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Some Results on Semilocal Rings

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Abstract We first give simple proofs to some main results on semilocal rings of Zhang (Proc. Amer. Math. Soc. 137, 845–852, 2009). Then, as an extension, we prove that the $R/J(R)$-module $\overline{aR}$ is semisimple if and only if every descending chain in $aR : a_1R \supseteq \cdots \supseteq a_nR \supseteq \cdots$ with $a_{i+1} = a_i - a_ib_ia_i$ $(i = 1, 2, \ldots)$ eventually terminates.

Keywords Semilocal ring · Endomorphism ring · Semisimple module

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1 Introduction

Semilocal rings are characterized in several amazing ways by Camps and Dicks [2], then based on their results, two new conceptions "hollow chain" and "hollow length" are introduced in [6]. Let R be a ring, a descending chain

$$a_0R \supsetneq a_1R \supsetneq \cdots \supsetneq a_nR \supsetneq \cdots \supsetneq a_{n+1}R \cdots \quad \text{with } a_{n+1} = a_n - a_nb_na_n \text{ for some } b_n \in R$$

is called a hollow chain of $a_0 \in R$. Let $r = \sup\{n \in \mathbb{Z} \mid a_0R \supsetneq a_1R \supsetneq \cdots \supsetneq a_{n-1}R \supsetneq a_nR$ is a hollow chain of $a_0\}$, then r is called the right hollow length of a_0, denoted as $h.\text{length}(a_0) = r$. And it is proved in [6] that R is a semilocal ring with $u.\dim(R/J(R)) = n$ if and only if $h.\text{length}(1_R) = n$. Several other equivalent conditions of semilocal rings are obtained in [6]. However, some proofs in [6] are not concise. Recently, Camillo and Nielsen gave a short and intuitive proof of Camps–Dicks's following result in [1].

Theorem 1 *Let R be a ring. The following conditions are equivalent.*

(1) *R is semilocal;*

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(2) *There exist a ring S and a R-S-bimodule ${}_RM_S$ such that M_S has finite uniform dimension, and for $r \in R$ the equation* $\mathrm{ann}_M(r) = (0)$ *implies* $r \in U(R)$.

Motivated by the method used in [1], we first give short and intuitive proofs of some other characterizations of semilocal rings of [2, 6] in Theorem 2.

For a ring R, write $\overline{R} = R/J(R)$, we then generalized some results in [6] by discussing when will $\overline{aR}$ be a semisimple $\overline{R}$-module in Theorem 3. For a module M, write $E = \mathrm{End}(M)$. Generally, it is not easy to find the relations between the structure of E and the submodules of M, but based on Theorem 3, we find that for all $f \in E$, the condition $\overline{fE}$ is a semisimple E-module is decided by chain conditions of submodules of M.

All rings are associative with 1, and modules are unital. Let R be a ring. $U(R)$ and $r.U(R)$ denote respectively the set of invertible and right invertible elements of R. $u.\dim(M)$ denotes the uniform dimension of module M. For more results on semilocal rings, please refer to [3, 5]. We refer to [4] for all undefined notions used in the text.

2 Main Results

The following result contains some important results of [2, Theorem 1] and [6, Corollary 10], we now provide a short and intuitive proof.

Theorem 2 *Let R be a ring. The following are equivalent:*

1) *R is semilocal;*
2) $h.\mathrm{length}(1_R) < \infty$;
3) *Every descending chain of principal right ideals of R*

$$a_0R \supseteq a_1R \supseteq a_2R \supseteq \cdots \supseteq a_nR \supseteq \cdots \quad \text{with } a_{i+1} = a_i - a_ib_ia_i$$

eventually terminates.

4) *Every strict descending chain*

$$a_0R \supsetneq a_1R \supsetneq a_2R \supsetneq \cdots \supsetneq a_nR \supsetneq \cdots \quad \text{with } a_{i+1} = a_i - a_ib_ia_i$$

eventually terminates.

5) *There exists a partial order $\geq$ on R, satisfying the minimum conditions such that for any $a, b \in R$, if $1 - ab \notin U(R)$, then $a > a - aba$.*

5') *There exists a partial order $\geq$ on R, satisfying the minimum conditions such that for any $a, b \in R$, if $1 - ab \notin r.U(R)$, then $a > a - aba$.*

Proof 3) $\Rightarrow$ 4), 5) $\Rightarrow$ 5$'$) are trivial.

