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A Fixed Point Theorem for New Type Contractions on Weak Partial Metric Spaces

Özlem Acar · Ishak Altun · Gonca Durmaz

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Abstract Recently, Karapinar and Romaguera (Filomat 27:1305–1314, 2013) have introduced a new type contraction on partial metric spaces, and they have obtained a nonunique fixed point result. Then Romaguera (Math. Sci. Appl. E-Notes 1:1–8, 2013) used this contraction to obtain some multivalued fixed point results on partial metric spaces. In the present work, we give a fixed point result on weak partial metric spaces using this new idea.

Keywords Fixed point · Partial metric space · Weak partial metric space

Mathematics Subject Classification (2010) 54H25 · 47H10

1 Introduction and Preliminaries

Fixed point theory concerns itself with a very basic mathematical setting. It is also well known that one of the fundamental and most useful results in fixed point theory is the Banach fixed point theorem. This result has been extended in many directions, and there are many generalizations of it in different spaces. One of the most interesting is given in a partial metric space, which was introduced by Matthews [20] as a part of the study of denotational semantics of dataflow networks with the interesting property "nonzero self-distance" in the space.

Ö. Acar (✉) · I. Altun · G. Durmaz
Department of Mathematics, Faculty of Science and Arts, Kirikkale University, 71450 Yahsihan, Kirikkale, Turkey
e-mail: acarozlem@ymail.com

I. Altun
e-mail: ishakaltun@yahoo.com

G. Durmaz
e-mail: gncmatematik@hotmail.com

Namely, let (X, p) be a complete partial metric space and let $T : X \to X$ be a contraction mapping, that is, there exists $\lambda \in [0, 1)$ such that

$$p(Tx, Ty) \le \lambda p(x, y)$$

for all $x, y \in X$. Then T has a unique fixed point $z \in X$. Moreover, $p(z, z) = 0$.

Later on, Abdeljawad et al. [1], Acar et al. [2, 3], Altun et al. [6–8], Karapinar and Erhan [18], Oltra and Valero [21], and Valero [26] gave some generalizations to the result of Matthews. The best two generalizations of it were given by Romaguera [23, 24].

On the other hand, Berinde [9–11] defined a weak contraction in a metric space, and he showed in [12] and [13] that any Banach, Kannan, Chatterjea, and Zamfirescu mappings are weak contractions. Also, Altun and Acar [4] introduced the concepts of weak and weak φ-contractions in the sense of Berinde on a partial metric space; showed that any Banach, Kannan, Chatterjea, and Zamfirescu mappings are weak contractions; and proved some fixed point theorems in this space.

Recently, Romaguera and Karapınar [19] have proved nonunique fixed point theorems for new type contractions which include weak contractions in the sense of Berinde on partial metric spaces and their results generalize and improve some earlier results on the topic in literature.

At this point, it is natural to ask whether it is possible to give nonunique type fixed point theorems on weak partial metric spaces. In this work, as an answer of the mentioned question, we will focus on fixed point result on weak partial metric spaces in the light of the aforementioned techniques, and we will give an example illustrating more effectiveness of our result.

First, we recall some definitions of partial and weak partial metric spaces and some of their properties. See [16, 17, 20–22, 26] for details.

A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ (nonnegative real numbers) such that for all $x, y, z \in X$,

(p_1) $x = y \iff p(x, x) = p(x, y) = p(y, y)$ (T_0-separation axiom),
(p_2) $p(x, x) \le p(x, y)$ (small self-distance axiom),
(p_3) $p(x, y) = p(y, x)$ (symmetry),
(p_4) $p(x, y) \le p(x, z) + p(z, y) - p(z, z)$ (modified triangular inequality).

A partial metric space (for short, PMS) is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x, y) = 0$, then, from (p_1) and (p_2), $x = y$. But if $x = y$, $p(x, y)$ may not be 0. A basic example of a PMS is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. For another example, let I denote the set of all intervals $[a, b]$ for any real numbers $a \le b$. Let $p : I \times I \to \mathbb{R}^+$ be the function such that $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$. Then (I, p) is a PMS. Other examples of PMS which are interesting from a computational point of view may be found in [15, 20].

