Vietnam J. Math. (2015) 43:501–513
DOI 10.1007/s10013-015-0127-1

Subgroup Structure and Representations of Finite and Algebraic Groups

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Received: 8 June 2014 / Accepted: 6 September 2014 / Published online: 7 March 2015

Abstract What can one say about maximal subgroups, or, more generally, the subgroup structure of simple, finite, or algebraic groups? In this survey, we will discuss how group representation theory helps us study this classical problem. These results have been applied to various problems, particularly in group theory, number theory, and algebraic geometry.

Keywords Representation theory · Subgroup structure · Maximal subgroups · Classical groups

Mathematics Subject Classification (2010) 20C15 · 20C20 · 20C30 · 20C33 · 20E28 · 20G40

1 Introduction

1.1 Why the Subgroup Structure?

Why would one be interested in the subgroup structure of finite and algebraic groups? To facilitate the discussion, let us begin with two examples, an elementary one and another one more technical.

Example 1 Can we solve the equation $f(x) = x^5 - 6x + 3 = 0$ *by radicals?*

This kind of questions was of great interest to mathematicians, at least until the nineteenth century. Of course, nowadays, everybody knows that the question can be answered by *Galois theory*, and the answer is "yes" if and only if the *Galois group*

This author was Plenary speaker at the Vietnam Congress of Mathematicians 2013.

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$G := \mathrm{Gal}(f) := \mathrm{Gal}(\mathbb{Q}(f)/\mathbb{Q})$ is solvable. So, one needs to find what G is. An elementary inspection reveals that

(i) f has three real roots α_1, α_2, and α_3 and two non-real (complex-conjugate) roots α_4 and α_5, and furthermore,
(ii) f is irreducible over $\mathbb{Q}$ by *Eisenstein's criterion*.

Now, (ii) implies that G is a transitive subgroup of the symmetric group $\mathsf{S}_5 = \mathrm{Sym}(\{\alpha_1, \dots, \alpha_5\})$. On the one hand, by (i), the complex conjugation induces the transposition $(4, 5)$ as an element of G. At this point, *the knowledge on subgroups of* S_5 allows us to conclude that $G = \mathsf{S}_5$, and so, it is not solvable. Hence, the answer to the stated question is "no"!

Example 2 Let $V = \mathbb{C}^d$ and let $G < GL(V)$ be a finite subgroup. Let us consider the quotient variety $Y = V/G$. Formally, one would consider homogeneous polynomials $f_1, \dots, f_m$ that generate the ring of all G-invariant polynomials in variables $x_1, \dots, x_d$:

$$\mathbb{C}[x_1, \dots, x_d]^G = \mathbb{C}[f_1, \dots, f_m].$$

Then V/G is defined to be $\varphi(V)$, where

$$\varphi : V \to \mathbb{C}^m, \quad \varphi(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_m(\mathbf{x})).$$

Unless G is generated by *complex reflections*, the variety V/G is singular. So, let us consider a *resolution* $f : X \to Y$ (i.e., X is a non-singular variety, and the morphism f is birational). Now one can relate the first Chern classes of X and Y via

$$K_X = f^* K_Y + \sum_i a_i E_i,$$

where $a_i \in \mathbb{Q}$ and the sum runs over irreducible exceptional divisors E_i. In this case, Y is called *canonical* if $a_i \geq 0$ for all i, and *terminal* if $a_i > 0$ for all i. Furthermore, the resolution f is called *crepant* if $a_i = 0$ for all i, cf. [52, 53].

Question 1 When does V/G admit a crepant resolution? When is V/G not terminal?

A partial answer to this question is given in [25], which relies on the *subgroup structure* of $GL(V)$.

We have seen that the subgroup structure of certain finite and algebraic groups plays a crucial role in Examples 1 and 2. These and other applications show that *it is important to understand the subgroup structure of G, where G is a (simple) group, finite or algebraic*. This subject has a rich history, dating back at least to Galois' 1832 letter to Chevalier [17]. It has become much more active since the *classification of finite simple groups* was announced to be completed in 1982 (indeed, by the Jordan–Hölder theorem, finite simple groups are building blocks of any finite group).

