Vietnam J. Math. (2016) 44:329–338
DOI 10.1007/s10013-015-0137-z

G-Morphic Rings and G-regular Rings

Zhanmin Zhu

Received: 6 April 2014 / Accepted: 22 October 2014 / Published online: 5 March 2015

Abstract A ring R is called *left G-morphic* if $\mathbf{l}(a)$ is a principal left ideal for each $a \in R$. A ring R is called *left G-regular* if R is left G-morphic and left P-injective. Several properties of the two classes of rings are investigated, conditions under which left G-regular rings are regular rings as well as semisimple artinian rings are given, respectively.

Keywords G-morphic ring · G-regular ring · P-injective module · P-flat module

Mathematics Subject Classification (2010) 16D40 · 16D50 · 16E50

1 Introduction

Throughout this paper, R denotes an associative ring with identity and all modules considered are unitary, m, n are positive integers unless otherwise specified. We call a ring *regular* if it is *von Neumann regular*. For any module M, M^* denotes $\mathrm{Hom}_R(M, R)$, and M^+ denotes $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Q}$ is the set of rational numbers, and $\mathbb{Z}$ is the set of integers. For an element a of the ring R, the right and left annihilators of a are denoted by $\mathbf{l}(a)$ and $\mathbf{r}(a)$, respectively.

First, we recall some concepts.

(1) R is called *left coherent* if every finitely generated left ideal of R is finitely presented.
(2) R is called *left n-coherent* [9] if every n-generated left ideal of R is finitely presented. Clearly, R is left 1-coherent if and only if $\mathbf{l}(a)$ is finite generated for each $a \in R$. Left 1-coherent rings are also called left $(1, 1)$-coherent in [11].
(3) An element a in R is called *left morphic* if $\mathbf{l}(a) \cong R/Ra$; the ring R is called a *left morphic ring* if every element in R is left morphic [6].

Z. Zhu (✉)
Department of Mathematics, Jiaxing University,
Jiaxing, 31400 Zhejiang, People's Republic of China
e-mail: zhanmin_zhu@hotmail.com

(4) An element a in R is called *left generalized morphic* if $\mathbf{l}(a) \cong R/Rb$ for some $b \in R$; the ring R is called a *left generalized morphic ring* if every element in R is left generalized morphic [12]. By [12, Corollary 2.3], a ring R is a left generalized morphic ring if and only if $\mathbf{l}(a)$ is a principal left ideal for each $a \in R$.

(5) A ring R is called left PP [5] if every principal left ideal of R is projective. Clearly, left PP rings are left generalized morphic, but left generalized morphic rings need not be left PP. For example, the ring $R = \mathbb{Z}_4$ is a commutative generalized morphic ring, but it is not PP (see Example 2).

(6) A left R-module M is called *P-injective* [7] if every R-homomorphism from a principal left ideal of R to M extends to a homomorphism of R to M; the ring R is called left P-injective if ${}_RR$ is P-injective. The P-injective dimension $P\text{-}id({}_RM)$ of a module ${}_RM$ is defined to be the smallest integer $n \geq 0$ such that $\text{Ext}_R^{n+1}(R/Ra, M) = 0$ for all $a \in R$. If no such n exists, set $P\text{-}id({}_RM) = \infty$. $l.P\text{-}i$ $\dim(R)$ is defined as $\sup\{P\text{-}id(M) \mid M \in R\text{-Mod}\}$ [12].

(7) A right R-module M is called *n-flat* [3] if the canonical map $M_R \otimes I \to M$ is monic for every n-generated left ideal I of R. 1-flat modules are also called P-flat in some literatures such as [12]. It is easy to see that M_R is P-flat if and only if $\text{Tor}_1^R(M, R/Ra) = 0$ for all $a \in R$. The P-flat dimension $P\text{-}fd(M_R)$ of a module M_R is defined to be the smallest integer $n \geq 0$ such that $\text{Tor}_{n+1}^R(M, R/Ra) = 0$ for all $a \in R$. If no such n exists, set $P\text{-}fd({}_RM)$=∞. $r.P\text{-}f$ $\dim(R)$ is defined to be $\sup\{P\text{-}fd(M) \mid M \in \text{Mod-}R\}$ [12].

(8) A left R-module M is called *(m, n)-presented* [11, 13] if there exists an exact sequence of left R-modules $0 \to K \to R^m \to M \to 0$, where K is n-generated.

