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Order Continuous Probabilistic Riesz Norms

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Abstract The concepts of order continuous norm, σ-order continuous norm, and Fatou norm defined on ordinary normed Riesz spaces are very important in the study of Riesz spaces. In this paper, we introduce the probabilistic analogues of such norms on a topological probabilistic normed Riesz (TPNR) space, and investigate their basic properties. In this context, some well-known theorems of the classical theory of topological Riesz spaces are proved in the setting of TPNR spaces, but now using the tools of probabilistic normed (PN) spaces. However, an interesting and different point here is that, although the classical order continuous Riesz norms are order preserving, the probabilistic Riesz norms considered in this work are order reversing mappings due to the nature of probabilistic distances.

Keywords Topological probabilistic normed Riesz space · Order continuous probabilistic Riesz norm · σ-order continuous probabilistic Riesz norm · Probabilistic Fatou norm

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1 Introduction

The concept of Riesz space, also called vector lattice or K-lineal, was first introduced by Riesz in [18]. The first contributions to the theory came from Freudenthal [6] and Kantorovich [10]. Since then, many others have developed the subject. Most of the spaces considered in mathematical analysis are Riesz spaces, and they have many applications in measure theory and operator theory.

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A probabilistic normed (PN) space is a generalization of an ordinary normed linear space. In a PN space, the norms of the vectors are uncertain due to randomness, therefore such norms are represented by probability distribution functions instead of nonnegative real numbers. Such a generalization of normed spaces may well be adapted to the setting of physical quantities (see [15]), and it has an important role in probabilistic analysis. PN spaces were first introduced by Šerstnev in [22]. Since then, some of the most deepest advances in this theory were obtained in [7, 17, 23, 24]. In 1993, Alsina et al. [3] presented a new definition of a PN space which includes Šerstnev's definition in [22] as a special case. Here, we will adopt this new definition. Following [3], many papers investigating the properties of PN spaces have appeared (see, for instance, [12, 14]). A detailed history and the development of the subject up to 2006 can be found in [20].

Starting out from the importance of the theories of Riesz spaces and PN spaces in functional analysis, we introduced the concepts of probabilistic normed Riesz (PNR) space and probabilistic Banach lattice (PBL), and studied their certain properties in [21]. Our aim was to integrate the theories of PN spaces and Riesz spaces, and thus to obtain a PN space endowed with the lattice structure. A more general treatment which does not consider a Riesz space structure explicitly, but is related to partial orders on probabilistic metric (PM) spaces can be found in [11].

In the current work, we continue the investigation of PNR spaces and PBLs. We discuss the connections between the topological and order structures of a PNR space. In this context, the probabilistic analogues of the concepts of order continuous norm, σ-order continuous norm, and Fatou norm of classical normed Riesz spaces are introduced in a topological probabilistic normed Riesz (TPNR) space, and their certain properties are examined. In this context, some well-known theorems of the classical theory of topological Riesz spaces (see, for instance, [1, 2, 5, 16, 25]) are proved in this special probabilistic setting, that is, in the setting of TPNR spaces, but here the tools of PN spaces are used to prove these theorems. We should also point out that, although the classical order continuous Riesz norms are order preserving, the probabilistic Riesz norms considered in this work are order reversing mappings due to the nature of probabilistic distances.

2 Preliminaries

In this section, to make the paper self-contained, we recall some of the basic concepts related to the theory of PN spaces, Riesz spaces, and PNR spaces, which will be used throughout the rest of the paper. First, we cast a glance at the theory of PN spaces and we refer to [3, 19, 20] for more details.

A *distance distribution function* is a non-decreasing function F that is left-continuous on $(-\infty, \infty)$, equals to zero on $[-\infty, 0]$ and $F(+\infty) = 1$. The set of all distance distribution functions is denoted by Δ^+. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, and has both a maximal element ε_0 and a minimal element ε_∞, defined by

$$\varepsilon_0(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0 \end{cases} \quad \text{and} \quad \varepsilon_\infty(x) = \begin{cases} 0, & x < +\infty, \\ 1, & x = \infty, \end{cases}$$

respectively. The subset $\mathcal{D}^+ = \{F \in \Delta^+ : \lim_{x\to\infty} F(x) = 1\}$ is called the set of *proper distance distribution functions*.

Now let $F, G \in \Delta^+$ and $h \in (0, 1]$. If we denote the condition

$$G(x) \leq F(x+h) + h \quad \text{for } x \in \left(0, \frac{1}{h}\right)$$

by $[F, G; h]$, then the function d_L defined on $\Delta^+ \times \Delta^+$ by

$$d_L(F, G) = \inf\{h : \text{ both } [F, G; h] \text{ and } [G, F; h] \text{ hold}\}$$

is called the *modified Lévy metric* on Δ^+. Convergence with respect to this metric is equivalent to the weak convergence of distribution functions, i.e., for any sequence (F_n) in Δ^+ and any F in Δ^+, we have $d_L(F_n, F) \longrightarrow 0$ if and only if the sequence $(F_n(x))$ converges to $F(x)$ at each continuity point x of F. Moreover, the metric space (Δ^+, d_L) is compact.