1.2 Aschbacher's Theorem

To understand the subgroup structure of a given group G, one could try to employ an "inductive" approach and go along a descending chain

$$G = G_0 \overset{\max}{>} G_1 \overset{\max}{>} G_2 \overset{\max}{>} \dots,$$

where each G_i is a *maximal* subgroup of G_{i-1}. Certainly, if G is not finite, then such a chain may not terminate. On the other hand, the importance of maximal subgroups of G can also be seen from the viewpoint of the *primitive permutation group theory*: if G acts transitively on a set Ω, then the action is primitive if and only if the point stabilizer G_α for $\alpha \in \Omega$ is maximal in G. If G is an algebraic group, then, in many of the situations under consideration, one may restrict oneself to *Zariski closed* subgroups and so the maximality of G_i is interpreted as maximal among the Zariski closed subgroups of G_{i-1}.

Thus, one would like to focus on understanding maximal subgroups of finite or algebraic groups. In fact, in most problems involving a finite primitive permutation group G, the *Aschbacher–O'Nan–Scott theorem* [2] allows one to concentrate on the case where G is *almost quasi-simple*, i.e., $S \lhd G/Z(G) \leq \mathrm{Aut}(S)$ for a non-abelian simple group S. Ignoring technicalities, in this survey, we will refer to any almost quasi-simple group as a *simple* group. The results of Liebeck–Praeger–Saxl [42] and Liebeck–Seitz [43] then allow one to assume furthermore that G is a finite classical group. By a *classical group* $\mathrm{Cl}(V)$ *over a field* $\mathbb{F}$ we usually mean

- the group $GL(V)$ of all invertible linear transformations of a vector space $V = \mathbb{F}^d$ over $\mathbb{F}$,
- the subgroup $Sp(V)$, respectively $O(V)$, of all $f \in GL(V)$ that preserve a non-degenerate alternating, respectively quadratic, form on V,
- the subgroup $U(V)$ of all $f \in GL(V)$ that preserve a non-degenerate Hermitian form on V, as well as
- the commutator subgroup $[G, G]$, where G is one of the above groups.

We will use the notation $\mathrm{Cl}(V)$ to denote one of the classical groups $GL(V)$, $Sp(V)$, $O(V)$, and $U(V)$. Taking $\mathbb{F} = \mathbb{C}$, we see that classical groups account for most of *complex simple Lie groups*, whereas the classical groups over finite fields $\mathbb{F} = \mathbb{F}_q$, together with their *exceptional* and *twisted* analogs, form the main source of *finite simple groups*.

Theorem 1 (Classification of finite simple groups) *Any finite non-abelian simple group is either*

(i) *an alternating group* A_n, $n \geq 5$;
(ii) *a simple group of Lie type (like* $PSL_n(\mathbb{F}_q)$*); or*
(iii) *one of the 26 sporadic finite simple groups.*

A fundamental theorem concerning maximal subgroups of classical groups was proved by M. Aschbacher in [1]. For the exposition's purposes, we will state Aschbacher's theorem in a very simplified version, referring the interested reader to [1] for its full version:

Theorem 2 (Aschbacher [1]) *Let* $\mathcal{G} = \mathrm{Cl}(V)$ *be a classical group over an algebraically closed field* $\mathbb{F}$*, and let* $M < \mathcal{G}$ *be a maximal Zariski closed subgroup of* $\mathcal{G}$*. Then*

$$M \in \bigcup_{i=1}^{4} \mathcal{C}_i \cup \mathcal{S},$$

where $\mathcal{C}_i$, $1 \leq i \leq 4$, *are collections of certain "natural" subgroups of* $\mathcal{G}$*, and* $\mathcal{S}$ *consists of the subgroups of the form* $M = N_{\mathcal{G}}(S)$ *for some simple closed subgroup* $S < \mathcal{G}$ *such that the restriction* $V{\downarrow_S}$ *is irreducible.*

Here,

- $\mathcal{C}_1$ consists of stabilizers $Stab_{\mathcal{G}}(U)$ of nonzero proper subspaces U of V (in particular, it includes the *parabolic* subgroups of $\mathcal{G}$);
- $\mathcal{C}_2$ consists of stabilizers of direct sum decompositions $V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$, where $V_1 \cong V_2 \cong \cdots \cong V_m$;
- $\mathcal{C}_3$ consists of normalizers $N_{\mathcal{G}}(P)$ of "special" finite p-subgroups $P < \mathcal{G}$; and
- $\mathcal{C}_4$ consists of stabilizers of tensor decompositions $V = V_1 \otimes V_2$, or $V = V_1 \otimes V_2 \otimes \cdots \otimes V_m$ with $V_1 \cong V_2 \cong \cdots \cong V_m$.