Each partial metric p on X generates a T_0 topology τ_p on X which has, as a base, the family of open p-balls:

$$\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\},$$

where

$$B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then the functions $p^s, p^w : X \times X \to \mathbb{R}^+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{1}$$

and

$$p^w(x, y) = \max\{p(x, y) - p(x, x), p(x, y) - p(y, y)\} \\ = p(x, y) - \min\{p(x, x), p(y, y)\} \tag{2}$$

are ordinary metrics on X. It is easy to see that p^s and p^w are equivalent metrics on X.

Definition 1

(i) A sequence $\{x_n\}$ in a PMS (X, p) converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.
(ii) A sequence $\{x_n\}$ in a PMS (X, p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n, x_m)$.
(iii) A PMS (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.
(iv) A mapping $F : X \to X$ is said to be continuous at $x_0 \in X$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $F(B_p(x_0, \delta)) \subseteq B_p(Fx_0, \varepsilon)$.

The following lemma plays an important role to give fixed point results on a PMS.

Lemma 1 ([20, 21]) *Let (X, p) be a PMS.*

(a) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^w).*
(b) *(X, p) is complete if and only if (X, p^w) is complete. Furthermore,*

$$\lim_{n\to\infty} p^w(x_n, x) = 0$$

if and only if

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

By omitting the small self-distance axiom, Heckmann [16] introduced the concept of weak partial metric space (for short, WPMS), which is a generalized version of Matthews' partial metric space. That is, the function $p : X \times X \to \mathbb{R}^+$ is called weak partial metric on X if it satisfies T_0 separation axiom, symmetry, and modified triangular inequality. Heckmann also shows that if p is a weak partial metric on X, then we have the following weak small self-distance property:

$$p(x, y) \geq \frac{p(x, x) + p(y, y)}{2} \tag{3}$$

for all $x, y \in X$. Weak, small self-distance property shows that WPMS are not far from small self-distance axiom. It is clear that PMS is a WPMS, but the converse may not be true. A basic example of a WPMS but not a PMS is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \frac{x+y}{2}$ for all $x, y \in \mathbb{R}^+$. For another example, let I denote the set of all intervals $[a, b]$ for any real numbers $a \leq b$. Let $p : I \times I \to \mathbb{R}^+$ be the function such that $p([a, b], [c, d]) = \frac{b+d-a-c}{2}$. Then (I, p) is a WPMS but not PMS. Again, for $x, y \in \mathbb{R}$, the function $p(x, y) = \frac{e^x+e^y}{2}$ is a weak partial metric, but not a partial metric, on $\mathbb{R}$.

Then, in 2012, Altun and Durmaz [5] proved the Banach, Kannan, and Reich-type fixed point theorems on weak partial metric spaces, and Durmaz et al. [14] obtained some fixed point results on weak partial metric spaces including a new extension of Banach's contraction principle.

Remark 1 As mentioned in [5], if (X, p) is a WPMS, but not PMS, then the function p^s as in (1) may not be an ordinary metric on X, but p^w as in (2) is still an ordinary metric on X.

The concepts of convergence of a sequence, Cauchy sequence, and completeness in WPMS are defined as in PMS. The following lemma, which is very important for fixed point theory on WPMS, was given in [5] without using a small self-distance axiom.

