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Degeneracy of Holomorphic Curves into Algebraic Varieties II

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Abstract In our former paper (Noguchi et al. in J. Math. Pures Appl. 88:293–306, 2007) we proved an algebraic degeneracy of entire holomorphic curves into a variety X which carries a finite morphism to a semi-abelian variety, but which is not isomorphic to a semi-abelian variety by itself. The finiteness condition of the morphism is necessary in general by example. In this paper we improve that finiteness condition under an assumption such that some open subset of non-singular points of X is of log-general type, and simplify the proof in (Noguchi et al. in J. Math. Pures Appl. 88:293–306, 2007), which was rather involved. As a corollary it implies that every entire holomorphic curve $f : \mathbb{C} \to V$ into an algebraic variety V with $\bar{q}(V) \geq \dim V = \bar{\kappa}(V)$ is algebraically degenerate, which is due to Winkelmann ($\dim V = 2$) (Winkelmann in Ann. Inst. Fourier 61:1517–1537, 2011) and Lu–Winkelmann (Lu and Winkelmann in Forum Math. 24:399–418, 2012).

Keywords Holomorphic curve · Entire curve · log-Bloch–Ochiai Theorem · Green–Griffiths Conjecture

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1 Introduction

The purpose of this paper is to prove the algebraic degeneracy of entire holomorphic curves into a normal algebraic variety X carrying a finite map onto a semi-abelian variety by assuming only some non-singular part of X to be of log-general type (see Theorem 1). The proof gives a simplification of that of the degeneracy theorem in [5]. The result also gives another proof of a degeneracy theorem due to Winkelmann [8] for surfaces and Lu–Winkelmann [2] in general, whose proofs rely on the main results of [6] and [5] (cf. Theorem 2).

We continue to use the terms and notation of [5] (cf. Sect. 3); moreover, in this paper, "degenerate" means "algebraically degenerate" for simplicity.

2 Finite Case

Theorem 1 *Let X be a complex normal algebraic variety and let $\pi : X \to A$ be a surjective and finite morphism onto a semi-abelian variety A. Let E be a reduced Weil divisor on X. Assume that $X \backslash (E \cup X_{\mathrm{sing}})$ is of log-general type. Let $f : \mathbb{C} \to X$ be an entire holomorphic curve such that*

$$N_1\big(r; f^*E\big) \leq \varepsilon T_f(r) \,\|_{\varepsilon} \tag{1}$$

for all $\varepsilon > 0$. Then $f : \mathbb{C} \to X$ is degenerate.

Proof Since A is quasi-projective, so is X; henceforth, a "variety" means a quasi-projective algebraic one. We assume the non-degeneracy of f to derive a contradiction.

Step 1. Let $R \subset X$ be the ramification divisor of π, and put $D_0 = \pi(R + E)_{\mathrm{red}}$. By [6], there is a smooth equivariant compactification $\bar{A}$ of A such that

- $\partial A = \bar{A} \backslash A$ is a simple normal crossing divisor;
- the second main theorem holds for D_0 and $\pi \circ f$.

Let $D_1 = \partial A$ and let $D_2 \subset \bar{A}$ be the Zariski closure of D_0. Put $D = D_1 + D_2$. Then the above second main theorem takes the form

$$T_{\pi\circ f}\big(r; K_{\bar{A}}(D)\big) \leq N_1\big(r; (\pi \circ f)^* D\big) + \varepsilon T_{\pi\circ f}(r) \,\|_{\varepsilon}, \quad \forall \varepsilon > 0.$$

We extend π to $\bar{\pi} : \bar{X} \to \bar{A}$, where $\bar{X}$ is normal and $\bar{\pi}$ is finite. Then $\bar{\pi}$ is unramified outside D.

Step 2. We apply the following embedded resolution Lemma 1 for $\bar{A}$ and D to get a smooth modification $\varphi : \tilde{A} \to \bar{A}$ and a simple normal crossing divisor $\tilde{D} := \varphi^{-1} D \subset \tilde{A}$, such that the modification φ is an isomorphism over $\bar{A} \backslash D_2$. Here we note that D_1 is simple normal crossing by Step 1.