Example 3 Let us describe the classes $\mathcal{C}_i$ for $\mathcal{G} = GL_n(\mathbb{F})$. First, $\mathcal{C}_1$ consists of parabolic subgroups, which are $\mathcal{G}$-conjugate of the upper block-triangular subgroups

$$\left\{ \begin{pmatrix} *_m & * \\ 0 & *_{n-m} \end{pmatrix} \right\}, \quad 1 \leq m \leq n-1.$$

Next, the members of $\mathcal{C}_2$ are $\mathcal{G}$-conjugate to block-monomial subgroups

$$GL_d(\mathbb{F}) \wr \mathsf{S}_{n/d} = \langle \{\mathrm{diag}(*_d, *_m, \ldots, *_d)\},\ \mathsf{S}_{n/d} \rangle,$$

where $\mathsf{S}_{n/d}$ permutes $V_1, \ldots, V_{n/d}$, and $d|n$.

The class $\mathcal{C}_3$ is non-empty only when $n = p^s$ for some prime p, in which case $N_{\mathcal{G}}(P) = Z(\mathcal{G})P \cdot Sp_{2n}(p)$ belongs to $\mathcal{C}_3$ (if we assume $\mathbb{F} = \overline{\mathbb{F}}$), where $P = p^{1+2n}$ is an *extraspecial p-group* of order p^{1+2n}.

Finally, the members of $\mathcal{C}_4$ are $\mathcal{G}$-conjugate to $GL_a(\mathbb{F}) \otimes GL_b(\mathbb{F})$ with $ab = n$, and $GL_a(\mathbb{F}) \wr \mathsf{S}_b$ with $n = a^b$.

The proof of Aschbacher's Theorem 2 can be outlined as follows. Consider the M-module $V\downarrow_M$. If $V\downarrow_M$ is reducible, then $M \in \mathcal{C}_1$. So we assume $V\downarrow_M$ is irreducible. If $V\downarrow_M$ is imprimitive then $M \in \mathcal{C}_2$, and if $V\downarrow_M$ is tensor decomposable or tensor induced then $M \in \mathcal{C}_4$. Otherwise one can show that $M \in \mathcal{C}_3 \cup \mathcal{S}$.

As mentioned above, more precise versions of Aschbacher's Theorem 2 are given for finite classical groups in [1] (for which the union $\cup_{i=1}^4 \mathcal{C}_i$ is replaced by $\cup_{i=1}^8 \mathcal{C}_i$), as well as for the symmetric group S_n (which might be viewed as "$GL_n(\mathbb{F}_1)$").

1.3 The Converse?

Given the hypothesis of Aschbacher's Theorem 2 (so that $\mathcal{G}$ is a classical group over $\mathbb{F}$, and $\mathbb{F}$ is either algebraically closed or finite), assume now that $M \in \cup_{i=1}^4 \mathcal{C}_i \cup \mathcal{S}$. When can one say that M is indeed a maximal subgroup of $\mathcal{G}$?

If $M \in \cup_{i=1}^4 \mathcal{C}_i$, then the maximality of M has been determined by Kleidman and Liebeck [31].

So we may assume that $M = N_{\mathcal{G}}(S) \in \mathcal{S}$. Suppose in addition that M is *not* maximal. Then $M < N < \mathcal{G}$, where N is a maximal subgroup of G, and we can again apply Aschbacher's Theorem 2 to N: $N \in \cup_{i=1}^4 \mathcal{C}_i \cup \mathcal{S}$. Note that $V\downarrow_M$ is irreducible (as $M \in \mathcal{S}$), whence $N \notin \mathcal{C}_1$. Hence, we arrive at one of the following cases.

Case I. $N \in \mathcal{C}_2$. This case is being analyzed by Hiss, Husen, and Magaard [27].
Case II. $N \in \mathcal{C}_3$. This case is handled by Magaard and Tiep [48].
Case III. $N \in \mathcal{C}_4$. Since the tensor-induced subcase is treated in [48], we may assume for the simple subgroup $S < GL(V)$ that the S-module $V\downarrow_S = A \otimes B$ is irreducible

and tensor decomposable. Henceforth we will use the convention that all representations of algebraic groups in question are assumed to be *rational*.