(9) A left R-module M is called *(m, n)-injective* [2] if every R-homomorphism from an n-generated submodule of R^m to M extends to the one from R^m to M. Clearly, a module is P-injective if and only if it is (1, 1)-injective.

(10) A left R-module M is called *(m, n)-flat* [11, 13] if the canonical map $I \otimes_R M \to R^m \otimes_R M$ is monic for every n-generated submodule I of the right R-module R^m. Clearly, a module is P-flat if and only if it is (1, 1)-flat.

(11) If U is a right R-module and U' is a submodule of U, then U' is called an *(m, n)-pure submodule* of U if the canonical map $U' \otimes_R V \to U \otimes_R V$ is monic for every (m, n)-presented left R-module V. In this case, the exact sequence $0 \to U' \to U \to U/U' \to 0$ is called (m, n)-pure [13].

In this paper, we shall further investigate left generalized morphic rings; several of properties of this class of rings will be given. Especially, left generalized morphic left P-injective rings will be studied; relations between this class of rings and regular rings, strongly regular rings, as well as semisimple artinian rings will be given.

Using the standard techniques, one can prove the following propositions.

Proposition 1 [13, Theorem 1.5] *Let U' be a submodule of the right R-module U, then the following statements are equivalent:*

(1) *U' is (m, n)-pure in U.*

(2) *For every (n, m)-presented left R-module V, the canonical map* $\text{Hom}_R(V, U) \to \text{Hom}_R(V, U/U')$ *is epic.*

(3) *$(U')^m \cap U^n C = (U')^n C$ for all $C \in R^{n\times m}$.*

Proposition 2 [13, Theorem 2.4] *Suppose that $A_R \leq B_R$ and B_R is (m, n)-injective, then A is (m, n)-injective if and only if A is (n, m)-pure in B.*

Proposition 3 [13, Theorem 3.6] *Let $U'_R \leq U_R$. Then,*

(1) *If U/U' is (m, n)-flat, then U' is (m, n)-pure in U.*
(2) *If U' is (m, n)-pure in U and U is (m, n)-flat, then also U/U' is (m, n)-flat.*

Proposition 4 *Let M be a left R-module. Then, the following statements are equivalent:*

(1) *M is P-injective.*
(2) $\mathrm{Ext}^1_R(R/Ra, M) = 0$ *for all $a \in R$.*
(3) *Every exact sequence of left R-modules $0 \to M \to M' \to M'' \to 0$ is $(1, 1)$-pure.*
(4) *There exists a $(1, 1)$-pure exact sequence of left R-modules $0 \to M \to M' \to M'' \to 0$, where M' is P-injective.*
(5) $\mathbf{r}_M\mathbf{l}(a) = aM$ *for all $a \in R$.*

Proof (1) $\Rightarrow$ (3) by [13, Theorem 2.2]. (3) $\Rightarrow$ (4) is clear. (4) $\Rightarrow$ (1) by Proposition 2. The other implications are easy. □

2 G-Morphic Rings

In order to facilitate, we call a ring R *left G-morphic* if it is left generalized morphic. Clearly, left G-morphic rings are left 1-coherent. However, the converse is false. For example, Zhu and Ding give a left and right artinian ring which is not right G-morphic [12, Example 2.7]. In fact, even if R is a commutative artinian local ring, it need not be G-morphic.

Example 1 Let F be a field, and let

$$R = \left\{ \begin{bmatrix} a & b & c \\ & a & 0 \\ & & a \end{bmatrix} \middle| \; a, b, c \in F \right\}.$$

Then, R is a commutative artinian local ring, but R is not G-morphic.

Proof It is obvious that R is a commutative local ring with unit element

$$1_R = \begin{bmatrix} 1_F & 0 & 0 \\ & 1_F & 0 \\ & & 1_F \end{bmatrix},$$

the set of zero divisors of R is

$$D = \left\{ \begin{bmatrix} 0 & b & c \\ & 0 & 0 \\ & & 0 \end{bmatrix} \middle| \; b, c \in F \right\}, \quad D^2 = 0,$$

D is the unique maximal ideal of R. Noting that

$${}_RD = R \begin{bmatrix} 0 & 1_F & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix} \oplus R \begin{bmatrix} 0 & 0 & 1_F \\ & 0 & 0 \\ & & 0 \end{bmatrix}$$

and that $R\begin{bmatrix} 0 & 1_F & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix}$ and $R\begin{bmatrix} 0 & 0 & 1_F \\ & 0 & 0 \\ & & 0 \end{bmatrix}$ are simple R-modules, we have that R is artinian. We assert that

$$\mathbf{l}\begin{bmatrix} 0 & 1_F & 1_F \\ & 0 & 0 \\ & & 0 \end{bmatrix} = D \neq Rd \quad \text{for all } \ d \in D.$$