Lemma 1 (The embedded resolution; Szabó [7]) *Let V be a smooth variety, and let $F \subset V$ be a reduced divisor. Then there is a smooth modification $p : \tilde{V} \to V$ with the following two properties*:

(i) *$p^{-1}F$ is a simple normal crossing divisor.*
(ii) *p is an isomorphism outside the locus of F where F is not simple normal crossing.*

Step 3. Let $\widetilde{\pi \circ f} : \mathbb{C} \to \tilde{A}$ be the lifting of $\pi \circ f$. Let Σ be a divisor on $\tilde{A}$ such that $K_{\tilde{A}}(\tilde{D}) = \varphi^* K_{\bar{A}}(D) + \Sigma$. The codimension of $\varphi(\mathrm{Supp}\,\Sigma)$ is greater than one. Hence by [6] we have

$$N\big(r; (\widetilde{\pi \circ f})^* \mathrm{Supp}\,\Sigma\big) \le \varepsilon T_{\widetilde{\pi \circ f}}(r) \,\|_{\varepsilon}, \quad \forall \varepsilon > 0.$$

Since the modification $\tilde{A} \to \bar{A}$ is an isomorphism outside D_2, we have $\varphi(\mathrm{Supp}\,\Sigma) \subset D_2$. Therefore by the estimate in Step 1, we have

$$m_{\widetilde{\pi \circ f}}(r; \mathrm{Supp}\,\Sigma) \le \varepsilon T_{\widetilde{\pi \circ f}}(r) \,\|_{\varepsilon}, \quad \forall \varepsilon > 0.$$

Hence we get

$$T_{\widetilde{\pi \circ f}}(r; \Sigma) \le \varepsilon T_{\widetilde{\pi \circ f}}(r) \,\|_{\varepsilon}, \quad \forall \varepsilon > 0.$$

Thus by Step 1, we have the estimate

$$T_{\widetilde{\pi \circ f}}\big(r; K_{\tilde{A}}(\tilde{D})\big) \le N_1\big(r; (\widetilde{\pi \circ f})^* \tilde{D}\big) + \varepsilon T_{\widetilde{\pi \circ f}}(r) \,\|_{\varepsilon}, \quad \forall \varepsilon > 0.$$

Step 4. By base change and normalization, we get the following:

$$\begin{array}{ccc} \bar{X} & \xleftarrow{\;\psi\;} & \tilde{X} \\ {\scriptstyle \bar{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \bar{A} & \xleftarrow[\;\varphi\;]{} & \tilde{A} \end{array}$$

where $\tilde{X}$ is a normal projective variety and p is a finite map. Then p is unramified outside the simple normal crossing divisor $\tilde{D}$. Thus by the following lemma, $\tilde{X}$ is $\mathbb{Q}$-factorial.

Lemma 2 *Let V be a normal variety, let Y be a smooth variety, let $F \subset Y$ be a simple normal crossing divisor, and let $p : V \to Y$ be a finite morphism. Assume that p is unramified outside F. Then V is a $\mathbb{Q}$-factorial variety, i.e., all Weil divisors on V are $\mathbb{Q}$-Cartier divisors.*

Proof Replacing $p : V \to Y$ by its compactification $\bar{p} : \bar{V} \to \bar{Y}$, where $F \cup \partial Y$ is simple normal crossing, if necessary, we may assume that Y is projective. Let B be a prime divisor on V. We shall show that some multiple of B is a Cartier divisor.