A *triangle function* is a binary operation τ on Δ^+, $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$, that is associative, commutative, non-decreasing in each place, and has ε_0 as identity. A triangle function is said to be *Archimedean* provided that $\tau(F, F) = F$ implies that $F = \varepsilon_0$ or $F = \varepsilon_\infty$. If τ is continuous on $\Delta^+ \times \Delta^+$, then it is uniformly continuous.

Definition 1 [3, 12] A *probabilistic normed space* (briefly, a PN space) is a quadruple (V, ν, τ, τ^*) where V is a real linear space, τ and τ^* are continuous triangle functions with $\tau \leq \tau^*$, and ν is a mapping (the *probabilistic norm*) from V into the space of distribution functions Δ^+ such that—writing ν_p for $\nu(p)$—for all p, q in V, the following conditions hold:

(N1) $\nu_p = \varepsilon_0$ if and only if $p = \theta$, the null vector in V,
(N2) $\nu_{-p} = \nu_p$,
(N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$,
(N4) $\nu_p \leq \tau^*(\nu_{\alpha p}, \nu_{(1-\alpha)p})$ for all $\alpha \in [0, 1]$.

A *Menger PN space* under T is a PN space (V, ν, τ, τ^*) in which $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$ for some continuous t-norm T and its t-conorm T^*; it is denoted by (V, ν, T).

For $p \in V$ and $t > 0$, the *strong t-neighborhood* of p is defined by the set

$$\mathcal{N}_p(t) = \left\{q \in V : d_L(\nu_{p-q}, \varepsilon_0) < t\right\} = \left\{q \in V : \nu_{p-q}(t) > 1 - t\right\}.$$

Since τ is continuous, the system of neighborhoods $\{\mathcal{N}_p(t) : p \in V \text{ and } t > 0\}$ determines a Hausdorff and first countable topology on V, called the *strong topology*.

A sequence (p_n) in (V, ν, τ, τ^*) is said to be *strongly convergent* (*convergent with respect to the probabilistic norm*) to a point p in V, and we will write $p_n \stackrel{\text{PN}}{\longrightarrow} p$, if for any $t > 0$, there is a positive integer N such that p_n is in $\mathcal{N}_p(t)$ whenever $n \geq N$. Thus, $p_n \stackrel{\text{PN}}{\longrightarrow} p$ if and only if $\lim_{n\to\infty} d_L(\nu_{p_n - p}, \varepsilon_0) = 0$. We will call p the *strong limit* of (p_n).

A sequence (p_n) in (V, ν, τ, τ^*) is said to be *strong Cauchy*, if for any $t > 0$, there is an integer N such that p_n is in $\mathcal{N}_{p_m}(t)$ whenever $n, m \geq N$. If every strong Cauchy sequence is strongly convergent to a point p in V, then we say that (V, ν, τ, τ^*) is complete in the strong topology.

In the sequel, when we consider a PN space (V, ν, τ, τ^*), we will assume that it is endowed with the strong topology.

Now, we list some of the basic concepts and notations related to the theory of Riesz spaces, and we refer to [25] for more details.

Definition 2 A real vector space E (with elements $f, g, \ldots$) with a partial order "$\leq$" is called an *ordered vector space* if E is partially ordered in such a manner that the vector space structure and the order structure are compatible, that is to say,

(i) $f \leq g$ implies $f + h \leq g + h$ for every $h \in E$,
(ii) $f \geq \theta$ implies $\alpha f \geq \theta$ for every $\alpha \geq 0$ in $\mathbb{R}$, where θ is the null element with respect to the vector addition. If, in addition, E is a lattice with respect to the partial ordering, then E is called a *Riesz space* or also a *vector lattice*. We will denote a Riesz space E by $(E, \leq)$.

Now let E be a Riesz space. For any $f \in E$, we write $f^+ = f \vee \theta$, $f^- = (-f) \vee \theta$ and $|f| = f \vee (-f)$. If $f, g \in E$ satisfy the equality $|f| \wedge |g| = \theta$, then they are said to be *disjoint*. A subset A of a Riesz space E is said to be *solid* if $|g| \leq |f|$ and $f \in A$ imply $g \in A$. If A is a solid linear subspace of E, then A is called an *ideal* in E. An ideal A is called a *band* if, whenever a subset of A possesses a supremum in E, this supremum is a member of A.

The subset $E^+ = \{f \in E : f \geq \theta\}$ is called the *positive cone* of E. A Riesz space E is said to be *Archimedean* provided that given $f, g \in E^+$ such that $\theta \leq nf \leq g$ for every $n \in \mathbb{N}$, it follows that $f = \theta$.

A sequence (f_n) in E is said to be *increasing* if $f_1 \leq f_2 \leq \cdots$, and *decreasing* if $f_1 \geq f_2 \geq \cdots$. This is denoted by $f_n \uparrow$ or $f_n \downarrow$, respectively. If $f_n \uparrow$ and $\sup f_n = f$ exists in E, we write $f_n \uparrow f$. Similarly, if $f_n \downarrow$ and $\inf f_n = f$ exists, we write $f_n \downarrow f$. If $f_n \uparrow f$ or $f_n \downarrow f$, we say that (f_n) *converges monotonically* to f as $n \to \infty$.