Otherwise, if $D = Rd$ for some $d = \begin{bmatrix} 0 & b & c \\ & 0 & 0 \\ & & 0 \end{bmatrix}$, let

$$\begin{bmatrix} 0 & 1_F & 1_F \\ & 0 & 0 \\ & & 0 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ & a_1 & 0 \\ & & a_1 \end{bmatrix} \begin{bmatrix} 0 & b & c \\ & 0 & 0 \\ & & 0 \end{bmatrix},$$

then $b = c$. Observing that

$$\begin{bmatrix} 0 & 0 & 1_F \\ & 0 & 0 \\ & & 0 \end{bmatrix} \in D = Rd,$$

we have $0 = 1_F$, a contradiction. Therefore, R is not G-morphic. □

Theorem 1 *Let R be a left G-morphic ring, and let M be a* $(1, 1)$*-presented left R-module. Then, $fd(M) = pd(M)$.*

Proof Clearly, $fd(M) \leq pd(M)$. Conversely, suppose that $fd(M) = n < \infty$. Since M is $(1, 1)$-presented, there exists an exact sequence $0 \to Ra_1 \to R \to M \to 0$. But R is left G-morphic, we have an exact sequence $0 \to Ra_2 \to R \to Ra_1 \to 0$. Continue in this way, we obtain an exact sequence $0 \to Ra_n \to R \to \cdots \to R \to M \to 0$. Since $fd(M) = n,\ Ra_n$ is flat. Note that Ra_n is finite presented, it is projective. Therefore, $pd(M) \leq n$. □

Theorem 2 *Let R be a left G-morphic ring, $a \in R,\ n \geq 0$. Then, $pd(R/Ra) \leq n$ if and only if $\mathrm{Ext}_R^{n+1}(R/Ra, R/Rb) = 0$ for all $b \in R$.*

Proof "⇒" It is clear.

"⇐" We use induction on n. If $n = 0$, by the exact sequence

$$0 \to Ra \to R \to R/Ra \to 0 \tag{1}$$

we have an exact sequence $\mathrm{Hom}_R(R, Ra) \to \mathrm{Hom}_R(Ra, Ra) \to \mathrm{Ext}_R^1(R/Ra, Ra)$. Since R is left G-morphic, Ra is $(1, 1)$-presented, and then $\mathrm{Ext}_R^1(R/Ra, Ra) = 0$ by hypothesis. Hence, the homomorphism $\mathrm{Hom}_R(R, Ra) \to \mathrm{Hom}_R(Ra, Ra)$ is epic. Thus, the exact sequence (1) is split, this follows that R/Ra is projective, that is, $pd(R/Ra) = 0$. If $n \geq 1$, for every $(1, 1)$-presented left R-module B, we have an exact sequence $0 = \mathrm{Ext}_R^n(R, B) \to \mathrm{Ext}_R^n(Ra, B) \to \mathrm{Ext}_R^{n+1}(R/Ra, B) \to \mathrm{Ext}_R^{n+1}(R, B) = 0$, this follows that $\mathrm{Ext}_R^n(Ra, B) \cong \mathrm{Ext}_R^{n+1}(R/Ra, B) = 0$. Since R is left G-morphic, Ra is $(1, 1)$-presented, hence $pd(Ra) \leq n - 1$ by induction hypothesis, and whence $pd(R/Ra) \leq n$. □

Corollary 1 *Let R be a left G-morphic ring, $a \in R,\ n \geq 0$. If $pd(R/Ra) = n$, then* $\mathrm{Ext}_R^n(R/Ra, R) \neq 0$.

