge\beta^{m-1} + \cdots + \left(dd^c u_j\right)^m\right]\wedge\beta^{n-m} < \infty.$$

Let $h \in \mathcal{E}_{m-1}^0(\Omega)$ be chosen. For each $j > 0$, take $m_j > 0$ such that $u_j \ge m_j h$ on Ω'. Set $v_j = \max(u_j, m_j h) \in \mathcal{E}_{m-1}^0(\Omega)$ and $v_j = u_j$ on Ω'. Note that $v_j \searrow u$ on Ω' and $(dd^c v_j)^p \wedge \beta^q = (dd^c u_j)^p \wedge \beta^q$ on Ω' for $1 \le p \le m-1$ and $1 \le q \le n-m+1$. We may assume that $u|_{\Omega'} \le -1$. By Hartogs' lemma (see Theorem 3.2.13 in [16]), we conclude that $v_j|_{\Omega'} \le -1$ for $j \ge j_0$ with some j_0. Without loss of generality, we may assume that $v_j|_{\Omega'} \le -1$ for $j \ge 1$. Hence, $|v_j|^m \ge |v_j|^{m-1}$ on Ω' for all $j \ge 1$. Now, we have

$$\int_{\Omega'} \chi(u_j)\left[|u_j|^m\beta^m + |u_j|^{m-1}dd^c u_j\wedge\beta^{m-1} + \cdots + |u_j|\left(dd^c u_j\right)^{m-1}\wedge\beta + \left(dd^c u_j\right)^m\right]\wedge\beta^{n-m}$$
$$\ge \int_{\Omega'} \chi(u_j)\left[|u_j|^m\beta^m + |u_j|^{m-1}dd^c u_j\wedge\beta^{m-1} + \cdots + |u_j|\left(dd^c u_j\right)^{m-1}\wedge\beta\right]\wedge\beta^{n-m}$$
$$= \int_{\Omega'} \chi(v_j)\left[|v_j|^m\beta^m + |v_j|^{m-1}dd^c v_j\wedge\beta^{m-1} + \cdots + |v_j|\left(dd^c v_j\right)^{m-1}\wedge\beta\right]\wedge\beta^{n-m}$$
$$= \int_{\Omega'} \chi(v_j)\left[|v_j|^m\beta^{m-1} + |v_j|^{m-1}dd^c v_j\wedge\beta^{m-2} + \cdots + |v_j|\left(dd^c v_j\right)^{m-1}\right]\wedge\beta^{n-m+1}$$
$$\ge \int_{\Omega'} \chi(v_j)\left[|v_j|^{m-1}\beta^{m-1} + |v_j|^{m-2}dd^c v_j\wedge\beta^{m-2} + \cdots + \left(dd^c v_j\right)^{m-1}\right]\wedge\beta^{n-m+1}.$$

Note that $v_j \searrow u$ on Ω' and

$$\sup_j \int_{\Omega'} \chi(v_j)\left[|v_j|^{m-1}\beta^{m-1}+|v_j|^{m-2}dd^c v_j\wedge\beta^{m-2}+\cdots+\left(dd^c v_j\right)^{m-1}\right]\wedge\beta^{n-m+1}<\infty.$$

Moreover, by Theorem 1, we get $u \in \mathcal{E}_{m-1,\chi}(\Omega)$. □

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