Theorem 3 (Steinberg, Seitz) *Suppose S is a simple, finite or algebraic group over $\mathbb{F}_q$ with q a power of $p = \mathrm{char}(F)$. Suppose that a $\mathbb{F}S$-module V is irreducible and tensor decomposable. Then either*

(i) *V is described by Steinberg's tensor product theorem, or*
(ii) *$p = 2$ and S is of type C_n or F_4, or*
(iii) *$p = 3$ and S is of type G_2.*

Moreover, in cases (ii) and (iii)*, the module V is known explicitly.*

Aside from the situation considered in Theorem 3, we have three more possible scenarios, according to Theorem 1:

- S is one of the 26 sporadic simple groups—this case is the subject of ongoing work of the GAP-team [18];
- S is a (covering group) of S_n or A_n. This case is mostly resolved by work of Bessenrodt–Kleshchev [4–6] and Kleshchev–Tiep [37].
- S is a finite simple group of Lie type defined over a field of characteristic different from $\mathrm{char}(\mathbb{F})$. This case is the subject of the following theorem:

Theorem 4 (Magaard–Tiep [47]) *Suppose S is a finite simple group defined over $\mathbb{F}_q$ with q coprime to $p = \mathrm{char}(F)$, and suppose that the $\mathbb{F}S$ module V is irreducible and tensor decomposable. Then, modulo a few open cases, V is a Weil module of $S = Sp_{2n}(3)$ or $SU_n(2)$.*

Example 4 [47] The symplectic group $S = Sp_{2n}(3)$ (with $n \geq 2$) admits two complex Weil modules, A of dimension $(3^n - 1)/2$, and B of dimension $(3^n + 1)/2$, such that $A \otimes B$ is irreducible.

2 The Irreducible Restriction Problem

2.1 The Setup

The analysis of the maximality of $M \in \cup_{i=1}^4 \mathcal{C}_i \cup \mathcal{S}$ in Section 1.3 still leaves out the case $N \in \mathcal{S}$. In this, arguably the most challenging case, we have that

$$\mathcal{G} = \mathrm{Cl}(V) > N = N_{\mathcal{G}}(R) > M = N_{\mathcal{G}}(S),$$

where $R, S < \mathcal{G}$ are simple and $V\downarrow_R$, $V\downarrow_S$ are irreducible. The *Schreier hypothesis* (which is a consequence of Theorem 1) implies that in fact S embeds in R. Relabeling R and S by G and H, we thus arrive at the following problem:

Problem 1 (Irreducible restriction problem) Let G be a simple, finite, or algebraic, group, and let $\mathbb{F} = \overline{\mathbb{F}}$. Classify pairs (V, H), where V is an $\mathbb{F}G$-module and H is a proper simple subgroup of G such that $V\downarrow_H$ is irreducible.

Problem 1 turns out to be a deep problem encompassing many important questions in the representation theory of finite and algebraic groups.

2.2 Problem 1 for Simple Algebraic Groups

Let us first consider the situation where both G and H in Problem 1 "arise" from simple algebraic groups defined over $\mathbb{F}$.

- **Connected case**, i.e., G and H are both simple algebraic groups. Thus we have a rational $\mathbb{F}G$-module V (labeled by its *highest weight*) that is irreducible over H. In this case, Problem 1 is solved by Dynkin when $\mathrm{char}(\mathbb{F}) = 0$ [14], and Seitz [56] and Testerman [60] when $\mathrm{char}(\mathbb{F}) > 0$.
- **Disconnected case**. Here, we assume only that the connected components (of the identity) G° and H° are simple. This case is still incomplete by now, but significant results on it have been obtained by Ford [15, 16] and more recently by Testerman and her collaborators [9, 10].

2.3 Problem 1 for Symmetric and Alternating Groups

Now, we turn our attention to the case G is a finite simple group, and consider various cases according to Theorem 1.

Example 5 Let us consider Problem 1 for $G = \mathsf{S}_n$, $H = \mathsf{S}_{n-1}$.

Suppose first that we are in the **complex case**, i.e., $\mathbb{F} = \mathbb{C}$. Then the irreducible S_n module $V = S^\lambda$ is labeled by a partition $\lambda \vdash n$. The classical *branching rule* tells us that

$$V\downarrow_{\mathsf{S}_{n-1}} = \bigoplus_{X \text{any removable node}} S^{\lambda \setminus X}.$$

It follows that $V\downarrow_H$ is irreducible if and only if the Young diagram $Y(\lambda)$ has only one removable node, i.e., $\lambda = (a, a, \dots, a)$ with $n = ab$.

The **modular case**, i.e., when $0 < \mathrm{char}(\mathbb{F}) = p \le n$, turns out to be much more complicated. The list of $\mathbb{F}\mathsf{S}_n$-modules that are irreducible over S_{n-1} was predicted by the *Jantzen–Seitz conjecture*, which was subsequently proved by Kleshchev [33].

