

# A Unified Approach to Convex and Convexified Generalized Differentiation of Nonsmooth Functions and Set-Valued Mappings

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**Abstract** The primary goal of this paper is to present a unified approach and shed new light on convex and Clarke generalized differentiation theories using the concepts and techniques from Mordukhovich's developments. We show that the concepts and techniques used by Mordukhovich are important, not only to his generalized differentiation theory itself, but also to many other aspects of nonsmooth analysis. In particular, they can be used to derive convex subdifferential calculus rules as well as many important calculus rules of Clarke subdifferentials.

**Keywords** Subgradient · Subdifferential · Coderivative · Generalized differentiation

**Mathematics Subject Classification (2010)** 49J53 · 49J52 · 90C31

## 1 Introduction

For centuries, differential calculus has served as an indispensable tool for science and technology, but the rise of more complex models requires new tools to deal with nondifferentiable data. Geometric properties of convex functions and sets were the topics of study for

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Dedicated to Professor Boris Mordukhovich on the occasion of his 65th birthday

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many accomplished mathematicians such as Fenchel and Minkowski in the early twentieth century. However, the beginning era of *nonsmooth/variational analysis* did not start until the 1960s when Moreau and Rockafellar independently introduced a concept called *subgradient* for convex functions and, together with many other mathematicians, started developing the theory of generalized differentiation for convex functions and sets. The choice to start with convex functions and sets comes from the fact that they have several interesting properties that are important for applications to optimization. The theory is now called *convex analysis*, a mature field serving as the mathematical foundation for *convex optimization*.

The beauty and tremendous applications of convex analysis motivated the search for a new theory to deal with broader classes of functions and sets where convexity is not assumed. This work was initiated by Clarke, a student of Rockafellar. In the early 1970s, Clarke began to develop a generalized differentiation theory for the class of Lipschitz continuous functions. This theory revolves around a notion called *generalized gradient*, a concept that has led to numerous works, developments, and applications.

The theory of generalized differentiation for nonsmooth functions relies on the interaction between the analytic and geometric properties of functions and sets. In the mid 1970s, the work of Mordukhovich brought this key idea to a very high level of generality and beauty as reflected in two important objects of his theory of generalized differentiation: *nonsmooth functions* and *set-valued mappings*. The generalized derivative notions for nonsmooth functions and set-valued mappings introduced by Mordukhovich are now called under the names *Mordukhovich subdifferentials* and *coderivatives*, respectively. Mordukhovich's generalized differentiation theory is effective for many applications, especially to optimization theory. In spite of the nonconvexity of the Mordukhovich generalized derivative constructions, they possess well-developed calculus rules in many important classes of Banach spaces including reflexive spaces.

Our main goal in this paper is to develop a unified approach for generalized differential calculus of convex subdifferentials and Clarke subdifferentials. We show that the concepts and techniques used by Mordukhovich are important, not only to his generalized differentiation theory itself, but also to many other aspects of nonsmooth analysis.

The paper is organized as follows. In Section 2, we introduce important concepts of nonsmooth analysis that are used throughout the paper. In Section 3, we obtain a subdifferential formula in the sense of convex analysis for the optimal value function under convexity, and then use it to derive many important subdifferential rules of convex analysis. Section 4 is devoted to deriving coderivative and subdifferential calculus rules in the sense of Clarke using Mordukhovich's constructions.

## 2 Definitions and Preliminaries

In this section, we present basic concepts and results of nonsmooth analysis to be used in the next sections.

Let  $X$  be a Banach space. For a function  $\varphi : X \rightarrow \mathbb{R}$  which is Lipschitz continuous around  $\bar{x} \in X$  with Lipschitz modulus  $\ell \geq 0$ , the *Clarke generalized directional derivative* of  $\varphi$  at  $\bar{x}$  in direction  $v \in X$  is defined by

$$\varphi^\circ(\bar{x}; v) := \limsup_{x \rightarrow \bar{x}, t \downarrow 0} \frac{\varphi(x + tv) - \varphi(x)}{t}.$$

The generalized directional derivatives of  $\varphi$  at  $\bar{x}$  are employed to define the Clarke subdifferential of the function at this point:

$$\partial_C \varphi(\bar{x}) := \{x^* \in X^* \mid \langle x^*, v \rangle \leq \varphi^\circ(\bar{x}; v) \text{ for any } v \in X\}.$$

Given a nonempty closed set  $\Omega$  and given a point  $\bar{x} \in \Omega$ , the *Clarke normal cone* to  $\Omega$  at  $\bar{x}$  is a subset of the dual space  $X^*$  defined by

$$N_C(\bar{x}; \Omega) := \text{cl}^* \left\{ \bigcup_{\lambda > 0} \lambda \partial_C \text{dist}(\bar{x}; \Omega) \right\},$$

where  $\text{dist}(\cdot; \Omega)$  is the *distance function* to  $\Omega$  with the representation

$$\text{dist}(x; \Omega) := \inf\{\|x - w\| \mid w \in \Omega\}, \quad x \in X.$$

With the definition of Clarke normal cones to nonempty closed sets in hand, the *Clarke subdifferential*  $\partial_C \varphi(\bar{x})$  for a lower semicontinuous extended-real-valued function  $\varphi : X \rightarrow (-\infty, \infty]$  at  $\bar{x} \in \text{dom } \varphi := \{x \in X \mid \varphi(x) < \infty\}$  can be defined in terms of Clarke normal cones to the epigraph of the function by

$$\partial_C \varphi(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N_C((\bar{x}, \varphi(\bar{x})); \text{epi } \varphi)\}.$$

Similarly, the *Clarke singular subdifferential* of  $\varphi$  at  $\bar{x}$  is defined by

$$\partial_C^\infty \varphi(\bar{x}) := \{x^* \in X^* \mid (x^*, 0) \in N_C((\bar{x}, \varphi(\bar{x})); \text{epi } \varphi)\}.$$

Another important normal cone structure of nonsmooth analysis is called the *Fréchet normal cone* to  $\Omega$  at  $\bar{x} \in \Omega$  and is defined by

$$\widehat{N}(\bar{x}; \Omega) := \left\{ x^* \in X^* \mid \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \right\}.$$

We also define the set of Fréchet  $\varepsilon$ -normals to  $\Omega$  at  $\bar{x}$  by

$$\widehat{N}_\varepsilon(\bar{x}; \Omega) := \left\{ x^* \in X^* \mid \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq \varepsilon \right\}.$$

If  $\bar{x} \notin \Omega$ , we put  $\widehat{N}(\bar{x}; \Omega) := \emptyset$  and  $\widehat{N}_\varepsilon(\bar{x}; \Omega) := \emptyset$ .

The *Mordukhovich normal cone* to  $\Omega$  at  $\bar{x}$  is defined in terms of the Fréchet  $\varepsilon$ -normals to the set around  $\bar{x}$  using the *Kuratowski upper limit*:

$$N(\bar{x}; \Omega) := \text{Lim sup}_{x \rightarrow \bar{x}; \varepsilon \downarrow 0} \widehat{N}_\varepsilon(x; \Omega).$$

If  $X$  is an Asplund space (see [3]) and  $\Omega$  is locally closed around  $\bar{x}$ , then the limiting normal cone can be equivalently represented as

$$N(\bar{x}; \Omega) := \text{Lim sup}_{x \rightarrow \bar{x}} \widehat{N}(x; \Omega).$$

The *Mordukhovich subdifferential* and *singular subdifferential* of an extended-real-valued lower semicontinuous function  $\varphi : X \rightarrow (-\infty, \infty]$  at  $\bar{x} \in \text{dom } \varphi$  are then, respectively, defined by

$$\begin{aligned} \partial \varphi(\bar{x}) &:= \{x^* \in X^* \mid (x^*, -1) \in N((\bar{x}, \varphi(\bar{x})); \text{epi } \varphi)\}, \\ \partial^\infty \varphi(\bar{x}) &:= \{x^* \in X^* \mid (x^*, 0) \in N((\bar{x}, \varphi(\bar{x})); \text{epi } \varphi)\}. \end{aligned}$$

It is well-known that when  $\varphi$  is a convex function, both subdifferential constructions reduce to the subdifferential in the sense of convex analysis

$$\partial_C \varphi(\bar{x}) = \partial \varphi(\bar{x}) = \{x^* \in X^* \mid \langle x^*, x - \bar{x} \rangle \leq \varphi(x) - \varphi(\bar{x}) \text{ for all } x \in X\}.$$

Moreover, if the set  $\Omega$  is convex, both normal cone structures reduce to the normal cone in the sense of convex analysis

$$N_C(\bar{x}; \Omega) = N(\bar{x}; \Omega) = \{x^* \in X^* \mid \langle x^*, x - \bar{x} \rangle \leq 0 \text{ for all } x \in \Omega\}.$$

The relation between the Mordukhovich normal cone and subdifferential constructions can also be represented by

$$N(\bar{x}; \Omega) = \partial \delta(\bar{x}; \Omega), \quad \bar{x} \in \Omega,$$

where  $\delta(\cdot; \Omega)$  is the indicator function associated with  $\Omega$  given by  $\delta(x; \Omega) = 0$  if  $x \in \Omega$ , and  $\delta(x; \Omega) = +\infty$  otherwise. Similar relations also hold true for Clarke and Fréchet normal cones and subdifferentials.

Let  $F : X \rightrightarrows Y$  be a *set-valued mapping* between two Banach spaces  $X$  and  $Y$ . That means  $F(x)$  is a subset of  $Y$  for every  $x \in X$ . The *domain* and *graph* of  $F$  are given, respectively, by

$$\text{dom } F := \{x \in X \mid F(x) \neq \emptyset\} \quad \text{and} \quad \text{gph } F := \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

We say that  $F$  has convex graph if its graph is a convex set in  $X \times Y$ . Similarly, we say that  $F$  has closed graph if its graph is a closed set in  $X \times Y$ . It is easy to see that if  $A : X \rightarrow Y$  is a bounded linear mapping, then  $A$  has closed convex graph.

