Evaluation of the bound state energies of some diatomic molecules from the approximate solutions of the Schrodinger equation with Eckart plus inversely quadratic Yukawa potential

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Abstract
We have obtained analytically the bound state solutions for the non-relativistic Schrodinger equation for the Eckart plus inversely quadratic Yukawa potential (EIQYP) using the parametric Nikiforov-Uvarov (NU) method. In order to validate our approximation, the bound state energies were computed and predicted for some selected diatomic molecules at different adjustable screening parameters from the available spectroscopic model parameters. The fact-finding obtained are in agreement with previously reported results available in literature. Furthermore, the graphs of the effective potential against inter-nuclear distance for low and high values of the screening parameters were reported. From our graphs, we observed that the approximation is best fit for very low values of the screening parameter $\alpha \ll 1$.

Introduction

Recently, much attention has been drawn towards the Eckart-type potential model as a molecular potential model due to its wide applications in physics and chemical physics [1]. It is expressed as

$$-\eta \frac{e^{-\alpha r}}{(1-e^{-\alpha r})} + \zeta \frac{e^{-\alpha r}}{(1-e^{-\alpha r})^2} \qquad (1)$$

where ζ and η are the potential depths of the Eckart potential and α is an adjustable positive screening parameter. Sari et al. in 2015 studied the analytical approximations to the bound state solutions of the Dirac equation with the Eckart potential by using the asymptotic method [2]. Also the l-wave scattering state solutions of the Schrödinger equation for the Eckart potential has been studied analytically by using this same approximation scheme proposed by Greene and Aldrich [3, 4]. Ikhdair and colleagues solved the D-dimensional radial Klein-Gordon equation for any orbital angular momentum quantum number and for the scalar and vector Eckart-type exponential potentials using a general mathematical model of the Nikiforov-Uvarov method [5]. Hassanabadi et al. in 2013 obtained the bound state solutions by studying the s-wave Klein–Gordon equation with equally mixed Eckart potentials [6]. Zhang in 2008 investigated the bound state solutions of the Klein–Gordon equation with equal vector and scalar Eckart potentials using the approximation for the centrifugal term proposed by Greene and Aldrich for any orbital angular momentum quantum number [7]. Taskin and Kocak investigated the solution of the Schrödinger equation for the Eckart potential by using the approximation for the centrifugal term given by Greene and Aldrich for any arbitrary l state [8]. Akpan et al. studied the Klein-Gordon equation subject to equal q-deformed scalar and vector Eckart potentials to obtain the bound state solutions using an improved approximation scheme proposed by Qiang and Dong [9, 10]. Surpami et al. investigated the bound state solutions to the three-dimensional Schrödinger equation for Eckart plus trigonometric Poschl-Teller non-central

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potential approximately using Romanovski polynomials with suitable coordinate transformation [11]. Unfortunately, analytical solutions for the s-wave Eckart potential is only possible for zero angular momentum state because of the presence of the centrifugal term.

Another potential that has attracted attention recently is the inversely quadratic Yukawa potential expressed as

$$-\frac{V_0 e^{-\alpha r}}{r^2} \tag{2}$$

Where V_0 represents the potential depth. This potential was first studied by Hamzavi and coworkers where they obtained the approximate analytic spin and pseudospin solutions of the Dirac equation for the inversely quadratic Yukawa potential with a Coulomb-like tensor interaction through the Nikiforov-Uvarov method [12]. The bound state analytical solutions of both the Schrodinger and the Klein-Gordon equations with Manning-Rosen potential plus a class of Yukawa potential of the Pekeris-type approximation of the Coulomb term via the Nikiforov-Uvarov method have been also studied in the literature [13, 14]. The solutions of the Schrodinger equation with inversely quadratic Yukawa plus Woods-Saxon potential (IQYWSP) have been also presented using the parametric Nikiforov-Uvarov (NU) method. The bound state energy eigenvalues and the corresponding un-normalized eigen functions were obtained in terms of Jacobi polynomials [15].

The sum of these potentials given as EIQYP is expressed as

$$V(r) = -\eta \frac{e^{-\alpha r}}{(1-e^{-\alpha r})} + \zeta \frac{e^{-\alpha r}}{(1-e^{-\alpha r})^2} - \frac{V_0 e^{-\alpha r}}{r^2}. \tag{3}$$

In this present work, a mixed potential known as the Eckart plus Inversely Quadratic Yukawa potential (EIQYP) is investigated through the Nikiforov-Uvarov quantum formalism. The motivation for the superposition of Yukawa and Eckart potentials is to see how the eigenvalues vary with those of the pure Yukawa and Eckart potentials and what information could be obtained from the combination of potentials. We organize the work as follows; firstly, a brief introduction in "Introduction" (this section). In "Theoretical approach," the parametric Nikiforov-Uvarov method was introduced and briefly discussed. The approximate solutions of the Schrodinger equation was explicitly obtained in "Solutions of Eckart plus inversely quadratic Yukawa potential." "Results and discussion" is dedicated to the numerical analysis of the obtained bound state solutions. Finally, the conclusion was given in "Conclusions."