A sequence (f_n) in E is said to *converge in order* to f if there exists a sequence $p_n \downarrow \theta$ such that $|f_n - f| \leq p_n$ holds for all $n \in \mathbb{N}$. In this case, we will write $f_n \xrightarrow{\text{ord}} f$. Note that monotone convergence is a particular case of convergence in order (order convergence).

A sequence (f_n) in E is said to be *order bounded* if there exists an order interval $[f, g]$ such that $f \leq f_n \leq g$ for all $n \in \mathbb{N}$.

A non-empty subset D of a Riesz space E is said to be *upwards directed* if for any $f, g \in D$ there exists an element $h \in D$ such that $h \geq f \vee g$. In this case, we write $D \uparrow$. If $D \uparrow$ and D has the supremum $f_0 \in E$, then we write $D \uparrow f_0$. A *downwards directed set* is defined similarly.

If E is a (real) Riesz space equipped with a norm $\|\cdot\|$ such that $|f| \leq |g|$ in E implies $\|f\| \leq \|g\|$, then the norm on E is called a *Riesz norm*. Any Riesz space equipped with a Riesz norm is called a *normed Riesz space*. We will denote a normed Riesz space E by $(E, \|\cdot\|, \leq)$.

Finally, we recall from [21] some basic concepts related to PNR spaces.

Definition 3 [21] Let $(E, \leq)$ be a (real) Riesz space equipped with a probabilistic norm ν, and continuous triangle functions τ and τ^* such that $\tau \leq \tau^*$. The probabilistic norm on E is a *probabilistic Riesz norm* provided that $|f| \leq |g|$ in E implies $\nu_f \geq \nu_g$. Any Riesz space, equipped with a probabilistic Riesz norm is a *probabilistic normed Riesz space* (PNR space, briefly). If a PNR space E is complete with respect to the strong topology, then E is a *probabilistic Banach lattice* (PBL, in short). We will denote a PNR space by the quintuple $(E, \nu, \tau, \tau^*, \leq)$ or just E, if the context is clear.

Example 1 Let $(\Omega, \mathfrak{a}, P)$ be a probability measure space, $(X, \mathcal{B})$ be a measurable space where X is a separable Banach lattice $(X, \|\cdot\|, \leq)$, and $\mathcal{B}$ is the σ-algebra of all Borel subsets of X. Let us consider the set $L^0(\mathfrak{a}, X)$ of all equivalance classes of X-valued random variables $f : \Omega \longrightarrow X$, where $f^{-1}(B) \in \mathfrak{a}$ for all $B \in \mathcal{B}$ (see [8, 21]). In this example, given an element in $L^0(\mathfrak{a}, X)$, we will consider a refined measurement for its norm. For instance, let us define a function $\nu : L^0(\mathfrak{a}, X) \longrightarrow \Delta^+$ by

$$\nu_f(t) = P\{\omega \in \Omega : \|f(\omega)\| < t\}$$

for any $f \in L^0(\mathfrak{a}, X)$ and $t \in \mathbb{R}$. Here we can interpret the number $\nu_f(t)$ as the probability that the norm of f is less than t. Then $(L^0(\mathfrak{a}, X), \nu, \tau_W, \tau_M)$ is an E-normed space which is a special type of PN space [23]. Here, the continuous triangle functions τ_W and τ_M are defined by

$$(\tau_W(F, G))(t) = \sup\{\max\{F(u) + G(v) - 1, 0\} : u + v = t\}$$

and

$$(\tau_M(F, G))(t) = \sup\{\min\{F(u), G(v)\} : u + v = t\},$$

where $F, G \in \Delta^+$ and $t \in \mathbb{R}$. If we define a partial order "$\leq$" on $L^0(\mathfrak{a}, X)$ as

$$f \leq g \quad \text{if and only if} \quad f^0(\omega) \leq g^0(\omega) \quad \text{a.s.},$$

where f^0 and g^0 are arbitrarily chosen representatives of f and g, respectively, then $(L^0(\mathfrak{a}, X), \nu, \tau_W, \tau_M, \leq)$ is a PBL.

Definition 4 [21] Let D be an upwards directed set in a PNR space E. Then D is *strongly convergent* to some $f_0 \in E$ provided that for any $t > 0$ there is an $f_t \in D$ such that $f \in \mathcal{N}_{f_0}(t)$ for all $f \in D$ satisfying $f \geq f_t$. The strong convergence of a downwards directed set is defined similarly.

Definition 5 [21] Let D be an upwards directed set in a PNR space E. Then D is a *probabilistic norm Cauchy system* provided that for any $t > 0$ there exists an element $f_t \in D$ such that $f_1 \in \mathcal{N}_{f_2}(t)$ for all $f_1, f_2 \in D$ satisfying $f_1, f_2 \geq f_t$. The definition for a downwards directed set is analogous.