Let us continue with generalized derivative concepts for set-valued mappings introduced by Mordukhovich. The set-valued mapping  $D^*F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$  defined by

$$D^*F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N((\bar{x}, \bar{y}); \text{gph } F)\}$$

is called the *Mordukhovich coderivative* of  $F$  at  $(\bar{x}, \bar{y})$ . By convention,  $D^*F(\bar{x}, \bar{y})(y^*) = \emptyset$  for all  $(\bar{x}, \bar{y}) \notin \text{gph } F$  and  $y^* \in Y^*$ . When  $F$  is single-valued, one writes  $D^*F(\bar{x})$  instead of  $D^*F(\bar{x}, \bar{y})$ , where  $\bar{y} = F(\bar{x})$ . The corresponding *Clarke coderivative* is similarly defined by

$$D_C^*F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N_C((\bar{x}, \bar{y}); \text{gph } F)\}.$$

In general, one has

$$D^*F(\bar{x}, \bar{y})(y^*) \subseteq D_C^*F(\bar{x}, \bar{y})(y^*) \quad \text{for all } (\bar{x}, \bar{y}) \in \text{gph } F, y^* \in Y^*,$$

where the inclusion holds as equality if  $F$  has convex graph.

The following theorem which establishes the relation between Clarke and Mordukhovich generalized differentiation constructions will play an important role in our paper; see [3] for the definition and proof.

**Theorem 1** *Let  $X$  be an Asplund space. The following assertions hold:*

(i) *Let  $\Omega \subseteq X$  be a closed set. Then*

$$N_C(\bar{x}; \Omega) = \text{cl}^* \text{co } N(\bar{x}; \Omega).$$

(ii) *Let  $\varphi : X \rightarrow (-\infty, \infty]$  be lower semicontinuous and let  $\bar{x} \in \text{dom } \varphi$ . Then*

$$\partial_C \varphi(\bar{x}) = \text{cl}^* \text{co}[\partial \varphi(\bar{x}) + \partial^\infty \varphi(\bar{x})].$$

*In particular, if  $\varphi$  is Lipschitz continuous around  $\bar{x}$ , then*

$$\partial_C \varphi(\bar{x}) = \text{cl}^* \text{co } \partial \varphi(\bar{x}).$$

### 3 Convex Subdifferential Calculus via Mordukhovich Subgradients

Our goal in this section is to develop a unified approach for convex subdifferential calculus in infinite dimensions. We show that the concepts and techniques of nonsmooth analysis involving Mordukhovich subdifferential and coderivative constructions can be employed to shed new light on a fundamental subject of convex analysis: convex subdifferential calculus.

Let us start with some well-known examples of computing coderivatives of convex set-valued mappings. We provide the details for the convenience of the reader. Throughout this section, we assume  $X, Y,$  and  $Z$  are Banach spaces unless otherwise stated.

*Example 1* Let  $K$  be a nonempty convex set in  $Y$ . Define  $F : X \rightrightarrows Y$  by  $F(x) = K$  for every  $x \in X$ . Then

$$\text{gph}F = X \times K.$$

For any  $(\bar{x}, \bar{y}) \in X \times K$ , one has  $N((\bar{x}, \bar{y}); \text{gph} F) = \{0\} \times N(\bar{y}; K)$ . Thus, for  $y^* \in Y^*$ ,

$$D^*F(\bar{x}, \bar{y})(y^*) = \begin{cases} \{0\} & \text{if } -y^* \in N(\bar{y}; K), \\ \emptyset & \text{otherwise.} \end{cases}$$

In particular, if  $K = X$ , then

$$D^*F(\bar{x}, \bar{y})(y^*) = \begin{cases} \{0\} & \text{if } y^* = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

The example below shows that the coderivative concept is a generalization of the *adjoint mapping* well known in functional analysis.

*Example 2* Let  $F : X \rightrightarrows Y$  be given by  $F(x) = \{A(x) + b\}$ , where  $A : X \rightarrow Y$  is a bounded linear mapping and  $b \in Y$ . For  $(\bar{x}, \bar{y}) \in \text{gph} F$  with  $\bar{y} = A(\bar{x}) + b$ , one has

$$D^*F(\bar{x}, \bar{y})(y^*) = \{A^*y^*\} \quad \text{for all } y^* \in Y^*.$$

Let us first prove that

$$N((\bar{x}, \bar{y}); \text{gph} F) = \{(x^*, y^*) \in X^* \times Y^* \mid x^* = -A^*y^*\}.$$

Obviously,  $\text{gph}F$  is a convex set. By the definition,  $(x^*, y^*) \in N((\bar{x}, \bar{y}); \text{gph}F)$  if and only if

$$\langle x^*, x - \bar{x} \rangle + \langle y^*, F(x) - F(\bar{x}) \rangle \leq 0 \quad \text{for all } x \in X. \tag{1}$$

In our case, we have

$$\begin{aligned} \langle x^*, x - \bar{x} \rangle + \langle y^*, F(x) - F(\bar{x}) \rangle &= \langle x^*, x - \bar{x} \rangle + \langle y^*, A(x) - A(\bar{x}) \rangle \\ &= \langle x^*, x - \bar{x} \rangle + \langle A^*y^*, x - \bar{x} \rangle \\ &= \langle x^* + A^*y^*, x - \bar{x} \rangle. \end{aligned}$$

Thus, Eq. (1) is equivalent to  $\langle x^* + A^*y^*, x - \bar{x} \rangle \leq 0$  for all  $x \in X$ , which holds if and only if  $x^* = -A^*y^*$ . As a result,  $x^* \in D^*F(\bar{x}, \bar{y})(y^*)$  if and only if  $x^* = A^*y^*$ .

The example below establishes the relationship between subdifferential and coderivative.

*Example 3* Let  $f : X \rightarrow (-\infty, \infty]$  be a convex function. Define

$$F(x) := [f(x), \infty).$$

Then  $\text{gph } F = \text{epi } f$  is a convex set. For  $\bar{y} = f(\bar{x}) \in \mathbb{R}$ , one has

$$D^*F(\bar{x}, \bar{y})(\lambda) = \begin{cases} \lambda \partial f(\bar{x}) & \text{if } \lambda > 0, \\ \partial^\infty f(\bar{x}) & \text{if } \lambda = 0, \\ \emptyset & \text{if } \lambda < 0. \end{cases}$$

First, we have the following

$$\text{gph } F = \{(x, \lambda) \in X \times \mathbb{R} \mid \lambda \in F(x)\} = \{(x, \lambda) \in X \times \mathbb{R} \mid \lambda \geq f(x)\} = \text{epi } f.$$

For  $\lambda > 0$ , by the definition,  $v \in D^*F(\bar{x}, \bar{y})(\lambda)$  if and only if  $(v, -\lambda) \in N((\bar{x}, \bar{y}); \text{gph } F)$ , which is equivalent to  $(\frac{v}{\lambda}, -1) \in N((\bar{x}, \bar{y}); \text{epi } f)$  or  $v \in \lambda \partial f(\bar{x})$ . Similarly, for  $\lambda = 0$ ,  $v \in D^*F(\bar{x}, \bar{y})(0)$  if and only if  $(v, 0) \in N((\bar{x}, \bar{y}); \text{epi } f)$  or  $v \in \partial^\infty f(\bar{x})$ . Observe that if  $(v, -\mu) \in N((\bar{x}, f(\bar{x})); \text{epi } f)$ , then  $\mu \geq 0$ . Thus, the last part of the formula follows.

Let  $F : X \rightrightarrows Y$  be a set-valued mapping between Banach spaces, and let  $\varphi : X \times Y \rightarrow (-\infty, \infty]$  be an extended-real-valued function. The *optimal value function* built on  $F$  and  $\varphi$  is given by

$$\mu(x) := \inf\{\varphi(x, y) \mid y \in F(x)\}. \tag{2}$$

We adopt the convention that  $\inf \emptyset = \infty$ . Thus,  $\mu$  is an extended-real-valued function. Under convexity assumptions on  $F$  and  $\varphi$ , we will show that convex subdifferentials of the optimal value function  $\mu$  can be represented as an equality in terms of convex subdifferentials of the function  $\varphi$  and coderivatives of the mapping  $F$ . Note that the result cannot be derived directly from [3].

The following obvious proposition guarantees that  $\mu(x) > -\infty$ , which is the *standing assumption* for this section.

**Proposition 1** *Consider the optimal value function  $\mu$  given by Eq. (2). Suppose that  $\varphi(x, \cdot)$  is bounded below on  $F(x)$ . Then  $\mu(x) > -\infty$ . In particular, suppose that there exist  $b \in \mathbb{R}$  and a function  $g : X \rightarrow (-\infty, \infty]$  such that for any  $(x, y) \in X \times Y$  with  $y \in F(x)$ , one has*

$$\varphi(x, y) \geq g(x) + b.$$

*Then  $\mu(x) > -\infty$  for all  $x \in X$ .*

**Proposition 2** *Suppose that  $F$  has convex graph and  $\varphi$  is a convex function. Then the optimal value function  $\mu$  defined by Eq. (2) is a convex function.*

*Proof* Fix  $x_1, x_2 \in \text{dom } \mu$  and  $\lambda \in (0, 1)$ . For any  $\varepsilon > 0$ , by the definition, there exist  $y_i \in F(x_i)$  such that

$$\varphi(x_i, y_i) < \mu(x_i) + \varepsilon \quad \text{for } i = 1, 2.$$

It follows that

$$\begin{aligned} \lambda \varphi(x_1, y_1) &< \lambda \mu(x_1) + \lambda \varepsilon, \\ (1 - \lambda) \varphi(x_2, y_2) &< (1 - \lambda) \mu(x_2) + (1 - \lambda) \varepsilon. \end{aligned}$$

Adding these inequalities and applying the convexity of  $\varphi$  yield

$$\begin{aligned} \varphi(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2) &\leq \lambda \varphi(x_1, y_1) + (1 - \lambda) \varphi(x_2, y_2) \\ &< \lambda \mu(x_1) + (1 - \lambda) \mu(x_2) + \varepsilon. \end{aligned}$$

Since  $\text{gph } F$  is convex,

$$(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2) = \lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2) \in \text{gph } F.$$

Therefore,  $\lambda y_1 + (1 - \lambda)y_2 \in F(\lambda x_1 + (1 - \lambda)x_2)$ , and hence

$$\begin{aligned} \mu(\lambda x_1 + (1 - \lambda)x_2) &\leq \varphi(\lambda x_1 + (1 - \lambda)x_2, \lambda y_1 + (1 - \lambda)y_2) \\ &< \lambda \mu(x_1) + (1 - \lambda)\mu(x_2) + \varepsilon. \end{aligned}$$

Letting  $\varepsilon \rightarrow 0$ , we derive the convexity of  $\mu$ . □

The optimal value function (2) can be used as a *convex function generator* in the sense that many operations that preserve convexity can be reduced to this function.