Theoretical approach

The generalization of the parametric NU method is very similar to the more generalized expression in Eq. (4). The

parametric form is simply using parameters to obtain explicitly energy eigenvalues and it is still based on the solutions of a generalized second-order linear differential equation with special orthogonal functions. The hypergeometric NU method has shown its power in calculating the exact energy levels of all bound states for some solvable quantum systems.

The equation below is a Schrodinger-like second-order differential equation of the form

$$\psi''(s) + \frac{\overline{\tau}(s)}{\sigma(s)} \psi'(s) + \frac{\overline{\sigma}(s)}{\sigma^2(s)} \psi(s) = 0 \tag{4}$$

Where $\sigma(s)$ and $\sigma(s)$ are both polynomials at most second degree and $\widetilde{\tau}(s)$ is a first-degree polynomial. The parametric generalization of the N-U method is given by the generalized hypergeometric-type Eq. (5)

$$\Psi''(s) + \frac{c_1 - c_2 s}{s(1-c_3 s)} \Psi'(s) \tag{5}$$

$$+ \frac{1}{s^2(1-c_3 s)^2} \left[-\epsilon_1 s^2 + \epsilon_2 s - \epsilon_3\right] \Psi(s) = 0.$$

Thus, Eq. (4) can be solved by comparing it with Eq. (5) and the following polynomials are obtained

$$\overline{\tau}(s) = (c_1 - c_2 s), \sigma(s) = s(1-c_3 s), \ \ \overline{\sigma}(s)$$
$$= -\epsilon_1 s^2 + \epsilon_2 s - \epsilon_3 \tag{6}$$

The parameters obtainable from Eq. (5) serve as important tools to finding the energy eigenvalues and eigenfunctions.

The function $\pi(s)$ and the parameters λ required for the NU method are given as follows:

$$\pi(s) = \frac{\sigma' - \widetilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \widetilde{\tau}}{2}\right)^2 - \overline{\sigma} + k\sigma} \tag{7}$$

$$\lambda = k + \pi(s). \tag{8}$$

Now substituting Eq. (6) into Eq. (7), we find

$$\overline{\sigma}(s) = c_4 + c_5 s \pm \left[(c_6 - c_3 k_{\pm})s^2 + (c_7 + k_{\pm})s + c_8\right]^{\frac{1}{2}} \tag{9}$$

where

$$c_4 = \frac{1}{2}(1-c_1), c_5 = \frac{1}{2}(c_2 - 2c_3), c_6 = c_5{}^2 + \epsilon_1, c_7$$
$$= 2c_4 c_5 - \epsilon_2, c_8 = c_4{}^2 + \epsilon_3, c_9 = c_3 c_7 + c_3{}^2 c_8 + c_6 \tag{10}$$

The resulting value of k in Eq. (9) is obtained from the condition that the function under the square root be square of a polynomial and it yields,

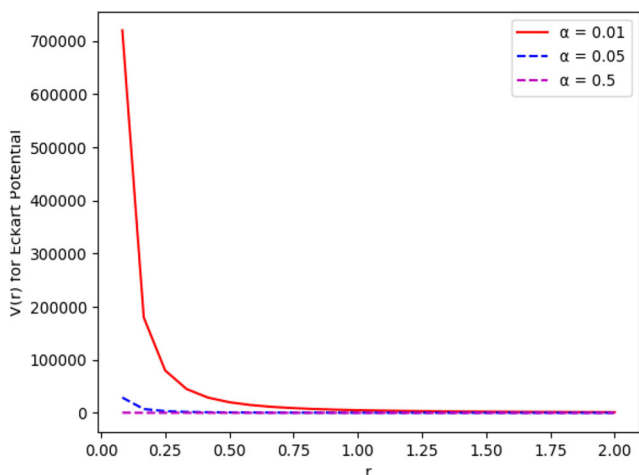$$k_{\pm} = -(c_7 + 2c_3 c_8) \pm 2\sqrt{c_8 c_9}. \tag{11}$$

