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On Strongly Regular Graphs of Order $3(2p+1)$ and $4(2p+1)$ where $2p+1$ is a Prime Number

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Abstract We say that a regular graph G of order n and degree $r \geq 1$ (which is not the complete graph) is strongly regular if any two distinct vertices have τ common neighbors if they are adjacent and have θ common neighbors if they are not adjacent. We here describe the parameters n, r, τ, and θ for strongly regular graphs of order $3(2p+1)$ and $4(2p+1)$, where $2p+1$ is a prime number.

Keywords Strongly regular graph · Conference graph · Integral graph

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1 Introduction

Let G be a simple graph of order n. The spectrum of G consists of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of its (0,1) adjacency matrix A and is denoted by $\sigma(G)$. We say that a regular graph G of order n and degree $r \geq 1$ (which is not the complete graph K_n) is strongly regular if any two distinct vertices have τ common neighbors if they are adjacent and have θ common neighbors if they are not adjacent. Besides, we say that a regular connected graph G is strongly regular if and only if it has exactly three distinct eigenvalues [1]. Let $\lambda_1 = r$, λ_2 and λ_3 denote the distinct eigenvalues of G. Let $m_1 = 1$, m_2 and m_3 denote the multiplicity of r, λ_2, and λ_3, respectively. The results obtained in this work are based on the following assertion [2, 3].

Theorem 1 *Let G be a connected strongly regular graph of order n and degree r. Then $m_2 m_3 \delta^2 = n r \overline{r}$ where $\delta = \lambda_2 - \lambda_3$ and $\overline{r} = (n-1) - r$.*

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Further, let $\overline{r} = (n-1) - r$, $\overline{\lambda}_2 = -\lambda_3 - 1$, and $\overline{\lambda}_3 = -\lambda_2 - 1$ denote the distinct eigenvalues of the strongly regular graph $\overline{G}$, where $\overline{G}$ denotes the complement of G. It is known that $\overline{\tau} = n - 2r - 2 + \theta$ and $\overline{\theta} = n - 2r + \tau$ where $\overline{\tau} = \tau(\overline{G})$ and $\overline{\theta} = \theta(\overline{G})$.

Remark 1 (i) A strongly regular graph G of order $4n+1$ and degree $r = 2n$ with $\tau = n-1$ and $\theta = n$ is called the conference graph; (ii) a strongly regular graph is the conference graph if and only if $m_2 = m_3$; and (iii) if $m_2 \neq m_3$, then G is an integral[1] graph.

Remark 2 If G is a disconnected strongly regular graph of degree r, then $G = mK_{r+1}$, where mH denotes the m-fold union of the graph H. We know that G is a disconnected strongly regular graph if and only if $\theta = 0$.

Due to Theorem 1, we have recently obtained the following results [3]: (i) there is no strongly regular graph of order $4p+3$ if $4p+3$ is a prime number, and (ii) the only strongly regular graphs of order $4p+1$ are conference graphs if $4p+1$ is a prime number. Besides, in the same work, we have described the parameters n, r, τ, and θ for strongly regular graphs of order $2(2p+1)$, where $2p+1$ is a prime number. We now proceed to establish the parameters of strongly regular graphs of order $3(2p+1)$ and $4(2p+1)$ where $2p+1$ is a prime number, as follows. First,

Proposition 1 (Elzinga [1]) *Let G be a connected or disconnected strongly regular graph of order n and degree r. Then,*

$$r^2 - (\tau - \theta + 1)r - (n-1)\theta = 0. \tag{1}$$

Proposition 2 (Elzinga [1]) *Let G be a connected strongly regular graph of order n and degree r. Then,*

$$2r + (\tau - \theta)(m_2 + m_3) + \delta(m_2 - m_3) = 0, \tag{2}$$

where $\delta = \lambda_2 - \lambda_3$.

Second, in what follows, (x, y) denotes the greatest common divisor of integers $x, y \in \mathbb{N}$ while $x \mid y$ means that x divides y.

2 Main Results

Remark 3 In the following two Theorems 2 and 3, the complements of strongly regular graphs appear in pairs in (k^0) and $(\overline{k}^0)$ classes, where k denotes the corresponding number of a class.

Proposition 3 *Let G be a connected strongly regular graph of order $3(2p+1)$ and degree r, where[2] $2p+1$ is a prime number. If $p \geq 2$, then G is a conference graph if and only if $\delta^2 = 3(2p+1)$.*

[1] We say that a connected or disconnected graph G is integral if its spectrum $\sigma(G)$ consists of integral values.

[2] The connected strongly regular graphs of order 9 are (i) the conference graph of degree $r = 4$ with $\tau = 1$ and $\theta = 2$. Its eigenvalues are $\lambda_2 = 1$ and $\lambda_3 = -2$ with $m_2 = 4$ and $m_3 = 4$ and (ii) $\overline{3K_3}$ of degree $r = 6$ with $\tau = 3$ and $\theta = 6$. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -3$ with $m_2 = 6$ and $m_3 = 2$.

Proof We note first that if G is a conference graph, then $\delta^2 = 3(2p+1)$. Conversely, let us assume that $\delta^2 = 3(2p+1)$. Since $3 \nmid (2p+1)$, it follows that δ^2 is not a perfect square. Since $\delta = \lambda_2 - \lambda_3 \notin \mathbb{N}$, it turns out that G is not integral, which proves the statement. □

Remark 4 Since the strongly regular graphs of order $n = 9$ are completely described, in the sequel, it will be assumed that $p \geq 2$.

Proposition 4 *Let G be a connected strongly regular graph of order* $3(2p+1)$ *and degree r, where* $2p+1$ *is a prime number. If* $\delta = 2p+1$, *then G belongs to the class* (1^0) *represented in Theorem 2.*

Proof Using Theorem 1, we have $(2p+1)m_2m_3 = 3r\,\overline{r}$, which means that $(2p+1) \mid r$ or $(2p+1) \mid \overline{r}$. Without loss of generality, we may consider only the case when $(2p+1) \mid r$.

Case 1 ($r = 2p+1$). Then, $m_2m_3 = 3(4p+1)$ and $m_2 + m_3 = 6p+2$, which provides that m_2 and m_3 are the roots of the quadratic equation $m^2 - (6p+2)m + 3(4p+1) = 0$. So we find that $m_2, m_3 = \frac{6p+2\pm\Delta}{2}$ where $\Delta^2 = (6p-2)^2 - 12$, a contradiction because Δ^2 is not a perfect square for $p \geq 2$.

Case 2 ($r = 2(2p+1)$). Then $m_2m_3 = 12p$ which yields that $m_2 = 6p$ and $m_3 = 2$ or $m_2 = 2$ and $m_3 = 6p$. Consider first the case when $m_2 = 6p$ and $m_3 = 2$. Using (2), we obtain $\tau - \theta = -(2p+1)$. Since $\lambda_{2,3} = \frac{\tau-\theta\pm\delta}{2}$, we get easily $\lambda_2 = 0$ and $\lambda_3 = -(2p+1)$, which proves that G is the strongly regular graph $\overline{3K_{2p+1}}$ of degree $r = 4p+2$ with $\tau = 2p+1$ and $\theta = 4p+2$. Consider the case when $m_2 = 2$ and $m_3 = 6p$. Using (2), we obtain $\tau - \theta = \frac{3(p-1)(2p+1)}{3p+1}$, a contradiction because $(3p+1) \nmid 3(p-1)$. □

Proposition 5 *There is no connected strongly regular graph G of order* $3(2p+1)$ *and degree r with* $\delta = 2(2p+1)$, *where* $2p+1$ *is a prime number.*