**Proposition 3** Consider the optimal value function (2), where  $F : X \rightrightarrows Y$  has convex graph and  $\varphi : X \times Y \rightarrow (-\infty, \infty]$  is a convex function. Let

$$S(\bar{x}) := \{\bar{y} \in F(\bar{x}) \mid \mu(\bar{x}) = \varphi(\bar{x}, \bar{y})\}. \tag{3}$$

Assume that  $\mu(\bar{x}) < \infty$  and  $S(\bar{x}) \neq \emptyset$ . For any  $\bar{y} \in S(\bar{x})$ , one has

$$\bigcup_{(u,v) \in \partial\varphi(\bar{x}, \bar{y})} [u + D^*F(\bar{x}, \bar{y})(v)] \subseteq \partial\mu(\bar{x}). \tag{4}$$

*Proof* Fix any  $w$  that belongs to the left side. Then there exists  $(u, v) \in \partial\varphi(\bar{x}, \bar{y})$  such that

$$w - u \in D^*F(\bar{x}, \bar{y})(v).$$

Thus,  $(w - u, -v) \in N((\bar{x}, \bar{y}); \text{gph } F)$ . Then

$$\langle w - u, x - \bar{x} \rangle - \langle v, y - \bar{y} \rangle \leq 0 \quad \text{for all } (x, y) \in \text{gph } F.$$

This implies

$$\langle w, x - \bar{x} \rangle \leq \langle u, x - \bar{x} \rangle + \langle v, y - \bar{y} \rangle \leq \varphi(x, y) - \varphi(\bar{x}, \bar{y}) = \varphi(x, y) - \mu(\bar{x})$$

whenever  $y \in F(x)$ . It follows that

$$\langle w, x - \bar{x} \rangle \leq \inf_{y \in F(x)} \varphi(x, y) - \mu(\bar{x}) = \mu(x) - \mu(\bar{x}).$$

Therefore,  $w \in \partial\mu(\bar{x})$ . □

The opposite inclusion in Eq. (4) also holds true under the sequentially normally compactness and the qualification condition presented below.

A nonempty closed subset  $\Omega$  of a Banach space is said to be *sequentially normally compact* (SNC) at  $\bar{x} \in \Omega$  if for  $x_k \xrightarrow{\Omega} \bar{x}$  and  $x_k^* \in N(x_k; \Omega)$ , the following implication holds:

$$\left[ x_k^* \xrightarrow{w^*} 0 \right] \implies \left[ x_k^* \xrightarrow{\|\cdot\|} 0 \right].$$

In this definition,  $x_k \xrightarrow{\Omega} \bar{x}$  means that  $x_k \rightarrow \bar{x}$  and  $x_k \in \Omega$  for every  $k$ . Obviously, every subset of a finite dimensional Banach space is SNC. An extended-real-valued function  $\varphi : X \rightarrow (-\infty, \infty]$  is called *sequentially normally epi-compact* (SNEC) at  $\bar{x}$  if its epigraph is SNC at  $(\bar{x}, \varphi(\bar{x}))$ .

Let us give a simple proof in the proposition below that every convex set with nonempty interior is SNC at any point of the set.

**Proposition 4** Let  $\Omega$  be a convex set with nonempty interior of a Banach space. Then  $\Omega$  is SNC at any point  $\bar{x} \in \Omega$ .

*Proof* Let  $\bar{u} \in \text{int } \Omega$  and let  $\delta > 0$  satisfy  $\mathbb{B}(\bar{u}; 2\delta) \subseteq \Omega$ . Fix sequences  $\{x_k\}$  and  $\{x_k^*\}$  with  $x_k \xrightarrow{\Omega} \bar{x}$  and  $x_k^* \in N(x_k; \Omega)$ , and  $x_k^* \xrightarrow{w^*} 0$ . Choose  $k_0$  such that  $\|x_k - \bar{x}\| < \delta$  for all  $k \geq k_0$ . It is not hard to see that for any  $e \in \mathbb{B}$ , one has

$$\tilde{x}_k := \bar{u} + \delta e + x_k - \bar{x} \in \Omega \quad \text{for all } k \geq k_0.$$

Thus,

$$\langle x_k^*, \tilde{x}_k - x_k \rangle \leq 0.$$

It follows that

$$\delta \langle x_k^*, e \rangle \leq \langle x_k^*, \bar{x} - \bar{u} \rangle,$$

so  $\delta \|x_k^*\| \leq \langle x_k^*, \bar{x} - \bar{u} \rangle \rightarrow 0$ . Therefore,  $\|x_k^*\| \rightarrow 0$ , and hence  $\Omega$  is SNC at  $\bar{x}$ . □

**Theorem 2** *Let  $X$  and  $Y$  be Asplund spaces. Consider the optimal value function (2), where  $F : X \rightrightarrows Y$  has closed convex graph and  $\varphi : X \times Y \rightarrow (-\infty, \infty]$  is a lower semicontinuous convex function. Assume that  $\mu(\bar{x}) < \infty$  and  $S(\bar{x}) \neq \emptyset$ , where  $S$  is the solution mapping (3). For any  $\bar{y} \in S(\bar{x})$ , one has*

$$\partial\mu(\bar{x}) = \bigcup_{(u,v) \in \partial\varphi(\bar{x}, \bar{y})} [u + D^*F(\bar{x}, \bar{y})(v)]$$

under the qualification condition:

$$\partial^\infty\varphi(\bar{x}, \bar{y}) \cap [-N((\bar{x}, \bar{y}); \text{gph } F)] = \{(0, 0)\}, \tag{5}$$

and either  $\varphi$  is SNEC at  $(\bar{x}, \bar{y})$  or  $\text{gph } F$  is SNC at this point.

*Proof* By Proposition 3, we only need to prove the inclusion  $\subseteq$ . Fix any  $w \in \partial\mu(\bar{x})$  and  $\bar{y} \in S(\bar{x})$ . Then

$$\begin{aligned} \langle w, x - \bar{x} \rangle &\leq \mu(x) - \mu(\bar{x}) \\ &= \mu(x) - \varphi(\bar{x}, \bar{y}) \\ &\leq \varphi(x, y) - \varphi(\bar{x}, \bar{y}) \quad \text{for all } y \in F(x). \end{aligned}$$

Thus, for any  $(x, y) \in X \times Y$ , one has

$$\langle w, x - \bar{x} \rangle + \langle 0, y - \bar{y} \rangle \leq \varphi(x, y) + \delta((x, y); \text{gph } F) - [\varphi(\bar{x}, \bar{y}) + \delta((\bar{x}, \bar{y}); \text{gph } F)].$$

Let  $f(x, y) := \varphi(x, y) + \delta((x, y); \text{gph } F)$ . By the subdifferential sum rule (see [3, Theorem 3.36]), one has

$$(w, 0) \in \partial f(\bar{x}, \bar{y}) = \partial\varphi(\bar{x}, \bar{y}) + N((\bar{x}, \bar{y}); \text{gph } F),$$

under the qualification condition (5). Thus,

$$(w, 0) = (u_1, v_1) + (u_2, v_2),$$

where  $(u_1, v_1) \in \partial\varphi(\bar{x}, \bar{y})$  and  $(u_2, v_2) \in N((\bar{x}, \bar{y}); \text{gph } F)$ . Then  $v_2 = -v_1$ , and hence  $(u_2, -v_1) \in N((\bar{x}, \bar{y}); \text{gph } F)$ . It follows that  $u_2 \in D^*F(\bar{x}, \bar{y})(v_1)$  and

$$w = u_1 + u_2 \in u_1 + D^*F(\bar{x}, \bar{y})(v_1),$$

where  $(u_1, v_1) \in \partial\varphi(\bar{x}, \bar{y})$ . □

A function  $g : \mathbb{R}^n \rightarrow (-\infty, \infty]$  is called nondecreasing componentwise if the following implication holds:

$$[x_i \leq y_i \text{ for all } i = 1, \dots, n] \implies [g(x_1, \dots, x_n) \leq g(y_1, \dots, y_n)].$$



**Proposition 5** Let  $f_i : X \rightarrow \mathbb{R}$  for  $i = 1, \dots, n$  be convex functions and let  $h : X \rightarrow \mathbb{R}^n$  be defined by  $h(x) = (f_1(x), \dots, f_n(x))$ . Suppose that  $g : \mathbb{R}^n \rightarrow (-\infty, \infty]$  is a convex function that is nondecreasing componentwise. Then  $g \circ h : X \rightarrow (-\infty, \infty]$  is a convex function.