Fig. 1 Eckart potentials versus r with $\eta = 0.01$ MeV and $\zeta = 0.5$ MeV for parameter $\alpha = 0.01, 0.05, 0.5$

The new $\pi(s)$ for k_- becomes

$$\pi(s) = c_4 + c_5 s - [(\sqrt{c_9} + c_3\sqrt{c_8})s - \sqrt{c_8}] \quad (12)$$

for the k_- value,

$$k_- = -(c_7 + 2c_3 c_8) - 2\sqrt{c_8 c_9} \quad (13)$$

Using Eq. (15), we obtain

$$\tau(s) = c_1 + 2c_4 - (c_2 - 2c_5)s - 2[(\sqrt{c_9} + c_3\sqrt{c_8})s - \sqrt{c_8}] \quad (14)$$

The physical condition for the bound state solution is $\tau' < 0$ and thus

$$\tau'(s) = -2c_3 - 2(\sqrt{c_9} + c_3\sqrt{c_8}) < 0 \quad (15)$$

with the aid of Eq. (9), we obtain the energy equation as

$$(c_2 - c_3)n + c_3 n^2 - (2n+1)c_5 + (2n+1)(\sqrt{c_9} + c_3\sqrt{c_8}) \quad (16)$$
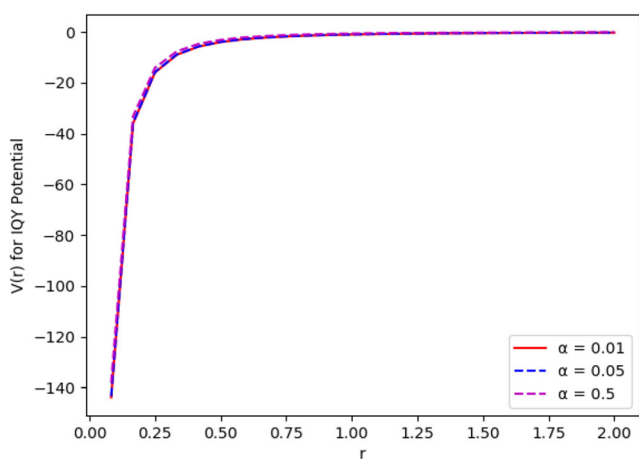$$+ c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0$$

$$\psi(s) = N_n s^{c_{12}} (1 - c_3 s)^{-c_{12} - \frac{c_{13}}{c_3}} P_n^{\left(c_{10}-1, \frac{c_{11}}{c_3} - c_{10} - 1\right)} (1 - 2c_3 s) \quad (17)$$

Solutions of Eckart plus inversely quadratic Yukawa potential

Solutions to the radial part of the Schrodinger equation (SE) with Eckart plus inversely quadratic Yukawa potential

To obtain the bound state solutions of Eq. (3), we consider Eq. (17) and obtain

$$R_{nl}''(r) + \frac{2\mu}{\hbar^2}\left[E - V(r) - \frac{\lambda\hbar^2}{2\mu r^2}\right] R_{nl}(r) = 0, \quad (18a)$$

Where $\lambda = l(l+1)$ and $V(r)$ is the potential energy function. Eq. (18) can also be expressed as

$$\frac{d^2 R_{nl}(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left[E - V_{eff}(r)\right] R_{nl}(r) \quad (18b)$$

where

$$V_{eff}(r) = V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2} = -\eta\frac{e^{-\alpha r}}{(1-e^{-\alpha r})} + \zeta\frac{e^{-\alpha r}}{(1-e^{-\alpha r})^2} - \frac{V_0 e^{-\alpha r}}{r^2} + \frac{l(l+1)\hbar^2}{2\mu r^2} \quad (19a)$$

However, since the SE with the Eckart plus inversely quadratic Yukawa potential has no exact solution, we then use the Greene and Aldrich approximation [3] for the centrifugal term as Hamzavi et al. [16].
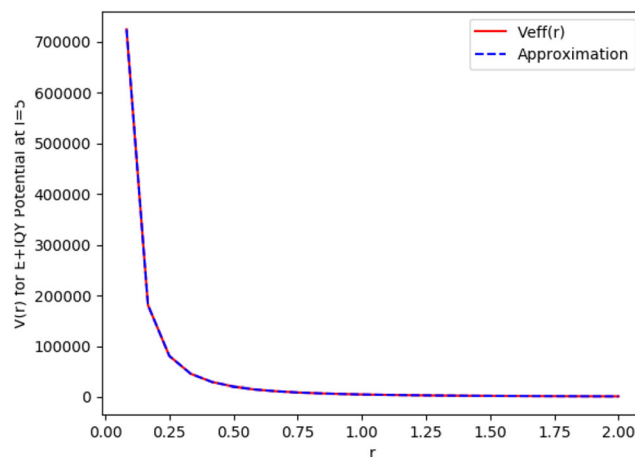

