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Analysis of a stabilized finite element approximation of the transient convection-diffusion-reaction equation using orthogonal subscales

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Abstract. In this paper we analyze a stabilized finite element method to solve the transient convection-diffusion-reaction equation based on the decomposition of the unknowns into resolvable and subgrid scales. We start from the time-discrete form of the problem and obtain an evolution equation for both components of the decomposition. A closed-form expression is proposed for the subscales which, when inserted into the equation for the resolvable scale, leads to the stabilized formulation that we analyze. Optimal error estimates in space are provided for the first order, backward Euler time integration.

1 Introduction

In this paper we describe and analyze a finite element formulation to solve the transient convection-diffusion-reaction problem

$$\partial_t u + \mathcal{L}u = f \quad \text{in } \Omega \times (0,T)\,, \tag{1}$$
$$u = 0 \quad \text{on } \partial\Omega \times (0,T)\,, \tag{2}$$
$$u = u_0 \quad \text{on } \Omega \times \{0\}\,, \tag{3}$$

where $\mathcal{L}$ is the convection-diffusion-reaction operator

$$\mathcal{L}u := \boldsymbol{a}\cdot\nabla u - \nu\Delta u + \sigma u\,. \tag{4}$$

Here, $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is a bounded and polyhedral domain, $\boldsymbol{a}$ is a divergence-free velocity field, $\nu > 0$ is the diffusion coefficient and $\sigma \geq 0$ is the (constant) reaction coefficient.

The functional setting for the problem can be defined in terms of the spaces $\mathcal{V} := H^1(\Omega)$ and $\mathcal{V}_0 := H_0^1(\Omega)$. For $f \in L^2(0,T;H^{-1}(\Omega))$, $u_0 \in L^2(\Omega)$, and $\boldsymbol{a} \in (L^\infty(\Omega))^d$ problem (1)–(3) admits a unique solution u which belongs to $L^2(0,T;\mathcal{V}_0) \cap \mathcal{C}^0(0,T;L^2(\Omega))$, and which also verifies $\partial_t u \in L^2(0,T;H^{-1}(\Omega))$. The norm in $L^2(\Omega)$ is denoted by $\|\cdot\|$, whereas $\|\cdot\|_m$ stands for the norm in $H^m(\Omega)$, $m = -1, 0, 1, 2, \ldots$

The variational formulation of the problem that we need reads: find $u \in L^2(0,T;\mathcal{V}_0)$ such that

$$\begin{aligned} &(\partial_t u, v) + b(u,v) = l(v) \quad \forall\, v \in \mathcal{V}_0\,,\\ &b(u,v) := \nu(\nabla u, \nabla v) + (\boldsymbol{a}\cdot\nabla u, v) + \sigma(u,v)\,,\\ &l(v) := \langle f, v\rangle\,, \end{aligned} \tag{5}$$

where $(\cdot,\cdot)$ stands for the $L^2(\Omega)$-inner product and $\langle\cdot,\cdot\rangle$ for the duality pairing in $H^{-1}(\Omega)\times H^1(\Omega)$.

Many schemes can be employed to discretize (5) in time. Since our main concern is the space discretization, we will use the simple backward Euler approximation described in the following section. Our ideas extend easily to any other algorithm, as well as to a space-time finite element formulation (case in which the variational equation (5) has to be modified).

Our aim is to present a finite element method free of the oscillations associated to the standard Galerkin formulation when diffusion is small. Many of such methods have been developed during the last two decades, among which the SUPG, the GLS, the characteristic Galerkin, the Taylor Galerkin, the residual free bubble stabilization, and others, can be named (see [3] for a comparison between these methods).

Our starting point will be the decomposition of the unknown into resolvable and subgrid scales. The former are intuitively associated to the component of the solution which can be represented by the finite element mesh, whereas the latter can not be captured. Nevertheless, its effect onto the resolvable scale needs to be accounted for.

The approach we will follow is similar to that presented in [10]. However, it is interesting to note that other methods share very similar concepts, as for example the nonlinear Galerkin method (see, e.g., [13]).

The main conceptual ingredients of the formulation to be presented here are the following. First, we let the subscales vary in time, and thus they need to be tracked. Second, we propose a closed-form expression for them at each time step. This is necessarily heuristic. However, we describe a Fourier analysis that provides some rationale to our proposal. A very important point is that we take the subscales orthogonal to the finite element space [4]. Finally, we add some additional approximations that lead to a method which is computationally feasible, in the sense that its cost is similar (very often smaller) to that of other stabilization methods.

The plan of the paper is the following. In Sect. 2 we present the backward Euler scheme and provide an error estimate which shows that this method is first order accurate in time. We note that it is most common to give this kind of estimates once the problem has been already discretized in space. Section 3 describes our numerical formulation from a purely heuristic point of view. This formulation is then analyzed in Sect. 4, where it is shown that it yields optimal rates of convergence in space. The key point is that the error estimates hold uniformly in the diffusion coefficient ν.

2 Semidiscrete problem

First of all, let us re-scale the time variable as $t \leftarrow t/T$, so that the new time interval is [0, 1] and the coefficient $1/T$ has to be inserted in front of the time derivatives. This will allow us to explicitly display which terms in our stability and error estimates disappear as the steady-state is reached, that is, as $T \to \infty$.

Let us consider a uniform partition of [0, 1] into N time intervals of length δt. Assuming for simplicity that f is continuous in time and denoting by a superscript the time level at which the algorithmic solution is computed, the backward Euler discretization of (5) is:

$$\frac{1}{T}(\delta_t u^n, v) + b(u^{n+1}, v) = l^{n+1}(v) \quad \forall\, v \in \mathcal{V}_0\,, \tag{6}$$

and $u^0 = u_0$ in $L^2(\Omega)$, where

$$\delta_t u^n := \frac{1}{\delta t}(u^{n+1} - u^n)\,, \qquad l^{n+1}(v) := \langle f^{n+1}, v\rangle\,.$$