Proof Since $pd(R/Ra) = n$, by Theorem 2, there exists $b \in R$ such that $\mathrm{Ext}_R^n(R/Ra, R/Rb) \neq 0$. But Rb is (1, 1)-presented because R is left G-morphic, $\mathrm{Ext}_R^{n+1}(R/Ra, Rb) = 0$ again by Theorem 2. Then, we get an exact sequence

$$\mathrm{Ext}_R^n(R/Ra, R) \to \mathrm{Ext}_R^n(R/Ra, R/Rb) \to 0,$$

it shows that $\mathrm{Ext}_R^n(R/Ra, R) \neq 0$. □

Lemma 1 *Every* (m, n)*-presented* (n, m)*-flat left* R*-module is projective.*

Proof Let A be an (m, n)-presented (n, m)-flat left R-module. Since A is (m, n)-presented, there exists an exact sequence of left R-modules $0 \to K \to R^m \to A \to 0$, where K is n-generated. But A is (n, m)-flat, by Proposition 3(1), K is (n, m)-pure in R^m. So by Proposition 1, the canonical map $\mathrm{Hom}_R(A, R^m) \to \mathrm{Hom}_R(A, A)$ is epic, which implies that A is isomorphic to a direct summand of R^m, and hence A is projective. □

Corollary 2 [8, Corollary 3.58] *Every finitely presented flat module is projective.*

Theorem 3 *The following statements are equivalent for a ring* R.

(1) *R is left PP.*
(2) *R is left G-morphic and every principal left ideal of R is P-flat.*

Proof (1) ⇒ (2) is trivial.

(2) ⇒ (1). Assume (2). Then, for each $a \in R$, Ra is (1, 1)-presented and P-flat, so Ra is projective by Lemma 1. □

Corollary 3 *If R is a domain, then R is left PP if and only if every principal left ideal of R is P-flat.*

Proposition 5 *Let R be a left G-morphic ring. If N_1 is a P-injective submodule of a P-injective left R-module N, then N/N_1 is P-injective.*

Proof Let $a \in A$. Since R is left G-morphic, $Ra \cong R/Rb$ for some $b \in R$, and so $\mathrm{Ext}_R^1(Ra, N_1) = 0$. Noting that $\mathrm{Ext}_R^1(Ra, N_1) \cong \mathrm{Ext}_R^2(R/Ra, N_1)$, we have $\mathrm{Ext}_R^2(R/Ra, N_1) = 0$. Thus, from the exact sequence

$$0 = \mathrm{Ext}_R^1(R/Ra, N) \to \mathrm{Ext}_R^1(R/Ra, N/N_1) \to \mathrm{Ext}_R^2(R/Ra, N_1) = 0,$$

we get that $\mathrm{Ext}_R^1(R/Ra, N/N_1) = 0$. Hence, N/N_1 is P-injective. □

3 G-Regular Rings

In this section, we study left G-morphic left P-injective rings. Let M be a right R-module. The group $M^* = \mathrm{Hom}_R(M, R)$ becomes a left R-module which we call the dual of M. If we do it again to get $M^{**} = \mathrm{Hom}_R(M^*, R)$, which is a right R-module. There is a natural homomorphism of M to M^{**}, $M \xrightarrow{\sigma} M^{**}$ caused by considering the elements of M as homomorphisms of M^* into R. Following the terminology of Bass [1], M is said to be *torsionless* if σ is a monomorphism, *reflexive* if σ is an isomorphism.

Theorem 4 *Let R be a left G-morphic ring. Then, the following statements are equivalent:*

(1) *R is left P-injective.*
(2) $\mathrm{Ext}^1_R(R/Ra, R) = 0$ *for all $a \in R$.*
(3) *Every injective right R-module is P-flat.*
(4) *Every P-flat left R-module is P-injective.*
(5) *Every projective left R-module is P-injective.*
(6) *Every right R-module is a submodule of a P-flat right R-module.*
(7) *The right R-module R/aR is reflexive for each $a \in R$.*
(8) *The right R-module R/aR is torsionless for each $a \in R$.*

Proof (1) ⇒ (2); (4) ⇒ (5) ⇒ (1); (3) ⇒ (6); and (7) ⇒ (8) are obvious.

(2) ⇒ (3). Let E be an injective right R-module. Then, for any $a \in R$, since R is left G-morphic, R/Ra has a projective resolution each of whose terms is R, by a remark of [8, Theorem 9.51], we have

$$\mathrm{Tor}^R_1(E, R/Ra) \cong \mathrm{Tor}^R_1(\mathrm{Hom}_R(R, E), R/Ra) \cong \mathrm{Hom}_R(\mathrm{Ext}^1_R(R/Ra, R), E).$$

By (2), $\mathrm{Ext}^1_R(R/Ra, R) = 0$, so $\mathrm{Tor}^R_1(E, R/Ra) = 0$, i.e., E is P-flat.