Proof Contrary to the statement, assume that G is a strongly regular graph with $\delta = 2(2p+1)$. Using Theorem 2, we have $4(2p+1)m_2m_3 = 3r\,\overline{r}$ which means that $(2p+1) \mid r$ or $(2p+1) \mid \overline{r}$. Consider the case when $r = 2p+1$ and $\overline{r} = 4p+1$. Then ,$4m_2m_3 = 3(4p+1)$, a contradiction because $4 \nmid (4p+1)$. Consider the case when $r = 2(2p+1)$ and $\overline{r} = 2p$. Then, $m_2 + m_3 = 6p+2$ and $m_2m_3 = 3p$, a contradiction. □

Proposition 6 *Let G be a connected strongly regular graph of order* $3(2p+1)$ *and degree r, where* $2p+1$ *is a prime number. If* $m_2 = 2p+1$ *and* $m_3 = 4p+1$, *then G belongs to the class* (6^0) *or* $(\overline{7}^0)$ *represented in Theorem 2.*

Proof Using (2), we obtain $p\delta = r + (\tau-\theta)(3p+1)$. Since $\delta = \lambda_2 - \lambda_3$ and $\tau - \theta = \lambda_2 + \lambda_3$, we arrive at $2p(2|\lambda_3| - \lambda_2) = \tau - \theta + r$. Since $r \leq 6p+1, \theta \leq r$ and $\tau < r$, it follows that $0 \leq \tau - \theta + r \leq 12p$. Let $2|\lambda_3| - \lambda_2 = t$ where $t = 0, 1, \ldots, 6$. Let $\lambda_3 = -k$ where k is a positive integer. Then (i) $\lambda_2 = 2k - t$; (ii) $\tau - \theta = k - t$; (iii) $\delta = 3k - t$; and (iv) $r = (2p+1)t - k$. Since $\delta^2 = (\tau-\theta)^2 + 4(r-\theta)$ (see [1]), we obtain (v) $\theta = (2p+1)t - (2k^2 - (t-1)k)$. Using (ii), (iv), and (v), it is not difficult to see that (1) is transformed into

$$2(p+1)t^2 - 3(2p+1)t + 6k^2 - 3k(2t-1) = 0. \tag{3}$$

Case 1 ($t = 0$). Using (i), (ii), (iii), and (iv), we find that $\lambda_2 = 2k$ and $\lambda_3 = -k$, $\tau - \theta = k$, $\delta = 3k$, and $r = -k$, a contradiction.

Case 2 ($t = 1$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = 2k - 1$ and $\lambda_3 = -k$, $\tau - \theta = k - 1$, $\delta = 3k - 1$, $r = (2p + 1) - k$, and $\theta = (2p + 1) - 2k^2$. Using (3), we find that $4p + 1 = 3k(2k - 1)$. Replacing k with $4k - 1$, we arrive at $p = 24k^2 - 15k + 2$, where k is a positive integer. So we obtain that G is a strongly regular graph of order $3(48k^2 - 30k + 5)$ and degree $r = 2(3k - 1)(8k - 3)$ with $\tau = (2k - 1)(8k - 1)$ and $\theta = (2k - 1)(8k - 3)$.

Case 3 ($t = 2$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = 2(k - 1)$ and $\lambda_3 = -k$, $\tau - \theta = k - 2$, $\delta = 3k - 2$, $r = 2(2p + 1) - k$, and $\theta = 2(2p + 1) - (2k^2 - k)$. Using (3), we find that $4p + 1 = 3(k - 1)(2k - 1)$. Replacing k with $4k + 2$, we arrive at $p = 24k^2 + 15k + 2$, where k is a non-negative integer. So we obtain that G is a strongly regular graph of order $3(48k^2 + 30k + 5)$ and degree $r = 8(3k + 1)(4k + 1)$ with $\tau = 4(4k + 1)^2 + 4k$ and $\theta = 4(4k + 1)^2$.

Case 4 ($t = 3$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = 2k - 3$ and $\lambda_3 = -k$, $\tau - \theta = k - 3$, $\delta = 3(k - 1)$, $r = 3(2p + 1) - k$, and $\theta = 3(2p + 1) - (2k^2 - 2k)$. Using (3), we find that $(k - 1)(2k - 3) = 0$. So we obtain that G is the complete graph, a contradiction.

Case 5 ($t = 4, 5, 6$). Using (3), we find that (x) $8p + 6k^2 - 21k + 20 = 0$; (y) $20p + 6k^2 - 27k + 35 = 0$ and (z) $12p + 2k^2 - 11k + 18 = 0$ for $t = 4$, $t = 5$ and $t = 6$, respectively, a contradiction. □

Proposition 7 *Let G be a connected strongly regular graph of order $3(2p + 1)$ and degree r, where $2p + 1$ is a prime number. If $m_2 = 2(2p + 1)$ and $m_3 = 2p$, then G belongs to the class (2^0) or (4^0) or $(\bar{5}^0)$ represented in Theorem 2.*

Proof Using (2), we obtain $2p(|\lambda_3| - 2\lambda_2) = (\tau - \theta) + \delta + r$. Since $(\tau - \theta) + \delta = 2\lambda_2$ and $\lambda_2 \leq \lfloor \frac{6p+3}{2} \rfloor - 1$ (see [3]), it follows that $0 < (\tau - \theta) + \delta + r \leq 12p$. Let $|\lambda_3| - 2\lambda_2 = t$ where $t = 1, 2, \ldots, 6$. Let $\lambda_2 = k$ where k is a non-negative integer. Then (i) $\lambda_3 = -(2k + t)$; (ii) $\tau - \theta = -(k + t)$; (iii) $\delta = 3k + t$; (iv) $r = 2(pt - k)$; and (v) $\theta = 2pt - (2k^2 + (t + 2)k)$. Using (ii), (iv), and (v), we can easily see that (1) is transformed into

$$t(t - 3)p + 3k(k + 1) = 0. \tag{4}$$

Case 1 ($t = 1$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = k$ and $\lambda_3 = -(2k + 1)$, $\tau - \theta = -(k + 1)$, $\delta = 3k + 1$, $r = 2(p - k)$, and $\theta = 2p - (2k^2 + 3k)$. Using (4), we find that $2p = 3k(k + 1)$. So we obtain that G is a strongly regular graph of order $3(3k^2 + 3k + 1)$ and degree $r = k(3k + 1)$ with $\tau = k^2 - k - 1$ and $\theta = k^2$, where $k \geq 2$.

Case 2 ($t = 2$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = k$ and $\lambda_3 = -2(k + 1)$, $\tau - \theta = -(k + 2)$, $\delta = 3k + 2$, $r = 2(2p - k)$, and $\theta = 4p - (2k^2 + 4k)$. Using (4), we find that $2p = 3k(k + 1)$. So we obtain that G is a strongly regular graph of order $3(3k^2 + 3k + 1)$ and degree $r = 2k(3k + 2)$ with $\tau = 4k^2 + k - 2$ and $\theta = 2k(2k + 1)$.

Case 3 ($t = 3$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = k$ and $\lambda_3 = -(2k + 3)$, $\tau - \theta = -(k + 3)$, $\delta = 3(k + 1)$, $r = 2(3p - k)$, and $\theta = 6p - (2k^2 + 5k)$. Using (4), we find that $k(k+1)=0$. So we obtain that G is a strongly regular graph $\overline{(2p+1)K_3}$ of degree $r=6p$ with $\tau=6p-3$ and $\theta=6p$.