Let $\mathbb{D}^n \subset Y$ be an open polydisc, where $F \cap \mathbb{D}^n$ is written as $z_1 \cdots z_l = 0$ with a coordinate $(z_1, \dots, z_n)$ of $\mathbb{D}^n$. Let U be an irreducible component of $p^{-1}\mathbb{D}^n$ and let $p_U : U \to \mathbb{D}^n$ be the restriction. Then p_U is finite and unramified over $\mathbb{D}^n \setminus F$. Note that $\pi_1(\mathbb{D}^n \setminus F) = \mathbb{Z}^l$ and $(p_U)_* \pi_1(U \setminus p_U^{-1} F) \subset \pi_1(\mathbb{D}^n \setminus F)$ is a finite index subgroup. Hence there exists a positive integer μ such that $\mu \mathbb{Z}^l \subset (p_U)_* \pi_1(U \setminus p_U^{-1} F)$. Let $\varphi : \mathbb{D}^n \to \mathbb{D}^n$ be defined by $\varphi(w_1, \dots, w_n) = (w_1^{\mu}, \dots, w_l^{\mu}, w_{l+1}, \dots, w_n)$. Then $\varphi_{\mathbb{D}^n \setminus F}$ factors $p_{U \setminus p_U^{-1} F}$; i.e., there is an unramified covering map $\psi_{\mathbb{D}^n \setminus F} : \mathbb{D}^n \setminus F \to U \setminus p_U^{-1} F$ satisfying $\varphi_{\mathbb{D}^n \setminus F} = p_U \circ \psi_{\mathbb{D}^n \setminus F}$. Since p_U is finite (so, proper), $\psi_{\mathbb{D}^n \setminus F}$ extends holomorphically to $\psi : \mathbb{D}^n \to U$ by Riemann's extension theorem. Hence we have a finite map $\psi : \mathbb{D}^n \to U$, which is unramified over $U \setminus p_U^{-1} F$.

Now there exists a holomorphic function g on $\mathbb{D}^n$ such that $\psi^{-1}(B \cap U)$ is defined by $g = 0$. We define a holomorphic function G on $U \setminus p_U^{-1} F$ by

$$G(x) = \prod_{w : \psi(w) = x} g(w).$$

Since G is locally bounded around $p_U^{-1}F$ and U is normal as complex analytic space (cf. [1, (36.7) p. 332]), G has a holomorphic extension over U. By the construction of G, the zero set of G is equal to $B \cap U$.

Since V is compact, we may cover V by a finite number of U as above. Thus some multiple of B is a Cartier divisor in the sense of analytic space. Hence by the GAGA principle, we complete the proof of our lemma. □

Step 5. We define a reduced divisor H on $\tilde{X}$ as follows. We note that $\psi^{-1}(X \setminus (E \cup X_{\text{sing}})) \subset \tilde{X}$ is a Zariski open subset, since $X \setminus (E \cup X_{\text{sing}}) \subset \bar{X}$ is so. We define H to be the union of the codimension one components of $\tilde{X} \setminus \psi^{-1}(X \setminus (E \cup X_{\text{sing}}))$. Then we have $\psi^{-1}(X \setminus (E \cup X_{\text{sing}})) = \tilde{X} \setminus (H \cup S)$, where $S \subset \tilde{X}$ is a Zariski closed subset with codimension greater than one. By the assumption of $X \setminus (E \cup X_{\text{sing}})$ being of log-general type, we deduce that $\psi^{-1}(X \setminus (E \cup X_{\text{sing}}))$ is also of log-general type. Thus by the following lemma, $K_{\tilde{X}}(H)$ is big.

Lemma 3 *Let V be a normal variety, and let $\bar{V}$ be a compactification of V such that $\partial V = \bar{V} \setminus V$ is a (Weil) divisor. Assume that $\bar{V}$ is normal and $\mathbb{Q}$-factorial. Let $S \subset \bar{V}$ be a Zariski closed set with codimension greater than one. If $V \setminus S$ is of log-general type, then $\mathbb{Q}$-line bundle $K_{\bar{V}}(\partial V)$ is big.*