It is convenient for notation purposes to introduce the sequence $U := \{u^0, u^1, ..., u^N\} \in L^2(\Omega) \times \mathcal{V}_0^N$, as well as the norm $|||\cdot|||$ defined on such sequences by

$$\begin{aligned}|||V|||^2 &:= \frac{1}{T}\|V\|^2_{\ell^\infty(L^2)} + \nu\|V\|^2_{\ell^2(H_0^1)} \\ &\quad + \delta t\| - \nu\Delta V + \boldsymbol{a}\cdot\nabla V\|^2_{\ell^2(L^2)} + \sigma\|V\|^2_{\ell^2(L^2)} \\ &:= \frac{1}{T}\max_{n=0,...,N}\|v^n\|^2 + \sum_{n=0}^{N-1}\delta t\Big[\nu\|\nabla v^{n+1}\|^2 \\ &\quad + \delta t\| - \nu\Delta v^{n+1} + \boldsymbol{a}\cdot\nabla v^{n+1}\|^2 + \sigma\|v^{n+1}\|^2\Big].\end{aligned} \tag{7}$$

This is the norm in which stability and convergence can be proved (note that we have identified the $L^2(\Omega)$-norm of ∇v with the $H_0^1(\Omega)$-norm of v). The distinctive feature of our analysis is that *we include the sum of the convective and viscous terms in* $|||\cdot|||$. In particular, our results apply to any value of the physical coefficients ν, $\|\boldsymbol{a}\|_{L^\infty(\Omega)}$ and σ. In general, C, possibly with subscripts, will denote a generic constant independent of these parameters, and also independent of δt (and of h in Sect. 4).

Problem (6), incorporating the initial condition, can be written as: find a sequence of algorithmic solutions U such that

$$B(U, V) = L(V) \tag{8}$$

for all sequences V, where

$$\begin{aligned}B(U, V) &:= \frac{1}{2T}\left(u^0, v^0\right) \\ &+ \sum_{n=0}^{N-1}\Big[\frac{1}{T}\left(u^{n+1} - u^n, v^{n+1}\right) + \delta t\, b\left(u^{n+1}, v^{n+1}\right)\Big],\end{aligned} \tag{9}$$

$$L(V) := \frac{1}{2T}\left(u_0, v^0\right) + \sum_{n=0}^{N-1}\delta t\left\langle f^{n+1}, v^{n+1}\right\rangle\,, \tag{10}$$

It is seen that the initial condition $u^0 = u_0$ is incorporated in the variational problem (8).

Both stability and convergence are stated in the following result:

Theorem 1. *There is a unique solution to problem* (8)*, which is bounded in the norm* $|||\cdot|||$*. Moreover, assume that the solution of* (5) *verifies the regularity requirement* $\partial^2_{tt}u \in L^2(0, T; L^2(\Omega))$*. Then, for* δt *small enough:*

$$|||U_{\rm ex} - U||| \le C\delta t\,, \tag{11}$$

where $U_{\rm ex} = \{u_0, u(t^1), u(t^2), ..., u(t^N)\}$ *is the exact solution* ($t^n := n\delta t$)*.*

We postpone to sketch the proof until the end of Sect. 4, since the concepts involved are similar to those employed in the analysis of the fully discrete problem.

3 Finite element approximation using orthogonal subscales

3.1 Orthogonal subscales for the stationary advection-diffusion-reaction equation

In this section we describe the numerical formulation that we will use. First, we consider the stationary equation $\mathcal{L}u = f$, and later on we will move to the transient case. The variational formulation of the problem is now: find $u \in \mathcal{V}_0$ such that

$$b(u, v) = l(v) \quad \forall\, v \in \mathcal{V}_0\,. \tag{12}$$

The basic idea is to split the continuous spaces $\mathcal{V}$ and $\mathcal{V}_0$ into the spaces of resolvable and subgrid scales:

$$\mathcal{V}_0 = \mathcal{V}_{h,0} \oplus \tilde{\mathcal{V}}_0\,, \quad \mathcal{V} = \mathcal{V}_h \oplus \tilde{\mathcal{V}}\,,$$

where $\mathcal{V}_h$ is the finite element space built from a partition $\mathcal{P}_h = \{K\}$ of Ω in the classical way, and $\mathcal{V}_{h,0}$ is the subspace of $\mathcal{V}_h$ made up of functions vanishing on $\partial\Omega$.

Problem (12) can be equivalently written as: find $u_h \in \mathcal{V}_{h,0}$, $\tilde{u} \in \tilde{\mathcal{V}}_0$ such that

$$b(u_h, v_h) + b(\tilde{u}, v_h) = l(v_h) \quad \forall\, v_h \in \mathcal{V}_{h,0}\,, \tag{13}$$

$$b(u_h, \tilde{v}) + b(\tilde{u}, \tilde{v}) = l(\tilde{v}) \quad \forall\, \tilde{v} \in \tilde{\mathcal{V}}_0\,. \tag{14}$$

After integration by parts of some terms and assuming that the diffusive fluxes of $u = u_h + \tilde{u}$ are continuous across interelement boundaries, these two equations yield:

$$b(u_h, v_h) + \sum_K \int_K \tilde{u}\, \mathcal{L}^* v_h \, \mathrm{d}\Omega + \sum_K \int_{\partial K} \tilde{u}\, \nu \frac{\partial v_h}{\partial n} \, \mathrm{d}\Gamma = l(v_h) , \tag{15}$$

$$\sum_K \int_K \tilde{v}\, \mathcal{L}\tilde{u} \, \mathrm{d}\Omega = \sum_K \int_K \tilde{v}\, (f - \mathcal{L}u_h) \, \mathrm{d}\Omega , \tag{16}$$

where $\mathcal{L}^* v := -\nu\Delta v - \boldsymbol{a} \cdot \nabla v + \sigma v$ is the adjoint of $\mathcal{L}$.