Case 4 ($t=4,5,6$). Using (4), we find that (x) $4p+3k^2+3k=0$; (y) $10p+3k^2+3k=0$ and (z) $6p+k^2+k=0$ for $t=4$, $t=5$ and $t=6$, respectively, a contradiction. □

Proposition 8 *Let G be a connected strongly regular graph of order* $3(2p+1)$ *and degree r, where* $2p+1$ *is a prime number. If* $m_3=2p+1$ *and* $m_2=4p+1$*, then G belongs to the class* $(\overline{6}^0)$ *or* (7^0) *represented in Theorem 2.*

Proof Using (2), we obtain $2p(|\lambda_3|-2\lambda_2)=\tau-\theta+r$. Let $|\lambda_3|-2\lambda_2=t$ where $t=0,1,\ldots,6$. Let $\lambda_2=k$ where k is a non-negative integer. Then, (i) $\lambda_3=-(2k+t)$; (ii) $\tau-\theta=-(k+t)$; (iii) $\delta=3k+t$; (iv) $r=(2p+1)t+k$; and (v) $\theta=(2p+1)t-(2k^2+(t-1)k)$. Using (ii), (iv), and (v), we can easily see that (1) is reduced to

$$2(p+1)t^2-3(2p+1)t+6k^2+3k(2t-1)=0. \tag{5}$$

Case 1 ($t=0$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2=k$ and $\lambda_3=-2k$, $\tau-\theta=-k$, $\delta=3k$, $r=k$ and $\theta=-k(2k-1)$, a contradiction.

Case 2 ($t=1$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2=k$ and $\lambda_3=-(2k+1)$, $\tau-\theta=-(k+1)$, $\delta=3k+1$, $r=(2p+1)+k$, and $\theta=(2p+1)-2k^2$. Using (5), we find that $4p+1=3k(2k+1)$. Replacing k with $4k+1$, we arrive at $p=24k^2+15k+2$, where k is a non-negative integer. So we obtain that G is a strongly regular graph of order $3(48k^2+30k+5)$ and degree $r=2(3k+1)(8k+3)$ with $\tau=(2k+1)(8k+1)$ and $\theta=(2k+1)(8k+3)$.

Case 3 ($t=2$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2=k$ and $\lambda_3=-2(k+1)$, $\tau-\theta=-(k+2)$, $\delta=3k+2$, $r=2(2p+1)+k$, and $\theta=2(2p+1)-(2k^2+k)$. Using (5), we find that $4p+1=3(k+1)(2k+1)$. Replacing k with $4k-2$, we arrive at $p=24k^2-15k+2$, where k is a positive integer. So we obtain that G is a strongly regular graph of order $3(48k^2-30k+5)$ and degree $r=8(3k-1)(4k-1)$ with $\tau=4(4k-1)^2-4k$ and $\theta=4(4k-1)^2$.

Case 4 ($t=3,4,5,6$). Using (5), we find that (x) $2k^2+5k+3=0$; (y) $8p+6k^2+21k+20=0$; (z) $20p+6k^2+27k+35=0$ and (w) $12p+2k^2+11k+18=0$ for $t=3,4,5,6$, respectively, a contradiction. □

Proposition 9 *Let G be a connected strongly regular graph of order* $3(2p+1)$ *and degree r, where* $2p+1$ *is a prime number. If* $m_3=2(2p+1)$ *and* $m_2=2p$*, then G belongs to the class* $(\overline{4}^0)$ *or* (5^0) *represented in Theorem 2.*

Proof Using (2), we obtain $2p(2|\lambda_3|-\lambda_2)=(\tau-\theta)-\delta+r$. Since $(\tau-\theta)-\delta=2\lambda_3$ and $|\lambda_3|\le\lfloor\frac{6p+3}{2}\rfloor$ (see [3]), it follows that $-6p\le(\tau-\theta)-\delta+r\le 6p$. Let $2|\lambda_3|-\lambda_2=t$ where $t=0,\pm1,\pm2,\pm3$. Let $\lambda_3=-k$ where k is a positive integer. Then (i) $\lambda_2=2k-t$; (ii) $\tau-\theta=k-t$; (iii) $\delta=3k-t$; (iv) $r=2(pt+k)$; and (v) $\theta=2pt-(2k^2-(t+2)k)$. Using (ii), (iv), and (v), we can easily see that (1) is reduced to

$$t(t-3)p+3k(k-1)=0. \tag{6}$$

Case 1 ($t = 0$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = 2k$ and $\lambda_3 = -k$, $\tau - \theta = k$, $\delta = 3k$, $r = 2k$, and $\theta = -2k^2 + 2k$. Using (6), we find that $k(k-1) = 0$. So we obtain that G is disconnected, a contradiction.

Case 2 ($t = 1$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = 2k - 1$ and $\lambda_3 = -k$, $\tau - \theta = k - 1$, $\delta = 3k - 1$, $r = 2(p + k)$, and $\theta = 2p - (2k^2 - 3k)$. Using (6), we find that $2p = 3k(k-1)$. Replacing k with $k+1$, we obtain that G is a strongly regular graph of order $3(3k^2 + 3k + 1)$ and degree $r = (k+1)(3k+2)$ with $\tau = (k+1)^2 + k$ and $\theta = (k+1)^2$.

Case 3 ($t = 2$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = 2(k-1)$ and $\lambda_3 = -k$, $\tau - \theta = k - 2$, $\delta = 3k - 2$, $r = 2(2p + k)$, and $\theta = 4p - (2k^2 - 4k)$. Using (6), we find that $2p = 3k(k-1)$. Replacing k with $k + 1$, we obtain that G is a strongly regular graph of order $3(3k^2 + 3k + 1)$ and degree $r = 2(k+1)(3k+1)$ with $\tau = 4k^2 + 7k + 1$ and $\theta = 2(k+1)(2k+1)$, where [3] $k \geq 2$.

Case 4 ($t = 3$). Using (i), (ii), (iii), (iv) and (v) we find that $\lambda_2 = 2k - 3$ and $\lambda_3 = -k$, $\tau - \theta = k - 3$, $\delta = 3(k-1)$, $r = 2(3p + k)$, and $\theta = 6p - (2k^2 - 5k)$. Using (6) we find that $k(k-1) = 0$. So we obtain that G is the complete graph, a contradiction.

Case 5 ($t = -1, -2, -3$). Using (v), we find that (x) $\theta = -2p - 2k^2 + k$; (y) $\theta = -4p - 2k^2$; and (z) $\theta = -6p - 2k^2 - k$ for $t = -1$, $t = -2$, and $t = -3$, respectively, a contradiction. □

Theorem 2 *Let G be a connected strongly regular graph of order $3(2p+1)$ and degree r, where $2p + 1$ is a prime number. Then G is one of the following strongly regular graphs:*

(1^0) *G is the strongly regular graph $\overline{3K_{2p+1}}$ of order $n = 3(2p+1)$ and degree $r = 4p+2$ with $\tau = 2p + 1$ and $\theta = 4p + 2$, where $p \in \mathbb{N}$ and $2p + 1$ is a prime number. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -(2p+1)$ with $m_2 = 6p$ and $m_3 = 2$;*

(2^0) *G is the strongly regular graph $\overline{(2p+1)K_3}$ of order $n = 3(2p+1)$ and degree $r = 6p$ with $\tau = 6p - 3$ and $\theta = 6p$, where $p \in \mathbb{N}$ and $2p + 1$ is a prime number. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -3$ with $m_2 = 2(2p+1)$ and $m_3 = 2p$;*

(3^0) *G is the conference graph of order $n = 3(4k-1)$ and degree $r = 6k - 2$ with $\tau = 3k - 2$ and $\theta = 3k - 1$, where $k \in \mathbb{N}$ and $4k - 1$ is a prime number. Its eigenvalues are $\lambda_2 = \frac{-1+\sqrt{3(4k-1)}}{2}$ and $\lambda_3 = \frac{-1-\sqrt{3(4k-1)}}{2}$ with $m_2 = 6k - 2$ and $m_3 = 6k - 2$;*

(4^0) *G is the strongly regular graph of order $n = 3(3k^2+3k+1)$ and degree $r = k(3k+1)$ with $\tau = k^2 - k - 1$ and $\theta = k^2$, where $k \geq 2$ and $3k^2 + 3k + 1$ is a prime number. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(2k+1)$ with $m_2 = 2(3k^2 + 3k + 1)$ and $m_3 = 3k(k+1)$;*

($\overline{4}^0$) *G is the strongly regular graph of order $n = 3(3k^2 + 3k + 1)$ and degree $r = 2(k+1)(3k+1)$ with $\tau = 4k^2 + 7k + 1$ and $\theta = 2(k+1)(2k+1)$, where $k \geq 2$ and $3k^2 + 3k + 1$ is a prime number. Its eigenvalues are $\lambda_2 = 2k$ and $\lambda_3 = -(k+1)$ with $m_2 = 3k(k+1)$ and $m_3 = 2(3k^2 + 3k + 1)$;*

[3] The case when $k = 1$ is impossible. Indeed, in this case, we have $n = 21$, $r = 16$ and $\theta = 12$, which yields that $\overline{\tau} = -1$, a contradiction.