*Proof* Define the set-valued mapping  $F : X \rightrightarrows \mathbb{R}^n$  by

$$F(x) = [f_1(x), \infty) \times [f_2(x), \infty) \times \dots \times [f_n(x), \infty).$$

Then

$$\text{gph } F = \{(x, t_1, \dots, t_n) \in X \times \mathbb{R}^n \mid t_i \geq f_i(x)\}.$$

Since  $f_i$  is convex for  $i = 1, \dots, n$ ,  $\text{gph } F$  is convex. Define  $\varphi : X \times \mathbb{R}^n \rightarrow (-\infty, \infty]$  by  $\varphi(x, y) = g(y)$ . Since  $g$  is nondecreasing componentwise, it is obvious that

$$\inf\{\varphi(x, y) \mid y \in F(x)\} = g(f_1(x), \dots, f_n(x)) = (g \circ h)(x).$$

Thus,  $g \circ h$  is convex by Proposition 2. □

**Proposition 6** Let  $X$  be an Asplund space. Let  $f : X \rightarrow \mathbb{R}$  be a convex function and let  $\varphi : \mathbb{R} \rightarrow (-\infty, \infty]$  be a nondecreasing convex function. Let  $\bar{x} \in X$  and let  $\bar{y} := f(\bar{x})$ . Then

$$\partial(\varphi \circ f)(\bar{x}) = \cup_{\lambda \in \partial\varphi(\bar{y})} \lambda \partial f(\bar{x}),$$

under the assumption that  $\partial^\infty\varphi(\bar{y}) = \{0\}$  or  $0 \notin \partial f(\bar{x})$ .

*Proof* It has been proved that  $\varphi \circ f$  is a convex function. Define

$$F(x) = [f(x), \infty).$$

Since  $\varphi$  is a nondecreasing function, one has

$$(\varphi \circ f)(x) = \inf_{y \in F(x)} \varphi(y).$$

By Theorem 2,

$$\partial(\varphi \circ f)(\bar{x}) = \cup_{\lambda \in \partial\varphi(\bar{y})} [D^*F(\bar{x}, \bar{y})(\lambda)].$$

Since  $\varphi$  is nondecreasing,  $\lambda \geq 0$  for every  $\lambda \in \partial\varphi(\bar{y})$ . By Proposition 3,

$$\partial(\varphi \circ f)(\bar{x}) = \cup_{\lambda \in \partial\varphi(\bar{y})} [D^*F(\bar{x}, \bar{y})(\lambda)] = \cup_{\lambda \in \partial\varphi(\bar{y})} \lambda \partial f(\bar{x}).$$

Note that the condition  $\partial^\infty\varphi(\bar{y}) = \{0\}$  or  $0 \notin \partial f(\bar{x})$  guarantees qualification condition (5), and  $\varphi$  is automatically SNEC at  $\bar{y}$  since its epigraph is a subset in finite dimensions. □

The same approach can be applied for the general composition in Proposition 7. A simplified version of the proposition below in finite dimensions can be found in [2]. Note that our result is new in infinite dimensions and more general in finite dimensions. Moreover, our proof is much simpler than the proof in [2].

**Proposition 7** Let  $X$  be an Asplund space. Let  $f_i : X \rightarrow \mathbb{R}$  for  $i = 1, \dots, n$  be convex functions that are locally Lipschitz around  $\bar{x}$ , and let  $h : X \rightarrow \mathbb{R}^n$  be defined by  $h(x) = (f_1(x), \dots, f_n(x))$ .

Suppose that  $g : \mathbb{R}^n \rightarrow (-\infty, \infty]$  is a convex function that is nondecreasing componentwise. Then  $g \circ h : X \rightarrow (-\infty, \infty]$  is a convex function. Moreover, for any  $\bar{x} \in \text{dom}(g \circ h)$ , one has

$$\partial(g \circ h)(\bar{x}) = \left\{ \sum_{i=1}^m \lambda_i x_i^* \mid (\lambda_1, \dots, \lambda_m) \in \partial g(h(\bar{x})), x_i^* \in \partial f_i(\bar{x}) \text{ for } i = 1, \dots, m \right\}$$

under the condition that whenever  $(\lambda_1, \dots, \lambda_m) \in \partial^\infty g(h(\bar{x}))$ ,  $x_i^* \in \partial f_i(\bar{x})$  for  $i = 1, \dots, m$ , the following implication holds

$$\left[ \sum_{i=1}^m \lambda_i x_i^* = 0 \right] \Rightarrow [\lambda_i = 0 \text{ for } i = 1, \dots, m].$$

*Proof* Define the set-valued mapping  $F : X \rightrightarrows \mathbb{R}^n$  by

$$F(x) = [f_1(x), \infty) \times [f_2(x), \infty) \times \dots \times [f_n(x), \infty).$$

Then

$$\text{gph } F = \{(x, t_1, \dots, t_n) \in X \times \mathbb{R}^n \mid t_i \geq f_i(x) \text{ for all } i = 1, \dots, n\}.$$

The set  $\text{gph } F$  is convex since  $f_i$  is convex for  $i = 1, \dots, n$ . Define  $\varphi : X \times \mathbb{R}^n \rightarrow (-\infty, \infty]$  by  $\varphi(x, y) = g(y)$ . Since  $g$  is increasing componentwise, it is obvious that

$$\inf\{\varphi(x, y) \mid y \in F(x)\} = g(f_1(x), \dots, f_n(x)) = (g \circ h)(x).$$

Define

$$\Omega_i = \{(x, \lambda_1, \lambda_2, \dots, \lambda_n) \mid \lambda_i \geq f_i(x)\}.$$

Then

$$\text{gph } F = \bigcap_{i=1}^n \Omega_i.$$

Using the Lipschitz continuity of  $f_i$  for  $i = 1, \dots, n$  and the structure of the set  $\Omega_i$ , one can apply the intersection rule (see [3, Corollary 3.5]), to get that:  $(x^*, -\lambda_1, \dots, -\lambda_n) \in N((\bar{x}, f_1(\bar{x}), \dots, f_n(\bar{x})); \text{gph } F)$  if and only if

$$(x^*, -\lambda_1, \dots, -\lambda_n) \in N((\bar{x}, f_1(\bar{x}), \dots, f_n(\bar{x})); \bigcap_{i=1}^n \Omega_i) = \sum_{i=1}^n N((\bar{x}, f_1(\bar{x}), \dots, f_n(\bar{x})); \Omega_i).$$

If this is the case, then

$$x^* = \sum_{i=1}^n x_i^*,$$

where  $(x_i^*, -\lambda_i) \in N(\bar{x}, f_i(\bar{x}))$ . Using Theorem 2, one has that  $x^* \in \partial(g \circ h)(\bar{x})$  if and only if there exists  $(\lambda_1, \dots, \lambda_m) \in \partial g(h(\bar{x}))$  such that

$$(x^*, -\lambda_1, \dots, -\lambda_n) \in N((\bar{x}, f_1(\bar{x}), \dots, f_n(\bar{x})); \text{gph } F).$$

This is equivalent to the fact that  $x^* = \sum_{i=1}^n x_i^*$ , where  $x_i^* \in \lambda_i \partial f_i(\bar{x})$ . In other words,

$$x^* = \sum_{i=1}^m \lambda_i x_i^*,$$

where  $(\lambda_1, \dots, \lambda_m) \in \partial g(h(\bar{x}))$ ,  $x_i^* \in \partial f_i(\bar{x})$  for  $i = 1, \dots, n$ . The proof is now complete. □

**Proposition 8** *Let  $X$  and  $Y$  be Asplund spaces. Let  $\varphi : X \times Y \rightarrow (-\infty, \infty]$  be an extended-real-valued function, and let  $K$  be a nonempty convex subset of  $Y$ . Define*

$$\mu(x) := \inf\{\varphi(x, y) \mid y \in K\}$$

and

$$S(x) := \{y \in K \mid \varphi(x, y) = \mu(x)\}.$$

*Suppose that for every  $x$ ,  $\varphi(x, \cdot)$  is bounded below on  $K$ . Let  $\bar{x} \in X$  with  $\mu(\bar{x}) < \infty$  and suppose that  $S(\bar{x}) \neq \emptyset$ . Then for every  $\bar{y} \in S(\bar{x})$ , one has*

$$\partial \mu(\bar{x}) = \{u \in X^* \mid (u, v) \in \partial \varphi(\bar{x}, \bar{y}), -v \in N(\bar{y}; K)\}$$

under the qualification condition

$$[(0, v) \in \partial^\infty \varphi(\bar{x}, \bar{y}), -v \in N(\bar{y}; K)] \Rightarrow [v = 0].$$

In particular, if  $K = Y$ , then this qualification condition holds automatically and

$$\partial\mu(\bar{x}) = \{u \in X^* \mid (u, 0) \in \partial\varphi(\bar{x}, \bar{y})\}.$$

*Proof* The results follow directly from Example 1 and Theorem 2 for the mapping  $F(x) = K$ . □

**Proposition 9** *Let  $X$  and  $Y$  be Asplund spaces. Let  $B : Y \rightarrow X$  be an affine mapping with  $B(y) = A(y) + b$ , where  $A$  is a bounded linear mapping and  $b \in X$ . Let  $\varphi : Y \rightarrow (-\infty, \infty]$  be a convex function so that for every  $x \in X$ ,  $\varphi$  is bounded below on  $B^{-1}(x)$ . Define*

$$\mu(x) := \inf\{\varphi(y) \mid B(y) = x\} = \inf\{\varphi(y) \mid y \in B^{-1}(x)\}$$

and

$$S(x) := \inf\{y \in B^{-1}(x) \mid \varphi(y) = \mu(x)\}.$$

Fix  $\bar{x} \in X$  with  $\mu(\bar{x}) < \infty$  and  $S(\bar{x}) \neq \emptyset$ . For any  $\bar{y} \in S(\bar{x})$ , one has

$$\partial\mu(\bar{x}) = (A^*)^{-1}(\partial\varphi(\bar{y})).$$

*Proof* Let us apply Theorem 2 for  $F(x) = B^{-1}(x)$  (preimage) and  $\varphi(x, y) \equiv \varphi(y)$ . Then

$$N((\bar{x}, \bar{y}); \text{gph } F) = \{(u, v) \in X^* \times Y^* \mid -A^*u = v\}.$$

Thus,

$$D^*F(\bar{x}, \bar{y})(v) = \{u \in X^* \mid A^*u = v\} = (A^*)^{-1}(v).$$

It follows that

$$\partial\mu(\bar{x}) = \bigcup_{v \in \partial\varphi(\bar{y})} [D^*F(\bar{x}, \bar{y})(v)] = (A^*)^{-1}(\partial\varphi(\bar{y})).$$