Remark 1 In classical Riesz space theory, it is known that every normed Riesz space is Archimedean. In general, a PNR space E need not be Archimedean. Nevertheless, if the condition that the triangle function τ^* of the PNR space E is Archimedean and $\nu_f \neq \varepsilon_\infty$ for all $f \in E$ is satisfied, then E is also Archimedean [21]. Also in this case, the PNR space E becomes a topological vector (TV) space (see [4]), but the condition mentioned above is not necessary for a PNR space to be a TV space [13]. If a PNR space is a TV space, then we will call it a *topological PNR space* (TPNR space, briefly). On the other hand, since the lattice operations on an arbitrary PNR space are uniformly continuous with respect to the strong topology [21], the strong topology becomes *locally solid* on a TPNR space E, that is, it has a base at θ consisting of solid neighborhoods. Hence E becomes a *locally solid Riesz space* (*topological Riesz space*), which is also Hausdorff. Thus, we can say that a *TPNR space is always Archimedean*. In this paper, we will particularly focus on TPNR spaces because local solidness is a natural topological condition related to the vector ordering. For more details about locally solid Riesz spaces, we refer to [1].

3 Main Results

In this section, we will treat the interplay between the probabilistic Riesz norm and the order of a TPNR space. For this purpose, on a TPNR space, we introduce the probabilistic analogues of the classical notions of order continuous norm, σ-order continuous norm, and Fatou norm. A common property of these analogous concepts that will be introduced below is that, they play a role similar to that of order continuous norms of ordinary normed Riesz spaces, but they are order-reversing mappings due to the nature of the probabilistic distances.

Before starting, let us recall the concept of order continuous norm from classical Riesz space theory. A normed Riesz space $(E, \|\cdot\|, \leq)$ is said to have *order continuous norm* if for any subset $D \downarrow \theta$ in E (that is, D is downwards directed and $\inf D = \theta$), we have $\inf\{\|f\| : f \in D\} = 0$ (see [25]). Note also that a locally solid linear topology $\mathcal{O}$ on a Riesz space E is *order continuous* if and only if $x_\alpha \downarrow \theta$ (i.e., (x_α) is decreasing and $\inf_\alpha x_\alpha = \theta$ in E) implies $x_\alpha \xrightarrow{\mathcal{O}} \theta$, where (x_α) is a net in E (see [2]). Although upwards (resp., downwards) directed sets and increasing (resp., decreasing) nets are equivalent for all practical purposes, we will employ directed sets since they are more convenient than nets in certain situations.

Now, in view of the foregoing concepts, we first introduce the following.

Definition 6 Let E be a TPNR space. We say that the probabilistic norm ν on E is *order continuous*, provided that for any subset $D \downarrow \theta$ in E, we have $\sup_{f \in D} \nu_f = \varepsilon_0$.

Note that if ν is an order continuous probabilistic norm, then it is an order-reversing mapping from $D \subset E^+$ into Δ^+, and hence the set D is strongly convergent to θ.

Example 2 Let $(L_1, \|\cdot\|, \leq)$ be the normed Riesz space with

$$\|f\| = \int_X |f(x)| d\mu,$$

where $f \in L_1$ and μ is a σ-finite measure in the point set X. Let us consider the simple space $(L_1, \|\cdot\|, G, M)$, where $G \in \mathcal{D}^+$, $G \neq \varepsilon_0$, and the probabilistic norm $\nu : L_1 \longrightarrow \Delta^+$ is defined by $\nu_\theta = \varepsilon_0$ and

$$\nu_f(x) = G\left(\frac{x}{\|f\|}\right) \quad (x > 0)$$

if $f \neq \theta$, and M is the t-norm defined by $M(x, y) = \min\{x, y\}$. Such a simple space is a TV space since $G \in \mathcal{D}^+$ (see [14]), hence $(L_1, \|\cdot\|, G, M, \leq)$ is a Menger TPNR space under M and the probabilistic norm ν on L_1 is order continuous.

To investigate the basic properties of a TPNR space having order continuous probabilistic norm, let us first consider the following important lemmas.

Lemma 1 ([25]) *If D is an upwards directed set in an Archimedean Riesz space $(E, \leq)$ and D is bounded from above with G as the set of its upper bounds, then $(G - D) \downarrow \theta$.*

Lemma 2 [21] *If D is an upwards directed set in a PNR space E such that D is strongly convergent to some f_0 then* $\sup D = f_0$.

Lemma 3 [21] *Every probabilistic norm Cauchy system in a PBL is strongly convergent. If the system is upwards directed, the strong limit is the supremum of the system. In other words, if $D \uparrow$ and D is a probabilistic norm Cauchy system, then D is strongly convergent to some f_0, and $D \uparrow f_0$. Similarly, if D is downwards directed.*

Theorem 1 *Let E be a TPNR space which has an order continuous probabilistic norm ν. Then the strong convergence of an upwards (resp., downwards) directed set D in E to f_0 is equivalent to* $\sup D = f_0$ *(resp.,* $\inf D = f_0$*).*