From (16) it follows that

$$\begin{aligned} &\mathcal{L}\tilde{u} = r \quad \text{in } K \in \mathcal{P}_h , \\ &r := f - \mathcal{L}u_h + v_{h,\mathrm{ort}} , \end{aligned} \tag{17}$$

where $v_{h,\mathrm{ort}}$ is the element that makes $\mathcal{L}\tilde{u} - r$ belong to $\tilde{\mathcal{V}}_0$, *which is the space in which* (17) *has to be understood.* Obviously, this equation for the subscales is as complicated as the original one, except for the fact that it is defined on each element domain. Note also that the values of $\tilde{u}$ on ∂K should be known to solve it, and these are unknown. However, rather than solving for $\tilde{u}$ we propose *to model* it, that is to say, we propose to give a closed-form expression for $\tilde{u}$ that, hopefully, will have a similar effect as the exact one when plugged in (15). The model we suggest is

$$\tilde{u}(\boldsymbol{x}) \approx \tau\, r(\boldsymbol{x}) \quad \text{in } K \in \mathcal{P}_h , \tag{18}$$

with τ a constant on each K. Moreover, we assume that

$$\sum_K \int_{\partial K} \tilde{u}\, \nu \frac{\partial v_h}{\partial n} \, \mathrm{d}\Gamma \approx 0 . \tag{19}$$

Equations (18) and (19) are our two modeling assumptions. In the following section we come back to (18). For the moment, let us accept it and concentrate on the function $v_{h,\mathrm{ort}}$ in (17). It is associated to how the space of subgrid scales is chosen. *Our particular option is to take*

$$\tilde{\mathcal{V}} = \mathcal{V}_h^{\perp} \cap \mathcal{V} ,$$

where orthogonality is understood in $L^2(\Omega)$. Note that $\mathcal{V}_h^{\perp}$ is not closed in $\mathcal{V}$, but it will be a closed subspace of the final approximating space.

In order to obtain a feasible numerical method we need to introduce two further approximations. These are:

$$\tilde{\mathcal{V}} \approx \mathcal{V}_h^{\perp} \approx \tilde{\mathcal{V}}_0 . \tag{20}$$

The first approximation means that we drop the requirement $\tilde{\mathcal{V}} \subset H^1(\Omega)$, that is to say, we allow the subscales to be *non-conforming*. The second approximation implies that we assume that elements in $\mathcal{V}_h^{\perp}$ are small on $\partial\Omega$.

Using approximations (20) we have that

$$v_{h,\mathrm{ort}} \in \tilde{\mathcal{V}}_0^{\perp} \approx \mathcal{V}_h , \quad \tilde{u} \in \tilde{\mathcal{V}}_0 \approx \mathcal{V}_h^{\perp} .$$

In particular, we note that $v_{h,\mathrm{ort}}$ *is a finite element function*, and therefore numerically computable. Due to assumption (18), the equation to obtain it is simply to impose that $(\tilde{u}, v_h) = 0$ for all $v_h \in \mathcal{V}_h$, that is,

$$\sum_K \left[(\tau(f - \mathcal{L}u_h), v_h)_K + (\tau v_{h,\mathrm{ort}}, v_h)_K \right] = 0 .$$

for all $v_h \in \mathcal{V}_h$. If we introduce the weighted $L^2(\Omega)$-inner product $(v, w)_\tau := \sum_K \tau(v, w)_K$, and call Π_τ the associated projection, we see that $v_{h,\mathrm{ort}}$ and the subgrid scale can be expressed as:

$$\begin{aligned} &v_{h,\mathrm{ort}} = -\Pi_\tau(f - \mathcal{L}u_h) , \\ &\tilde{u} = \tau\Pi_\tau^{\perp}(f - \mathcal{L}u_h), \quad \Pi_\tau^{\perp} := I - \Pi_\tau . \end{aligned} \tag{21}$$

It is understood in these expressions that $\mathcal{L}u_h$ and the parameters τ are evaluated elementwise. This closes the modeling process. Once τ is known, $\tilde{u}$ can be computed from (21) and inserted in (15) (using also (19)).

There are other strategies to model the subscales. For example, in [9, 11] it is proposed to add artificial diffusion only in their subspace, and in the nonlinear Galerkin method they are treated with a coarse numerical approximation [7, 13]. Likewise, when subscales are approximated by bubble functions, as in [1, 2], solving for them is what can be considered the modeling step.

3.2 Behavior of the stabilization parameters from a Fourier analysis

If $\tilde{u}$ is a solution of the equation in $\tilde{\mathcal{V}}_0$

$$-\nu\Delta\tilde{u} + \boldsymbol{a} \cdot \nabla\tilde{u} + \sigma\tilde{u} = r \quad \text{in } K \in \mathcal{P}_h , \tag{22}$$

we want to understand in which sense $\tilde{u}(\boldsymbol{x}) \approx \tau\, r(\boldsymbol{x})$ (both functions belonging to $\tilde{\mathcal{V}}_0$) and to give an expression for τ. Now we will assume that $\boldsymbol{a}$ is constant within each element K.

Let us consider the following Fourier transform defined on each K:

$$\widehat{g}(\boldsymbol{k}) := \int_K \mathrm{e}^{-\mathrm{i}\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{h}} g(\boldsymbol{x}) \, \mathrm{d}\Omega_x ,$$

where $\boldsymbol{k} = (k_1, \ldots, k_d)$ is the dimensionless wave number and h the diameter of K. If n_j is the j-th component of the normal exterior to K, it is easily checked that

$$\widehat{\frac{\partial g}{\partial x_j}}(\boldsymbol{k}) = \int_{\partial K} n_j \mathrm{e}^{-\mathrm{i}\frac{\boldsymbol{k}\cdot\boldsymbol{x}}{h}} g(\boldsymbol{x}) \, \mathrm{d}\Gamma_x + \mathrm{i}\frac{k_j}{h}\widehat{g}(\boldsymbol{k}) .$$

We assume that the subscales only contain *high wave numbers*, and thus

$$\widehat{\frac{\partial \tilde{u}}{\partial x_j}}(\boldsymbol{k}) \approx \mathrm{i}\frac{k_j}{h}\widehat{\tilde{u}}(\boldsymbol{k}) , \quad \widehat{\frac{\partial^2 \tilde{u}}{\partial x_i \partial x_j}}(\boldsymbol{k}) \approx -\frac{k_i k_j}{h^2}\widehat{\tilde{u}}(\boldsymbol{k}) . \tag{23}$$

Note that all the results valid for Fourier transforms of functions of rapid decay in $\mathbb{R}^d$ will apply to $\widehat{\tilde{u}}$.