It is not hard to verify that the qualification condition (5) satisfies automatically in this case. □

**Proposition 10** *Let  $X$  be an Asplund space. Let  $f_i : X \rightarrow (-\infty, \infty]$  for  $i = 1, 2$  be convex functions. Define the convolution of  $f_1$  and  $f_2$  by*

$$(f_1 \oplus f_2)(x) = \inf\{f_1(x_1) + f_2(x_2) \mid x_1 + x_2 = x\}.$$

Suppose that  $(f_1 \oplus f_2)(x) > -\infty$  for all  $x \in X$  and let  $\bar{x} \in \text{dom } f_1 \oplus f_2$ . Fix  $\bar{x}_1, \bar{x}_2 \in X$  such that  $\bar{x} = \bar{x}_1 + \bar{x}_2$  and  $(f_1 \oplus f_2)(\bar{x}) = f_1(\bar{x}_1) + f_2(\bar{x}_2)$ . Then

$$\partial(f_1 \oplus f_2)(\bar{x}) = \partial f_1(\bar{x}_1) \cap \partial f_2(\bar{x}_2).$$

*Proof* Let us apply Proposition 9 for  $\varphi : X \times X \rightarrow (-\infty, \infty]$  with  $\varphi(y_1, y_2) = f_1(y_1) + f_2(y_2)$  and  $A : X \times X \rightarrow X$  with  $A(y_1, y_2) = y_1 + y_2$ . Then  $A^*(v) = (v, v)$  for any  $v \in X^*$  and  $\partial\varphi(\bar{y}_1, \bar{y}_2) = (\partial f_1(\bar{y}_1), \partial f_2(\bar{y}_2))$ . So  $v \in \partial(f_1 \oplus f_2)(\bar{x})$  if and only if  $A^*(v) = (v, v) \in \partial\varphi(\bar{y}_1, \bar{y}_2)$ , i.e.,  $v \in \partial f_1(\bar{x}_1) \cap \partial f_2(\bar{x}_2)$ . □

*Remark 1* The optimal value function (2) covers many other convex operations. Thus, based on Theorem 2, it is possible to derive many other convex subdifferential calculus rules. Some examples are given below.

- (i) Let  $f_1, f_2 : X \rightarrow (-\infty, \infty]$  be convex functions. Define

$$\varphi(x, y) = f_1(x) + y,$$

and  $F(x) = [f_2(x), \infty)$ . For any  $x \in X$ , one has

$$f_1(x) + f_2(x) = \inf_{y \in F(x)} \varphi(x, y).$$

- (ii) Let  $B : X \rightarrow Y$  be an affine function, and let  $f : Y \rightarrow (-\infty, \infty]$  be a convex function. Define  $F(x) = \{B(x)\}$  and  $\varphi(x, y) = f(y)$  for  $x \in X$  and  $y \in Y$ . Then

$$(f \circ B)(x) = \inf_{y \in F(x)} \varphi(x, y).$$

- (iii) Let  $f_1, f_2 : X \rightarrow (-\infty, \infty]$  be convex functions. Define  $F(x) = [f_1(x), \infty) \times [f_2(x), \infty)$  and

$$\varphi(y_1, y_2) = \max\{y_1, y_2\} = \frac{|y_1 - y_2| + y_1 + y_2}{2}.$$

Then

$$\max\{f_1, f_2\}(x) = \inf_{(y_1, y_2) \in F(x)} \varphi(y_1, y_2).$$

#### 4 Convexified Coderivative and Subdifferential Calculus

Throughout this section, we assume that *all Banach spaces under consideration are reflexive*. Under this assumption, the definition of sequential normal compactness can be rewritten using weak sequential convergence. A subset  $\Omega \subseteq X$  is sequentially normally compact (or shortly SNC) at  $\bar{x} \in \Omega$  if and only if, for any sequences involved, we have the implication

$$\left[ x_k \xrightarrow{\Omega} \bar{x}, x_k^* \xrightarrow{w} 0, x_k^* \in \hat{N}(x_k; \Omega) \right] \implies [\|x_k^*\| \rightarrow 0 \text{ as } k \rightarrow \infty].$$

A subset  $\Omega$  in the product space  $X \times Y$  is said to be *partially sequentially normally compact* (PSNC) at  $(\bar{x}, \bar{y}) \in \Omega$  (with respect to  $X$ ) if and only if for any sequences  $(x_k, y_k) \subseteq \Omega$  and  $\{(x_k^*, y_k^*)\} \subseteq X^* \times Y^*$  such that  $(x_k^*, y_k^*) \in \hat{N}((x_k, y_k); \Omega)$ ,  $x_k^* \xrightarrow{w} 0$  and  $y_k^* \xrightarrow{\|\cdot\|} 0$ , we have  $x_k^* \xrightarrow{\|\cdot\|} 0$ .

Accordingly, a set-valued mapping  $G : X \rightrightarrows Y$  is SNC (PSNC) at  $(\bar{x}, \bar{y}) \in \text{gph } G$  if and only if its graph is SNC (PSNC) at this point, and an extended-real-valued function  $\varphi : X \rightarrow (-\infty, \infty]$  is *sequentially normally epi-compact* (SNEC) at  $\bar{x} \in \text{dom } \varphi$  if and only if its epigraph is SNC at  $(\bar{x}, \varphi(\bar{x}))$ . These properties and their partial variants are comprehensively studied and applied in [3, 4].

For the purposes of this paper, we need the following modifications of the above properties.

**Definition 1** (i) A set  $\Omega \subseteq X$  is *sequentially convexly normally compact* (SCNC) at  $\bar{x} \in \Omega$  if and only if we have the implication

$$\left[ x_k \xrightarrow{\Omega} \bar{x}, x_k^* \xrightarrow{w} 0, x_k^* \in \text{co } N(x_k; \Omega) \right] \implies [\|x_k^*\| \rightarrow 0 \text{ as } k \rightarrow \infty] \quad (6)$$

for any sequences involved in Eq. (6). A mapping  $G : X \rightrightarrows Y$  is SCNC at  $(\bar{x}, \bar{y}) \in \text{gph } G$  if and only if its graph is SCNC at this point. A function  $\varphi : X \rightarrow (-\infty, \infty]$  is *sequentially convexly epi-compact* (SCNEC) at  $\bar{x} \in \text{dom } \varphi$  if and only if its epigraph is SCNC at  $(\bar{x}, \varphi(\bar{x}))$ .

(ii) A subset  $\Omega$  of the product space  $X \times Y$  is said to be *partially sequentially convex-ifically normally compact* (PSCNC) at  $(\bar{x}, \bar{y}) \in \Omega$  with respect to  $X$  if and only if for any sequences  $(x_k, y_k) \xrightarrow{\Omega} (\bar{x}, \bar{y})$ ,  $\{(x_k^*, y_k^*)\} \subseteq X^* \times Y^*$  with  $(x_k^*, y_k^*) \in \text{co}N((x_k, y_k); \Omega)$ ,  $x_k^* \xrightarrow{w} 0$  and  $y_k^* \xrightarrow{\|\cdot\|} 0$ , we have  $x_k^* \xrightarrow{\|\cdot\|} 0$ . A mapping  $G : X \rightrightarrows Y$  is PSCNC at  $(\bar{x}, \bar{y}) \in \text{gph } G$  if and only if its graph is PSCNC at this point.

It is easy to check that the SCNC property holds at every point of a convex set with nonempty interior. Let us extend this result to a broad class of nonconvex sets. Given  $\Omega \subseteq X$  with  $\bar{x} \in \Omega$ , recall from [7] that  $v \in X$  is a *hypertangent* to  $\Omega$  at  $\bar{x}$  if for some  $\delta > 0$  we have

$$x + tw \in \Omega \text{ for all } x \in (\bar{x} + \delta\mathbb{B}) \cap \Omega, w \in v + \delta\mathbb{B}, \text{ and } t \in (0, \delta).$$

From the definition, we see that if  $\Omega \subseteq X$  admits a hypertangent at  $\bar{x}$ , then  $\Omega$  is SCNC at this point. Moreover, if  $\varphi : X \rightarrow (-\infty, \infty]$  is locally Lipschitz around  $\bar{x} \in \text{dom } \varphi$ , then it is SCNEC at this point; see [5] for more details.

**Lemma 1** *Let  $X$  be a reflexive space and let  $\Omega$  be a subset of  $X$  with  $\bar{x} \in \Omega$ . Then*

$$N_C(\bar{x}; \Omega) = \text{clco}N(\bar{x}; \Omega). \tag{7}$$

*Proof* By [3, Theorem 3.57], we have

$$N_C(\bar{x}; \Omega) = \text{cl}^*\text{co}N(\bar{x}; \Omega).$$

Since in the reflexive spaces, the weak\* topology and weak topology on  $X^*$  coincide, we obtain by the celebrated Mazur’s theorem that

$$N_C(\bar{x}; \Omega) = \text{cl}^*\text{co}N(\bar{x}; \Omega) = \text{clco}N(\bar{x}; \Omega),$$

which completes the proof. □

Let us start with a new sum rule for Clarke coderivatives that allow us to obtain many calculus rules for Clarke subdifferentials and normal cones. Given  $F_i : X \rightrightarrows Y, i = 1, 2$ , we define a multifunction  $S : X \times Y \rightrightarrows Y \times Y$  by

$$S(x, y) := \left\{ (y_1, y_2) \in Y^2 \mid y_1 \in F_1(x), y_2 \in F_2(x), y_1 + y_2 = y \right\}. \tag{8}$$

$S$  is said to be *inner semicontinuous* at  $(\bar{x}, \bar{y}, \bar{y}_1, \bar{y}_2) \in \text{gph } S$  if for any sequence  $\{(x_k, y_k)\}$  converging to  $(\bar{x}, \bar{y})$  with  $S(x_k, y_k) \neq \emptyset$  for each  $k \in \mathbb{N}$ , there exists  $(y_{1k}, y_{2k}) \in S(x_k, y_k)$  such that  $\{(y_{1k}, y_{2k})\}$  contains a convergent subsequence to  $(\bar{y}_1, \bar{y}_2)$ .