(3) ⇒ (1). Let C_R be an injective cogenerator and $a \in R$. Since R is left G-morphic, by a remark of [8, Theorem 9.51] again, we have

$$\mathrm{Tor}^R_1(C, R/Ra) \cong \mathrm{Tor}^R_1(\mathrm{Hom}_R(R, C), R/Ra) \cong \mathrm{Hom}_R(\mathrm{Ext}^1_R(R/Ra, R), C).$$

By (3), $\mathrm{Tor}^R_1(C, R/Ra) = 0$. Thus, $\mathrm{Ext}^1_R(R/Ra, R) = 0$, and then R is left P-injective.

(1) ⇒ (4). Let U be a P-flat left R-module. Take an exact sequence of left R-modules $0 \to K \to F \to U \to 0$, where F is a free module. Since ${}_RR$ is P-injective, F is also P-injective. But U is P-flat, by Proposition 3(1), K is (1, 1)-pure in F. And hence K is also P-injective by Proposition 2. Since R is left G-morphic, by Proposition 5, U is P-injective.

(6) ⇒ (3). Let E be an injective right R-module. By (6), we have an exact sequence of right R-modules $0 \to E \to U \to U/E \to 0$ with U P-flat. Hence, $U \cong E \oplus U/E$, and then E is P-flat.

(1) ⇒ (7). Since R is left G-morphic and left P-injective, by [12, Lemma 3.6], $\mathrm{Ext}^n_R(R/Ra, R) = 0$ for all $a \in R$ and all positive integers n. Let $R \stackrel{d_1}{\to} R \stackrel{d_0}{\to} R/aR \to 0$ be exact, by [4, Lemma 2.2], we have an exact sequence

$$0 \to \mathrm{Ext}^1_R(N, R) \to R/aR \to (R/aR)^{**} \to \mathrm{Ext}^2_R(N, R) \to 0,$$

where $N = R^*/\mathrm{im}(d_1^*)$ is a (1, 1)-presented left R-module. Thus

$$\mathrm{Ext}^1_R(N, R) = \mathrm{Ext}^2_R(N, R) = 0,$$

and then $R/Ra \to (R/Ra)^{**}$ is an isomorphism, as required.

(8) ⇒ (1). To prove (1), we need only to prove that if $a, b \in R$ with $\mathbf{l}(a) \subseteq \mathbf{l}(b)$ then $b = ac$ for some $c \in R$ by [7, Lemma 1.1]. Indeed, if $b \notin aR$, then $0 \neq b + aR \in R/aR$. Since R/aR is torsionless, it embeds in a direct product of R and so there exists a right R-homomorphism $g : R/aR \to R$ such that $g(b + aR) \neq 0$. Then, $g(1 + aR) \notin \mathbf{l}(b)$ but $g(1 + aR) \in \mathbf{l}(a)$, a contradiction. □

Recall that a ring R is regular if and only if R is left PP and left P-injective.

Definition 1 A ring R is called left G-regular, if R is left G-morphic and left P-injective.

Clearly, regular rings are left G-regular, but the inverse implication is not true.

Example 2 The ring $R = \mathbb{Z}_4 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$ is a commutative G-regular ring, but it is not regular.

Proof It is obvious that R is a commutative G-regular ring. Since $\mathbf{l}(\overline{2}) = \{\overline{0}, \overline{2}\}$ contains no nonzero idempotent elements, it is not a direct summand of ${}_RR$, so that $R\overline{2}$ is not projective, and thus R is not PP. Therefore, R is not regular. □

Example 3

(1) Let R be the ring of 2×2 upper triangular matrices over the field $\mathbb{Z}_2$. Then, R is left G-morphic, but it is not left G-regular.
(2) The ring $\mathbb{Z}$ of integers is G-morphic but not G-regular.

Proof (1). Let e_{ij} be the 2×2 matrices over the field $\mathbb{Z}_2$ having a lone 1 as its (i, j)-entry and all other entries $0, i, j = 1, 2$. Then, by a routine computation, we get $\mathbf{l}(0) = R, \mathbf{l}(e_{11} + e_{22}) = \mathbf{l}(e_{11} + e_{12} + e_{22}) = 0, \mathbf{l}(e_{11}) = Re_{22}, \mathbf{l}(e_{12}) = Re_{22}, \mathbf{l}(e_{22}) = Re_{11}, \mathbf{l}(e_{11} + e_{12}) = Re_{22}, \mathbf{l}(e_{12} + e_{22}) = R(e_{11} + e_{12})$; hence, R is left G-morphic. But since

$$\mathbf{rl}(e_{12}) = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix} = e_{12}R,$$

R is not left P-injective by Proposition 4, and so R is not left G-regular.