Equation (23) allows us to obtain an expression for τ. Indeed, if we take the Fourier transform of (22) we have:

$$\begin{aligned} &\widehat{\tilde{u}}(\boldsymbol{k}) \approx \mathcal{T}(\boldsymbol{k})\widehat{r}(\boldsymbol{k}) , \\ &\mathcal{T}(\boldsymbol{k}) := \left(\nu\frac{|\boldsymbol{k}|^2}{h^2} + \mathrm{i}\frac{\boldsymbol{a}\cdot\boldsymbol{k}}{h} + \sigma \right)^{-1} . \end{aligned}$$

Plancherel's formula and the mean value theorem imply

$$\begin{aligned}\|\tilde{u}\|^2_{L^2(K)} &\approx \frac{1}{(2\pi)^d}\|\widehat{\tilde{u}}\|^2_{L^2(\mathbb{R}^d)}\\ &\approx \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\mathcal{T}(\boldsymbol{k})|^2|\widehat{r}(\boldsymbol{k})|^2\,\mathrm{d}k\\ &= \frac{1}{(2\pi)^d}|\mathcal{T}(\boldsymbol{k}_0)|^2\int_{\mathbb{R}^d}|\widehat{r}(\boldsymbol{k})|^2\,\mathrm{d}k\\ &= |\mathcal{T}(\boldsymbol{k}_0)|^2\|r\|^2_{L^2(K)}\,,\end{aligned}$$

for a certain $\boldsymbol{k}_0$. If we identify τ with $|\mathcal{T}(\boldsymbol{k}_0)|$, it allows us to conclude that *if we take*

$$\tau = \left[\left(c_1\frac{\nu}{h^2}+\sigma\right)^2+\left(c_2\frac{|\boldsymbol{a}|}{h}\right)^2\right]^{-1/2}$$

there exist values of c_1 and c_2 independent of h, ν, $|\boldsymbol{a}|$ and σ for which both $\tilde{u}$ and τr have (approximately) the same L^2-norm on element K.

Since we are interested in the asymptotic behavior of τ in terms of h, ν, $|\boldsymbol{a}|$ and σ, another possibility (slightly simpler) is

$$\tau = \left(c_1\frac{\nu}{h^2}+c_2\frac{|\boldsymbol{a}|}{h}+\sigma\right)^{-1}. \tag{24}$$

This is the expression extended to the transient case. When $\boldsymbol{a}$ is variable, $|\boldsymbol{a}|$ is replaced by $\|\boldsymbol{a}\|_{L^\infty(K)}$.

3.3 Tracking of subscales for the transient problem

Let us consider now the transient problem (1)–(3), discretized in time using the backward Euler method. The variational formulation is given in (6). To this variational problem we can apply now the same decomposition as for the stationary problem. Neglecting again the integral over the element boundaries and using the same reasoning as before, the problem for the finite element approximation is:

$$T^{-1}(\delta_t u_h^n, v_h)+T^{-1}(\delta_t\tilde{u}^n, v_h)+b\left(u_h^{n+1}, v_h\right) + \sum_K\int_K \tilde{u}^{n+1}\,\mathcal{L}^* v_h\,\mathrm{d}\Omega = l^{n+1}(v_h)\,, \tag{25}$$

for all $v_h\in\mathcal{V}_{h,0}$, and the equation for the subscales is

$$\begin{aligned}&T^{-1}\delta_t\tilde{u}^n+\mathcal{L}\tilde{u}^{n+1} = r_t^{n+1}\,,\\ &r_t^{n+1} := f^{n+1}-\left(T^{-1}\delta_t u_h^n+\mathcal{L}u_h^{n+1}\right)+v_{h,\mathrm{ort}}\end{aligned} \tag{26}$$

in $K\in\mathcal{P}_h$ and for $v_{h,\mathrm{ort}}$ such that (26) holds in $\tilde{\mathcal{V}}_0$. To start the algorithm we take $\tilde{u}^0\approx 0$, which means that u_0 is resolvable by $\mathcal{V}_h$. Equation (26) can also be written as

$$\left(\frac{1}{T\delta t}+\mathcal{L}\right)\tilde{u}^{n+1} = \frac{1}{T\delta t}\tilde{u}^n + r_t^{n+1}\,.$$

Calling

$$\tau_t := \left(\frac{1}{T\delta t}+\frac{1}{\tau}\right)^{-1} \tag{27}$$

and using the same strategy as for the stationary problem, we obtain

$$\tilde{u}^{n+1} = \tau_t\frac{1}{T\delta t}\tilde{u}^n+\tau_t r_t^{n+1}\,. \tag{28}$$

Imposing that the subscales be orthogonal to $\mathcal{V}_h$ and using the same approximations as for the stationary case, it is found that

$$\begin{aligned}v_{h,\mathrm{ort}} &= -\Pi_{\tau_t}\left[\frac{1}{T\delta t}\tilde{u}^n+f^{n+1}-(T^{-1}\delta_t u_h^n+\mathcal{L}u_h^{n+1})\right],\\ \tilde{u}^{n+1} &= \tau_t\Pi^\perp_{\tau_t}\left[\frac{1}{T\delta t}\tilde{u}^n+f^{n+1}-(T^{-1}\delta_t u_h^n+\mathcal{L}u_h^{n+1})\right],\end{aligned}$$

where $\mathcal{L}u_h^{n+1}$ and τ_t are evaluated within each $K\in\mathcal{P}_h$ and Π_{τ_t} is the projection computed with τ_t instead of τ.

This completes the description of the method: the subscale $\tilde{u}^{n+1}$ obtained can be inserted in (25), resulting in a finite element problem which is expected to have better stability properties than the original Galerkin formulation. From the mathematical point of view, this means that sharp stability estimates uniform in ν can be derived. However, to simplify further the numerical method we will use the following additional approximations, again within each element domain:

$$\Pi_{\tau_t}\approx\Pi \quad (\text{standard } L^2(\Omega)\text{-projection})\,, \tag{29}$$

$$\tau_t\Pi^\perp f\approx 0\,, \tag{30}$$

$$\tau_t\Pi^\perp\Delta u_h\approx 0\,. \tag{31}$$

Approximation (29) depends on the variation of the coefficients τ_t from element to element. From the computational point of view, it is very convenient to use Π, since L^2 projections can be computed very efficiently. Taking $\tau_t\Pi^\perp f\approx 0$ means that f belongs to the finite element space $\mathcal{V}_h$ or it is approximated by an element of this space. In any case, the term $\tau_t\Pi^\perp f$ is of the same order as the optimal error that can be expected, since for $f\in H^m(\Omega)$, $m=-1,0,\ldots$, we may expect $u\in H^{m+2}(\Omega)$ and an L^2 error of order $\mathcal{O}(h^r)$, with $r=\min\{m+2,\,p+1\}$ and p the degree of the finite element interpolation, and this is precisely the order of $\tau_t\Pi^\perp f$. Referring to approximation (31), it greatly simplifies the numerical implementation of the method. Moreover, we are interested precisely in the case in which the diffusive term is small. What we can not neglect is the orthogonal projection of the convective term, which is what has to provide the enhanced stability.