Fig. 2 Inversely quadratic Yukawa potentials versus r with $V_0 = 1.0$ MeV for parameter $\alpha = 0.01, 0.05, 0.5$

Fig. 3 Effective potential and its approximation for Eckart potentials plus inversely quadratic Yukawa potentials with $\eta = 0.01$ MeV and $\zeta = 0.5$ MeV, $V_0 = 1.0$ MeV for parameter $\alpha = 0.01$ at $l = 5$

Fig. 4 Effective potential and its approximation for Eckart potentials plus inversely quadratic Yukawa potentials with $\eta = 0.01$ MeV and $\zeta = 0.5$ MeV, $V_0 = 1.0$ MeV for parameter $\alpha = 0.5$ at $l = 5$
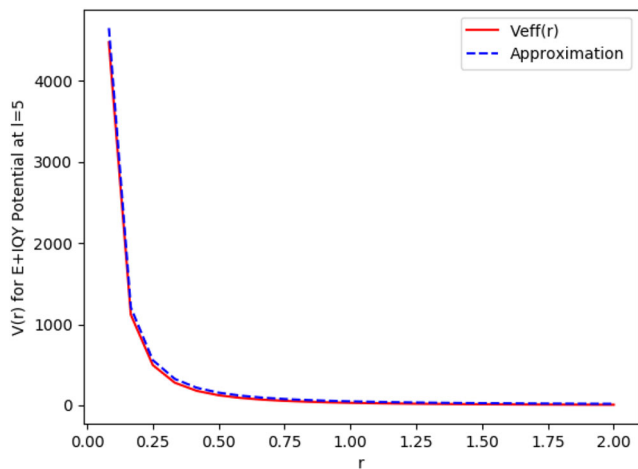
$$\frac{1}{r^2} \approx \frac{\alpha^2}{(1-e^{-\alpha r})^2} \tag{19b}$$

which is valid for $\alpha r \ll 1$. Therefore the EIQYP becomes

$$
\begin{aligned}
V_{eff}(r) &= V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2} \\
&= -\eta \frac{e^{-\alpha r}}{(1-e^{-\alpha r})} + \zeta \frac{e^{-\alpha r}}{(1-e^{-\alpha r})^2} - \frac{V_0 e^{-\alpha r} \alpha^2}{(1-e^{-\alpha r})^2} \\
&\quad + \frac{l(l+1)\hbar^2 \alpha^2}{2\mu(1-e^{-\alpha r})^2}.
\end{aligned} \tag{19c}
$$

To test the accuracy of our approximation, we plotted the Eckart plus inversely quadratic Yukawa potential (3) and its approximation (19a) with parameters $\eta = 0.01$ MeV, $\zeta = 0.5$ MeV, and $V_0 = 1.0$ MeV at different screening parameters
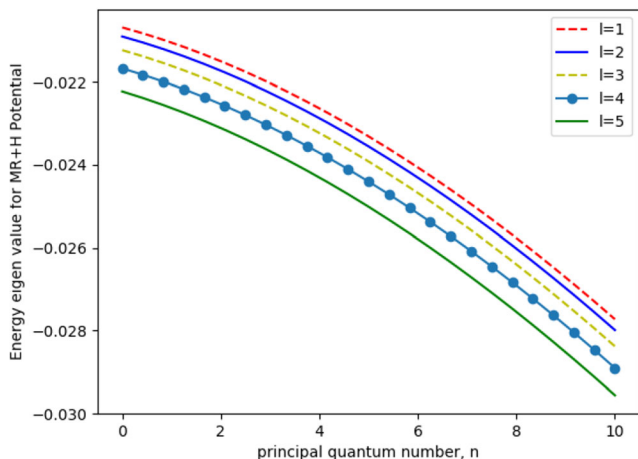
$\alpha = 0.01, 0.05,$ and 0.5 in Fig. 3. In Fig. 1 and Fig. 2, a plot of the Eckart potential and inversely Yukawa potential against internuclear distance 'r' at different screening parameters was reported. Additionally, in Fig. 3, Fig. 4, and Fig. 5, the effective potential and its approximation (19c) for angular momentum numbers $l = 5$, at different screening parameters is demonstrated.