1) $\Rightarrow$ 3). By Theorem 1, there is a R-S-bimodule ${}_RM_S$ satisfying condition 2) of Theorem 1. Since $\mathrm{ann}_M(a - ada) = \mathrm{ann}_M(a) \oplus \mathrm{ann}_M(1 - da)$, we know from the descending chain in condition 3) that for any $n > 1$, $\mathrm{ann}_M(a_n) = \mathrm{ann}_M(a_0) \oplus \mathrm{ann}_M(1 - b_0a_0) \oplus \cdots \oplus \mathrm{ann}_M(1 - b_{n-1}a_{n-1})$. So for any $n > 1$, $u.\dim(\mathrm{ann}_M(a_n)) = u.\dim(\mathrm{ann}_M(a_0)) \oplus u.\dim(\mathrm{ann}_M(1 - b_0a_0)) + \cdots + u.\dim(\mathrm{ann}_M(1 - b_{n-1}a_{n-1})) \leq u.\dim(M)$, thus, all but finite $u.\dim(\mathrm{ann}_M(1 - b_ia_i)) = 0$. By condition 2) of Theorem 1, it means that $1 - b_ia_i \in U(R)$, thus $1 - a_ib_i \in U(R)$, so by [6, Lemma 3], all but finite $a_{i+1}R = a_i(1 - b_ia_i)R = a_iR$. It proves 3).

1) $\Rightarrow$ 2). Similar to the 1) $\Rightarrow$ 3),we can get $h.\mathrm{length}(1_R) \leq u.\dim(M)$.

Let $A_{\overline{R}}$ be a maximal submodule of $\overline{R}_{\overline{R}}$. We will indicate in the following 2) $\Rightarrow$ 1) and 5′) $\Rightarrow$ 1) that A is a direct summand. Since a ring is semisimple if and only if every maximal left ideal is a summand, we can get 1).

2) $\Rightarrow$ 1). First, for any $b \in R$, we have h.length$(b) \le h$.length$(1) < \infty$. In fact, if $b \notin U(R)$, then $1 - 1(1-b) = b \notin U(R)$. So by [6, Lemma 3], we get $1R \gneq (1 - 1(1-b)1)R = bR$, thus h.length$(1) > h$.length(b); If $b \in U(R)$ and $bR \gneq (b - bdb)R$ for some $d \in R$, then $(1 - bd) \notin r.U(R)$, so $1R \gneq (1-bd)R = (1-bd)bR = (b-bdb)R$, thus h.length$(1) > h$.length$(b-bdb)$, so h.length$(1) \ge h$.length(b).

Let $\Phi = \{b \in R | \overline{b} \notin A\}$, then $1_R \in \Phi$. Since for any $b \in R$, h.length$(b) \le h$.length$(1_R) < \infty$, there exists an element $b \in \Phi$ such that h.length$(b) = \text{Min}\{h\text{.length}(b) \mid b \in \Phi\}$. We have $\overline{bR} \cap A = 0$. In fact, for any $\overline{bx} \in \overline{bR} \cap A$, if $\overline{b - bxb} \in A$, then $\overline{b} = \overline{(bx)b} + \overline{b - bxb} \in A$, a contradiction. Whence $\overline{b - bxb} \in \Phi$. The conditions h.length$(b - bxb) \leqslant h$.length(b) and h.length$(b) = \text{Min}\{h\text{.length}(b) \mid b \in \Phi\}$ indicate that $\overline{bR} = \overline{(b - bxb)R}$. [6, Lemma 3] shows $1 - bx \in r.U(R)$. We can prove similarly that for any $y \in R$, $1 - bxy \in r.U(R)$, so $bx \in J(R)$. Therefore $\overline{bR} \cap A = 0$. Since A is a maximal left ideal of $\overline{R}$, $\overline{R} = \overline{bR} \oplus A$.

5′) $\Rightarrow$ 1). Write $\Phi = \{b \in R \mid \overline{b} \notin A\}$. By 5’), there is a minimum element $b \in \Phi$. Similar to the proof of 2) $\Rightarrow$ 1), we can get $\overline{R} = \overline{Rb} \oplus A$.