(2). It is obvious. □

Theorem 5 *For a ring R, the following statements are equivalent:*

(1) *R is left G-regular.*
(2) *For any $a \in R$, there exists $b \in R$ such that $\mathbf{l}(a) = Rb$ and $\mathbf{r}(b) = aR$.*

Proof (1)⇒ (2). Let $a \in R$. Since R is left G-morphic, by [12, Corollary 2.3], there exists $b \in R$ such that $\mathbf{l}(a) = Rb$, which implies that $\mathbf{rl}(a) = \mathbf{r}(b)$. But R is left P-injective, by Proposition 4, we have $\mathbf{rl}(a) = aR$, and hence $\mathbf{r}(b) = aR$.

(2)⇒ (1). Assume (2), then it is clear that R is left G-morphic. For any $a \in R$, by hypothesis, there exists $b \in R$ such that $\mathbf{l}(a) = Rb$ and $\mathbf{r}(b) = aR$, so $\mathbf{rl}(a) = \mathbf{r}(b) = aR$, and thus R is left P-injective by Proposition 4. Hence, R is left G-regular. □

Theorem 6 *For a left G-regular ring R, the following statements are equivalent:*

(1) *R is regular.*
(2) *M^* is P-flat for every left R-module M.*
(3) *R is left PP.*
(4) *R is right PP.*
(5) *Every principal right ideal of R is P-flat.*
(6) *Every principal left ideal of R is P-flat.*

Proof (1)⇒ (2) through (6) are clear. (6)⇒ (3) by Theorem 3.

(2)⇒ (1). For any $a \in R$, since R is left G-regular, by Theorem 4, $R/aR \cong (R/aR)^{**} = ((R/aR)^*)^*$. But $((R/aR)^*)^*$ is P-flat by hypothesis, so R/aR is P-flat and hence projective by Lemma 1. It follows that aR is a direct summand of R_R. Consequently, R is regular.

(3)⇒ (1). By [10, Theorem 3], left P-injective left PP ring is regular.

(4)⇒ (3). Since R is right PP, every principal right ideal of R is projective and hence flat. By [5, Theorem 2.2], every principal left ideal of R is flat. Note that R is left G-morphic , by Theorem 3, R is left PP.

(5)⇒ (1). For any $a \in R$, since R is left P-injective, we have $\mathbf{rl}(a) = aR$. Since R is left G-morphic, $\mathbf{l}(a) = Rb$ for some $b \in R$. Thus $\mathbf{r}(b) = aR$, and hence $R/aR = R/\mathbf{r}(b) \cong bR$ is P-flat by hypothesis. By Lemma 1, R/aR is projective, so that aR is a direct summand of R_R and (1) follows. □

Theorem 7 *The following statements are equivalent for a ring R:*

(1) *R is a strongly regular ring.*
(2) *R is a reduced left G-regular ring.*

Proof (1) ⇒ (2). It is obvious.

(2) ⇒ (1). Let $a \in R$. Since R is reduced, we have $\mathbf{l}(a^2) = \mathbf{l}(a)$. But R is left P-injective, by Proposition 4, we have $a^2R = aR$, and so $a = a^2b$ for some $b \in R$, as required. □

Lemma 2 *If R is a left G-regular ring with ACC on principal left ideals, then R is left perfect.*

Proof Suppose that $a_1R \supseteq a_2R \supseteq \cdots$. Then, $\mathbf{l}(a_1) \subseteq \mathbf{l}(a_2) \subseteq \cdots$. Since R is a left G-regular ring with ACC on principal left ideals, by Theorem 5, there exists a positive integer n such that $\mathbf{l}(a_{n+1}) = \mathbf{l}(a_{n+2}) = \cdots$. Noting that R is left P-injective, by [7, Lemma 1.1], we have $a_{n+1}R = a_{n+2}R = \cdots$. Therefore, R is left perfect. □

Lemma 3 *If R is a left perfect semiprime ring, then any nonzero right ideal of R is not nil.*

Proof Let I be any nonzero right ideal of R. We claim that there exists $0 \neq b \in I$ such that bR is simple. If not, let $0 \neq a_1 \in I$, then there is $a_2 \in R$ such that $a_1a_2 \neq 0$ and $a_1R \neq a_1a_2R$. Since $0 \neq a_1a_2 \in I$, there exists $a_3 \in R$ such that $a_1a_2a_3 \neq 0$ and $a_1a_2R \neq a_1a_2a_3R$. Continuing in this way, we get a strictly descending chain of right ideals of R

$$a_1R \supsetneqq a_1a_2R \supsetneqq a_1a_2a_3R \supsetneqq \cdots,$$

a contradiction. This proves the claim.