Making the transformation $s = e^{-\alpha r}$ eq. (19) becomes

$$V(s) = -\eta \frac{s}{(1-s)} + \zeta \frac{s}{(1-s)^2} - \frac{V_0 \alpha^2 s}{(1-s)^2}. \tag{20}$$

Again, applying the same transformation to get the form that NU method is applicable, Eq. (21) gives a generalized hypergeometric-type equation as

$$
\begin{aligned}
&\frac{d^2 R(s)}{ds^2} + \frac{(1-s)}{(1-s)s}\frac{dR(s)}{ds} + \frac{1}{(1-s)^2 s^2} \\
&\left[-(\beta^2 + A)s^2 + (2\beta^2 + A + B - C)s - (\beta^2 + \lambda) \right] R(s) = 0,
\end{aligned} \tag{21}
$$

Fig. 5 Plot of energy eigenvalues against principal quantum number 'n' for $1 \le l \le 5$ at $C = -0.05$ MeV, $D = -0.1875$ MeV, $a = 2$ MeV, $b = -1$ MeV, and $\alpha = 0.01$
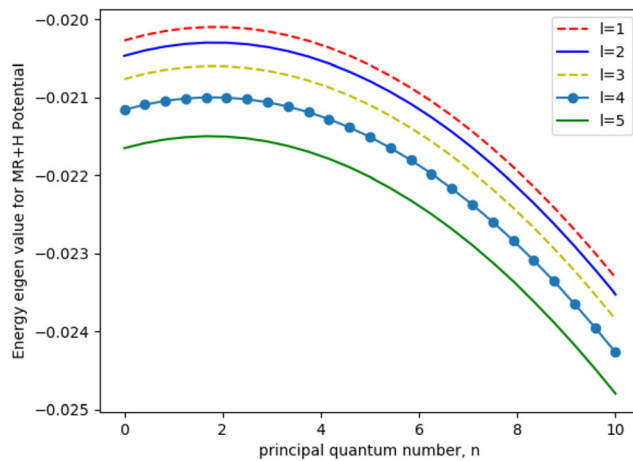

Fig. 6 Plot of energy eigenvalues against principal quantum number 'n' for $1 \le l \le 5$ at $C = 0$ MeV, $D = -3.1875$ MeV, $a = 2$ MeV, $b = -4$ MeV, and $\alpha = 0.01$
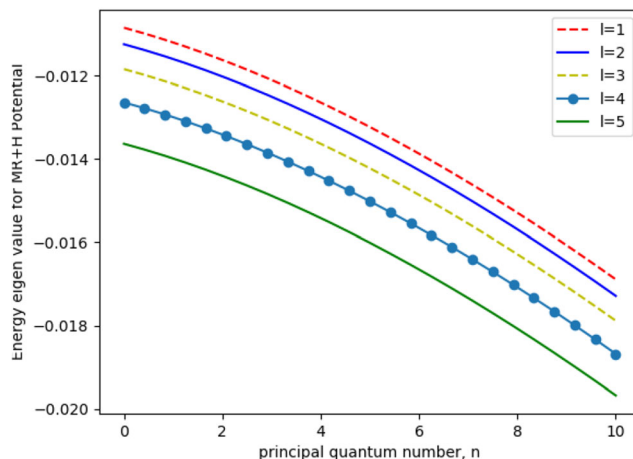

Fig. 7 Plot of energy eigenvalues against principal quantum number 'n' for $1 \le l \le 5$ at $C = -0.05$ MeV, $D = 0$ MeV, $a = -3$ MeV, $b = 1$ MeV, and $\alpha = 0.01$

Table 1 The energy eigenvalues $E_{n,l}$ (in eV) corresponding to the Eckart plus Inversely Quadratic Yukawa Potential (EIQYP) for various principal (n) and angular momentum (l) quantum number