Proof First note that E is Archimedean (see Remark 1). Now suppose that $D \subset E$ is non-empty, upwards directed and has a supremum f_0. Then the set $B = \{f_0 - f : f \in D\}$ is downwards directed and has infimum θ by Lemma 1. Since ν is order continuous, we have $\sup_{f \in D} \nu_{f_0 - f} = \varepsilon_0$, which implies that $\sup_{f \in D} \nu_{f_0 - f}(t) = 1$ for each $t > 0$. Hence, for each $t > 0$, there exists an $f_t \in D$ such that $\nu_{f_0 - f_t}(t) > 1 - t$, namely, $d_L(\nu_{f_0 - f_t}, \varepsilon_0) < t$. Now let $f \in D$ be such that $f \geq f_t$. Hence we have $\nu_{f_0 - f} \geq \nu_{f_0 - f_t}$, which yields

$$d_L(\nu_{f_0 - f}, \varepsilon_0) \leq d_L(\nu_{f_0 - f_t}, \varepsilon_0) < t$$

for every $f \geq f_t$. This shows that, for each $t > 0$, there exists an $f_t \in D$ such that $d_L(\nu_{f_0 - f}, \varepsilon_0) < t$ for every $f \geq f_t$. Hence, D is strongly convergent to f_0 by Definition 4. Now let $D \subset E$ be a non-empty downwards directed set with an infimum f_0. Then the set $C = \{f - f_0 : f \in D\}$ is downwards directed and has infimum θ. Since ν is order continuous, we have

$$\sup_{f \in D} \nu_{f - f_0} = \sup_{f \in D} \nu_{f_0 - f} = \varepsilon_0,$$

which shows that D is strongly convergent to f_0.

Conversely, in any PNR space, if $D \uparrow$ (resp., $D \downarrow$) and D is strongly convergent to f_0, then $\sup D = f_0$ (resp., $\inf D = f_0$) by Lemma 2. Hence the proof is complete. □

Now we will present a result regarding topological probabilistic Banach lattices (TPBL, briefly) with order continuous probabilistic norms.

Theorem 2 *Let E be a TPBL having order continuous probabilistic norm ν. Then E is Dedekind complete, that is, every set which is bounded from above in E has a supremum.*

Proof Let D be an upwards directed set in E such that D is bounded from above. Let us denote the set of all upper bounds of D by G. Then by Lemma 1, we have

$$G - D = \{g - f : g \in G,\ f \in D\} \downarrow \theta.$$

Since ν is order continuous, we can write

$$\sup \left\{ \nu_{g-f} : g \in G,\ f \in D \right\} = \varepsilon_0.$$

Hence, for each $t > 0$, there exist $g \in G$ and $f_1 \in D$ such that

$$d_L\left(\nu_{g - f_1}, \varepsilon_0\right) < t.$$

Since $\theta \leq f_2 - f_1 \leq g - f_1$ for all $f_2 \in D$ satisfying $f_2 \geq f_1$, it follows that $\nu_{f_2 - f_1} \geq \nu_{g - f_1}$, and hence we get $d_L(\nu_{f_2 - f_1}, \varepsilon_0) \leq d_L(\nu_{g - f_1}, \varepsilon_0) < t$ for all $f_2 \geq f_1$ in D. This shows that D is a probabilistic norm Cauchy system by Definition 5. Since E is a PBL, D is strongly convergent to some $f_0 \in E$ and $f_0 = \sup D$ by Lemma 3. Hence the proof is complete. □

In what follows, we characterize order continuous probabilistic Riesz norms.

Theorem 3 *Let E be a TPNR space with a probabilistic norm ν. Then ν is an order continuous probabilistic norm if and only if every ideal closed with respect to the strong topology in E is a band.*

Proof Let the probabilistic norm ν be order continuous, and A be a closed ideal in E. To show that A is a band, assume that D is an upwards directed subset of A^+ with supremum f_0, so $D \uparrow f_0$. We will show that $f_0 \in A$. Since the probabilistic norm on E is order continuous, $D \uparrow f_0$ implies that D is strongly convergent to f_0 by Theorem 1. Now let (t_n) be a number sequence such that $t_n \downarrow 0$. Since D is strongly convergent to f_0, there exists an element $f_1 \in D$ such that $d_L(\nu_{f-f_0}, \varepsilon_0) < t_1$ for every $f \geq f_1$. Similarly, there exists an $f_2^* \in D$ such that $d_L(\nu_{f-f_0}, \varepsilon_0) < t_2$ for every $f \geq f_2^*$. Now choose $f_2 \in D$ such that $f_2 \geq f_1 \vee f_2^*$. Then $f_2 \geq f_1$ and all f with $f \geq f_2$ in D satisfy $d_L(\nu_{f-f_0}, \varepsilon_0) < t_2$. Hence we obtain an increasing sequence $\theta \leq f_1 \leq f_2 \leq \cdots$ in D such that $d_L(\nu_{f_n-f_0}, \varepsilon_0) \longrightarrow 0$ as $n \longrightarrow \infty$, i.e., $f_n \overset{\text{PN}}{\longrightarrow} f_0$. Since $f_n \in A$ for all $n \in \mathbb{N}$ and A is closed with respect to the strong topology, we have $f_0 \in A$. Hence A is a band.