(5^0) *G is the strongly regular graph of order* $n = 3(3k^2 + 3k + 1)$ *and degree* $r = (k+1)(3k+2)$ *with* $\tau = (k+1)^2 + k$ *and* $\theta = (k+1)^2$, *where* $k \in \mathbb{N}$ *and* $3k^2 + 3k + 1$ *is a prime number. Its eigenvalues are* $\lambda_2 = 2k+1$ *and* $\lambda_3 = -(k+1)$ *with* $m_2 = 3k(k+1)$ *and* $m_3 = 2(3k^2 + 3k + 1)$;

($\overline{5}^0$) *G is the strongly regular graph of order* $n = 3(3k^2 + 3k + 1)$ *and degree* $r = 2k(3k+2)$ *with* $\tau = 4k^2 + k - 2$ *and* $\theta = 2k(2k+1)$, *where* $k \in \mathbb{N}$ *and* $3k^2 + 3k + 1$ *is a prime number. Its eigenvalues are* $\lambda_2 = k$ *and* $\lambda_3 = -2(k+1)$ *with* $m_2 = 2(3k^2 + 3k + 1)$ *and* $m_3 = 3k(k+1)$;

(6^0) *G is the strongly regular graph of order* $n = 3(48k^2 - 30k + 5)$ *and degree* $r = 2(3k-1)(8k-3)$ *with* $\tau = (2k-1)(8k-1)$ *and* $\theta = (2k-1)(8k-3)$, *where* $k \in \mathbb{N}$ *and* $48k^2 - 30k + 5$ *is a prime number. Its eigenvalues are* $\lambda_2 = 8k-3$ *and* $\lambda_3 = -(4k-1)$ *with* $m_2 = 48k^2 - 30k + 5$ *and* $m_3 = 3(4k-1)(8k-3)$;

($\overline{6}^0$) *G is the strongly regular graph of order* $n = 3(48k^2 - 30k + 5)$ *and degree* $r = 8(3k-1)(4k-1)$ *with* $\tau = 4(4k-1)^2 - 4k$ *and* $\theta = 4(4k-1)^2$, *where* $k \in \mathbb{N}$ *and* $48k^2 - 30k + 5$ *is a prime number. Its eigenvalues are* $\lambda_2 = 4k-2$ *and* $\lambda_3 = -2(4k-1)$ *with* $m_2 = 3(4k-1)(8k-3)$ *and* $m_3 = 48k^2 - 30k + 5$;

(7^0) *G is the strongly regular graph of order* $n = 3(48k^2 + 30k + 5)$ *and degree* $r = 2(3k+1)(8k+3)$ *with* $\tau = (2k+1)(8k+1)$ *and* $\theta = (2k+1)(8k+3)$, *where* $k \geq 0$ *and* $48k^2 + 30k + 5$ *is a prime number. Its eigenvalues are* $\lambda_2 = 4k+1$ *and* $\lambda_3 = -(8k+3)$ *with* $m_2 = 3(4k+1)(8k+3)$ *and* $m_3 = 48k^2 + 30k + 5$;

($\overline{7}^0$) *G is the strongly regular graph of order* $n = 3(48k^2 + 30k + 5)$ *and degree* $r = 8(3k+1)(4k+1)$ *with* $\tau = 4(4k+1)^2 + 4k$ *and* $\theta = 4(4k+1)^2$, *where* $k \geq 0$ *and* $48k^2 + 30k + 5$ *is a prime number. Its eigenvalues are* $\lambda_2 = 2(4k+1)$ *and* $\lambda_3 = -(4k+2)$ *with* $m_2 = 48k^2 + 30k + 5$ *and* $m_3 = 3(4k+1)(8k+3)$.

Proof We note first that if G is a strongly regular graph with $\delta^2 = 3(2p+1)$, according to Proposition 3, it belongs to the class (3^0). Consequently, assume that G is an integral (non-conference) strongly regular graph. Using Theorem 1, we have $m_2 m_3 \delta^2 = 3(2p+1) r \overline{r}$. We shall now consider the following three cases.

Case 1 ($(2p+1) \mid \delta^2$). In this case, $(2p+1) \mid \delta$ because G is an integral graph. Since $\delta = \lambda_2 + |\lambda_3| < 6p + 3$ (see [3]), it follows that $\delta = 2p+1$ or $\delta = 2(2p+1)$. Using Propositions 4 and 5, it turns out that G belongs to the class (1^0).

Case 2 ($(2p+1) \mid m_2$). Since $m_2 + m_3 = 6p + 2$, it follows that $m_2 = 2p+1$ and $m_3 = 4p+1$ or $m_2 = 2(2p+1)$ and $m_3 = 2p$. Using Propositions 6 and 7, it turns out that G belongs to the class (2^0) or (4^0) or ($\overline{5}^0$) or (6^0) or ($\overline{7}^0$).

Case 3 ($(2p+1) \mid m_3$). Since $m_3 + m_2 = 6p + 2$, it follows that $m_3 = 2p+1$ and $m_2 = 4p+1$ or $m_3 = 2(2p+1)$ and $m_2 = 2p$. Using Propositions 8 and 9, it turns out that G belongs to the class ($\overline{4}^0$) or (5^0) or ($\overline{6}^0$) or (7^0). □

Proposition 10 *Let G be a connected strongly regular graph of order* $4(2p+1)$ *and degree r, where* $2p+1$ *is a prime number. If* $\delta = 2p+1$, *then G belongs to the class* (2^0) *represented in Theorem 3.*

Proof Using Theorem 1, we have $(2p+1)m_2 m_3 = 4r\overline{r}$, which means that $(2p+1) \mid r$ or $(2p+1) \mid \overline{r}$. It is sufficient to consider only the case when $(2p+1) \mid r$.

Case 1 ($r = 2p + 1$). Then, $m_2 m_3 = 8(3p + 1)$ and $m_2 + m_3 = 8p + 3$. So we find that $m_2, m_3 = \frac{8p+3\pm\Delta}{2}$ where $\Delta^2 = (8p - 3)^2 - 32$, a contradiction because Δ^2 is not a perfect square.

Case 2 ($r = 2(2p + 1)$). Then $m_2 m_3 = 8(4p + 1)$ which yields that $m_2, m_3 = \frac{8p+3\pm\Delta}{2}$ where $\Delta^2 = (8p - 3)^2 - 32(p + 1)$ and $\Delta^2 = (8p - 6)^2 + 16p - 59$. We can easily verify that $\Delta^2 = -39, 73, 313$ for $p = 1, 2, 3$, respectively. Since Δ^2 is not a perfect square for $p = 1, 2, 3$, we can assume $p \geq 4$. So we obtain $(8p - 6) < \Delta < (8p - 3)$ for $p \geq 4$, which provides that $\Delta = 8p - 5$. Using this fact, we find that $m_2 = 8p - 1$ and $m_3 = 4$ or $m_2 = 4$ and $m_3 = 8p - 1$. Thus, we have $4(8p - 1) = 8(4p + 1)$, a contradiction.