n	l	H₂	CO	NO	ScH	N₂	CH	Ref[19]CO	Ref[19]NO	Ref[19]N₂	Ref[19]CH
0	0	0.387225	0.10188	0.082516	0.077683	0.109754	0.168743	0.10193061	0.08248827	0.1091559	0.1686344
1	0	1.133242	0.30468	0.247334	0.233368	0.325224	0.505062	0.30567217	0.24735916	0.32734304	0.50500718
	1	1.152167	0.306479	0.247603	0.2346	0.327141	0.508806	0.30615078	0.24778171	0.32734304	0.50859034
2	0	1.886129	0.508199	0.412356	0.389343	0.544497	0.841663	0.50941373	0.41223005	0.54553018	0.84137996
	1	1.909516	0.510504	0.4127	0.390863	0.546955	0.845116	0.50989234	0.4126526	0.54602881	0.84496312
	2	1.93204	0.51095	0.413743	0.393025	0.547927	0.852378	0.51084953	0.41349768	0.54702603	0.85212458
3	0	2.652514	0.715916	0.510117	0.544045	0.743041	1.209041				
	1	2.66051	0.718621	0.533788	0.545749	0.745883	1.218552				
	2	2.695333	0.72132	0.5458	0.548141	0.748718	1.219715				
	3	2.741565	0.724011	0.576411	0.552243	0.749257	1.222456				
4	0	3.407709	0.917188	0.742251	0.700596	0.982149	1.515482	0.91689685	0.74197183	0.98190446	1.5141255
	1	3.416307	0.917782	0.742469	0.702426	0.982459	1.519325	0.91737546	0.74239438	0.98240309	1.5177087
	2	3.41966	0.918361	0.743369	0.714022	0.983079	1.522649	0.91833265	0.43293946	0.98340031	1.5248701
	3	3.421218	0.920145	0.744721	0.71613	0.985307	1.53201	0.91976835	0.744507	0.98489606	1.5356002
	4	3.424428	0.921915	0.746525	0.718934	0.987178	1.551845	0.92168247	0.74619689	0.98689026	1.5498843
5	0	4.160899	1.120292	0.906961	0.855358	1.200522	1.851085	1.1206384	0.90684272	1.2000916	1.8504983
	1	4.166178	1.121608	0.907452	0.856052	1.20086	1.855174	1.121117	0.90726527	1.2005902	1.8540815
	2	4.169244	1.12224	0.908435	0.857438	1.201536	1.86099	1.1220742	0.90811035	1.2015875	1.8612429
	3	4.170085	1.123187	0.9094	0.858297	1.203261	1.873281	1.1235099	0.90937789	1.2030832	1.8719729
	4	4.176167	1.125119	0.911369	0.859839	1.205326	1.88185	1.125424	0.91106778	1.2050774	1.8862571
	5	4.179987	1.128037	0.91332	0.860844	1.207018	1.905507	1.1278165	0.9131799	1.2075699	1.9040761

Where

$$\lambda = l(l+1), -\beta^2 = \frac{2\mu E}{\alpha^2 \hbar^2}, A = \frac{2\mu\eta}{\alpha^2 \hbar^2}, B = \frac{2\mu V_0}{\hbar^2}, C = \frac{2\mu\zeta}{\alpha^2 \hbar^2}. \quad (22)$$

Hence, comparing Eq. (21) with Eq. (5) yields the following parameters below:

$$c_1 = c_2 = c_3 = 1, c_4 = 0, c_5 = -\frac{1}{2}, c_6 = \frac{1}{4} + \beta^2 + A, \quad (23)$$

$$c_7 = -2\beta^2 - A - B + C, c_8 = \beta^2 + \lambda, c_9 = \frac{1}{4} + C - B + \lambda,$$

$$c_{10} = 1 + 2\sqrt{\beta^2 + \lambda}, c_{11} = 2 + 2\left(\sqrt{\frac{1}{4} + C - B + \lambda} + \sqrt{\beta^2 + \lambda}\right),$$

$$c_{12} = \sqrt{\beta^2 + \lambda}, c_{13} = -\frac{1}{2} - \left(\sqrt{\frac{1}{4} + C - B + \lambda} + \sqrt{\beta^2 + \lambda}\right),$$

$$\epsilon_1 = (\beta^2 + A), \epsilon_2 = 2\beta^2 + A - B - C, \epsilon_3 = \beta^2 + \lambda.$$

Now using Eqs. (5), (22), and (23), we obtain the energy eigen spectrum of the EIQYP as

$$\beta^2 = \left[\frac{A + B - C - 2\lambda - \left(n^2 + n + \frac{1}{2}\right) - (2n+1)\sqrt{\frac{1}{4} + C - B + \lambda}}{(2n+1) + 2\sqrt{\frac{1}{4} + C - B + \lambda}}\right]^2 - \lambda \quad (24)$$