Since R is semiprime, there exists $c \in R$ such that $bcb \neq 0$, and so $bR = bcbR$ because bR is simple. Since $cb \neq 0$, $c \notin \mathbf{l}(b) = \mathbf{l}(bcb)$; hence, $(cb)^2 \neq 0$. Continuing this process, we have that $(cb)^n \neq 0$ for every positive integer n. Therefore, I is not nil. □

Theorem 8 *The following statements are equivalent for a ring R:*

(1) *R is a semisimple artinian ring.*
(2) *R is a left G-regular semiprime ring with ACC on principal left ideals.*

Proof (1) ⇒ (2). It is clear.

(2) ⇒ (1). Since R is a left G-regular ring with ACC on principal left ideals, by Lemma 2, R is left perfect. This follows that R is semilocal and $J(R)$ is nil. But R is semiprime, by Lemma 3, $J(R) = 0$. Therefore, R is semisimple artinian. □

Proposition 6 *Let R be a left and right G-morphic ring. Then, the following statements are equivalent:*

(1) *R is left P-injective.*
(2) *Every P-injective right R-module is P-flat.*
(3) *Every injective right R-module is P-flat.*

Proof (2) ⇒ (3) is obvious. (3) ⇒ (1) by Theorem 4.

(1) ⇒ (2). Let M be a P-injective right R-module. Since R is right G-morphic and $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, by a remark of [8, Theorem 9.51], we have

$$\mathrm{Tor}_1^R\left(\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}), R/aR\right) \cong \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{Ext}_R^1(R/aR, M), \mathbb{Q}/\mathbb{Z}\right).$$

Since M is P-injective, $\mathrm{Ext}_R^1(R/aR, M) = 0$, so $\mathrm{Tor}_1^R(\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}), R/aR) = 0$, and hence $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a P-flat left R-module. But R is left G-morphic and left P-injective, by Theorem 4, M^+ is P-injective. Thus, for any $b \in R$, we have

$$0 = \mathrm{Ext}_R^1(R/Rb, \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{Tor}_1^R(M, R/Rb), \mathbb{Q}/\mathbb{Z}\right).$$

Hence, $\mathrm{Tor}_1^R(M, R/Rb) = 0$ because $\mathbb{Q}/\mathbb{Z}$ is a cogenerator, i.e., M is P-flat. □

Proposition 7 *Let* $0 \to A \to B \to C \to 0$ *be an exact sequence of left R-modules. Then,*

(1) *If A, C are P-flat, then B is also P-flat.*
(2) *If R is right G-morphic and B, C are P-flat, then A is also P-flat.*
(3) *If R is left G-regular and A, B are P-flat, then C is also P-flat.*

Proof (1). It is easy to prove by using the exact sequence

$$\mathrm{Tor}_1^R(R/aR, A) \to \mathrm{Tor}_1^R(R/aR, B) \to \mathrm{Tor}_1^R(R/aR, C),$$

where $a \in R$.

(2). For any $a \in R$, the exact sequence

$$0 = \mathrm{Tor}_2^R(R, C) \to \mathrm{Tor}_2^R(R/aR, C) \to \mathrm{Tor}_1^R(aR, C) \to \mathrm{Tor}_1^R(R, C) = 0$$

implies that $\mathrm{Tor}_2^R(R/aR, C) \cong \mathrm{Tor}_1^R(aR, C)$. Since R is right G-morphic, aR is $(1, 1)$-presented. Note that C is P-flat, we have $\mathrm{Tor}_1^R(aR, C) = 0$, and then $\mathrm{Tor}_2^R(R/aR, C) = 0$. This shows that $\mathrm{Tor}_1^R(R/aR, A) = 0$ from the exact sequence $0 = \mathrm{Tor}_2^R(R/aR, C) \to \mathrm{Tor}_1^R(R/aR, A) \to \mathrm{Tor}_1^R(R/aR, B) = 0$. Therefore, A is P-flat.