4) $\Rightarrow$ 5′). Define an order $\ge$ on R via $a \ge b$ if $a = b$ or $aR \supsetneq bR$ and $b = a - ada$ for some $d \in R$. Then a routine verification shows that this order satisfying condition 5’).

5′) $\Rightarrow$ 5). By 5′) $\Rightarrow$ 1), we know that R is semilocal, thus R is direct finite, so $r.U(R) = U(R)$. □

Let R and S be rings and let ${}_RM_S$ be a R-S-bimodule. If

(i) M_S has finite uniform dimension;
(ii) for $r \in R$, the equation $\text{ann}_M(r) = (0)$ implies $r \in U(R)$,

then we know that R is semilocal and $\dim(R/J(R)) = h$.length$(1_R) \le u.\dim(M_S)$. A natural question is:

When will $\dim(R/J(R)) = u.\dim(M_S)$?

We now give a characterization of this condition.

Proposition 1 *Suppose that R is a semilocal ring with* $\dim(R/J(R)) = n$, *${}_RM_S$ is a R-S-bimodule satisfying the above conditions* (i) *and* (ii). *Let*

$$a_0R \supsetneq a_1R \supsetneq a_2R \supsetneq \cdots \supsetneq a_nR \quad \textit{with } a_{i+1} = a_i - a_ib_ia_i$$

be a hollow chain of $a_0 = 1_R$.

Then $\dim(R/J(R)) = u.\dim(M_s)$ *if and only if*

1) $\text{ann}_M(a_n)$ *is an essential submodule of M, and*
2) *for any $b \in R$, if R/bR is local, then* $\text{ann}_M(b)$ *is uniform.*

Proof “ $\Leftarrow$ ” By [6, Proposition 8], $R/(1 - a_ib_i)R$ is local for each i. So by condition 2) and the equation $\text{ann}_M(a_n) = \text{ann}_M(1 - a_0b_0)) \oplus \cdots \oplus \text{ann}_M(1 - a_{n-1}b_{n-1}))$, $u.\dim(\text{ann}_M(a_n)) = \sum_{i=0}^{n-1} u.\dim(\text{ann}_M(1 - a_ib_i)) = n$, condition 1) shows $u.\dim(M) = u.\dim(\text{ann}_M(a_n)) = n$.

" $\Rightarrow$ " $a_iR \neq a_{i+1}R$ gives $1 - a_ib_i \notin r.U(R) = U(R)$, thus $1 - b_ia_i \notin U(R)$. Condition (ii) shows that $\mathrm{ann}_M(1 - b_ia_i) \neq 0$, whence $u.\dim(\mathrm{ann}_M(1 - b_ia_i)) \geq 1$. So $n = u.\dim(M) \geq u.\dim(\mathrm{ann}_M(a_n)) = \sum_{i=0}^{n-1} u.\dim(\mathrm{ann}_M(1 - b_ia_i)) \geq n$.

Thus $u.\dim(\mathrm{ann}_M(a_n)) = n = u.\dim(M)$, whence $\mathrm{ann}_M(a_n)$ is an essential submodule of M and each $u.\dim(\mathrm{ann}_M(1 - b_ia_i)) = 1$, i.e., $\mathrm{ann}_M(1 - b_ia_i)$ is uniform.

For any $b \in R$, suppose that R/bR is local, then $b \notin U(R)$. By [6, Proposition 8(1)], we know that $h.\mathrm{length}(b) = h.\dim(R) - 1 = n - 1$.

Therefore, we can find a hollow chain of $b = b_1$

$$b_1R \supsetneq b_2R \supsetneq \cdots \supsetneq b_nR \quad \text{with } b_{i+1} = b_i - b_ic_ib_i.$$

Write $b_0 = 1_R$ and $c_0 = 1 - b$, then $b_1 = b = 1 - 1(1 - b)1 = b_0 - b_0c_0b_0$. Since $1 - c_0b_0 = b \notin U(R)$, we have the following hollow chain of $b_0 = 1$,

$$b_0R \supsetneq b_1R \supsetneq b_2R \supsetneq \cdots \supsetneq b_nR \quad \text{with } b_{i+1} = b_i - b_ic_ib_i.$$

Then similar to the above proof, we can know that $\mathrm{ann}_M(1-c_ib_i)$ is uniform, so $\mathrm{ann}_M(b)$ $= \mathrm{ann}_M(1 - c_0b_0)$ is uniform, as asserted. □