More generally, we describe the current status of Problem 1 for G a covering group of S_n or A_n.

- $G = \mathsf{S}_n, \mathsf{A}_n$. Saxl [54] solved Problem 1 in the case $\mathrm{char}(\mathbb{F}) = 0$. In the case $\mathrm{char}(\mathbb{F}) > 3$, the problem has been solved by Brundan and Kleshchev in [8] for $G = \mathsf{S}_n$, and by Kleshchev and Sheth in [34, 35] for $G = \mathsf{A}_n$. The case $\mathrm{char}(\mathbb{F}) = 2, 3$ is still incomplete, but significant results have been obtained by Kleshchev and Sheth [34], and more recently by Kleshchev, Sin, and Tiep [36].
- $G = 2\mathsf{S}_n, 2\mathsf{A}_n$. The complex case $\mathrm{char}(\mathbb{F}) = 0$ of Problem 1 is completely solved by Kleidman and Wales in [32]. The modular case $\mathrm{char}(\mathbb{F}) > 0$ is mostly understood by work of Kleshchev and Tiep [37]. The difficulty of the latter case is explained in part by the complicated nature of the modular spin representations of G. Even the irreducible

modular spin representations of low degree have only been recently determined by Kleshchev and Tiep in [40].

2.4 Problem 1 for G a Sporadic Group

Here G is one of the 26 sporadic simple groups. The GAP-team [18] has been working over the last 20 years (or so) with the goal to determine the irreducible $\mathbb{F}G$ modules, and further progress on Problem 1, especially for the largest sporadic simple groups, will depend largely on this.

2.5 The Defining Characteristic Case of Problem 1: $G \in \mathrm{Lie}(p)$, $p = \mathrm{char}(\mathbb{F})$

Here, G is a simple group of Lie type, defined over a field $\mathbb{F}_q$ with q a power of the prime $p = \mathrm{char}(\mathbb{F})$. We distinguish several cases according to Theorem 1 applied to H:

- $H \in \mathrm{Lie}(p)$. The situation here is well understood by work of Liebeck, Seitz, and Testerman.
- $H = \mathsf{A}_n$. Husen analyzed this situation in his 1997 Ph. D. Thesis.
- H is one of the 26 sporadic simple groups. Again, further progress in this case depends heavily on the GAP-team [18].
- $H \in \mathrm{Lie}(\ell)$ with $\ell \neq p$. Recent work of Magaard, Röhrle, and Testerman [46] essentially reduce the problem in this case to

Problem 2 Let G be a Zariski closed subgroup of a simple classical group $\mathcal{G} = \mathrm{Cl}(V)$ and let W be the largest composition factor of the $\mathcal{G}$-module $V \otimes V^*$, $\mathrm{Sym}^k(V)$, or $\wedge^k(V)$, for some "small" $k > 1$. When can $W \downarrow_G$ be irreducible?

A particular instance of problem 2 turns out to have important implications, particularly on the holonomy groups of vector bundles on a smooth complex projective variety [3].

Problem 3 (Kollár–Larsen problem on symmetric powers) Let $\mathbb{F} = \overline{\mathbb{F}}$ and let $V = \mathbb{F}^d$ with $d \geq 5$. Which Zariski closed subgroups of $\mathcal{G} = GL(V)$ act irreducibly on some symmetric power $\mathrm{Sym}^k(V)$ of V for some $k \geq 4$?

Theorem 5 (Guralnick–Tiep [24]) *Assume a Zariski closed subgroup H of $\mathcal{G} := GL(V)$ acts irreducibly on $\mathrm{Sym}^k(V)$ for some $k \geq 4$. Then $L \lhd H \leq N_{\mathcal{G}}(L)$, and one of the following holds:*

(i) $L \in \{SL(V), Sp(V)\}$;
(ii) $\mathrm{char}(\mathbb{F}) = p$, $L = SL_d(q)$, $SU_d(q)$, *or* $Sp_d(q)$, $q = p^a$ *and* $d = \dim(V)$;
(iii) $k = 4, 5$, *and* $(\dim(V), L) = (6, 2J_2), (12, 2G_2(4)), (12, 6Suz)$;
(iv) $k = 4, 5$, $p = 5, 7$, *and* $L = \mathit{Monster}$.