Case 3 ($r = 3(2p + 1)$). In this situation, $m_2 m_3 = 24p$ and $m_2 + m_3 = 8p + 3$, which yields that $m_2 = 8p$ and $m_3 = 3$ or $m_2 = 3$ and $m_3 = 8p$. Consider first the case when $m_2 = 8p$ and $m_3 = 3$. Using (2), we obtain $\tau - \theta = -(2p + 1)$. Since $\lambda_{2,3} = \frac{(\tau-\theta)\pm\delta}{2}$, we get easily $\lambda_2 = 0$ and $\lambda_3 = -(2p + 1)$, which proves that G is the strongly regular graph $\overline{4K_{2p+1}}$ of degree $r = 6p + 3$ with $\tau = 4p + 2$ and $\theta = 6p + 3$. Consider the case when $m_2 = 3$ and $m_3 = 8p$. Using (2), we obtain $\tau - \theta = \frac{(2p+1)(8p-9)}{8p+3}$, a contradiction because $(8p + 3) \nmid (8p - 9)$. □

Proposition 11 *Let G be a connected strongly regular graph of order $4(2p+1)$ and degree r, where $2p + 1$ is a prime number. If $\delta = 2(2p + 1)$, then G belongs to the class (1^0) represented in Theorem 3.*

Proof Using Theorem 1, we have $(2p + 1)m_2 m_3 = r\,\overline{r}$, which means that $(2p + 1)|r$ or $(2p + 1) \mid \overline{r}$. We shall here consider only the case when $(2p + 1) \mid r$.

Case 1 ($r = 2p + 1$). In this situation, we have $m_2 m_3 = 6p + 2$ and $m_2 + m_3 = 8p + 3$, a contradiction.

Case 2 ($r = 2(2p + 1)$). Then, $m_2 m_3 = 8p + 2$ and $m_2 + m_3 = 8p + 3$, which means that $m_2 = 8p + 2$ and $m_3 = 1$ or $m_2 = 1$ and $m_3 = 8p + 2$. Consider first the case when $m_2 = 8p+2$ and $m_3 = 1$. Using (2), we obtain easily $\tau - \theta = -2(2p+1)$, which provides that $\lambda_2 = 0$ and $\lambda_3 = -2(2p + 1)$. So we obtain that G is the complete bipartite graph $K_{4p+2,4p+2}$ of degree $r = 2(2p + 1)$ with $\tau = 0$ and $\theta = 2(2p + 1)$. Consider the case when $m_2 = 1$ and $m_3 = 8p+2$. Using (2), we obtain $\tau - \theta = \frac{2(2p+1)(8p-1)}{8p+3}$, a contradiction because $(8p + 3) \nmid (8p - 1)$.

Case 3 ($r = 3(2p + 1)$). In this situation, we have $m_2 m_3 = 6p$ and $m_2 + m_3 = 8p + 3$, a contradiction. □

Proposition 12 *There is no connected strongly regular graph G of order $4(2p + 1)$ and degree r with $\delta = 3(2p + 1)$, where $2p + 1$ is a prime number.*

Proof Contrary to the statement, assume that G is a strongly regular graph with $\delta = 3(2p + 1)$. Using Theorem 2, we have $9(2p + 1)m_2 m_3 = 4r\,\overline{r}$. Consider first the case when $r = 2p + 1$ and $\overline{r} = 6p + 2$. Then, $9m_2 m_3 = 8(3p + 1)$ and $9(m_2 + m_3) = 9(8p + 3)$, a contradiction. Consider the case when $r = 2(2p + 1)$ and $\overline{r} = 4p + 1$. Then $9m_2 m_3 = 8(4p + 1)$ and $9(m_2 + m_3) = 9(8p + 3)$, a contradiction. Consider the case when $r = 3(2p+1)$ and $\overline{r} = 2p$. Then $3m_2m_3 = 8p$ and $m_2 + m_3 = 8p + 3$, a contradiction. □

Proposition 13 *Let G be a connected strongly regular graph of order $4(2p+1)$ and degree r, where $2p+1$ is a prime number. If $m_2 = 2p+1$ and $m_3 = 6p+2$, then G belongs to the class (6^0) or $(\overline{7}^0)$ or (8^0) represented in Theorem 3.*

Proof Using (2), we obtain $4p(3|\lambda_3| - \lambda_2) = 3(\tau - \theta) - \delta + 2r$. Since $3(\tau - \theta) - \delta = 2\lambda_2 + 4\lambda_3$, it follows that $-16p \leq 3(\tau - \theta) - \delta + 2r \leq 24p$. Let $3|\lambda_3| - \lambda_2 = t$ where $-4 \leq t \leq 6$. Let $\lambda_3 = -k$ where k is a positive integer. Then (i) $\lambda_2 = 3k - t$; (ii) $\tau - \theta = 2k - t$; (iii) $\delta = 4k - t$; (iv) $r = (2p+1)t - k$; and (v) $\theta = (2p+1)t - (3k^2 - (t-1)k)$. Using (ii), (iv), and (v), we can easily see that (1) is reduced to

$$(p+1)t^2 - 2(2p+1)t + 6k^2 - 2k(2t-1) = 0. \tag{7}$$

Case 1 ($t = 0$). Using (i), (ii), (iii), and (iv), we find that $\lambda_2 = 3k$ and $\lambda_3 = -k$, $\tau - \theta = k$, $\delta = 4k$, and $r = -k$, a contradiction.

Case 2 ($t = 1$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = 3k - 1$ and $\lambda_3 = -k$, $\tau - \theta = 2k - 1, \delta = 4k - 1, r = (2p+1) - k$, and $\theta = (2p+1) - 3k^2$. Using (7), we find that $3p + 1 = 2k(3k - 1)$. Replacing k with $3k + 1$, we arrive at $p = 18k^2 + 10k + 1$, where k is a positive integer. So we obtain that G is a strongly regular graph of order $4(36k^2 + 20k + 3)$ and degree $r = (4k+1)(9k+2)$ with $\tau = 9k^2 + 8k + 1$ and $\theta = k(9k+2)$.

Case 3 ($t = 2$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = 3k - 2$ and $\lambda_3 = -k$, $\tau - \theta = 2(k-1), \delta = 2(2k-1), r = 2(2p+1) - k$, and $\theta = 2(2p+1) - (3k^2 - k)$. Using (7), we find that $2p = 3k(k-1)$. Replacing k with $k+1$, we obtain that G is a strongly regular graph of order $4(3k^2 + 3k + 1)$ and degree $r = (2k+1)(3k+1)$ with $\tau = 3k(k+1)$ and $\theta = k(3k+1)$.

Case 4 ($t = 3$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = 3(k-1)$ and $\lambda_3 = -k$, $\tau - \theta = 2k - 3, \delta = 4k - 3, r = 3(2p+1) - k$, and $\theta = 3(2p+1) - (3k^2 - 2k)$. Using (7), we find that $3p - 3 = 2k(3k - 5)$. Replacing k with $3k$, we arrive at $p = 18k^2 - 10k + 1$, where k is a positive integer. So we obtain that G is a strongly regular graph of order $4(36k^2 - 20k + 3)$ and degree $r = 9(3k-1)(4k-1)$ with $\tau = 9(3k-1)^2 + 3(2k-1)$ and $\theta = 9(3k-1)^2$.

Case 5 ($t = 4$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = 3k - 4$ and $\lambda_3 = -k$, $\tau - \theta = 2(k-2), \delta = 4(k-1), r = 4(2p+1) - k$, and $\theta = 4(2p+1) - (3k^2 - 3k)$. Using (7), we find that $(k-1)(3k-4) = 0$. So we obtain that G is the complete graph, a contradiction.

Case 6 ($t = 5$ and $t = 6$). Using (7), we find that $5p + 6k^2 - 18k + 15 = 0$ and $6p + 3k^2 - 11k + 12 = 0$ for $t = 5$ and $t = 6$, respectively, a contradiction.