In the theorem below, we show that under the inner semicontinuity of  $S$ , the convexified coderivative of a sum of two set-valued mappings can be represented in terms of the coderivative of each set-valued mapping. However, this does not hold true under the so-called *inner semicompactness*; see [3, Definition 1.63]. We use the fact that every bounded sequence in a reflexive Banach space has a subsequence that is weakly convergent, and that every closed convex set is weakly closed.

**Theorem 3** *Let  $F_i : X \rightrightarrows Y$  for  $i = 1, 2$  be closed-graph mappings and  $(\bar{x}, \bar{y}) \in \text{gph}(F_1 + F_2)$ . Fix  $(\bar{y}_1, \bar{y}_2) \in S(\bar{x}, \bar{y})$  such that  $S$  is inner semicontinuous at  $(\bar{x}, \bar{y}, \bar{y}_1, \bar{y}_2)$ . Assume that either  $F_1$  is PCSNC at  $(\bar{x}, \bar{y}_1)$  or  $F_2$  is PCSNC at  $(\bar{x}, \bar{y}_2)$ , and that  $\{F_1, F_2\}$  satisfies the qualification condition*

$$D_C^*F_1(\bar{x}, \bar{y}_1)(0) \cap (-D_C^*F_2(\bar{x}, \bar{y}_2)(0)) = \{0\}.$$

Then

$$D_C^*(F_1 + F_2)(\bar{x}, \bar{y})(y^*) \subseteq D_C^*F_1(\bar{x}, \bar{y}_1)(y^*) + D_C^*F_2(\bar{x}, \bar{y}_2)(y^*).$$

*Proof* Define

$$\Omega_i := \{(x, y_1, y_2) \mid (x, y_i) \in \text{gph } F_i\} \quad \text{for } i = 1, 2.$$

Following the proof of [3, Theorem 3.10], one has

$$\{(x^*, y^*, y^*) \mid (x^*, y^*) \in N((\bar{x}, \bar{y}); \text{gph}(F_1 + F_2))\} \subseteq N((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_1 \cap \Omega_2).$$

It follows that

$$\{(x^*, y^*, y^*) \mid (x^*, y^*) \in \text{co } N((\bar{x}, \bar{y}); \text{gph}(F_1 + F_2))\} \subseteq \text{co } N((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_1 \cap \Omega_2).$$

Under the assumptions made, one can apply [3, Theorem 3.4] to obtain the following inclusion:

$$N((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_1 \cap \Omega_2) \subseteq N((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_1) + N((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_2).$$

This implies

$$\text{co } N((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_1 \cap \Omega_2) \subseteq \text{co } N((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_1) + \text{co } N((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_2).$$

By the definition of  $\Omega_1$  and  $\Omega_2$ ,

$$N((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_1) = N((\bar{x}, \bar{y}_1); \text{gph } F_1) \times \{0\},$$

and

$$N((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_2) = \{(x^*, 0, y^*) \mid (x^*, y^*) \in N((\bar{x}, \bar{y}_2); \text{gph } F_2)\}.$$

Now fix any  $x^* \in D_C^*(F_1 + F_2)(\bar{x}, \bar{y})(y^*)$ . Then

$$(x^*, -y^*) \in N_C((\bar{x}, \bar{y}); \text{gph}(F_1 + F_2)).$$

Let us show that

$$x^* \in D_C^*F_1(\bar{x}, \bar{y}_1)(y^*) + D_C^*F_2(\bar{x}, \bar{y}_2)(y^*).$$

Using Lemma 1, there exists a sequence  $\{(x_k^*, -y_k^*)\}$  in  $\text{co } N((\bar{x}, \bar{y}); \text{gph}(F_1 + F_2))$  such that  $(x_k^*, -y_k^*) \rightarrow (x^*, -y^*)$ , and hence  $(x_k^*, -y_k^*, -y_k^*) \rightarrow (x^*, -y^*, -y^*)$ .

Because  $(x_k^*, -y_k^*, -y_k^*) \in \text{co } N((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_1 \cap \Omega_2)$ , using the above results, there exist sequences  $\{(x_{1k}^*, -y_k^*, 0)\}$  in  $\text{co } N((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_1)$  and  $\{(x_{2k}^*, 0, -y_k^*)\}$  in  $\text{co } N((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_2)$  such that

$$x_{1k}^* + x_{2k}^* = x_k^* \quad \text{for every } k \in \mathbb{N}.$$

Without loss of generality, we assume that  $F_1$  is PCSNC at  $(\bar{x}, \bar{y}_1)$ .

We assume by a contradiction that  $\{x_{1k}^*\}$  is not bounded, so we can extract a subsequence, without relabeling, such that  $\|x_{1k}^*\| \rightarrow \infty$ . Then

$$\frac{1}{\|x_{1k}^*\|} (x_{1k}^* + x_{2k}^*) \rightarrow 0.$$

The bounded sequence  $\{z_k^*\}$  defined by  $z_k^* := \frac{x_{1k}^*}{\|x_{1k}^*\|}$  has a weak convergent subsequence, say,  $z_k^* \xrightarrow{w} z^*$ . Then  $(z^*, 0, 0) \in N_C((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_1)$ , and it is clear that  $(z^*, 0, 0) \in (-N_C((\bar{x}, \bar{y}_1, \bar{y}_2); \Omega_2))$ , which implies  $(z^*, 0) \in N_C((\bar{x}, \bar{y}_1); \Omega_1) \cap (-N_C((\bar{x}, \bar{y}_2); \Omega_2))$ . It follows that

$$z^* \in D_C^*F_1(\bar{x}, \bar{y}_1)(0) \cap (-D_C^*F_2(\bar{x}, \bar{y}_2)(0)) = \{0\},$$

so  $z^* = 0$ . Because  $F_1$  is PCSNC at  $(\bar{x}, \bar{y}_1)$ , one has that  $\|z_k^*\| \rightarrow 0$ . This is a contradiction since  $\|z_k^*\| = 1$  for every  $k \in \mathbb{N}$ .

Thus,  $\{x_{1k}^*\}$  is bounded, so we can extract a weak convergent subsequence. Suppose that  $x_{1k}^* \xrightarrow{w} x_1^*$  and  $x_{2k}^* \xrightarrow{w} x_2^*$ .

Using the fact that  $N_C((\bar{x}, \bar{y}_i); \text{gph}(F_i)) = \text{cl co } N((\bar{x}, \bar{y}_i); \text{gph}(F_i))$ , one obtains

$$(x_i^*, -y^*) \in N_C((\bar{x}, \bar{y}_i); \text{gph}(F_i)).$$

By definition,  $x_i^* \in D_C^*F_i(\bar{x}, \bar{y}_i)(y^*)$  for  $i = 1, 2$ . So we have

$$x^* = x_1^* + x_2^* \in D_C^*F_1(\bar{x}, \bar{y}_1)(y^*) + D_C^*F_2(\bar{x}, \bar{y}_2)(y^*).$$

The theorem has been proved. □

The PCSNC holds automatically in finite dimensions, so we obtain the following sum rule for Clarke coderivatives in finite dimensions.

**Corollary 1** *Let  $F_i : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$  for  $i = 1, 2$  be closed-graph mappings with  $(\bar{x}, \bar{y}) \in \text{gph}(F_1 + F_2)$ . Assume that  $S$  is inner semicontinuous at  $(\bar{x}, \bar{y}, \bar{y}_1, \bar{y}_2)$  and that  $\{F_1, F_2\}$  satisfies the qualification condition*

$$D_C^*F_1(\bar{x}, \bar{y}_1)(0) \cap (-D_C^*F_2(\bar{x}, \bar{y}_2)(0)) = \{0\}. \tag{9}$$

Then

$$D_C^*(F_1 + F_2)(\bar{x}, \bar{y})(y^*) \subseteq D_C^*F_1(\bar{x}, \bar{y}_1)(y^*) + D_C^*F_2(\bar{x}, \bar{y}_2)(y^*). \tag{10}$$

Next, we consider

$$\Phi(x) := F(x) + \Delta(x; \Omega), \quad x \in X,$$

where  $F : X \rightrightarrows Y$  and  $\Delta(x; \Omega) = \{0\} \subseteq X$  if  $x \in \Omega$  and  $\Delta(x; \Omega) = \emptyset$  otherwise.

**Proposition 11** *Let  $\Omega$  and  $\text{gph } F$  be closed with  $\bar{x} \in \Omega$  and  $(\bar{x}, \bar{y}) \in \text{gph } F$  such that either  $F$  is PCSNC at  $(\bar{x}, \bar{y})$  or  $\Omega$  is SCNC at  $\bar{x}$ . Assume that*

$$D_C^*F(\bar{x}, \bar{y})(0) \cap (-N_C(\bar{x}; \Omega)) = \{0\}.$$

Then

$$D_C^*(F + \Delta(\cdot; \Omega))(\bar{x}, \bar{y})(y^*) \subseteq D_C^*F(\bar{x}, \bar{y})(y^*) + N_C(\bar{x}; \Omega), \quad y^* \in Y^*.$$

*Proof* Let us apply Theorem 3 with  $F_1 = F$  and  $F_2 = \Delta(\cdot; \Omega)$ . It is not hard to see that for any  $(x, y) \in \text{gph}(F + \Delta(\cdot; \Omega))$ , we have  $S(x, y) = \{(y, 0)\}$ , which implies that  $S$  is inner semicontinuous at  $(\bar{x}, \bar{y}, \bar{y}, 0)$ . We also see that  $\text{gph } \Delta(\cdot; \Omega) = \Omega \times \{0\}$ , so it is PCSNC under the assumption that  $\Omega$  is SCNC at  $\bar{x}$ . We have  $N((\bar{x}, \bar{y}); \text{gph } F_2) = N((\bar{x}, \bar{y}); \Omega \times \{0\}) = N(\bar{x}; \Omega) \times Y^*$ , so  $N_C((\bar{x}, \bar{y}); \text{gph } F_2) = N((\bar{x}, \bar{y}); \Omega \times \{0\}) = N_C(\bar{x}; \Omega) \times Y^*$ . Therefore,  $D_C^*F(\bar{x}, \bar{y})(y^*) = N_C(\bar{x}, \Omega)$ . The rest follows directly Theorem 3. □