Balaji and Kollár have shown in [3] that Theorem 5 implies the following

Corollary 1 *Let E be a stable vector bundle on a smooth complex projective variety X of rank r different from* 2, 6, 12. *Then the following are equivalent:*

(i) $\mathrm{Sym}^k(E)$ *is stable for some* $k \geq 4$.

(ii) $\mathrm{Sym}^k(E)$ *is stable for every* $k \geq 4$.
(iii) *The commutator subgroup of the holonomy group is either* $SL(E_x)$ *or* $Sp(E_x)$.

Note that the condition $r \neq 2, 6, 12$ in Corollary 1 arises from the exceptions listed in Theorem 5. Guralnick and Tiep have been working on an analog of Theorem 5 for alternating powers $\wedge^k(V)$.

The obvious question arises: what is going on with "smal" symmetric/alternating powers, for instance with $k = 3$? We indicate one infinite series of examples:

Theorem 6 (Magaard–Tiep [48]) *Let* $G = Sp_{2n}(3)$ *with* $n \geq 3$, *and let* A, B *be complex Weil representations of* G *of degree* $(3^n - 1)/2$, *resp.* $(3^n + 1)/2$. *Then*

$$\mathrm{Sym}^3(A), \quad \wedge^3(B), \quad A \otimes \mathrm{Sym}^2(B), \quad B \otimes \wedge^2(A)$$

are all irreducible.

When $k = 2$, there are many more examples, and they are classified by work of Magaard, Malle, and Tiep [44, 45, 49].

2.6 The Cross Characteristic Case of Problem 1: $G \in \mathrm{Lie}(\ell)$, $\ell \neq p = \mathrm{char}(\mathbb{F})$

Recall that in this case, $\mathbb{F} = \overline{\mathbb{F}}$ is of characteristic p and G is a finite simple group of Lie type defined over a field $\mathbb{F}_q$ with q a power of some prime $\ell \neq p$.

Suppose in addition that G is an *exceptional* group of Lie type. Then the complex case of Problem 1 was completed in an unpublished work of Saxl. In general, Problem 1 for all smaller exceptional groups ${}^2B_2(q)$, ${}^2G_2(q)$, $G_2(q)$, ${}^3D_4(q)$, and ${}^2F_4(q)$, has been completed in [26, 50, 51]. The remaining, large exceptional groups are being analyzed by Saxl and Tiep.

Now, let us turn our attention to the case where G is a *classical* group. In the important paper [57], Seitz determined all possible simple subgroups $H \in \mathrm{Lie}(\ell)$ that could arise in Problem 1 (but the module V was not known). One particular such a pair is $(G, H) = (Sp_6(q), G_2(q))$ with $2|q$, and the possible modules V for this pair have been determined by Schaeffer Fry in [55]. In fact, Schaeffer Fry [55] completely classified all pairs (V, H), where H is a proper subgroup of $G = Sp_6(q)$ (still with $2|q$) and V an $\mathbb{F}G$-module with $\mathrm{char}(\mathbb{F}) \neq 2$ such that $V\downarrow_H$ is irreducible.

A major case where Problem 1 has been completely solved is where $G = GL_n(q)$, cf. [39]. Let us briefly recall the Dipper–James classification of irreducible $\mathbb{F}G$-modules [12, 13]. Suppose we are given some positive integers m, k_i, and d_i with $1 \leq i \leq m$ such that $\sum_{i=1}^m k_i d_i = n$. For $1 \leq i \leq m$, to each p' -element $s_i \in \overline{\mathbb{F}}_q$ (with a minimal polynomial) of degree d_i over $\mathbb{F}_q$ and a partition $\lambda_i \vdash k_i$, one can associate an irreducible $\mathbb{F}L_i$-module $V_i = L(s_i, \lambda_i)$, where $L_i \cong GL_{k_i d_i}(q)$. Moreover, since $\sum_{i=1}^m k_i d_i = n$, G contains a parabolic subgroup $P = U \rtimes L$ with unipotent radical U and Levi subgroup $L \cong L_1 \times \cdots \times L_m$. Hence, one can view $V_1 \otimes V_2 \otimes \cdots \otimes V_m$ as an $\mathbb{F}L$ module, inflate it to an $\mathbb{F}P$ module, and then induce up to an $\mathbb{F}G$ module

$$V = L(s_1, \lambda_1) \circ L(s_2, \lambda_2) \circ \cdots \circ L(s_m, \lambda_m),$$

(the so-called *Harish–Chandra induction* of $V_1 \otimes \cdots \otimes V_m$ from L to G). Under the extra assumption that s_i and s_j have different minimal polynomials whenever $i \neq j$, the resulting module V is irreducible, and furthermore any irreducible $\mathbb{F}G$ module can be obtained in this way.