Case 7 ($t \leq -1$). Using (7), we find that $(p+1)t^2 + 2|t|(2p+1) + 6k^2 + 2k(2|t|+1) = 0$, a contradiction. □

Proposition 14 *Let G be a connected strongly regular graph of order* $4(2p+1)$ *and degree r, where* $2p+1$ *is a prime number. If* $m_2 = 2(2p+1)$ *and* $m_3 = 4p+1$*, then G belongs to the class* (4^0) *represented in Theorem 3.*

Proof Using (2), we obtain $8p(|\lambda_3|-\lambda_2) = 3(\tau-\theta)+\delta+2r$. Since $3(\tau-\theta)+\delta = 4\lambda_2+2\lambda_3$, it follows that $-8p \le 3(\tau-\theta)+\delta+2r \le 32p$. Let $|\lambda_3|-\lambda_2 = t$ where $-1 \le t \le 4$. Let $\lambda_2 = k$ where k is a non-negative integer. Then (i) $\lambda_3 = -(k+t)$; (ii) $\tau-\theta = -t$; (iii) $\delta = 2k+t$; (iv) $r = (4p+1)t-k$; and (v) $\theta = (4p+1)t-(k^2+(t+1)k)$. Using (ii), (iv), and (v), we can easily see that (1) is reduced to

$$t(t-2)(4p+1)+2k(k+1) = 0. \tag{8}$$

Case 1 ($t = 0$). Using (i), (ii), (iii), and (iv), we find that $\lambda_2 = k$ and $\lambda_3 = -k$, $\tau-\theta = 0$, $\delta = 2k$, and $r = -k$, a contradiction.

Case 2 ($t = 1$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = k$ and $\lambda_3 = -(k+1)$, $\tau-\theta = -1$, $\delta = 2k+1$, $r = (4p+1)-k$, and $\theta = (4p+1)-(k^2+2k)$. Using (8), we find that $4p+1 = 2k(k+1)$, a contradiction because $2 \nmid (4p+1)$.

Case 3 ($t = 2$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = k$ and $\lambda_3 = -(k+2)$, $\tau-\theta = -2$, $\delta = 2(k+1)$, $r = 2(4p+1)-k$, and $\theta = 2(4p+1)-(k^2+3k)$. Using (8), we find that $k(k+1) = 0$. So we obtain that G is the cocktail-party graph $\overline{(4p+2)K_2}$ of degree $r = 8p+2$ with $\tau = 8p$ and $\theta = 8p+2$.

Case 4 ($t = 3, 4$ and $t = -1$). Using (8), we find that (x) $3(4p+1)+2k(k+1) = 0$; (y) $4(4p+1)+k(k+1) = 0$ and (z) $3(4p+1)+2k(k+1) = 0$ for $t = 3$, $t = 4$ and $t = -1$ respectively, a contradiction. □

Proposition 15 *Let G be a connected strongly regular graph of order* $4(2p+1)$ *and degree r, where* $2p+1$ *is a prime number. If* $m_2 = 3(2p+1)$ *and* $m_3 = 2p$*, then G belongs to the class* (3^0) *or* (5^0) *represented in Theorem 3.*

Proof Using (2), we obtain $4p(|\lambda_3|-3\lambda_2) = 3(\tau-\theta)+3\delta+2r$. Since $3(\tau-\theta)+3\delta = 6\lambda_2$, it follows that $0 < 3(\tau-\theta)+3\delta+2r \le 40p$. Let $|\lambda_3|-3\lambda_2 = t$ where $t = 1, 2, \ldots, 10$. Let $\lambda_2 = k$ where k is a non-negative integer. Then (i) $\lambda_3 = -(3k+t)$; (ii) $\tau-\theta = -(2k+t)$; (iii) $\delta = 4k+t$; (iv) $r = 2pt-3k$; and (v) $\theta = 2pt-(3k^2+(t+3)k)$. Using (ii), (iv), and (v), we can easily see that (1) is reduced to

$$t(t-4)p+6k(k+1) = 0. \tag{9}$$

Case 1 ($t = 1$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = k$ and $\lambda_3 = -(3k+1)$, $\tau-\theta = -(2k+1)$, $\delta = 4k+1$, $r = 2p-3k$, and $\theta = 2p-(3k^2+4k)$. Using (9), we find that $p = 2k(k+1)$ which yields that $2p+1 = (2k+1)^2$, a contradiction.

Case 2 ($t = 2$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = k$ and $\lambda_3 = -(3k+2)$, $\tau-\theta = -2(k+1)$, $\delta = 2(2k+1)$, $r = 4p-3k$, and $\theta = 4p-(3k^2+5k)$. Using (9), we find that $2p = 3k(k+1)$, where k is a positive integer. So we obtain that G is a strongly regular graph of order $4(3k^2+3k+1)$ and degree $r = 3k(2k+1)$ with $\tau = 3k^2-k-2$ and $\theta = k(3k+1)$.

Case 3 ($t = 3$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = k$ and $\lambda_3 = -3(k+1)$, $\tau - \theta = -(2k+3)$, $\delta = 4k+3$, $r = 3(2p-1)$, and $\theta = 6p - (3k^2 + 6k)$. Using (9), we find that $p = 2k(k+1)$ which yields that $2p+1 = (2k+1)^2$, a contradiction.

Case 4 ($t = 4$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = k$ and $\lambda_3 = -(3k+4)$, $\tau - \theta = -2(k+2)$, $\delta = 4(k+1)$, $r = 8p - 3k$, and $\theta = 8p - (3k^2 + 7k)$. Using (9), we find that $k(k+1) = 0$. So we obtain that G is the strongly regular graph $\overline{(2p+1)K_4}$ of degree $r = 8p$ with $\tau = 8p - 4$ and $\theta = 8p$.

Case 5 ($t \geq 5$). In this case, we find that $t(t-4)p + 6k(k+1) = 0$, a contradiction (see (9)). □

Proposition 16 *Let G be a connected strongly regular graph of order $4(2p+1)$ and degree r, where $2p+1$ is a prime number. If $m_3 = 2p+1$ and $m_2 = 6p+2$, then G belongs to the class $(\overline{6}^0)$ or (7^0) or $(\overline{8}^0)$ represented in Theorem 3.*

Proof Using (2) we obtain $4p(|\lambda_3| - 3\lambda_2) = 3(\tau - \theta) + \delta + 2r$. Let $|\lambda_3| - 3\lambda_2 = t$ where $-2 \leq t \leq 8$. Let $\lambda_2 = k$ where k is a non-negative integer. Then (i) $\lambda_3 = -(3k+t)$; (ii) $\tau - \theta = -(2k+t)$; (iii) $\delta = 4k+t$; (iv) $r = (2p+1)t + k$ and (v) $\theta = (2p+1)t - (3k^2 + (t-1)k)$. Using (ii), (iv) and (v) we can easily see that (1) is reduced to

$$(p+1)t^2 - 2(2p+1)t + 6k^2 + 2k(2t-1) = 0. \qquad (10)$$

Case 1 ($t = 0$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = k$ and $\lambda_3 = -3k$, $\tau - \theta = -2k$, $\delta = 4k$, $r = k$, and $\theta = -k(3k-1)$, which provides that $\theta = 0$. So we obtain that G is disconnected, a contradiction.

Case 2 ($t = 1$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = k$ and $\lambda_3 = -(3k+1)$, $\tau - \theta = -(2k+1)$, $\delta = 4k+1$, $r = (2p+1) + k$, and $\theta = (2p+1) - 3k^2$. Using (10) we find that $3p+1 = 2k(3k+1)$. Replacing k with $3k-1$, we arrive at $p = 18k^2 - 10k + 1$, where k is a positive integer. So we obtain that G is a strongly regular graph of order $4(36k^2 - 20k + 3)$ and degree $r = (4k-1)(9k-2)$ with $\tau = 9k^2 - 8k + 1$ and $\theta = k(9k-2)$.