Conversely, assume that every ideal closed with respect to the strong topology in E is a band. It is sufficient to prove that $D \uparrow f_0$ in E^+ implies that $\sup_{f\in D} \nu_{f_0-f} = \varepsilon_0$. To this end, we will use the uniform continuity of the triangle function τ. Note that $\tau : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$ is uniformly continuous if and only if for any $t > 0$ there is a $\lambda > 0$ such that $d_L(\tau(F, G), \varepsilon_0) < t$ whenever $d_L(F, \varepsilon_0) < \lambda$ and $d_L(G, \varepsilon_0) < \lambda$, where $F, G \in \Delta^+$. Now let $t > 0$. Then we can find a $\lambda > 0$ and an $\alpha \in (0, 1)$ such that

$$d_L\left(\nu_{(1-\alpha)f_0}, \varepsilon_0\right) < \lambda. \tag{1}$$

On the other hand, we can write

$$\theta \leq f_0 - f = (1-\alpha)f_0 + \alpha f_0 - f \leq (1-\alpha)f_0 + (\alpha f_0 - f)^+$$

for every $f \in D$. Thus we have $\nu_{f_0-f} \geq \tau(\nu_{(1-\alpha)f_0}, \nu_{(\alpha f_0-f)^+})$ and hence we get

$$d_L\left(\nu_{f_0-f}, \varepsilon_0\right) \leq d_L\left(\tau\left(\nu_{(1-\alpha)f_0}, \nu_{(\alpha f_0-f)^+}\right), \varepsilon_0\right). \tag{2}$$

Since the set $\{(\alpha f_0 - f)^+ : f \in D\}$ is downwards directed, now it is sufficient to prove that

$$d_L\left(\nu_{(\alpha f_0-f^*)^+}, \varepsilon_0\right) < \lambda \tag{3}$$

for some $f^* \in D$, since then the uniform continuity of τ will complete the proof via inequality (2) which will result in $d_L(\nu_{f_0-f}, \varepsilon_0) < t$. Observe that

$$\{f - \alpha f_0 : f \in D\} \uparrow (1-\alpha)f_0,$$

so $\{(f - \alpha f_0)^+ : f \in D\} \uparrow (1-\alpha)f_0$. Therefore, denoting by A the ideal generated by the set $\{(f - \alpha f_0)^+ : f \in D\}$, it is clear that f_0 is an element of the band B generated by A. By hypothesis, $\overline{A}$ is a band, so $B \subset \overline{A}$ (since B is the smallest band containing A). Therefore, $f_0 \in B$ implies that $f_0 \in \overline{A}$. Hence, for the given $\lambda > 0$ there exists an element $g \in A$ such that $d_L(\nu_{f_0-g}, \varepsilon_0) < \lambda$. Then $g^+ \in A$ and $|f_0 - g^+| = |f_0^+ - g^+| \leq |f_0 - g|$, so $d_L(\nu_{f_0-g^+}, \varepsilon_0) < \lambda$. We may assume that $\theta \leq g \in A$. Similarly, g may be replaced by $g \wedge f_0$, so we may assume that $\theta \leq g \leq f_0$ holds. By the definition of the ideal generated by a given set of elements, any element in the ideal is already contained in the ideal generated by a finite number of elements of the given set. This implies in our case (since the set $\{(f - \alpha f_0)^+ : f \in D\}$ is upwards directed) that there exists an $f^* \in D$ such that g is an element of the principal ideal generated by $(f^* - \alpha f_0)^+$. Hence

$$g \perp \left(f^* - \alpha f_0\right)^- = \left(\alpha f_0 - f^*\right)^+.$$

Each of the disjoint elements g and $(\alpha f_0 - f^*)^+$ is majorized by f_0, so

$$g + \left(\alpha f_0 - f^*\right)^+ \leq f_0.$$

Thus $d_L(\nu_{(\alpha f_0 - f^*)^+}, \varepsilon_0) \leq d_L(\nu_{f_0 - g}, \varepsilon_0) < \lambda$. Now the uniform continuity of τ implies that for every $t > 0$ there exists a $\lambda > 0$ and hence an

$$f_\lambda = f_t = f^* \in D$$

such that $d_L(\nu_{f_0 - f}, \varepsilon_0) < t$ for all $f \geq f^*$, which shows that

$$\sup_{f \in D} \nu_{f_0 - f} = \varepsilon_0.$$

Hence the probabilistic norm ν is order continuous, which completes the proof. □

In what follows, we introduce the probabilistic analogue of the σ-order continuous norm of an ordinary normed Riesz space, namely, the concept of σ-order continuous probabilistic norm on a TPNR space. Before this, let us recall that a normed Riesz space $(E, \|\cdot\|, \leq)$ is said to have σ-order continuous norm if, for any sequence $f_n \downarrow \theta$ in E, we have $\|f_n\| \downarrow 0$ (see [25]). Based on this, we introduce the following.

Definition 7 Let E be a TPNR space. The probabilistic norm ν on E is *σ-order continuous*, provided that for any sequence $f_n \downarrow \theta$ in E, we have that (ν_{f_n}) is increasing and has supremum ε_0, that is, $\nu_{f_n} \uparrow \varepsilon_0$.

Note that such a probabilistic norm preserves order convergence. Observe that if the probabilistic norm ν on E is σ-order continuous, then $f_n \uparrow f_0$ (or $f_n \downarrow f_0$) implies that $\nu_{f_n - f_0} \uparrow \varepsilon_0$, which yields $d_L(\nu_{f_n - f_0}, \varepsilon_0) \longrightarrow 0$ as $n \longrightarrow \infty$, namely, $f_n \xrightarrow{\text{PN}} f_0$. Conversely, in any PNR space, if $f_n \uparrow$ (or $f_n \downarrow$) and $f_n \xrightarrow{\text{PN}} f_0$, then $f_n \uparrow f_0$ (or $f_n \downarrow f_0$) ([21]). Hence order convergence and strong convergence for monotone sequences in a TPNR space E are equivalent if the probabilistic norm ν on E is σ-order continuous.