Proof Let $\bar{V}_{\text{sing}}$ be the set of singularities of $\bar{V}$. Let $(\partial V)_{\text{sing}}$ be the set of singularities of ∂V. Then $T = S \cup \bar{V}_{\text{sing}} \cup (\partial V)_{\text{sing}}$ is codimension greater than one in $\bar{V}$. There exists a smooth modification $p : \tilde{V} \to \bar{V}$ such that $F = p^{-1}(S \cup \partial V)$ is a simple normal crossing divisor and p is an isomorphism over $\bar{V} \setminus T$. By the assumption that $V \setminus S$ is of log-general type, $K_{\tilde{V}}(F)$ is big. Over $\bar{V} \setminus T$, $K_{\bar{V}}(\partial V)$ is isomorphic to $K_{\tilde{V}}(F)$, hence big. There exists a positive integer ν such that $\nu K_{\bar{V}}(\partial V)$ is a line bundle. For each positive integer l, we have $H^0(\bar{V} \setminus T, l\nu K_{\bar{V}}(\partial V)) = H^0(\bar{V}, l\nu K_{\bar{V}}(\partial V))$. Hence $K_{\bar{V}}(\partial V)$ is big on $\bar{V}$. □

Step 6. Note that the image of each irreducible component of H under ψ is contained in one of $\bar{X} \setminus X$, $\bar{E}$ (the closure of E in $\bar{X}$) or X_{sing}. Thus denoting by $\tilde{f} : \mathbb{C} \to \tilde{X}$ the canonically induced holomorphic map from f, we have

$$N_1\bigl(r; \tilde{f}^* H\bigr) \le N_1\bigl(r; f^*(\bar{X} \setminus X)\bigr) + N_1\bigl(r; f^*\bar{E}\bigr) + N_1\bigl(r; f^* X_{\text{sing}}\bigr).$$

By the assumption, we have $N_1(r; f^*(\bar{X} \setminus X)) = 0$ and

$$N_1\bigl(r; f^*\bar{E}\bigr) \le \varepsilon T_{\tilde{f}}(r) \,\|_{\varepsilon}, \quad \forall \varepsilon > 0.$$

Since X_{sing} is codimension greater than one, we have [6]

$$N_1\bigl(r; f^* X_{\text{sing}}\bigr) \le \varepsilon T_{\tilde{f}}(r) \,\|_{\varepsilon}, \quad \forall \varepsilon > 0.$$

Thus

$$N_1\bigl(r; \tilde{f}^* H\bigr) \le \varepsilon T_{\tilde{f}}(r) \,\|_{\varepsilon}, \quad \forall \varepsilon > 0.$$

Step 7. Put $G = p^*(\tilde{D})_{\text{red}}$. Then we have $H \subset G$. By the ramification formula, we have

$$K_{\tilde{X}}(G) = p^* K_{\tilde{A}}(\tilde{D}).$$

Thus we have

$$T_{\tilde{f}}\big(r;\, K_{\tilde{X}}(G)\big) = T_{\tilde{f}}\big(r;\, p^* K_{\tilde{A}}(\tilde{D})\big) + O(1) = T_{p\circ\tilde{f}}\big(r;\, K_{\tilde{A}}(\tilde{D})\big) + O(1).$$

Hence by Step 3 (note that $p \circ \tilde{f} = \widetilde{\pi \circ f}$), we have

$$T_{\tilde{f}}\big(r;\, K_{\tilde{X}}(G)\big) \le N_1\big(r;\, (p\circ\tilde{f})^*\tilde{D}\big) + \varepsilon T_{p\circ\tilde{f}}(r) \,\|_{\varepsilon}, \quad \forall \varepsilon > 0.$$

Thus by $N_1(r;\, (p\circ\tilde{f})^*\tilde{D}) = N_1(r;\, \tilde{f}^* G)$, we get

$$T_{\tilde{f}}\big(r;\, K_{\tilde{X}}(G)\big) \le N_1\big(r;\, \tilde{f}^* G\big) + \varepsilon T_{\tilde{f}}(r) \,\|_{\varepsilon}, \quad \forall \varepsilon > 0.$$