Lemma 2 *Let (X, p) be a WPMS.*

(a) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^w).*
(b) *(X, p) is complete if and only if (X, p^w) is complete. Furthermore,*

$$\lim_{n\to\infty} p^w(x_n, x) = 0$$

if and only if

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Definition 2 A map $T : X \to X$ is said to be orbitally continuous if $\{x_n\}$ is a sequence in X such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} p(x, x_n) = p(x, x)$ for some $x \in X$, then

$$\lim_{n\to\infty} p(Tx, x_n) = p(Tx, Tx).$$

Lemma 3 *Let (X, p) be a weak partial metric space. A sequence $\{x_n\}$ in X is a Cauchy sequence in (X, p) if and only if it satisfies the following condition:*

$$\text{For each } \epsilon > 0, \text{ there is } n_0 \in \mathbb{N} \text{ such that } p(x_n, x_m) - p(x_n, x_n) < \epsilon$$

whenever $n_0 \le n \le m$.

2 The Main Result

The following are our main results. We give two nonunique fixed point theorems in weak partial metric spaces; one of them is in partially ordered spaces.

Theorem 1 *Let (X, p) be a complete weak partial metric space and $T : X \to X$ be an orbitally continuous mapping such that*

$$p(Tx, Ty) \le k[p(x, y) - p(x, x)] + p(y, y) + L \min\left\{p^w(y, Tx), p^w(x, Ty)\right\} \quad (4)$$

for all $x, y \in X$, where $k \in (0, 1)$ and $L \in \mathbb{R}^+$. Then T has a fixed point in X.

Proof Let $x \in X$ and $\{x_n\}_{n=0}^{\infty}$ be a sequence defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. First, by taking $x = x_{n-1}$ and $y = x_n$ in (4), we get

$$\begin{aligned} p(x_n, x_{n+1}) &= p(Tx_{n-1}, Tx_n) \\ &\le k[p(x_{n-1}, x_n) - p(x_{n-1}, x_{n-1})] + p(x_n, x_n) \\ &\quad + L \min\left\{p^w(x_n, Tx_{n-1}), p^w(x_{n-1}, Tx_n)\right\} \\ &= k[p(x_{n-1}, x_n) - p(x_{n-1}, x_{n-1})] + p(x_n, x_n). \end{aligned}$$

Using this inequality, we have

$$\begin{aligned} p(x_n, x_{n+1}) - p(x_n, x_n) &\leq k[p(x_{n-1}, x_n) - p(x_{n-1}, x_{n-1})] \\ &\leq k^2[p(x_{n-2}, x_{n-1}) - p(x_{n-2}, x_{n-2})] \\ &\;\;\vdots \\ &\leq k^n[p(x_0, x_1) - p(x_0, x_0)]. \end{aligned}$$

This implies that

$$\begin{aligned} p(x_n, x_m) - p(x_n, x_n) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) - p(x_n, x_n) \\ &\leq k^n[p(x_0, x_1) - p(x_0, x_0)] + k^{n+1}[p(x_0, x_1) - p(x_0, x_0)] + \\ &\quad \vdots \\ &\quad +k^{m-1}[p(x_0, x_1) - p(x_0, x_0)] \\ &\leq \frac{k^n}{1-k}[p(x_0, x_1) - p(x_0, x_0)], \end{aligned}$$

then for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$p(x_n, x_m) - p(x_n, x_n) < \varepsilon$$

whenever $n_0 \leq n < m$. Thus, $\{x_n\}$ is a Cauchy sequence in (X, p) by Lemma 3. Then $\dot{X}$ is a complete WPMS, so there exists $z \in X$ such that

$$\lim_{n\to\infty} p(z, x_n) = \lim_{n\to\infty} p(x_n, x_n) = p(z, z)$$

or equivalently $\lim_{n\to\infty} p^w(z, x_n) = 0$.