Theorem 3 *Let R be a ring, $a \in R$. The following conditions are equivalent:*

1) *$\overline{aR}$ is a semisimple $\overline{R}$-module;*
2) *$h.\mathrm{length}(a) < \infty$;*
3) *Every descending chain in aR*

$$a_1R \supseteq \cdots \supseteq a_nR \supseteq \cdots \quad \textit{with } a_{i+1} = a_i - a_ib_ia_i \ (i = 1, 2, \ldots)$$

eventually terminates;

4) *Every strict descending chain in aR*

$$a_1R \supsetneq \cdots \supsetneq a_nR \supsetneq \cdots \quad \textit{with } a_{i+1} = a_i - a_ib_ia_i \ (i = 1, 2, \ldots)$$

eventually terminates;

5) *Every descending chain in aR*

$$a_1R \supseteq \cdots \supseteq a_nR \supseteq \cdots \quad \textit{with } a_1 = a \textit{ and } a_{i+1} = a_i - a_ib_ia_i \ (i = 1, 2, \ldots)$$

eventually terminates;

6) *Every strict descending chain in aR*

$$a_1R \supsetneq \cdots \supsetneq a_nR \supsetneq \cdots \quad \textit{with } a_1 = a \textit{ and } a_{i+1} = a_i - a_ib_ia_i \ (i = 1, 2, \ldots)$$

eventually terminates.

Proof 3) $\Rightarrow$ 4), 3) $\Rightarrow$ 5), 2) $\Rightarrow$ 6), 4) $\Rightarrow$ 6), 5) $\Rightarrow$ 6) are trivial.

1) $\Rightarrow$ 3). Given a descending chain in aR

$$aR \supseteq a_1R \supseteq a_2R \supseteq \cdots \supseteq a_nR \supseteq \cdots \quad \text{with } a_{i+1} = a_i - a_ib_ia_i \tag{1}$$

We have a descending chain in $\overline{aR}$

$$\overline{a_1R} \supseteq \cdots \supseteq \overline{a_nR} \supseteq \cdots \quad \text{with } \overline{a_{i+1}} = \overline{a_i - a_ib_ia_i}. \tag{2}$$

Since $\overline{aR}$ is semisimple and cyclic, $\overline{aR}$ is a direct sum of finite simple modules. So $\overline{aR}$ has a composition series of finite length. Thus, the descending chain (2) eventually terminates. By [6, Lemma 3], it means the chain (1) eventually terminates.

1) $\Rightarrow$ 2) is similar to the 1) $\Rightarrow$ 3).

6) $\Rightarrow$ 1) We first prove the following result.

Claim If $B \lneq \overline{aR}$, then there exists a submodule $C \neq 0$ of $\overline{aR}$ such that $B \cap C = 0$.

Write $\Phi = \{b \in R \mid b = a - aya$ for some $y \in R$ and $\overline{b} \notin B\}$. $a \in \Phi$ shows $\Phi \neq \emptyset$. Define an order $\leq$ on Φ via

$$x \leq y \text{ if } x = y \text{ or } xR \supsetneq yR \text{ and } y = x - xdx \text{ for some } d \in R.$$

Then it is easy to show that " $\leq$ " is a partial order for Φ. Condition 6) shows that $(\Phi, \leq)$ is inductive, so by Zorn lemma, there is a maximum element $b \in \Phi$.

We have $\overline{bR} \cap B = 0$. In fact, for any $\overline{bx} \in \overline{bR} \cap B$, we can get $1 - bx \in r.U(R)$. Otherwise, we have $(b - bxb)R \lneq bR$. $b \in \Phi$ shows that $b = a - aya$ for some $y \in R$, so $b - bxb = (a - aya) - (a - aya)x(a - aya) = a - a[y + (1 - ya)x(1 - ay)]a$. In addition, by $\overline{b} \notin B$ and $\overline{bx} \in B$, we know that $\overline{b - bxb} \notin B$, so $b - bxb \in \Phi$, thus $b - bxb > b$, but b is a maximal element in Φ, a contradiction. For $\overline{bx} \in \overline{bR} \cap B$, repeating the argument, we see that $1 - bxy \in r.U(R)$ for all $y \in R$, so $bx \in J(R)$, i.e., $\overline{bx} = \overline{0}$.