Theorem 4 *Let E be a TPBL with a probabilistic norm ν. Then the following are equivalent:*

(i) *The probabilistic norm ν is order continuous.*
(ii) *The probabilistic norm ν is σ-order continuous and E is Dedekind σ-complete.*
(iii) *Every monotone order bounded sequence in E is strongly convergent.*

Proof (i)$\Longrightarrow$(ii). This part follows from Definitions 6, 7 and Theorem 2.

(ii)$\Longrightarrow$(iii). Let (f_n) be an increasing sequence in E, which is bounded from above. Since E is Dedekind σ-complete, the supremum of (f_n) exists, say f_0, thus $f_n \uparrow f_0$. Hence $(f_0 - f_n) \downarrow \theta$, and so $\nu_{f_0 - f_n} \uparrow \varepsilon_0$, since E has σ-order continuous probabilistic norm. Hence (f_n) is strongly convergent to f_0.

(iii)$\Longrightarrow$(i). Suppose on the contrary that ν is not order continuous, that is, there exists a downwards directed set $D \subset E^+$ such that $D \downarrow \theta$ and $\sup_{f \in D} \nu_f \neq \varepsilon_0$. This means that there exists a $t > 0$ such that, for every $f \in D$ there exists an $f_0 \leq f$ with $d_L(\nu_{f_0}, \varepsilon_0) \geq t$. Namely, D is not strongly convergent to θ. Since E is a PBL, it follows that D cannot be a probabilistic norm Cauchy system, by Lemma 3. Thus there exists a decreasing sequence (f_n) in D, which fails to be strongly convergent. This contradicts (iii). Hence the proof is complete. □

Finally, we introduce the notion of probabilistic Fatou norm which plays a role similar to the order continuous probabilistic norm. But before this, we recall two concepts, that is, the Fatou norm on a Riesz space and the probabilistic radius of a set in a PN space. We will define the probabilistic Fatou norm via the notion of probabilistic radius.

A *Fatou norm* on a Riesz space $(E, \leq)$ is a Riesz norm such that whenever $D \subset E^+$ is non-empty, upwards directed, and has a supremum in E, then $\|\sup D\| = \sup_{f \in D} \|f\|$ (see [9]).

Given a non-empty set D in a PN space (V, ν, τ, τ^*), the *probabilistic radius* R_D of D is defined by

$$R_D(x) = \begin{cases} l^- \phi_D(x) & \text{if } x \in [0, +\infty), \\ 1 & \text{if } x = +\infty, \end{cases}$$

where $l^- \phi_D(x)$ denotes the left-hand limit of the function ϕ at the point x, and $\phi_D(x) = \inf\{\nu_f(x) : f \in D\}$ (see [12]).

Definition 8 Let E be a TPNR space and $D \subset E^+$ be a non-empty, upwards directed set with a supremum in E. Then the probabilistic norm ν on E is a *probabilistic Fatou norm*, provided that $\nu_{\sup D} = R_D$.

Theorem 5 *Let E be a TPNR space with an order continuous probabilistic norm ν. Then ν is a probabilistic Fatou norm.*

Proof Let $D \subset E^+$ be a non-empty, upwards directed set with $\sup D = f_0$ in E. Since ν is a probabilistic Riesz norm, we have $\nu_f \geq \nu_{f_0}$ for all $f \in D$. Hence we get $R_D \geq \nu_{f_0}$. Now let us show that $\nu_{f_0} \geq R_D$. Since ν is order continuous, we have $\sup_{f \in D} \nu_{f_0 - f} = \varepsilon_0$. Hence the inequality $\nu_{f_0} \geq \tau(\nu_{f_0 - f}, \nu_f)$ for each $f \in D$ implies that

$$\nu_{f_0} \geq \sup_{f \in D} \tau\left(\nu_{f_0 - f}, \nu_f\right) \geq R_D,$$

which completes the proof. □

The converse implication in Theorem 5 does not hold in general. To see this, let us consider the following example.

Example 3 Let $(E, \|\cdot\|, \leq)$ be the normed Riesz space of the real continuous functions f defined on [0, 1], where $\|\cdot\|$ is the supremum norm and "$\leq$" is the pointwise ordering. Let us consider the quintuple $(E, \nu, \tau, \mathbf{M}, \leq)$ where $\tau(\varepsilon_c, \varepsilon_d) \leq \varepsilon_{c+d}$ for every $c, d > 0$, $\mathbf{M}$ is the maximal triangle function defined by

$$[\mathbf{M}(F, G)](x) = \min\{F(x), G(x)\} \qquad (F, G \in \Delta^+,\ x \in \mathbb{R})$$

and ν is the mapping defined by

$$\nu : E \longrightarrow \Delta^+$$
$$\nu(f) = \nu_f = \varepsilon_{\frac{\|f\|}{1+\|f\|}}.$$