Case 3 ($t = 2$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = k$ and $\lambda_3 = -(3k+2)$, $\tau - \theta = -2(k+1)$, $\delta = 2(2k+1)$, $r = 2(2p+1) + k$, and $\theta = 2(2p+1) - (3k^2 + k)$. Using (10), we find that $2p = 3k(k+1)$, where k is a positive integer. So we obtain that G is a strongly regular graph of order $4(3k^2 + 3k + 1)$ and degree $r = (2k+1)(3k+2)$ with $\tau = 3k(k+1)$ and $\theta = (k+1)(3k+2)$.

Case 4 ($t = 3$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = k$ and $\lambda_3 = -3(k+1)$, $\tau - \theta = -(2k+3)$, $\delta = 4k+3$, $r = 3(2p+1) + k$, and $\theta = 3(2p+1) - (3k^2 + 2k)$. Using (10), we find that $3p - 3 = 2k(3k+5)$. Replacing k with $3k$, we arrive at $p = 18k^2 + 10k + 1$, where k is a positive integer. So we obtain that G is a strongly regular graph of order $4(36k^2 + 20k + 3)$ and degree $r = 9(3k+1)(4k+1)$ with $\tau = 9(3k+1)^2 - 3(2k+1)$ and $\theta = 9(3k+1)^2$.

Case 5 ($t \geq 4$). Using (i), (ii), (iii), and (iv), we find that $\lambda_2 = k$ and $\lambda_3 = -(3k+4)$, $\tau - \theta = -2(k+2)$, $\delta = 4(k+1)$, and $r = 4(2p+1) + k \geq 8p + 4$, a contradiction.

Case 6 ($t = -1, -2$). Using (10), we obtain $(p+1)t^2+2|t|(2p+1)+6k^2-2k(2|t|+1) = 0$, a contradiction. □

Proposition 17 *There is no connected strongly regular graph G of order $4(2p + 1)$ and degree r with $m_3 = 2(2p + 1)$ and $m_2 = 4p + 1$, where $2p + 1$ is a prime number.*

Proof Contrary to the statement, assume that G is a strongly regular graph with $m_3 = 2(2p + 1)$ and $m_2 = 4p + 1$. Using (2), we obtain $8p(|\lambda_3| - \lambda_2) = 3(\tau - \theta) - \delta + r$. Let $|\lambda_3| - \lambda_2 = t$ where $-2 \le t \le 3$. Let $\lambda_2 = k$ where k is a non-negative integer. Then (i) $\lambda_3 = -(k + t)$; (ii) $\tau - \theta = -t$; (iii) $\delta = 2k + t$; (iv) $r = 2t(2p + 1) + k$; and (v) $\theta = 2t(2p + 1) - (k^2 + (t - 1)k)$. Using (ii), (iv), and (v), we can easily see that (1) is reduced to

$$(4p + 3)t^2 - 4(2p + 1)t + 2k^2 + 2k(2t - 1) = 0. \tag{11}$$

Case 1 ($t = 0$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = k$ and $\lambda_3 = -k$, $\tau - \theta = 0$, $\delta = 2k$, $r = k$, and $\theta = -k^2 + k$, a contradiction.

Case 2 ($t = 1$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = k$ and $\lambda_3 = -(k + 1)$, $\tau - \theta = -1$, $\delta = 2k + 1$, $r = 2(2p + 1) + k$, and $\theta = 2(2p + 1) - k^2$. Using (11), we find that $4p + 1 = 2k(k + 1)$, a contradiction because $2 \nmid (4p + 1)$.

Case 3 ($t = 2$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = k$ and $\lambda_3 = -(k + 2)$, $\tau - \theta = -2$, $\delta = 2(k + 1)$, $r = 4(2p + 1) + k$, and $\theta = 4(2p + 1) - (k^2 + k)$. Using (11), we find that $(k + 1)(k + 2) = 0$, a contradiction.

Case 4 ($t = 3$ and $t = -1, -2$). Using (11), we find that (x) $12p + 2k^2 + 10k + 5 = 0$; (y) $12p + 2k^2 - 6k + 7 = 0$; and (z) $16p + k^2 - 5k + 10 = 0$ for $t = 3$, $t = -1$, and $t = -2$, respectively, a contradiction. □

Proposition 18 *Let G be a connected strongly regular graph of order $4(2p+1)$ and degree r, where $2p+1$ is a prime number. If $m_3 = 3(2p + 1)$ and $m_2 = 2p$, then G belongs to the class $(\overline{\overline{5}}^0)$ represented in Theorem 3.*

Proof Using (2), we obtain $4p(3|\lambda_3|-\lambda_2) = 3(\tau-\theta)-3\delta+r$. Since $3(\tau-\theta)-3\delta = 6\lambda_3$, it follows that $-16p \le 3(\tau - \theta) - 3\delta + 2r \le 16p$. Let $3|\lambda_3| - \lambda_2 = t$ where $-4 \le t \le 4$. Let $\lambda_3 = -k$ where k is a positive integer. Then (i) $\lambda_2 = 3k - t$; (ii) $\tau - \theta = 2k - t$; (iii) $\delta = 4k - t$; (iv) $r = 2pt + 3k$; and (v) $\theta = 2pt - (3k^2 - (t + 3)k)$. Using (ii), (iv), and (v), we can easily see that (1) is reduced to

$$t(t - 4)p + 6k(k - 1) = 0. \tag{12}$$

Case 1 ($t = 0$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = 3k$ and $\lambda_3 = -k$, $\tau-\theta = 2k$, $\delta = 4k$, $r = 3k$, and $\theta = -3k^2 + 3k$. Using (12), we find that $k(k - 1) = 0$, which yields that $\theta = 0$. So we obtain that G is disconnected, a contradiction.

Case 2 ($t = 1$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = 3k + 1$ and $\lambda_3 = -k$, $\tau - \theta = 2k - 1$, $\delta = 4k - 1$, $r = 2p + 3k$, and $\theta = 2p - (3k^2 - 4k)$. Using (12), we find that $p = 2k(k - 1)$, which yields that $2p + 1 = (2k - 1)^2$, a contradiction.

Case 3 ($t = 2$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = 3k - 2$ and $\lambda_3 = -k$, $\tau - \theta = 2(k-1)$, $\delta = 2(2k-1)$, $r = 4p + 3k$, and $\theta = 4p - (3k^2 - 5k)$. Using (12), we find that $2p = 3k(k-1)$. Replacing k with $k+1$, we obtain that G is the strongly regular graph of order $4(3k^2 + 3k + 1)$ and degree $r = 3(k+1)(2k+1)$ with $\tau = (k+2)(3k+1)$ and $\theta = (k+1)(3k+2)$.

Case 4 ($t = 3$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = 3(k-1)$ and $\lambda_3 = -k$, $\tau - \theta = 2k - 3$, $\delta = 4k - 3$, $r = 6p + 3k$, and $\theta = 6p - (3k^2 - 6k)$. Using (12), we find that $p = k(k-1)$, which yields that $2p + 1 = (2k-1)^2$, a contradiction.

Case 5 ($t = 4$). Using (i), (ii), (iii), (iv), and (v), we find that $\lambda_2 = 3k - 4$ and $\lambda_3 = -k$, $\tau - \theta = 2(k-2)$, $\delta = 4(k-1)$, $r = 8p + 3k$, and $\theta = 6p - (3k^2 - 7k)$. Using (12), we find that $k(k-1) = 0$, a contradiction.