We also have the exact relationships:

$$\Pi^\perp(T^{-1}\delta_t u_h^n)=0\,,\quad \Pi^\perp(\sigma u_h)=0\,,\quad T^{-1}(\delta_t\tilde{u}^n, v_h)=0\,.$$

Using these equations, approximations (29)–(31) and inserting the subscales in (25) we finally find the equation for the resolvable scales:

$$\begin{aligned}&\frac{1}{T}\left(\delta_t u_h^n, v_h\right)+b\left(u_h^{n+1}, v_h\right)+\left(\Pi^\perp\left(\boldsymbol{a}\cdot\nabla u_h^{n+1}\right), \boldsymbol{a}\cdot\nabla v_h\right)_{\tau_t}\\ &= \frac{1}{T\delta t}\left(\tilde{u}^n, \boldsymbol{a}\cdot\nabla v_h\right)_{\tau_t}+l(v_h)\quad \forall\, v_h\in\mathcal{V}_{h,0}\,.\end{aligned} \tag{32}$$

Once u_h^{n+1} is computed, $\tilde{u}^{n+1}$ can be updated by

$$\tilde{u}^{n+1} = \tau_t\frac{1}{T\delta t}\tilde{u}^n-\tau_t\Pi^\perp\left(\boldsymbol{a}\cdot\nabla u_h^{n+1}\right)\,. \tag{33}$$

within each element K.

This completes the definition of the formulation proposed in this paper and analyzed in the following section. However, before doing this there is an important remark to be made. It is observed from (32) that $\tilde{u}^n$ is needed, that is, *subscales need to tracked in time*. From the computational point of view, a possible way to do this is to evaluate the subscales using (33) and to store them at each integration point of the numerical scheme adopted in the evaluation of the integrals. This allows us to compute $(\tilde{u}^n, \boldsymbol{a} \cdot \nabla v_h)_{\tau_t}$ numerically.

Finally, let us mention that there is the possibility of avoiding the tracking of the subscales by assuming that they are "quasi-static", that is, by taking $\delta_t \tilde{u}^n \approx 0$. In this case, the equation for $\tilde{u}^{n+1}$ becomes

$$\tilde{u}^{n+1} = \tau_t \frac{1}{T\delta t} \tilde{u}^{n+1} - \tau_t \Pi_{\tau_t}^{\perp} \mathcal{L} u_h^{n+1} ,$$

from where it is found that $\tilde{u}^{n+1}$ can be computed as for the stationary problem:

$$\tilde{u}^{n+1} = -\tau \Pi_{\tau}^{\perp} (\mathcal{L} u_h^{n+1})$$

and the resulting simplified stabilized finite element formulation, after making the approximations described before, is

$$\frac{1}{T} \left(\delta_t u_h^n, v_h\right) + b\left(u_h^{n+1}, v_h\right) + \left(\Pi^{\perp}\left(\boldsymbol{a} \cdot \nabla u_h^{n+1}\right), \boldsymbol{a} \cdot \nabla v_h\right)_{\tau} = l(v_h) \quad \forall\, v_h \in \mathcal{V}_{h,0} ,$$

which replaces (32).

4 Analysis of the fully discrete problem

In this section we prove that the finite element formulation given by (32) approximates the semidiscrete problem (6) with optimal orders of convergence.

To keep our arguments as close as possible to those of the standard finite element analysis of *stationary* problems, let us consider that the test functions are taken independently within each time interval and define the sequences:

$$U_h = \{u_h^n\} , \quad \tilde{U} = \{\tilde{u}^n\} , \quad V_h = \{v_h^n\} .$$

Problem (32), incorporating the initial condition prescribed weakly, can be written as: find a sequence of algorithmic solutions U_h such that

$$B_h(U_h, V_h) = L(V_h) \tag{34}$$

for all sequences V_h, where

$$B_h\left(U_h, V_h\right) := B\left(U_h, V_h\right) - \sum_{n=0}^{N-1} \delta t \left(\tilde{u}^{n+1}, \boldsymbol{a} \cdot \nabla v_h^{n+1}\right) ,$$

and B and L are defined in (9) and (10), respectively. The equation for the subscales is

$$\frac{1}{T\delta t}\left(\tilde{u}^{n+1} - \tilde{u}^n\right) + \frac{1}{\tau}\tilde{u}^{n+1} + \Pi^{\perp}\left(\boldsymbol{a} \cdot \nabla u_h^{n+1}\right) = 0 . \tag{35}$$

In principle, this equation holds within each element domain. However, to simplify the analysis we will consider that the parameter τ computed with (24) is constant. This happens for example if $\boldsymbol{a}$ is constant and $\mathscr{P}_h$ is quasi-uniform, case in which h in (24) can be taken as the diameter of the finite element partition rather than the element diameter (see [6] for the techniques employed when $\mathscr{P}_h$ is not quasi-uniform). Assuming this quasi-uniformity, the inverse estimates

$$\|\nabla v_h\| \leq \frac{C_{\text{inv}}}{h} \|v_h\| , \quad \|\Delta v_h\| \leq \frac{C_{\text{inv}}}{h} \|\nabla v_h\| \tag{36}$$

hold true. Likewise, we will assume that the components of the semidiscrete solution U belong to $H^{p+1}(\Omega)$, where p is the degree of the polynomial defining the finite element space $\mathcal{V}_h$, and that for any function $v \in H^{p+1}(\Omega)$ there exists a finite element interpolant $\hat{v}_h$ such that

$$\|v - \hat{v}_h\|_m \leq C_{\text{int}} h^{p+1-m} \|v\|_{p+1} , \tag{37}$$

for $m = 0, 1, \ldots, p+1$ $(p \geq 1)$.

The expression of τ that we will use is (24). The constants c_1 and c_2 need to be related to the constant C_{inv} appearing in (36). In particular, we will take them as

$$c_1 = \frac{C_{\text{inv}}^2}{\alpha^2} , \quad c_2 = \frac{C_{\text{inv}}}{\alpha} , \tag{38}$$

where the constant $\alpha > 0$ measures the size of τ. For this parameter we also need to assume that

$$\tau \leq CT\delta t , \tag{39}$$

which in particular implies that we can not let $\delta t \to 0$ without refining the finite element mesh.