We decompose as $G = H + F$, where F is a reduced (Weil) divisor on $\tilde{X}$. Then we have

$$T_{\tilde{f}}\big(r;\, K_{\tilde{X}}(H+F)\big) \le N_1\big(r;\, \tilde{f}^* H\big) + N_1\big(r;\, \tilde{f}^* F\big) + \varepsilon T_{\tilde{f}}(r) \,\|_{\varepsilon}, \quad \forall \varepsilon > 0.$$

Applying the estimate in Step 6, we get

$$T_{\tilde{f}}\big(r;\, K_{\tilde{X}}(H+F)\big) \le N_1\big(r;\, \tilde{f}^* F\big) + \varepsilon T_{\tilde{f}}(r) \,\|_{\varepsilon}, \quad \forall \varepsilon > 0.$$

Step 8. Let k be a positive integer such that kF is a Cartier divisor. Since $F \cap (\tilde{X} \setminus \tilde{X}_{\mathrm{sing}})$ is a Cartier divisor on $\tilde{X} \setminus \tilde{X}_{\mathrm{sing}}$, we have

$$k \,\mathrm{ord}_z \tilde{f}^* F = \mathrm{ord}_z \tilde{f}^*(kF)$$

for $z \in \tilde{f}^{-1}(\tilde{X} \setminus \tilde{X}_{\mathrm{sing}})$, and hence

$$k \,\min\big\{1, \mathrm{ord}_z \tilde{f}^* F\big\} \le \mathrm{ord}_z \tilde{f}^*(kF).$$

Thus we get

$$kN_1\big(r;\, \tilde{f}^{-1}(F)\big) \le N\big(r;\, \tilde{f}^*(kF)\big) + kN_1\big(r;\, \tilde{f}^{-1}(\tilde{X}_{\mathrm{sing}})\big).$$

By the Nevanlinna inequality, we have

$$N\big(r;\, \tilde{f}^*(kF)\big) \le T_{\tilde{f}}(r; kF) + O(1).$$

Hence

$$kN_1\big(r;\, \tilde{f}^{-1}(F)\big) \le T_{\tilde{f}}(r; kF) + kN_1\big(r;\, \tilde{f}^{-1}(\tilde{X}_{\mathrm{sing}})\big) + O(1).$$

Since $\tilde{X}_{\mathrm{sing}}$ is codimension greater than one, we have

$$N_1\big(r;\, \tilde{f}^{-1}(\tilde{X}_{\mathrm{sing}})\big) \le \varepsilon T_{\tilde{f}}(r) \,\|_{\varepsilon}, \quad \forall \varepsilon > 0.$$

Hence we have

$$kN_1\big(r;\, \tilde{f}^{-1}(F)\big) \le T_{\tilde{f}}(r; kF) + \varepsilon T_{\tilde{f}}(r) \,\|_{\varepsilon}, \quad \forall \varepsilon > 0.$$

Step 9. Now we conclude the proof. By Step 7, we have

$$T_{\tilde{f}}\big(r; kK_{\tilde{X}}(H)\big) + T_{\tilde{f}}(r; kF) \le kN_1\big(r; \tilde{f}^*F\big) + \varepsilon T_{\tilde{f}}(r) \,\|_\varepsilon, \quad \forall \varepsilon > 0.$$

Thus by Step 8, we get

$$T_{\tilde{f}}\big(r; kK_{\tilde{X}}(H)\big) \le \varepsilon T_{\tilde{f}}(r) \,\|_\varepsilon, \quad \forall \varepsilon > 0.$$

However, since $kK_{\tilde{X}}(H)$ is big (cf. Step 5) and $\tilde{f}$ is non-degenerate, this is a contradiction. We completed the proof of the theorem. □

3 Non-finite Case

Theorem 1 implies the following statement due to [8] (dim = 2) and [2].