Example 6

(i) (Gelfand [19]) Taking $\mathbb{F} = \mathbb{C}$, $m = 1$, $(d_1, k_1) = (n, 1)$, one obtains a cuspidal $\mathbb{C}G$ module of G of dimension

$$(q-1)(q^2-1)\cdots(q^{n-1}-1),$$

which is irreducible over the stabilizer $\mathrm{Stab}_G(\langle v\rangle_{\mathbb{F}_q})$ of any 1-dimensional subspace of the natural G-module $\mathbb{F}_q^n$.

(ii) (Seitz [57]) Assume $2|n$ and $q > 3$. Taking $\mathbb{F} = \mathbb{C}$, $m = 1$, $(d_1, k_1) = (1, n)$, $s_1 \neq \pm 1$, one obtains a (Weil) $\mathbb{C}G$-module of dimension $(q^n - 1)/(q - 1)$, which is irreducible over $Sp_n(q)$.

Now we can describe the main result of [39], which solves Problem 1 for all groups G between $SL_n(q)$ and $GL_n(q)$:

Theorem 7 (Kleshchev–Tiep [39]) *Let $SL_n(q) \le G \le GL_n(q)$, $H < G$ be a proper subgroup not containing $SL_n(q)$, $\mathbb{F} = \overline{\mathbb{F}}$ be of characteristic $p \nmid q$, and let V be an irreducible $\mathbb{F}G$-module of dimension greater than 1. Let W be an irreducible $\mathbb{F}GL_n(q)$-module such that V is an irreducible constituent of $W\downarrow_G$. Then, $V\downarrow_H$ is irreducible if and only if one of the following holds:*

(i) *$H \le P$, where $P = UL$ is the stabilizer in $GL_n(q)$ of a 1-space or an $(n-1)$-space in the natural $GL_n(q)$-module $\mathbb{F}_q^n$, $W = L(s, (k))$ for some $s \in \bar{\mathbb{F}}_q^\times$ of degree $n/k > 1$, and one of the following holds:*

 (a) *$H \ge [P, P]$; furthermore, W has the same number of irreducible constituents over $SL_n(q)H$ and over G.*
 (b) *$G = SL_n(3)$, $s^2 = -1$ if $p \neq 2$, and $[L, L] \le H \le L$.*
 (c) *$G = SL_n(2)$, $s \neq 1 = s^3$ and $H = L$.*

(ii) *n is even, $W = L(s, (1)) \circ L(t, (n-1))$ for some p'-elements $s, t \in \mathbb{F}_q^\times$ with $s \neq t$* (in particular, $V = W\downarrow_G$ is a Weil representation of dimension $(q^n - 1)/(q - 1)$), *and one of the following holds:*

 (a) *$Sp_n(q)Z(H) < H \le CSp_n(q)$.*
 (b) *$H = Sp_n(q)Z(H)$ and $t \neq \pm s$.*
 (c) *$2|q$, $n = 6$, and $G_2(q)' \lhd H \le GL_n(q)$.*

(iii) *Small cases with $(n, q) = (4, 2), (3, 4), (3, 2)$, or with $n = 2$, occur.*

Some of the main tools of the proof of Theorem 7 are *quantum group* methods of [7], and the main result of [38] which gives a *branching rule* from $GL_n(\mathbb{F}_q)$ to $SL_n(\mathbb{F}_q)$. In particular, one has

Theorem 8 (Kleshchev–Tiep [38]) *Let $\mathbb{F} = \overline{\mathbb{F}}$ be of characteristic $p \nmid q$ and let $V = L(s_1, \lambda_1)\circ\cdots\circ L(s_m, \lambda_m)$ be an irreducible $\mathbb{F}GL_n(q)$-module, where the s_i are p'-elements. Then V is reducible over $SL_n(q)$ if and only if at least one of the following holds.*

(i) *There is some p'-element $1 \neq t \in \mathbb{F}_q^\times$ such that, for all $i = 1, \ldots, m$, the set $\{s_j \mid 1 \le j \le m, \lambda_j = \lambda_i\}$ is stable under the multiplication by t.*
(ii) *$p|\gcd(n, q-1)$, and for each i all parts of the partition λ_i' conjugate to λ_i are divisible by p.*