We shall show that z is a fixed point of T. Indeed, since by (4), we have

$$\begin{aligned} p(z, Tz) &\leq p(z, x_n) + p(x_n, Tz) - p(x_n, x_n) \\ &\leq p(z, x_n) + k\big[p(x_{n-1}, z) - p(x_{n-1}, x_{n-1})\big] + p(z, z) \\ &\quad +L\min\{p^w(z, x_{n-1}), p^w(x_{n-1}, Tz)\} - p(x_n, x_n). \end{aligned}$$

Passing to limit as $n \to \infty$, we get

$$p(z, Tz) \leq p(z, z).$$

On the other hand, there holds

$$\begin{aligned} p(Tz, Tz) &\leq k[p(z, z) - p(z, z)] + p(z, z) + L\min\{p^w(z, Tz), p^w(z, Tz)\} \\ &= p(z, z) + Lp^w(z, Tz) \\ &= p(z, z) + L\big[p(z, Tz) - \min\{p(z, z), p(Tz, Tz)\}\big]. \end{aligned}$$

If $\min\{p(z, z), p(Tz, Tz)\} = p(z, z)$, then we have

$$\begin{aligned} p(Tz, Tz) &\leq p(z, z) + L[p(z, Tz) - p(z, z)] \\ &\leq p(z, z) + L[p(z, z) - p(z, z)] \\ &= p(z, z). \end{aligned}$$

If $\min\{p(z, z), p(Tz, Tz)\} = p(Tz, Tz)$, then

$$\begin{aligned} p(Tz, Tz) &\leq p(z, z) + L[p(z, Tz) - p(Tz, Tz)] \\ &\leq p(z, z) + L[p(z, z) - p(Tz, Tz)] \end{aligned}$$

holds. Therefore,

$$p(Tz, Tz) \leq p(z, z).$$

Also, from (3), we have $p(z, z) \leq p(z, Tz)$. So we get $p(z, z) = p(z, Tz)$. On the other hand, since T is orbitally continuous, we have

$$\lim_{n\to\infty} p(x_n, Tz) = p(Tz, Tz)$$

and also

$$\frac{p(z, z) + p(Tz, Tz)}{2} \leq p(z, Tz) \leq p(z, x_n) + p(x_n, Tz) - p(x_n, x_n).$$

Passing to limit as $n \to \infty$, we get

$$\frac{p(z, z) + p(Tz, Tz)}{2} \leq p(z, Tz) \leq p(Tz, Tz)$$

and then $p(z, z) \leq p(Tz, Tz)$.

Hence, we have

$$p(z, z) = p(z, Tz) = p(Tz, Tz),$$

which completes the proof. □

Remark 2 In this theorem, we obtain the same result if the continuity of T replaces its orbital continuity.

Now, we give an illustrative example.

Example 1 Let $X = [0, 1]$ and

$$p(x, y) = \begin{cases} \frac{x+y}{2}, & x \neq y, \\ 0, & x = y. \end{cases}$$

It is clear that p is a weak partial metric and (X, p) is complete. Also, $p^w(x, y) = p(x, y)$. Define $T : X \to X$ by

$$Tx = \begin{cases} x^2, & x \in \left[0, \frac{1}{2}\right), \\ 0, & x \in \left[\frac{1}{2}, 1\right), \\ 1, & x = 1. \end{cases}$$

Now, we show that (4) is satisfied for $k = \frac{1}{2}$ and $L = 1$, that is,

$$p(Tx, Ty) \leq kp(x, y) + L \min\left\{p^w(y, Tx), p^w(x, Ty)\right\}$$

for all $x, y \in X$. Consider the following six cases:

Case 1 Let $x = y$, then $p(Tx, Ty) = 0$ and so the result is clear. Therefore, we will assume $x \neq y$ in the following cases.

Case 2 Let $x, y \in \left[\frac{1}{2}, 1\right)$, then $p(Tx, Ty) = 0$ and so the result is clear.