(3). Since R is left G-regular and A is a P-flat left R-module, by Theorem 4, A is P-injective, and so A is a $(1, 1)$-pure submodule of B by Proposition 4. But B is P-flat, by Proposition 3(2), C is P-flat. □

Proposition 8 *Let R be a right G-regular left G-morphic ring, and let M be a right R-module. Then,* $P\text{-}fd(M_R) = 0$ *or* ∞.

Proof If $P\text{-}fd(M_R) \leq 1$, since R is left G-morphic, by [12, Lemma 3.8], there exists an exact sequence of right R-modules $0 \to F_1 \to F_0 \to M \to 0$, where F_1, F_0 are P-flat. Since R is right G-regular, by Proposition 7(3), M is P-flat, i.e., $P\text{-}fd(M_R) = 0$. Assume that $1 < P\text{-}fd(M_R) = n$, then by [12, Lemma 3.8], there exists a P-flat resolution of M $0 \to F_n \stackrel{d_n}{\to} F_{n-1} \stackrel{d_{n-1}}{\to} \cdots \to F_0 \stackrel{d_0}{\to} M \to 0$. The exact sequence $0 \to F_n \stackrel{d_n}{\to}$

$F_{n-1} \stackrel{d_{n-1}}{\rightarrow} \mathrm{im}(d_{n-1}) \rightarrow 0$ implies that $\mathrm{im}(d_{n-1})$ is P-flat by Proposition 7(3). Note that the sequence $0 \rightarrow \mathrm{im}(d_{n-1}) \rightarrow F_{n-2} \stackrel{d_{n-2}}{\rightarrow} \cdots \rightarrow F_0 \stackrel{d_0}{\rightarrow} M \rightarrow 0$ is exact, we have that $P\text{-}fd(M_R) \leq n-1$ by [12, Lemma 3.8] because R is left G-morphic, a contradiction. So $P\text{-}fd(M_R) = 0$ or ∞. □

Clearly, R is a regular ring if and only if $r.P\text{-}f\ \dim(R) = 0$. By [12, Lemma 3.8], for a left G-morphic ring R, $r.P\text{-}f\ \dim(R) \leq n$ if and only if $\mathrm{Tor}_{n+1}^R(M, R/Ra) = 0$ for all right R-modules M and all $a \in R$.

Corollary 4 *Let R be a right G-regular left G-morphic ring. Then, $r.P\text{-}f\ \dim(R) = 0$ or ∞.*

Acknowledgments The author is very grateful to the referee for the useful comments.

References

1. Bass, H.: Finitistic dimension and a homological generalization of semi-primary rings. Trans. Am. Math. Soc. **95**, 466–488 (1960)
2. Chen, J.L., Ding, N.Q., Li, Y.L., Zhou, Y.Q.: On (m, n)-injectivity of modules. Commun. Algebra **29**, 5589–5603 (2001)
3. Dobbs, D.E.: On n-flat modules over a commutative ring. Bull. Aust. Math. Soc. **43**, 491–498 (1991)
4. Jain, S.: Flat and FP-injectivity. Proc. Am. Math. Soc. **41**, 437–442 (1973)
5. Jøndrup, S.: p.p. rings and finitely generated flat ideals. Proc. Am. Math. Soc. **28**, 431–435 (1971)
6. Nicholson, W.K., Sánchez Campos, E.: Rings with the dual of the isomorphism theorem. J. Algebra **271**, 391–406 (2004)
7. Nicholson, W.K., Yousif, M.F.: Principally injective rings. J. Algebra **174**, 77–93 (1995)
8. Rotman, J.J.: An Introduction to Homological Algebra. Academic, New York (1979)
9. Shamsuddin, A.: n-Injective and n-flat modules. Commun. Algebra **29**, 2039–2050 (2001)
10. Xue, W.M.: On PP rings. Kobe J. Math. **7**, 77–80 (1990)
11. Zhang, X.X., Chen, J.L., Zhang, J.: On (m, n)-injective modules and (m, n)-coherent rings. Algebra Colloq. **12**, 149–160 (2005)
12. Zhu, H.Y., Ding, N.Q.: Generalized morphic rings and their applications. Commun. Algebra **35**, 2820–2837 (2007)
13. Zhu, Z.M., Chen, J.L., Zhang, X.X.: On (m, n)-purity of modules. East-West J. Math. **5**, 35–44 (2003)