Let Π_0 be the L^2 projection onto $\mathcal{V}_{h,0}$. It will be shown in the proof of the following theorem that the terms in (32) added with respect to the standard Galerkin method provide control over $\Pi^{\perp}\left(\boldsymbol{a} \cdot \nabla u_h^{n+1}\right)$, whereas control over $\Pi_0(\boldsymbol{a} \cdot \nabla u_h^{n+1})$ is already inherent to the Galerkin method. In order to achieve full control over $\boldsymbol{a} \cdot \nabla u_h^{n+1}$ we assume that the following stability condition holds:

$$\|\boldsymbol{a} \cdot \nabla v_h\| \leq C_{\text{stab}} \left\| \left(\Pi^{\perp} + \Pi_0\right) \boldsymbol{a} \cdot \nabla v_h \right\| \quad \forall\, v_h \in \mathcal{V}_h . \tag{40}$$

This condition can be proved to hold for the most common finite elements using the technique of [5].

The final ingredient we need is the norm in which stability and convergence will be proven, which is

$$\begin{aligned} |\!|\!| V_h |\!|\!|_h^2 &:= \frac{1}{T} \|V_h\|_{\ell^\infty(L^2)}^2 + \nu \|V_h\|_{\ell^2\left(H_0^1\right)}^2 + \tau \|\boldsymbol{a} \cdot \nabla V_h\|_{\ell^2(L^2)}^2 + \sigma \|V_h\|_{\ell^2(L^2)}^2 \\ &:= \frac{1}{T} \max_{n=0,\ldots,N-1} \left\|v_h^{n+1}\right\|^2 + \nu \sum_{n=0}^{N-1} \delta t \left\|\nabla v_h^{n+1}\right\|^2 + \tau \sum_{n=0}^{N-1} \delta t \left\|\boldsymbol{a} \cdot \nabla v_h^{n+1}\right\|^2 + \sigma \sum_{n=0}^{N-1} \delta t \left\|v_h^{n+1}\right\|^2 . \end{aligned}$$

Stability is stated through the classical inf-sup condition:

Theorem 2 (Stability)**.** *There exists a constant $\beta > 0$, independent of h, such that*

$$\inf_{U_h} \sup_{V_h} \frac{B_h(U_h, V_h)}{|\!|\!| U_h |\!|\!|_h \, |\!|\!| V_h |\!|\!|_h} \geq \beta . \tag{41}$$

Proof. Taking $V_h = U_h$ in (34) and using the skew-symmetry of the convective term it is found that:

$$B_h\left(U_h, U_h\right) = \frac{1}{2T}\left\|u_h^N\right\|^2 + \frac{1}{2T}\sum_{n=0}^{N-1}\left\|u_h^{n+1} - u_h^n\right\|^2 + \sum_{n=0}^{N-1}\delta t\nu\left\|\nabla u_h^{n+1}\right\|^2 + \sum_{n=0}^{N-1}\delta t\sigma\left\|u_h^{n+1}\right\|^2 - \sum_{n=0}^{N-1}\delta t\left(\tilde{u}^{n+1}, \boldsymbol{a}\cdot\nabla u_h^{n+1}\right). \tag{42}$$

Using the equation for the subscales and condition (39) it is found that

$$\begin{aligned}
&-\sum_{n=0}^{N-1}\delta t\left(\tilde{u}^{n+1}, \boldsymbol{a}\cdot\nabla u_h^{n+1}\right) \\
&= -\sum_{n=0}^{N-1}\delta t\left(\tilde{u}^{n+1}, \Pi^{\perp}\left(\boldsymbol{a}\cdot\nabla u_h^{n+1}\right)\right) \\
&= \frac{1}{2T}\left\|\tilde{u}^N\right\|^2 + \frac{1}{2T}\sum_{n=0}^{N-1}\left\|\tilde{u}^{n+1} - \tilde{u}^n\right\|^2 + \sum_{n=0}^{N-1}\frac{\delta t}{\tau}\left\|\tilde{u}^{n+1}\right\|^2 \\
&\geq C\sum_{n=0}^{N-1}\delta t\tau\left\|\Pi^{\perp}\left(\boldsymbol{a}\cdot\nabla u_h^{n+1}\right)\right\|^2.
\end{aligned} \tag{43}$$

This already shows that (34) is well posed and admits a unique solution U_h.

Taking now $V_h = \left\{\tau\Pi_0^{n+1}\right\} \equiv \left\{\tau\Pi_0\left(\boldsymbol{a}\cdot\nabla u_h^{n+1}\right)\right\}$ in (34) and using the inverse estimates (36) yields:

$$\begin{aligned}
B_h\left(U_h, \left\{\tau\Pi_0^{n+1}\right\}\right) \geq \sum_{n=0}^{N-1}\Big[&-\frac{\tau}{T}\left\|u_h^{n+1} - u_h^n\right\|\,\left\|\Pi_0^{n+1}\right\| \\
&- \delta t\nu\tau\frac{C_{\text{inv}}}{h}\left\|\nabla u_h^{n+1}\right\|\,\left\|\Pi_0^{n+1}\right\| \\
&+ \delta t\tau\left\|\Pi_0^{n+1}\right\|^2 \\
&- \delta t\sigma\tau\left\|u_h^{n+1}\right\|\,\left\|\Pi_0^{n+1}\right\| \\
&- \delta t\tau|\boldsymbol{a}|\frac{C_{\text{inv}}}{h}\left\|\tilde{u}^{n+1}\right\|\,\left\|\Pi_0^{n+1}\right\|\Big].
\end{aligned} \tag{44}$$

From the expression (24) for τ and (38), it is found that

$$\nu\tau\frac{C_{\text{inv}}}{h} \leq \nu^{1/2}\tau^{1/2}\frac{1}{\alpha}, \quad \tau|\boldsymbol{a}|\frac{C_{\text{inv}}}{h} \leq \frac{1}{\alpha}, \quad \tau\sigma \leq \tau^{1/2}\sigma^{1/2}.$$