More generally, [38, Theorem 1.1] determines the number of irreducible constituents of each irreducible $\mathbb{F}GL_n(q)$-module over $SL_n(q)$. It also allows us to

- get a parametrization of irreducible $\mathbb{F}SL_n(q)$-modules [38, Corollary 1.2];
- classify in [38, Theorem 1.3] the complex representations of $SL_n(q)$ whose reductions modulo $p = \mathrm{char}(\mathbb{F})$ are irreducible, relying on a similar result for $GL_n(q)$ of [28]; and to
- exhibit an explicit subset of complex irreducible characters of $SL_n(q)$ and a partial order on the set of irreducible p-Brauer characters of $SL_n(q)$ such that the corresponding decomposition submatrix is lower unitriangular [38, Theorem 1.4].

It remains a big challenge to resolve the various problems in representation theory, posed by Problem 1 for the remaining simple groups of Lie type.

3 Applications

The aforementioned results have led to significant progress in several recent applications in group theory, number theory, and algebraic geometry. One such application, the *Kollár–Larsen problem* 3 *on symmetric powers*, has been described in Section 2.5. We refer the reader to Section 3 of [64] for a discussion of some other applications, including

(i) the *Katz–Larsen conjecture on moments* [23, 29, 30];
(ii) the *Kollár–Larsen problem on finite linear groups and crepant resolutions* [25, 41] (see also Example 2); and
(iii) *adequate subgroups and automorphy lifting* [20–22, 59, 62].

We conclude the survey with another open problem. For a Kähler manifold X and a compact subgroup $G \leq \mathrm{Aut}(X)$, Tian [63] defined an invariant $\alpha_G(X)$. In particular, Tian showed that a Fano variety X admits a G-invariant Kähler–Einstein metric if

$$\alpha_G(X) > \frac{\dim(X)}{\dim(X)+1}.$$

Consider the case a finite group $G < GL_{n+1}(\mathbb{C})$ acts on the projective space $\mathbb{P}^n$. Then Tian's invariant $\alpha_G(\mathbb{P}^n)$ is just the *log-canonical threshold* $\mathrm{lct}(\mathbb{P}^n, G)$. Moreover, as shown in [11, Theorem 1.17], $\mathbb{C}^{n+1}/G$ is exceptional if $\alpha_G(\mathbb{P}^n) > 1$, and *not exceptional* if $\alpha_G(\mathbb{P})^n < 1$, provided that G contains no complex reflection.

Recall that $G < GL(V)$ is said to have *a semi-invariant of degree k on V* if $\mathrm{Sym}^k(V)$ contains a one-dimensional G-submodule. Now the connection between $\alpha_G(\mathbb{P}^n)$ and semi-invariants of $G < GL_{n+1}(\mathbb{C})$ can be seen as follows

$$\alpha_G(\mathbb{P}^n) \leq \frac{\min\{k \mid G \text{ has a semi-invariant of degree } k \text{ on } \mathbb{C}^{n+1}\}}{n+1}.$$

It turns out that a strong upper bound on $\alpha_G(\mathbb{P}^n)$ was proved by J. G. Thompson some years before the invariant was formally defined:

Theorem 9 (Thompson [61]) *Suppose that $G < GL_{n+1}(\mathbb{C})$ is any finite group. If $p \nmid |G|$ is any prime then $\alpha_G(\mathbb{P}^n) \leq p-1$. In particular, $\alpha_G(\mathbb{P}^n) \leq 4(n+1)$.*

In fact, a much stronger bound should hold asymptotically:

Conjecture 1 (Thompson [61]) *There exists a constant* $C > 0$ *such that* $\alpha_G(\mathbb{P}^n) \leq C$ *for all finite subgroups* $G < GL_{n+1}(\mathbb{C})$.

Very recent progress on Conjecture 1 will be discussed in a forthcoming paper.

Acknowledgments This survey is based in part on the plenary address given by the author at the Eighth Congress of Vietnamese Mathematicians (Nha Trang, Vietnam, August 10–14, 2013), as well as the lectures given by the author at the Vietnam Institute of Advanced Study in Mathematics (July 2013) and the University of Southern California (March 2014). It is a pleasure to thank the National Science Foundation, the Vietnam Institute of Advanced Study in Mathematics, and the University of Southern California for partial support.

The author is grateful to the referee for careful reading and helpful comments on the paper.

The author gratefully acknowledges the support of the NSF (grant DMS-1201374) and the Simons Foundation Fellowship 305247.

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