Then the quadruple $(E, \nu, \tau, \mathbf{M})$ is a PN space which is also a TV space (see [13]), although the triangle function $\mathbf{M}$ is not Archimedean. Hence the quintuple $(E, \nu, \tau, \mathbf{M}, \leq)$ is a TPNR space. Now let $D \subset E^+$ be a non-empty, upwards directed set with supremum f_0 in E. Then we have (see [1]) $\|f_0\| = \sup_{f \in D} \|f\|$, and hence we can write

$$R_D = l^- \inf_{f \in D} \nu_f = l^- \inf \varepsilon_{\frac{\|f\|}{1+\|f\|}} = \varepsilon_{\frac{\|f_0\|}{1+\|f_0\|}} = \nu_{\sup D},$$

which shows that ν is a probabilistic Fatou norm. Now let us consider the sequence $(f_n) \subset E$ defined by $f_n(x) = x^n$ for all $x \in [0, 1]$ and $n \in \mathbb{N}$. Then we have $f_n \downarrow \theta$ as $n \longrightarrow \infty$, but $\nu_{f_n} = \varepsilon_{\frac{1}{2}}$ for every $n \in \mathbb{N}$. Thus ν is not σ-order continuous, and therefore, not order continuous.

4 Conclusion

This work is a brief introduction to the probabilistic analogues of order continuous norms and Fatou norms. Thus, many of the classical results of topological Riesz spaces can be investigated in this probabilistic setting to constitute a probabilistic lattice theory for probabilistic normed spaces.

References

1. Aliprantis, C.D., Burkinshaw, O.: Locally Solid Riesz Spaces with Applications to Economics. American Mathematical Society, Providence, R.I (2003)
2. Aliprantis, C.D., Border, K.C.: Infinite Dimensional Analysis. Springer-Verlag, Berlin–Heidelberg (2006)
3. Alsina, C., Schweizer, B., Sklar, A.: On the definition of a probabilistic normed space. Aequationes Math. **46**, 91–98 (1993)
4. Alsina, C., Schweizer, B., Sklar, A.: Continuity properties of probabilistic norms. J. Math. Anal. Appl. **208**, 446–452 (1997)
5. Fremlin, D.H.: Measure Theory, vol. 3. Torres Fremlin, Colchester (2002)
6. Freudenthal, H.: Teilweise geordnete modulen. Proc. Acad. Amst. **39**, 641–651 (1936)
7. Guo, T.-X., Ma, R.-P.: Some reviews on various definitions of a random conjugate space together with various kinds of boundedness of a random linear functional. Acta Anal. Funct. Appl. **6**, 16–38 (2004)
8. Guo, T.-X., Zeng, X.-L.: Random strict convexity and random uniform convexity in random normed modules. Nonlinear Anal. TMA **73**, 1239–1263 (2010)
9. Johnson, W.B., Lindenstrauss, J.: Handbook of the Geometry of Banach Spaces, vol. 1. Elsevier (2001)
10. Kantorovich, L.V.: Lineare halbgeordnete Raume. Receueil Math. **2**, 121–168 (1937)
11. Kent, D.C., Richardson, G.D.: Ordered probabilistic metric spaces. J. Aust. Math. Soc., Ser. A **46**, 88–99 (1989)
12. Lafuerza-Guillén, B., Rodríguez-Lallena, J.A., Sempi, C.: A study of boundedness in probabilistic normed spaces. J. Math. Anal. Appl. **232**, 183–196 (1999)
13. Lafuerza-Guillén, B.: D-bounded sets in probabilistic normed spaces and in their products. Rend. Mat. **21**, 17–28 (2001)
14. Lafuerza-Guillén, B., Rodríguez-Lallena, J.A., Sempi, C.: Normability of probabilistic normed spaces. Note Mat. **29**, 99–111 (2009)
15. Menger, K.: Probabilistic geometry. Proc. Natl. Acad. Sci. USA **37**, 226–229 (1951)
16. Meyer-Neiberg, P.: Banach Lattices. Springer-Verlag (1991)
17. Mustari, D.H.: On almost sure convergence in linear spaces of random variables. Theory Probab. Appl. **15**, 337–342 (1970)
18. Riesz, F.: Sur la décomposition des opérations linéaires. Atti. Congr. Int. Mat. Bologna **3**, 143–148 (1930)
19. Schweizer, B., Sklar, A.: Probabilistic Metric Spaces. Elsevier/North-Holland, New York, (1983). Reissued with an errata list, notes and supplemental bibliography by Dover Publications, New York (2005)
20. Sempi, C.: A short and partial history of probabilistic normed spaces. Mediterr. J. Math. **3**, 283–300 (2006)
21. Şençimen, C., Pehlivan, S.: Probabilistic normed Riesz spaces. Acta. Math. Sin. Engl. Ser. **28**, 1401–1410 (2012)
22. Šerstnev, A.N.: On the notion of a random normed space. Dokl. Akad. Nauk. SSSR **149**, 280–283 (1963)
23. Sherwood, H.: Isomorphically isometric probabilistic normed linear spaces. Stochastica **3**, 71–77 (1979)
24. Taylor, R.L.: Convergence of elements in random normed spaces. Bull. Austral. Math. Soc. **12**, 156–183 (1975)
25. Zaanen, A.C.: Introduction to Operator Theory in Riesz Spaces. Springer-Verlag, Berlin–Heidelberg (1997)