Case 3 Let $x, y \in \left[0, \frac{1}{2}\right)$, then

$$\begin{aligned} p(Tx, Ty) &= \frac{x^2 + y^2}{2} \leq \frac{1}{2}\frac{x+y}{2} = \frac{1}{2}p(x, y) \\ &\leq \frac{1}{2}p(x, y) + \min\left\{p^w(y, Tx), p^w(x, Ty)\right\}. \end{aligned}$$

Case 4 Let $x \in \left[\frac{1}{2}, 1\right)$ and $y = 1$, then

$$\begin{aligned} p(Tx, Ty) &= \frac{1}{2} < \frac{1}{2}\frac{x+1}{2} + \frac{1}{2} \\ &= \frac{1}{2}p(x, y) + \min\{p(1, 0), p(x, 1)\} \\ &= \frac{1}{2}p(x, y) + \min\{p(y, Tx), p(x, Ty)\} \\ &= \frac{1}{2}p(x, y) + \min\left\{p^w(y, Tx), p^w(x, Ty)\right\}. \end{aligned}$$

Case 5 Let $x \in \left[0, \frac{1}{2}\right)$ and $y = 1$, then

$$\begin{aligned} p(Tx, Ty) &= \frac{x^2 + 1}{2} < \frac{1}{2}\frac{x+1}{2} + \frac{1+x^2}{2} \\ &= \frac{1}{2}p(x, y) + \min\left\{p\left(1, x^2\right), p(x, 1)\right\} \\ &= \frac{1}{2}p(x, y) + \min\{p(y, Tx), p(x, Ty)\} \\ &= \frac{1}{2}p(x, y) + \min\left\{p^w(y, Tx), p^w(x, Ty)\right\}. \end{aligned}$$

Case 6 Let $x \in \left[0, \frac{1}{2}\right)$ and $y \in \left[\frac{1}{2}, 1\right)$, then

$$\begin{aligned} p(Tx, Ty) &= \frac{x^2}{2} \leq \frac{1}{2}\frac{x+y}{2} \leq \frac{1}{2}p(x, y) \\ &\leq \frac{1}{2}p(x, y) + \min\left\{p^w(y, Tx), p^w(x, Ty)\right\}. \end{aligned}$$

Finally, T is orbitally continuous. Therefore, all conditions of Theorem 1 are satisfied, so T has a fixed point. Note that since

$$p(T0, T1) = \frac{1}{2} = p(0, 1),$$

then T is not a contraction in the sense of Banach on this weak partial metric space.

Theorem 2 *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a partial metric p on X such that (X, p) is a complete weak partial metric space. Assume that $T : X \to X$ is an orbitally continuous and nondecreasing mapping such that*

$$p(Tx, Ty) \leq k[p(x, y) - p(x, x)] + p(y, y) + L \min\left\{p^w(y, Tx), p^w(x, Ty)\right\} \quad (5)$$

for all $x, y \in X$ with $y \preceq x$, where $k \in (0, 1)$ and $L \in \mathbb{R}^+$. If there exists an $x_0 \in X$ with $x_0 \preceq Tx_0$, then there exists $x \in X$ such that $x = Tx$.

Proof Let $x_0 \in X$. If $Tx_0 = x_0$, then the proof is clear. Now, suppose that $Tx_0 \neq x_0$ and let $x_n = Tx_{n-1}$ for $n = 1, 2, \ldots$. If $x_{n_0} = x_{n_{0-1}}$ for some $n_0 \in \mathbb{N}$, then it is clear that x_{n_0-1} is a fixed point of T. Thus, assume $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. Note that since $x_0 \preceq Tx_0$ and T is nondecreasing, we have

$$x_0 \preceq Tx_0 = x_1 \preceq Tx_1 = x_2 \preceq \cdots \preceq Tx_{n-1} = x_n \preceq Tx_n \cdots .$$

Now, since $x_{n-1} \preceq x_n$, we can use the inequality (5) for these points, and then we have

$$\begin{aligned} p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n) &\leq k[p(x_{n-1}, x_n) - p(x_{n-1}, x_{n-1})] + p(x_n, x_n) \\ &\quad +L \min\left\{p^w(x_n, Tx_{n-1}), p^w(x_{n-1}, Tx_n)\right\} \\ &= k\left[p(x_{n-1}, x_n) - p(x_{n-1}, x_{n-1})\right] + p(x_n, x_n) \end{aligned}$$