Equation (24) can be solved explicitly and the energy eigen spectrum of EIQYP becomes, for $l \neq 0$

$$E = \frac{\alpha^2 \hbar^2}{2\mu}\left\{\left[\frac{\frac{2\mu\eta}{\alpha^2 \hbar^2} + \frac{2\mu V_0}{\hbar^2} - \frac{2\mu\zeta}{\alpha^2 \hbar^2} - 2l(l+1) - \left(n^2 + n + \frac{1}{2}\right) - (2n+1)\sqrt{\frac{1}{4} + l(l+1) + \frac{2\mu\zeta}{\alpha^2 \hbar^2} - \frac{2\mu V_0}{\hbar^2}}}{(2n+1) + 2\sqrt{\frac{1}{4} + l(l+1) + \frac{2\mu\zeta}{\alpha^2 \hbar^2} - \frac{2\mu V_0}{\hbar^2}}}\right]^2 - l(l+1)\right\} \quad (25a)$$

also for s-wave state where $l = 0$

$$E = \frac{\alpha^2 \hbar^2}{2\mu}\left[\frac{\frac{2\mu\eta}{\alpha^2 \hbar^2} + \frac{2\mu V_0}{\hbar^2} - \frac{2\mu\zeta}{\alpha^2 \hbar^2} - \left(n^2 + n + \frac{1}{2}\right) - (2n+1)\sqrt{\frac{1}{4} + \frac{2\mu\zeta}{\alpha^2 \hbar^2} - \frac{2\mu V_0}{\hbar^2}}}{(2n+1) + 2\sqrt{\frac{1}{4} + \frac{2\mu\zeta}{\alpha^2 \hbar^2} - \frac{2\mu V_0}{\hbar^2}}}\right]^2 \quad (25b)$$

We now calculate the radial wave function of the EIQYP as follows.

The weight function $\rho(s)$ is given as

$$\rho(s) = s^{c_{10}-1}(1 - c_3 s)^{\frac{c_{11}}{c_3} - c_{10} - 1}. \quad (26)$$

Using Eq. (23), we get the weight function as

$$\rho(s) = s^U (1-s)^V, \quad (27)$$

Where $U = 2\sqrt{\beta^2 - \lambda}$ and $V = 2\sqrt{\frac{1}{4} + C - B + \lambda}$.
Also we obtain the wave function $\chi(s)$ as

$$\chi(s) = P_n^{c_{10}-1, \frac{c_{11}}{c_3} - c_{10} - 1}(1 - 2c_3 s). \quad (28)$$

Using Eq. (23), we get the function $\chi(s)$ as

$$\chi(s) = P_n^{(U,V)}(1-2s), \quad (29)$$

Where $P_n^{(U,V)}$ are Jacobi polynomials.
Lastly,

$$\varphi(s) = s^{c_{12}}(1 - c_3 s)^{-c_{12} - \frac{c_{13}}{c_3}}, \quad (30)$$

and using Eq. (23), we get

$$\varphi(s) = s^{U/2}(1-s)^{V/2}. \quad (31)$$

We then obtain the radial wave function from the equation

$$R_n(s) = N_n \varphi(s)\chi_n(s), \quad (32)$$

$$R_n(s) = N_n s^{U/2}(1-s)^{(V-1)/2} P_n^{(U,V)}(1-2s), \quad (33)$$

Where n is a positive integer and N_n is the normalization constant.

Results and discussion

Figures 6 and 7 show graphically the energy against principal quantum number 'n' for different choices of parameters C, D, a, and b at $\alpha = 0.01$ for $1 \leq l \leq 5$. As the principal quantum number increases, the energy at various l-states decreases. The explicit bound state energy expression obtained in this work (Eq. (25)) was used to calculate the numerical bound state energies of some selected diatomic molecules (H_2, CO, ScH, N_2, NO, and CH) using python programming language (version 3.7.7) in an Anaconda navigator environment. The spectroscopic parameters (mainly the dissociation energies and the molecular reduced mass) for the diatomic molecules selected for this work were obtained from previously reported works [17–20]. In comparison, our numerical results for some

of the selected diatomic molecules CO, N_2, NO, and CH are in good agreement with the results reported by Ramazan, Sever., and co [19]. The numerical computed bound state energies for the selected diatomic molecules at different n and l are shown in Table 1. From the calculated results, it can be observed that the bound state energies of the diatomic molecules increase as the values of n and l increase. The numerical values of the explicit bound state energies of some of these diatomic molecules for various values of n and l are obtained and compared with the Poisson summation approach [17] and the exact method [18] for H_2, CO, ScH, and NO.