Since $\overline{b} \notin B$, $\overline{bR} \neq 0$. Set $C = \overline{bR} \neq 0$, then $B \cap C = 0$. So the claim is proved.

Let A be a submodule of $\overline{aR}$. Suppose $A \lneq \overline{aR}$, if we can prove that A is a summand of $\overline{aR}$, then $\overline{aR}$ is semisimple.

Let X be the complement of A in $\overline{aR}$. If $A \oplus X \neq \overline{aR}$, applying the claim, we can find a $D \subseteq \overline{aR}$ such that $D \neq 0$ and $(A \oplus X) \cap D = 0$, and so $A \cap (X \oplus D) = 0$, a contradiction. So $A \oplus X = \overline{aR}$. □

Let S be a ring and M_S a module. Write $\mathrm{End}(M) = E$ and $\overline{E} = E/J(E)$. For any $f \in E$, as an application of Theorem 3, we can characterize the semisimplicity of $\overline{fE}$ by chain conditions of submodules of module M_S.

For $f, g \in E$, we proved in [6, Remark 4] that if $f(M) = (f - fgf)(M)$, then $1 - fg$ is surjective. Besides, since $(f - fgf)(M) = f(M) \cap (1 - fg)(M)$, we know that $f(M) = (f - fgf)(M)$ if and only if $1 - fg$ is surjective. Since $\ker(f - fgf) = \ker f \oplus \ker(1 - gf)$, and $\ker(1 - gf) \cong \ker(1 - fg)$, $\ker(f - fgf) = \ker f$ if and only if $1 - fg$ is injective.

We can get from Theorem 3 the following

Proposition 2 *Let M_S be a module and $f \in E = \mathrm{End}(M)$. If M is direct finite, then the following conditions are equivalent.*

(1) $\overline{fE}$ *is a semisimple* $\overline{E}$*-module;*
(2) *Write $f_0 = f$, for every sequence $g_0, g_1, g_2, \ldots$ of elements of* $\mathrm{End}(M)$*, if we set $f_{n+1} = f_n - f_n g_n f_n$ for every $n \geq 0$, then the chains*

 (i) $f_0(M) \supseteq f_1(M) \supseteq \cdots \supseteq f_n(M) \supseteq \cdots$,
 (ii) $\ker f_0 \subseteq \ker f_1 \subseteq \cdots \subseteq \ker f_n \subseteq \cdots$

 of submodules of M both eventually terminate.

By Theorem 3, we can get the following sufficient conditions for $\overline{fE}$ to be semisimple.

Proposition 3 *Let M_S be a module and $f \in E = \mathrm{End}(M)$. Write $f_0 = f$, for every sequence $g_0, g_1, g_2, \ldots$ of elements of* $\mathrm{End}(M)$*, setting $f_{n+1} = f_n - f_n g_n f_n$ for every $n \geq 0$, if the chains*

(i) $f_0(M) \supseteq f_1(M) \supseteq \cdots \supseteq f_n(M) \supseteq \cdots$,
(ii) $\ker f_0 \subseteq \ker f_1 \subseteq \cdots \subseteq \ker f_n \subseteq \cdots$

of submodules of M both eventually terminate, then $\overline{fE}$ is a semisimple $\overline{E}$-module.

The following result is a generalization of [6, Corollary 13], their proofs are similar.

Corollary 1 *Let M_S be a module for which every monomorphism $M \to M$ splits. $f_0 = f \in E = \mathrm{End}(M)$. The following conditions are equivalent*

(i) $\overline{fE}$ *is a semisimple* $\overline{E}$*-module.*
(ii) *For every sequence $g_0, g_1, g_2, \ldots$ of elements of* $\mathrm{End}(M)$*, if we set $f_{n+1} = f_n - f_n g_n f_n$ for every $n \geq 0$, then the chain* $\ker f_0 \subseteq \ker f_1 \subseteq \cdots \subseteq \ker f_n \subseteq \cdots$ *of submodules of M eventually terminates.*

Moreover, $h.\dim(fE) \leq \dim(M)$.

Corollary 2 *Let M_S be a module for which every monomorphism in* $\mathrm{End}(M)$ *splits. Write $\overline{E} = E/J(E)$. For any $f \in E$, if $M/\ker f$ has finite uniform dimension, then $\overline{fE}$ is a semisimple $\overline{E}$-module.*

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