and then

$$\begin{aligned} p(x_n, x_{n+1}) - p(x_n, x_n) &\leq k\left[p(x_{n-1}, x_n) - p(x_{n-1}, x_{n-1})\right] \\ &\leq k^2\left[p\left(x_{n-2}, x_{n-1}\right) - p\left(x_{n-2}, x_{n-2}\right)\right] \\ &\quad \vdots \\ &\leq k^n\left[p(x_0, x_1) - p(x_0, x_0)\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} p(x_n, x_m) - p(x_n, x_n) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) - p(x_n, x_n) \\ &\leq k^n[p(x_0, x_1) - p(x_0, x_0)] + k^{n+1}[p(x_0, x_1) - p(x_0, x_0)] + \\ &\quad \vdots \\ &\quad +k^{m-1}[p(x_0, x_1) - p(x_0, x_0)] \\ &\leq \frac{k^n}{1-k}[p(x_0, x_1) - p(x_0, x_0)]. \end{aligned}$$

Then for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$p(x_n, x_m) - p(x_n, x_n) < \varepsilon$$

whenever $n_0 \leq n < m$. Thus, $\{x_n\}$ is a Cauchy sequence in (X, p) by Lemma 3. Then $\dot{X}$ is a complete WPMS, so there exists $z \in X$ such that

$$\lim_{n\to\infty} p(z, x_n) = \lim_{n\to\infty} p(x_n, x_n) = p(z, z)$$

or equivalently $\lim_{n\to\infty} p^w(z, x_n) = 0$.

We shall show that z is a fixed point of T. Indeed, by (5), we have

$$\begin{aligned} p(z, Tz) &\leq p(z, x_n) + p(x_n, Tz) - p(x_n, x_n) \\ &\leq p(z, x_n) + k[p(x_{n-1}, z) - p(x_{n-1}, x_{n-1})] + p(z, z) \\ &\quad + L \min \left\{ p^w(z, x_{n-1}), p^w(x_{n-1}, Tz) \right\} - p(x_n, x_n) \end{aligned}$$

and passing to limit as $n \to \infty$, we get

$$p(z, Tz) \leq p(z, z).$$

On the other hand, we can write

$$\begin{aligned} p(Tz, Tz) &\leq k[p(z, z) - p(z, z)] + p(z, z) + L \min \left\{ p^w(z, Tz), p^w(z, Tz) \right\} \\ &= p(z, z) + Lp^w(z, Tz) \\ &= p(z, z) + L[p(z, Tz) - \min\{p(z, z), p(Tz, Tz)\}]. \end{aligned}$$

If $\min\{p(z, z), p(Tz, Tz)\} = p(z, z)$, then we have

$$p(Tz, Tz) \leq p(z, z) + L[p(z, Tz) - p(z, z)] \leq p(z, z) + L[p(z, z) - p(z, z)] = p(z, z).$$

If $\min\{p(z, z), p(Tz, Tz)\} = p(Tz, Tz)$, then

$$p(Tz, Tz) \leq p(z, z) + L[p(z, Tz) - p(Tz, Tz)] \leq p(z, z) + L[p(z, z) - p(Tz, Tz)]$$

holds, which implies

$$p(Tz, Tz) \leq p(z, z).$$

Also, we have $p(z, z) \leq p(z, Tz)$ from (3). Hence, we get $p(z, z) = p(z, Tz)$. On the other hand, since T is orbitally continuous, we have

$$\lim_{n \to \infty} p(x_n, Tz) = p(Tz, Tz).$$

Since

$$\frac{p(z, z) + p(Tz, Tz)}{2} \leq p(z, Tz) \leq p(z, x_n) + p(x_n, Tz) - p(x_n, x_n),$$

passing to limit as $n \to \infty$, we get

$$\frac{p(z, z) + p(Tz, Tz)}{2} \leq p(z, Tz) \leq p(Tz, Tz)$$

and then $p(z, z) \leq p(Tz, Tz)$.

So we deduce

$$p(z, z) = p(z, Tz) = p(Tz, Tz),$$

and the proof is complete. □

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