Using this, Young's inequality and assumption (39) in (44) leads to

$$\begin{aligned}
B_h\left(U_h, \left\{\tau\Pi_0^{n+1}\right\}\right) \geq \frac{1}{4}\sum_{n=0}^{N-1}\delta t\tau\left\|\Pi_0^{n+1}\right\|^2 &- C\sum_{n=0}^{N-1}\Big[\frac{1}{T}\left\|u_h^{n+1} - u_h^n\right\|^2 \\
&+ \delta t\nu\left\|\nabla u_h^{n+1}\right\|^2 \\
&+ \delta t\sigma\left\|u_h^{n+1}\right\|^2 + \frac{\delta t}{\tau}\left\|\tilde{u}^{n+1}\right\|^2\Big].
\end{aligned}$$

From this, the stability condition (40), (42) and (43), it follows that, for c sufficiently small,

$$B_h\left(U_h, U_h + c\left\{\tau\Pi_0^{n+1}\right\}\right) \geq C|||U_h|||_h^2. \tag{45}$$

In fact, if the maximum of $\|u_h^n\|$ is achieved at $n = N_0$, the sequence $\{u_h^0, u_h^1, \ldots, u_h^{N_0}, 0, \ldots, 0\}$ has to be added to $U_h + c\left\{\tau\Pi_0^{n+1}\right\}$, but this does not alter (45).

It only remains to show that

$$|||\{\tau\Pi_0^{n+1}\}|||_h \leq C|||U_h|||_h$$

which follows easily from the expression for τ and using again the inverse estimates (36).

Once stability has been proven, let us proceed to prove convergence. The objective is to show that

$$\varepsilon(\delta t, h) := \tau^{-1/2}h^{p+1}\left(\sum_{n=0}^{N}\delta t\|u^n\|_{p+1}^2\right)^{1/2} \tag{46}$$

is the error function of the formulation. We will show first that this is also the error function of the consistency error (which now is not zero) and the interpolation error.

Lemma 1 (Bound for the consistency error)**.** *There is a constant C such that*

$$B_h(U - U_h, V_h) \leq C\varepsilon(\delta t, h)|||V_h|||_h \tag{47}$$

for all sequences V_h.

Proof. Since U satisfies $B(U, V_h) = L(V_h)$ and U_h is solution of (34), we have that

$$\begin{aligned}
B_h(U - U_h, V_h) &= -\sum_{n=0}^{N-1}\delta t\left(\tilde{u}^{n+1}, \boldsymbol{a}\cdot\nabla v_h^{n+1}\right) \\
&\leq \left(\sum_{n=0}^{N-1}\frac{\delta t}{\tau}\left\|\tilde{u}^{n+1}\right\|^2\right)^{1/2} \times \left(\sum_{n=0}^{N-1}\delta t\tau\left\|\boldsymbol{a}\cdot\nabla v_h^{n+1}\right\|^2\right)^{1/2} \\
&\leq \left(\sum_{n=0}^{N-1}\frac{\delta t}{\tau}\left\|\tilde{u}^{n+1}\right\|^2\right)^{1/2}|||V_h|||_h,
\end{aligned} \tag{48}$$

where now $\tilde{u}^{n+1}$ is the solution of

$$\frac{1}{T}\delta_t\tilde{u}^n + \frac{1}{\tau}\tilde{u}^{n+1} = -\Pi^{\perp}\left(\boldsymbol{a}\cdot\nabla u^{n+1}\right).$$

Weighting this equation by $\tilde{u}^{n+1}$, adding up for n and using Young's inequality, it is found that

$$\begin{aligned}
&\frac{1}{2T}\left\|\tilde{u}^N\right\| + \frac{1}{2T}\sum_{n=0}^{N-1}\left\|\tilde{u}^{n+1} - \tilde{u}^n\right\|^2 + \sum_{n=0}^{N-1}\frac{\delta t}{\tau}\left\|\tilde{u}^{n+1}\right\|^2 \\
&\leq \sum_{n=0}^{N-1}\delta t\left(\frac{\tau}{2}\left\|\Pi^{\perp}\left(\boldsymbol{a}\cdot\nabla u^{n+1}\right)\right\|^2 + \frac{1}{2\tau}\left\|\tilde{u}^{n+1}\right\|^2\right).
\end{aligned}$$

Using this, the best approximation property of the L^2 projection with respect to its associated norm and the interpolation estimates (37), we have that

$$\sum_{n=0}^{N-1}\frac{\delta t}{\tau}\left\|\tilde{u}^{n+1}\right\|^2 \le C\sum_{n=0}^{N-1}\delta t\tau\left\|\boldsymbol{a}\cdot\nabla u^{n+1}-\Pi\left(\boldsymbol{a}\cdot\nabla u^{n+1}\right)\right\|^2 \le C\sum_{n=0}^{N-1}\delta t\tau|\boldsymbol{a}|^2h^{2p}\left\|u^{n+1}\right\|_{p+1}^2 .$$

The last term can be bounded by $C\varepsilon^2(\delta t, h)$. This, together with (48), completes the proof of Lemma 1.

Lemma 2 (Estimates for the interpolation error)**.** *Let U be the solution of* (6) *and $\hat{U}_h=\{\hat{u}_h^0,\hat{u}_h^1,\ldots,\hat{u}_h^N\}$ a finite element interpolant of U, each component satisfying* (37)*. Then, there is a constant C such that*

$$B_h(U-\hat{U}_h, V_h)\le C\varepsilon(\delta t,h)|||V_h|||_h , \tag{49}$$

$$|||U-\hat{U}_h|||_h\le C\varepsilon(\delta t,h) . \tag{50}$$

Proof. From the definition of B_h we have that

$$\begin{aligned} B_h(U-\hat{U}_h, V_h) =\sum_{n=0}^{N-1}\Bigg[&\frac{1}{T}\left(\left(u^{n+1}-\hat{u}_h^{n+1}\right)-\left(u^n-\hat{u}_h^n\right), v_h^{n+1}\right)\\ &+\nu\delta t\left(\nabla\left(u^{n+1}-\hat{u}_h^{n+1}\right),\nabla v_h^{n+1}\right)\\ &+\delta t\left(\boldsymbol{a}\cdot\nabla\left(u^{n+1}-\hat{u}_h^{n+1}\right), v_h^{n+1}\right)\\ &+\sigma\delta t\left(u^{n+1}-\hat{u}_h^{n+1}, v_h^{n+1}\right)\\ &-\delta t\left(\tilde{u}^{n+1},\boldsymbol{a}\cdot\nabla v_h^{n+1}\right)\Bigg], \end{aligned} \tag{51}$$

where now $\tilde{u}^{n+1}$ is the solution of

$$\frac{1}{T}\,\delta_t\tilde{u}^n+\frac{1}{\tau}\tilde{u}^{n+1}=-\Pi^{\perp}\left(\boldsymbol{a}\cdot\nabla(u^{n+1}-\hat{u}_h^{n+1})\right) . \tag{52}$$

To prove (49) we have to check that each of the terms in (51) can be bounded by $C\varepsilon(\delta t,h)|||V_h|||_h$. Let us prove this for the first and last terms, since the others can be bounded using similar arguments.