Case 6 ($t \le -1$). In this case, we find that $|t|(|t| + 4)p + 6k(k-1) = 0$, a contradiction (see (12)). □

Theorem 3 *Let G be a connected strongly regular graph of order $4(2p+1)$ and degree r, where $2p+1$ is a prime number. Then G is one of the following strongly regular graphs:*

(1^0) *G is the complete bipartite graph $K_{4p+2,4p+2}$ of order $n = 4(2p+1)$ and degree $r = 4p+2$ with $\tau = 0$ and $\theta = 4p+2$, where $p \in \mathbb{N}$ and $2p+1$ is a prime number. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -(4p+2)$ with $m_2 = 8p+2$ and $m_3 = 1$;*

(2^0) *G is the strongly regular graph $\overline{4K_{2p+1}}$ of order $n = 4(2p+1)$ and degree $r = 6p+3$ with $\tau = 4p+2$ and $\theta = 6p+3$, where $p \in \mathbb{N}$ and $2p+1$ is a prime number. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -(2p+1)$ with $m_2 = 8p$ and $m_3 = 3$;*

(3^0) *G is the strongly regular graph $\overline{(2p+1)K_4}$ of order $n = 4(2p+1)$ and degree $r = 8p$ with $\tau = 8p-4$ and $\theta = 8p$, where $p \in \mathbb{N}$ and $2p+1$ is a prime number. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -4$ with $m_2 = 3(2p+1)$ and $m_3 = 2p$;*

(4^0) *G is the cocktail-party graph $\overline{(4p+2)K_2}$ of order $n = 4(2p+1)$ and degree $r = 8p+2$ with $\tau = 8p$ and $\theta = 8p+2$, where $p \in \mathbb{N}$ and $2p+1$ is a prime number. Its eigenvalues are $\lambda_2 = 0$ and $\lambda_3 = -2$ with $m_2 = 2(2p+1)$ and $m_3 = 4p+1$;*

(5^0) *G is the strongly regular graph of order $n = 4(3k^2+3k+1)$ and degree $r = 3k(2k+1)$ with $\tau = 3k^2 - k - 2$ and $\theta = k(3k+1)$, where $k \in \mathbb{N}$ and $3k^2+3k+1$ is a prime number. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(3k+2)$ with $m_2 = 3(3k^2+3k+1)$ and $m_3 = 3k(k+1)$;*

($\overline{5}^0$) *G is the strongly regular graph of order $n = 4(3k^2+3k+1)$ and degree $r = 3(k+1)(2k+1)$ with $\tau = (k+2)(3k+1)$ and $\theta = (k+1)(3k+2)$, where $k \in \mathbb{N}$ and $3k^2+3k+1$ is a prime number. Its eigenvalues are $\lambda_2 = 3k+1$ and $\lambda_3 = -(k+1)$ with $m_2 = 3k(k+1)$ and $m_3 = 3(3k^2+3k+1)$;*

(6^0) *G is the strongly regular graph of order $n = 4(3k^2+3k+1)$ and degree $r = (2k+1)(3k+1)$ with $\tau = 3k(k+1)$ and $\theta = k(3k+1)$, where $k \in \mathbb{N}$ and $3k^2+3k+1$ is a prime number. Its eigenvalues are $\lambda_2 = 3k+1$ and $\lambda_3 = -(k+1)$ with $m_2 = 3k^2+3k+1$ and $m_3 = (3k+1)(3k+2)$;*

($\overline{6}^0$) *G is the strongly regular graph of order $n = 4(3k^2+3k+1)$ and degree $r = (2k+1)(3k+2)$ with $\tau = 3k(k+1)$ and $\theta = (k+1)(3k+2)$, where $k \in \mathbb{N}$ and $3k^2+3k+1$ is a prime number. Its eigenvalues are $\lambda_2 = k$ and $\lambda_3 = -(3k+2)$ with $m_2 = (3k+1)(3k+2)$ and $m_3 = 3k^2+3k+1$;*

(7^0) *G is the strongly regular graph of order* $n = 4(36k^2 - 20k + 3)$ *and degree* $r = (4k-1)(9k-2)$ *with* $\tau = 9k^2 - 8k + 1$ *and* $\theta = k(9k-2)$*, where* $k \in \mathbb{N}$ *and* $36k^2-20k+3$ *is a prime number. Its eigenvalues are* $\lambda_2 = 3k-1$ *and* $\lambda_3 = -(9k-2)$ *with* $m_2 = 4(3k-1)(9k-2)$ *and* $m_3 = 36k^2 - 20k + 3$*;*

($\overline{7}^0$) *G is the strongly regular graph of order* $n = 4(36k^2 - 20k + 3)$ *and degree* $r = 9(3k-1)(4k-1)$ *with* $\tau = 9(3k-1)^2 + 3(2k-1)$ *and* $\theta = 9(3k-1)^2$*, where* $k \in \mathbb{N}$ *and* $36k^2 - 20k + 3$ *is a prime number. Its eigenvalues are* $\lambda_2 = 3(3k-1)$ *and* $\lambda_3 = -3k$ *with* $m_2 = 36k^2 - 20k + 3$ *and* $m_3 = 4(3k-1)(9k-2)$*;*

(8^0) *G is the strongly regular graph of order* $n = 4(36k^2 + 20k + 3)$ *and degree* $r = (4k+1)(9k+2)$ *with* $\tau = 9k^2 + 8k + 1$ *and* $\theta = k(9k+2)$*, where* $k \in \mathbb{N}$ *and* $36k^2+20k+3$ *is a prime number. Its eigenvalues are* $\lambda_2 = 9k+2$ *and* $\lambda_3 = -(3k+1)$ *with* $m_2 = 36k^2 + 20k + 3$ *and* $m_3 = 4(3k+1)(9k+2)$*;*

($\overline{8}^0$) *G is the strongly regular graph of order* $n = 4(36k^2 + 20k + 3)$ *and degree* $r = 9(3k+1)(4k+1)$ *with* $\tau = 9(3k+1)^2 - 3(2k+1)$ *and* $\theta = 9(3k+1)^2$*, where* $k \in \mathbb{N}$ *and* $36k^2 + 20k + 3$ *is a prime number. Its eigenvalues are* $\lambda_2 = 3k$ *and* $\lambda_3 = -3(3k+1)$ *with* $m_2 = 4(3k+1)(9k+2)$ *and* $m_3 = 36k^2 + 20k + 3$.

Proof Using Theorem 1, we have $m_2 m_3 \delta^2 = 4(2p+1) r \overline{r}$. We shall now consider the following three cases.

Case 1 ($(2p+1) \mid \delta^2$). In this case, $(2p+1) \mid \delta$ because G is an integral graph. Since $\delta = \lambda_2 + |\lambda_3| < 8p+4$ (see [3]), it follows that $\delta = 2p+1$ or $\delta = 2(2p+1)$ or $\delta = 3(2p+1)$. Using Propositions 10, 11, and 12, it turns out that G belongs to the class (1^0) or (2^0).

Case 2 ($(2p+1) \mid m_2$). Since $m_2 + m_3 = 8p+3$, it follows that $m_2 = 2p+1$ and $m_3 = 6p+2$ or $m_2 = 2(2p+1)$ and $m_3 = 4p+1$ or $m_2 = 3(2p+1)$ and $m_3 = 2p$. Using Propositions 13, 14, and 15, it turns out that G belongs to the class (3^0) or (4^0) or (5^0) or (6^0) or ($\overline{7}^0$) or (8^0).

Case 3 ($(2p+1) \mid m_3$). Since $m_3 + m_2 = 8p+3$, it follows that $m_3 = 2p+1$ and $m_2 = 6p+2$ or $m_3 = 2(2p+1)$ and $m_2 = 4p+1$ or $m_3 = 3(2p+1)$ and $m_2 = 2p$. Using Propositions 16, 17, and 18, it turns out that G belongs to the class ($\overline{5}^0$) or ($\overline{6}^0$) or (7^0) or ($\overline{8}^0$). □

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