Theorem 2 *Let V be a smooth quasi-projective variety with log-irregularity $\bar{q}(V) = \dim V$. Assume that V is of log-general type. Then every entire holomorphic curve $f : \mathbb{C} \to V$ is degenerate.*

Remark 1 If $\bar{q}(V) > \dim V$, $f : \mathbb{C} \to V$ is degenerate by the log-Bloch–Ochiai Theorem [3, 4]. Therefore, f must be degenerate under the condition that $\bar{q}(V) \ge \dim V = \bar{\kappa}(V)$.

Proof Let $a : V \to A$ be a quasi-Albanese map to a semi-abelian variety A. We may assume that a is dominant by the log-Bloch–Ochiai Theorem. Let $\pi : X \to A$ be the normalization of A in V and let $\varphi : V \to X$ be the induced map. Then φ is birational. Let $\bar{V}$ be a smooth partial compactification such that φ extends to a projective morphism $\bar{\varphi} : \bar{V} \to X$ and $\bar{V} \setminus V$ is a divisor on $\bar{V}$. Since $\bar{\varphi}$ is birational and X is normal, there exists a Zariski closed subset $Z \subset X$ whose codimension is greater than one such that $\bar{\varphi}$ is an isomorphism over $X \setminus Z$. In particular $X \setminus Z$ is smooth. Let E be the Zariski closure of $(X \setminus Z) \cap \bar{\varphi}(\bar{V} \setminus V)$ in X. Then E is a reduced divisor on X and $X \setminus (Z \cup E)$ is of log-general type. Thus by Lemma 4 below, $X \setminus (X_{\mathrm{sing}} \cup E)$ is of log-general type. Let $F = E \cap Z$. Then $F \subset X$ has codimension greater than one. Let $f : \mathbb{C} \to V$ be non-degenerate. Then we have

$$N_1\big(r; (\varphi \circ f)^*E\big) = N_1\big(r; (\varphi \circ f)^*F\big) \le \varepsilon T_{\varphi \circ f}(r) \,\|_\varepsilon, \quad \forall \varepsilon > 0.$$

Hence we may apply Theorem 1 to conclude the degeneracy of $\varphi \circ f$, which contradicts the assumption for f being non-degenerate. Thus every $f : \mathbb{C} \to V$ is degenerate. □

Lemma 4 *Let V be a smooth, quasi-projective variety. Let $S \subset V$ be a Zariski closed set of codimension greater than one. If $V \setminus S$ is of log-general type, then so is V.*

Proof Let $\bar{V}$ be a smooth compactification of V such that $\partial V = \bar{V} \setminus V$ is simple normal crossing. Let $\bar{S} \subset \bar{V}$ be the compactification. Then by Lemma 3, $K_{\bar{V}}(\partial V)$ is big. Thus V is of log-general type. □

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References

1. Abhyankar, S.S.: Local Analytic Geometry. World Scientific, Singapore (2001)
2. Lu, S.S.Y., Winkelmann, J.: Quasiprojective varieties admitting Zariski dense entire holomorphic curves. Forum Math. **24**, 399–418 (2012)
3. Noguchi, J.: Lemma on logarithmic derivatives and holomorphic curves in algebraic varieties. Nagoya Math. J. **83**, 213–233 (1981)
4. Noguchi, J., Ochiai, T.: Geometric Function Theory in Several Complex Variables. Transl. Math. Mon., vol. 80. Am. Math. Soc., Providence (1990)
5. Noguchi, J., Winkelmann, J., Yamanoi, K.: Degeneracy of holomorphic curves into algebraic varieties. J. Math. Pures Appl. **88**, 293–306 (2007)
6. Noguchi, J., Winkelmann, J., Yamanoi, K.: The second main theorem for holomorphic curves into semi-abelian varieties II. Forum Math. **20**, 469–503 (2008)
7. Szabó, E.: Divisorial log terminal singularities. J. Math. Sci. Univ. Tokyo **1**, 631–639 (1994)
8. Winkelmann, J.: Degeneracy of entire curves in log surfaces with $\bar{q} = 2$. Ann. Inst. Fourier **61**, 1517–1537 (2011)