**Corollary 2** *Let  $\Omega_1$  and  $\Omega_2$  be two closed subsets of  $X$  and  $\bar{x} \in \Omega_1 \cap \Omega_2$ . Assume that  $\Omega_1$  or  $\Omega_2$  is SCNC at  $\bar{x}$  and the qualification condition*

$$N_C(\bar{x}; \Omega_1) \cap (-N_C(\bar{x}; \Omega_2)) = \{0\} \tag{11}$$

is satisfied. Then

$$N_C(\bar{x}; \Omega_1 \cap \Omega_2) \subseteq N_C(\bar{x}; \Omega_1) + N_C(\bar{x}; \Omega_2). \tag{12}$$

*Proof* This is a special case of Proposition 11 with  $F_1 \equiv \Delta(\cdot; \Omega_1)$ . □

Let us show that the qualification condition in Eq. (11) is weaker than a similar condition using Clarke tangent; see [1]. Let  $\Omega$  be a nonempty closed subset a Banach space  $X$ . For  $\bar{x} \in \Omega$ , the Clarke tangent cone  $T(\bar{x}; \Omega)$  contains all  $v \in X$  such that, whenever  $t_k \downarrow 0$  and  $x_k \xrightarrow{\Omega} \bar{x}$ , there exists  $w_k \rightarrow v$  with  $x_k + t_k w_k \in \Omega$  for all  $k$ .

**Proposition 12** *Let  $\Omega_1$  and  $\Omega_2$  be closed subsets of  $X$  and  $\bar{x} \in \Omega_1 \cap \Omega_2$ . Suppose that*

$$T(\bar{x}; \Omega_1) \cap \text{int } T(\bar{x}; \Omega_2) \neq \emptyset. \tag{13}$$

Then

$$N_C(\bar{x}; \Omega_1) \cap (-N_C(\bar{x}; \Omega_2)) = \{0\}.$$

*Proof* Fix any  $x^* \in N_C(\bar{x}; \Omega_1) \cap (-N_C(\bar{x}; \Omega_2))$ . Choose  $v \in T(\bar{x}; \Omega_1)$  such that  $v + 2\delta\mathbb{B} \subseteq T(\bar{x}; \Omega_2)$ . Then we have  $\langle x^*, v \rangle \leq 0$ , and  $\langle -x^*, v + \delta e \rangle \leq 0$  for any  $e \in \mathbb{B}$ . It follows that

$$\delta \langle x^*, -e \rangle \leq \langle x^*, v \rangle$$

for any  $e \in \mathbb{B}$  and hence  $\delta \|x^*\| \leq \langle x^*, v \rangle \leq 0$ . Therefore  $x^* = 0$ . □

*Remark 2* Note that the converse of the above proposition does not hold in general. For example, in  $\mathbb{R}^3$ , we consider  $\bar{x} = (0, 0, 0)$ ,  $\Omega_1 = \{(0, 0, z) \mid z \in \mathbb{R}\}$  and  $\Omega_2 = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$ . We have  $N_C(\bar{x}; \Omega_1) = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$  and  $N_C(\bar{x}; \Omega_2) = \{(0, 0, z) \mid z \in \mathbb{R}\}$ . Thus,

$$N_C(\bar{x}; \Omega_1) \cap (-N_C(\bar{x}; \Omega_2)) = \{0\},$$

but  $T(\bar{x}; \Omega_1) \cap \text{int } T(\bar{x}; \Omega_2) = \emptyset$  since  $\text{int } T(\bar{x}; \Omega_2) = \text{int } T(\bar{x}; \Omega_1) = \emptyset$ . Therefore, we obtain a stronger finite-dimensional version of Corollary 2.9.8 in [1].

**Definition 2** We say that an extended-real-valued function  $\varphi : X \rightarrow (-\infty, \infty]$  is lower regular at  $\bar{x} \in \text{dom } \varphi$  if  $\hat{\partial}\varphi(\bar{x}) = \partial_C\varphi(\bar{x})$ , where  $\hat{\partial}\varphi(\bar{x})$  is the Fréchet normal subdifferential of  $\varphi$  at  $\bar{x}$  defined by

$$\hat{\partial}\varphi(\bar{x}) := \left\{ x^* \in X^* \mid \liminf_{x \rightarrow \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}.$$

It is clear that any convex function is lower regular.

Now, we apply the results obtained from coderivative calculus in Theorem 3 to obtain calculus for Clarke subdifferential in reflexive Banach spaces.

**Theorem 4** *Let  $\varphi_i : X \rightarrow (-\infty, \infty]$  for  $i = 1, \dots, m$  be l.s.c. around  $\bar{x}$  and finite at this point. Assume all (except possibly one) of  $\varphi_i, i = 1, \dots, m$  are SCNEC at  $(\bar{x}, \varphi_i(\bar{x}))$  and*

$$[x_1^* + \dots + x_m^* = 0, x_i^* \in \partial_C^\infty \varphi_i(\bar{x})] \Rightarrow x_i^* = 0, i = 1, \dots, m. \tag{14}$$

Then

$$\partial_C(\varphi_1 + \dots + \varphi_m)(\bar{x}) \subseteq \partial_C\varphi_1(\bar{x}) + \dots + \partial_C\varphi_m(\bar{x}), \tag{15}$$

and

$$\partial_C^\infty(\varphi_1 + \dots + \varphi_m)(\bar{x}) \subseteq \partial_C^\infty\varphi_1(\bar{x}) + \dots + \partial_C^\infty\varphi_m(\bar{x}). \tag{16}$$

The equality in Eq. (15) holds if all  $\varphi_i$  for  $i = 1, \dots, m$  are upper regular at  $\bar{x}$ .

*Proof* We first consider the case where  $m = 2$ . Let us consider  $F_1(x) = [\varphi_1(x), \infty)$  and  $F_2(x) = [\varphi_2(x), \infty)$ . Obviously,  $\text{gph } F_i = \text{epi } (\varphi_i)$  for  $i = 1, 2$  and  $\text{gph } (F_1 + F_2) = \text{epi } (\varphi_1 + \varphi_2)$ . We see that at the point  $(\bar{x}, \bar{y})$  where  $\bar{y} = \varphi_1(\bar{x}) + \varphi_2(\bar{x})$ , we have  $S(\bar{x}, \bar{y}) =$



$\{(\varphi_1(\bar{x}), \varphi_2(\bar{x}))\}$ , and  $S$  is inner semicontinuous at  $(\bar{x}, \bar{y}, \varphi_1(\bar{x}), \varphi_2(\bar{x}))$ . Indeed, take any sequence  $\{(x_k, y_k)\}$  converging to  $(\bar{x}, \bar{y})$  with  $S(x_k, y_k) \neq \emptyset$ . Fix  $(\lambda_{1k}, \lambda_{2k}) \in S(x_k, y_k)$ . Then

$$\lambda_{1k} \geq \varphi_1(x_k), \quad \lambda_{2k} \geq \varphi_2(x_k), \quad \lambda_{1k} + \lambda_{2k} = y_k.$$

Since  $\varphi_1$  and  $\varphi_2$  are lower semicontinuous,

$$\liminf_{k \rightarrow \infty} \lambda_{ik} \geq \liminf_{k \rightarrow \infty} \varphi_i(x_k) \geq \varphi_i(\bar{x}) \quad \text{for } i = 1, 2.$$

We can see that  $\{\lambda_{1k}\}$  and  $\{\lambda_{2k}\}$  are bounded sequences. Let  $\lambda_1 := \liminf_{k \rightarrow \infty} \lambda_{1k}$  and  $\lambda_2 := \liminf_{k \rightarrow \infty} \lambda_{2k}$ . Then there exist subsequences of  $\{\lambda_{1k}\}$  and  $\{\lambda_{2k}\}$  that converge to  $\lambda_1$  and  $\lambda_2$ , respectively. Since  $\lambda_1 \geq \varphi_1(\bar{x})$ ,  $\lambda_2 \geq \varphi_2(\bar{x})$ , and  $\lambda_1 + \lambda_2 = \bar{y}$ , we see that  $\lambda_1 = \varphi_1(\bar{x})$  and  $\lambda_2 = \varphi_2(\bar{x})$ , so  $S$  is inner semicontinuous at  $(\bar{x}, \bar{y}, \varphi_1(\bar{x}), \varphi_2(\bar{x}))$ .

Because  $D_C^* F_i(\bar{x}, \varphi_i(\bar{x}))(0) = \partial_C^\infty \varphi_i(\bar{x})$ , the qualification condition (9) holds. Applying Theorem 10, we have

$$D_C^*(F_1 + F_2)(\bar{x}, \bar{y})(1) \subseteq D_C^* F_1(\bar{x}, \varphi_1(\bar{x}))(1) + D_C^* F_2(\bar{x}, \varphi_2(\bar{x}))(1),$$

and

$$D_C^*(F_1 + F_2)(\bar{x}, \bar{y})(0) \subseteq D_C^* F_1(\bar{x}, \varphi_1(\bar{x}))(0) + D_C^* F_2(\bar{x}, \varphi_2(\bar{x}))(0)$$

which imply Eqs. (15) and (16) for  $m = 2$ .