Conclusions

In this study, we investigate the solutions of the Schrodinger equation for the mixed proposed potential consisting of the Eckart and Inversely Quadratic Yukawa Potential (EIQYP) by using the Greene and Aldrich type approximation for the centrifugal term. We obtained the energy eigenvalues and the corresponding wave function for several values of the screening parameter for an arbitrary state. To show the accuracy of the approximation used in this study, we plotted graphs of the potentials and its approximation with the PYTHON 3.6 program.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest

References

1. Onate CA, Onyeaju MC, Ikot AN, Idiodi JOA, Ojonubah JO (2017) Eigen solutions, Shannon entropy and fisher information under the Eckart Manning Rosen potential model. Journal of the Korean Physical Society 70(4):339–347
2. Sari RA, Suparmi A, Cari C (2015) Solution of Dirac equation for Eckart potential and trigonometric manning Rosen potential using asymptotic iteration method. Chinese Physics B 25(1):010301
3. Greene RL, Aldrich C (1976) Variational wave functions for a screened Coulomb potential. Phys. Rev. A 14(6):2363
4. Chen CY, Sun DS, Lu FL (2008) Analytical approximations of scattering states to the l-wave solutions for the Schrödinger equation with the Eckart potential. J. Phys. A Math. Theor. 41(3): 035302
5. Ikhdair SM Bound states of the Klein-Gordon equation in D-dimensions with some physical scalar and vector exponential-type potentials including orbital centrifugal term. arXiv preprint arXiv: 1110.0943 (2011)
6. Hassanabadi H, Yazarloo BH, Zarrinkamar S (2013) Exact solution of Klein–Gordon equation for Hua plus modified Eckart potentials. Few-Body Systems 54(11):2017–2025
7. Zhang Y (2008) Approximate analytical solutions of the Klein–Gordon equation with scalar and vector Eckart potentials. Phys. Scr. 78(1):015006
8. Taskin F, Koçak GÖKHAN (2010) Approximate solutions of Schrödinger equation for Eckart potential with centrifugal term. Chinese Physics B 19(9):090314
9. Akpan IO, Antia AD, Ikot AN (2012) Bound-state solutions of the Klein-Gordon equation with-deformed equal scalar and vector Eckart potential using a newly improved approximation scheme. ISRN High Energy Physics
10. Qiang WC, Dong SH (2007) Arbitrary l-state solutions of the rotating Morse potential through the exact quantization rule method. Phys. Lett. A 363(3):169–176
11. Suparmi A, Cari C, Handhika J (2013) Approximate solution of Schrodinger equation for Eckart potential combined with Trigonormetric Poschl-teller non-central potential using Romanovski polynomials. Journal of Physics: Conference Series 423(1):012039
12. Hamzavi M, Ikhdair SM, Ita BI (2012) Approximate spin and pseudospin solutions to the Dirac equation for the inversely quadratic Yukawa potential and tensor interaction. Phys. Scr. 85(4): 045009
13. Ita BI, Louis H, Magu TO, Nzeata-Ibe NA (2017) Bound state solutions of the Schrödinger's equation with Manning-Rosen plus a class of Yukawa potential using Pekeris-like approximation of the Coulomb term and parametric Nikiforov-Uvarov. World Scientific News 70(2):312–319
14. Louis, B. I. Ita, PI Amos, OU Akakuru, MM Orosun, NA Nzeata-Ibe and M. Philip bound state solutions of Klein-Gordon equation with Manning-Rosen plus a class of Yukawa potential using pekeris-like approximation of the Coulomb term and parametric Nikiforov-Uvarov. International Journal of Chemical and Physical Sciences, 7(1), (2018) 33–37
15. Onate CA, Ojonubah JO (2016) Eigensolutions of the Schrödinger equation with a class of Yukawa potentials via supersymmetric approach. Journal of Theoretical and Applied Physics 10(1):21–26
16. Maireche A (2017) Effects of two-dimensional noncommutative theories on bound states Schrödinger diatomic molecules under new modified Kratzer-type interactions. International Letters of Chemistry, Physics and Astronomy 76:1–11
17. Hamzavi M, Thylwe KE, Rajabi AA (2013) Approximate bound states solution of the Hellmann potential. Commun. Theor. Phys. 60(1):1
18. Edet CO, Okorie US, Osobonye G, Ikot AN, Rampho GJ, Sever R (2020) Thermal properties of Deng–Fan–Eckart potential model using Poisson summation approach. J. Math. Chem.:1–25
19. Ramazan, Sever, Cevdet, Teczan, Metin, Aktas, Ozlem, Yesiltas. (2018). Exact solution of Schrodinger equation for Pseudoharmonic potential. arXiv:quant-ph/0701206v1
20. Oyewumi KJ, Sen KD (2012) Exact solutions of the Schrödinger equation for the pseudoharmonic potential: an application to some diatomic molecules. J. Math. Chem. 50(5):1039–1059