Let us prove (15) holds as an equality under the regularity conditions. Assume that all  $\varphi_i$  for  $i = 1, 2$  are lower regular at  $\bar{x}$ . Then

$$\begin{aligned} \partial_C(\varphi_1 + \varphi_2)(\bar{x}) &\subseteq \partial_C \varphi_1(\bar{x}) + \partial_C \varphi_2(\bar{x}) \\ &= \hat{\partial} \varphi_1(\bar{x}) + \hat{\partial} \varphi_2(\bar{x}) \\ &\subseteq \hat{\partial}(\varphi_1 + \varphi_2)(\bar{x}) \\ &\subseteq \partial_C(\varphi_1 + \varphi_2)(\bar{x}), \end{aligned}$$

which implies the equality. The proof for  $m > 2$  follows easily by induction. □

**Proposition 13** *Let  $\varphi_i : X \rightarrow \mathbb{R}$  satisfy*

$$\{v \in X \mid \varphi_1^\circ(\bar{x}; v) < \infty\} \cap \text{int}\{v \in X \mid \varphi_2^\circ(\bar{x}; v) < \infty\} \neq \emptyset.$$

*Then*

$$\partial_C^\infty \varphi_1(\bar{x}) \cap (-\partial_C^\infty \varphi_2(\bar{x})) = \{0\}.$$

*Proof* Fix any  $x^* \in \partial_C^\infty \varphi_1(\bar{x}) \cap (-\partial_C^\infty \varphi_2(\bar{x}))$ . Then  $(x^*, 0) \in N_C((\bar{x}, \varphi_1(\bar{x})); \text{epi } \varphi_1)$  and  $(-x^*, 0) \in N_C((\bar{x}, \varphi_2(\bar{x})); \text{epi } \varphi_2)$ . Fix an element  $v \in X$  such that  $\varphi_1^\circ(\bar{x}; v) < \infty$ , while  $v \in \text{int}\{v_1 \in X \mid \varphi_2^\circ(\bar{x}; v_1) < \infty\} \neq \emptyset$ . Let us show that  $x^* = 0$ . From the proof of [1, Theorem 2.9.5], one finds  $\beta \in \mathbb{R}$  such that  $(v, \beta) \in \text{int}T((\bar{x}, \varphi_2(\bar{x})); \text{epi } \varphi_2)$ . From [1, Theorem 2.9.1], we see that if  $\gamma \in \mathbb{R}$  is fixed such that  $\gamma > \varphi_1^\circ(\bar{x}; v)$ , then  $(v, \gamma) \in T((\bar{x}, \varphi_1(\bar{x})); \text{epi } \varphi_1)$ . Thus, there exists  $\delta > 0$  with

$$(v, \beta) + \delta \mathbb{B} \subseteq T((\bar{x}, \varphi_2(\bar{x})); \text{epi } \varphi_2), (v, \gamma) \in T((\bar{x}, \varphi_1(\bar{x})); \text{epi } \varphi_1).$$

Then

$$\langle (x^*, 0), (v, \beta) + \delta/2(e_1, e_2) \rangle \leq 0 \leq \langle (x^*, 0), (v, \gamma) \rangle$$

whenever  $\|(e_1, e_2)\| \leq 1$ . It follows that  $\langle x^*, \delta/2e_1 \rangle \leq 0$  whenever  $\|e_1\| \leq 1$ , which implies  $x^* = 0$ . □

The proposition below shows that the SCNEC condition for extended-real-valued functions holds under the *directional Lipschitz condition*; see [1, Definition 2.9.2].

**Proposition 14** *Assume that  $\varphi : X \rightarrow (-\infty, \infty]$  is directionally Lipschitz at  $\bar{x} \in \text{dom } \varphi$ . Then  $\varphi$  is SCNEC at  $(\bar{x}, \varphi(\bar{x}))$ .*

*Proof* Assume that  $\varphi$  is directionally Lipschitz at  $\bar{x}$ . Then by [1, Proposition 2.9.3], there exists  $\beta \in \mathbb{R}$  such that  $(v, \beta)$  is a hypertangent to  $\text{epi } \varphi$  at  $(\bar{x}, \varphi(\bar{x}))$ . Therefore,  $\text{epi } \varphi$  is SCNC at this point.  $\square$

**Theorem 5** *Let  $G : X \rightrightarrows Y, F : Y \rightrightarrows Z$  be closed-graph mappings with  $\bar{z} \in (F \circ G)(\bar{x})$ , and*

$$S(x, z) := G(x) \cap F^{-1}(z) = \{y \in G(x) \mid z \in F(y)\}.$$

*Given  $\bar{y} \in S(\bar{x}, \bar{z})$ , assume that  $S$  is inner semicontinuous at  $(\bar{x}, \bar{z}, \bar{y})$ , that either  $F$  is PSCNC at  $(\bar{y}, \bar{z})$  or  $G$  is PSCNC at  $(\bar{x}, \bar{y})$ , and that the qualification condition*

$$D_C^*F(\bar{y}, \bar{z})(0) \cap \ker D_C^*G(\bar{x}, \bar{y}) = \{0\}$$

*is fulfilled. Then*

$$D_C^*(F \circ G)(\bar{x}, \bar{z})(z^*) \subseteq D_C^*G(\bar{x}, \bar{y}) \circ D_C^*F(\bar{y}, \bar{z})(z^*) \tag{17}$$

*for any  $z^* \in Z^*$ .*

*Proof* Consider the set-valued mapping  $\Phi : X \times Y \rightrightarrows Z$  as follows

$$\Phi(x, y) = F(y) + \Delta((x, y); \text{gph } G).$$

Using [3, Theorem 1.64], because  $S$  is inner semicontinuous at  $(\bar{x}, \bar{z}, \bar{y})$ , we have

$$D^*(F \circ G)(\bar{x}, \bar{z})(z^*) \subseteq \{x^* \in X^* \mid (x^*, 0) \in D^*\Phi(\bar{x}, \bar{y}, \bar{z})(z^*)\}.$$

Thus,

$$\text{co } D^*(F \circ G)(\bar{x}, \bar{z})(z^*) \subseteq \{x^* \in X^* \mid (x^*, 0) \in \text{co } D^*\Phi(\bar{x}, \bar{y}, \bar{z})(z^*)\}.$$

Therefore,

$$D_C^*(F \circ G)(\bar{x}, \bar{z})(z^*) \subseteq \{x^* \in X^* \mid (x^*, 0) \in D_C^*\Phi(\bar{x}, \bar{y}, \bar{z})(z^*)\}.$$

It follows from Proposition 11 that

$$D_C^*\Phi(\bar{x}, \bar{y}, \bar{z})(z^*) \subseteq D_C^*F(\bar{y}, \bar{z})(z^*) + N_C((\bar{x}, \bar{y}); \text{gph } G).$$

Take  $x^* \in D_C^*(F \circ G)(\bar{x}, \bar{z})(z^*)$ . Then  $(x^*, 0) \in D_C^*\Phi(\bar{x}, \bar{y}, \bar{z})(z^*)$ . Since  $F = F(y)$ , there exists  $y^* \in Y^*$  such that  $(y^*, -z^*) \in N_C((\bar{y}, \bar{z}); \text{gph } F)$  and  $(x^*, -y^*) \in N_C((\bar{x}, \bar{y}); \text{gph } G)$ . Then  $y^* \in D_C^*F(\bar{y}, \bar{z})(z^*)$  and  $x^* \in D_C^*G(\bar{x}, \bar{y})(y^*)$ . Therefore,  $x^* \in D_C^*G(\bar{x}, \bar{y}) \circ D_C^*F(\bar{y}, \bar{z})(z^*)$ . The theorem has been proved.  $\square$

**Corollary 3** *Let  $F : X \rightrightarrows Y$  be a closed-graph mapping, and let  $\Omega \subseteq Y$  be a closed set. For  $(\bar{x}, \bar{y}) \in \text{gph } F$  and  $\bar{y} \in \Omega$ , define*

$$F^{-1}(\Omega) := \{x \in X \mid F(x) \cap \Omega \neq \emptyset\}.$$

*Assume that  $S(x) := F(x) \cap \Omega$  is inner semicontinuous at  $(\bar{x}, \bar{y})$ , and that either  $F$  is PSCNC at  $(\bar{x}, \bar{y})$  or  $\Omega$  is SCNC at  $\bar{y}$ . Under the qualification condition*

$$N_C(\bar{y}; \Omega) \cap \ker D_C^*F(\bar{x}, \bar{y}) = \{0\},$$

*one has*

$$N_C(\bar{x}; F^{-1}(\Omega)) \subseteq D_C^*F(\bar{x}, \bar{y})(N_C(\bar{y}; \Omega)).$$

*Proof* This follows from Theorem 5 with  $F_1 : Y \rightrightarrows \mathbb{R}$  and  $G_1 : X \rightrightarrows Y$ , where  $G_1 = F$  and  $F_1(\cdot) = \Delta(\cdot; \Omega)$ . Then  $\text{gph } F_1 = \Omega \times \{0\}$ . Obviously, if  $\Omega$  is SCNC at  $\bar{y}$ , then  $\text{gph } F_1$  is PSCNC at  $(\bar{y}, 0)$ . Applying Theorem 5, we obtain the result.  $\square$

**Theorem 6** *Let  $f := g \circ F$ , where  $F : X \rightarrow Y$  is a strictly differentiable mapping and  $g : Y \rightarrow (-\infty, \infty]$  is an extended-real-valued function. Assume that  $g$  is l.s.c around  $F(\bar{x})$ , and that*

$$\partial_C^\infty g(F(\bar{x})) \cap (\ker \nabla F(\bar{x})^*) = \{0\}.$$

*Then*

$$\partial_C f(\bar{x}) \subseteq \nabla F(\bar{x})^*(\partial_C g(F(\bar{x}))),$$

*and*

$$\partial_C^\infty f(\bar{x}) \subseteq \nabla F(\bar{x})^*(\partial_C^\infty g(F(\bar{x}))).$$

*The first inclusion holds as an equality if  $g$  is lower regular at  $F(\bar{x})$ .*

*Proof* Let us consider  $E_g(x) = [g(x), \infty)$ . Obviously,

$$E_f(x) = (E_g \circ F)(x).$$

Observe that  $D_C^* F(\bar{x}) = \nabla F(\bar{x})^*$ . The result then follows directly from Theorem 5 with  $z^* = 1$  and  $z^* = 0$ .  $\square$

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