Using (39) we have that $1/T\le C\delta t/\tau$. From this and the interpolation estimate (37) we get

$$\begin{aligned} &\frac{1}{T}\sum_{n=0}^{N-1}\left(\left(u^{n+1}-\hat{u}_h^{n+1}\right)-\left(u^n-\hat{u}_h^n\right), v_h^{n+1}\right)\\ &\le C\frac{1}{T^{1/2}}\|V_h\|_{\ell^\infty(L^2)}\frac{1}{T^{1/2}}\sum_{n=0}^{N}h^{p+1}\|u^n\|_{p+1}\\ &\le C|||V_h|||_h\sum_{n=0}^{N}\delta t^{1/2}\tau^{-1/2}h^{p+1}\|u^n\|_{p+1} . \end{aligned}$$

To bound the last term in (51) observe first that

$$\begin{aligned} &-\sum_{n=0}^{N-1}\delta t\left(\tilde{u}^{n+1},\boldsymbol{a}\cdot\nabla v_h^{n+1}\right)\\ &\le\sum_{n=0}^{N-1}\delta t^{1/2}\tau^{-1/2}\left\|\tilde{u}^{n+1}\right\|\delta t^{1/2}\tau^{1/2}\left\|\boldsymbol{a}\cdot\nabla v_h^{n+1}\right\|\\ &\le|||V_h|||_h\left(\sum_{n=0}^{N-1}\frac{\delta t}{\tau}\left\|\tilde{u}^{n+1}\right\|^2\right)^{1/2} . \end{aligned} \tag{53}$$

Similarly to the proof of the previous lemma, weighting (52) by $\tilde{u}^{n+1}$, adding up for n and using Young's inequality we obtain

$$\begin{aligned} &\sum_{n=0}^{N-1}\frac{\delta t}{\tau}\left\|\tilde{u}^{n+1}\right\|^2\\ &\le\sum_{n=0}^{N-1}\left(\frac{\delta t}{2\tau}\left\|\tilde{u}^{n+1}\right\|^2+C\delta t\tau\left\|\Pi^{\perp}\left(\boldsymbol{a}\cdot\nabla\left(u^{n+1}-\hat{u}_h^{n+1}\right)\right)\right\|^2\right) . \end{aligned}$$

Since the norm of $\Pi^{\perp}$ is ≤ 1, it follows that

$$\sum_{n=0}^{N-1}\frac{\delta t}{\tau}\left\|\tilde{u}^{n+1}\right\|^2\le C\sum_{n=0}^{N-1}\delta t\tau\left\|\boldsymbol{a}\cdot\nabla\left(u^{n+1}-\hat{u}_h^{n+1}\right)\right\|^2 \le C\sum_{n=0}^{N-1}\delta t\tau|\boldsymbol{a}|^2h^{2p}\left\|u^{n+1}\right\|_{p+1}^2 ,$$

which is bounded by $C\varepsilon^2(\delta t,h)$. Using this in (53) shows that the last term in (51) is bounded by $C\varepsilon(\delta t,h)|||V_h|||_h$. The rest of the terms in (51) can be bounded in a similar manner. Also similar arguments can be used to prove (50). We omit the details.

Theorem 3 (Convergence)**.** *There is a constant C such that*

$$|||U-U_h|||_h\le C\varepsilon(\delta t,h) .$$

Proof. The proof is standard: from Theorem 2 and using Lemma 1 and 2 (estimate (49)), there exists a sequence V_h such that

$$\begin{aligned} \beta|||\hat{U}_h-U_h|||_h|||V_h|||_h&\le B_h(\hat{U}_h-U,V_h)+B_h(U-U_h,V_h)\\ &\le C\varepsilon(\delta t,h)|||V_h|||_h , \end{aligned}$$

and therefore $|||\hat{U}_h-U_h|||_h\le C\varepsilon(\delta t,h)$. The result follows now from Lemma 2 (estimate (50)) and the triangle inequality.

This convergence estimate is optimal. For the stationary case, it reduces to the same error estimate as for the Galerkin/least-square method [8], and is similar to what is found in [12] for the transient problem.

Appendix: Sketch of the proof of Theorem 1

Even though it is not the classical approach, Theorem 1 can be proved using exactly the same steps as in the analysis of the fully discrete problem. Let us sketch how this can be done.

The first step is to give a stability estimate in the form (41), using now the bilinear form (9) that defines the semidiscrete problem (8) and the norm (7).

Taking $V = U$ in the definition (9) of B, we obtain

$$B(U, U) = \frac{1}{2T}\left\|u^N\right\|^2 + \frac{1}{2T}\sum_{n=0}^{N-1}\left\|u^{n+1} - u^n\right\|^2 + \sum_{n=0}^{N-1}\delta t \nu \left\|\nabla u^{n+1}\right\|^2 + \sum_{n=0}^{N-1}\delta t \sigma \left\|u^{n+1}\right\|^2 . \tag{A.1}$$

This already proves that problem (8) is well posed.

In the proof of Theorem 2 we got control on the convective term in two steps. First, we saw that the subscales provide control on the component orthogonal to $\mathcal{V}_h$ (see (43)), and then we showed that the projection of the convective term onto $\mathcal{V}_h$ can be controlled by taking the test function equal to this projection multiplied by τ. For the semidiscrete problem we can obtain directly control on the viscous plus convective term by using the density of $\ell^2(H_0^1(\Omega))$ in $\ell^2(L^2(\Omega))$ (which is a consequence of the density of $H_0^1(\Omega)$ in $L^2(\Omega)$). This guarantees that for all $\epsilon > 0$ there exists a sequence $V_c = \{v_c^0, v_c^1, \ldots, v_c^N\}$ such that $v_c^n \in H_0^1(\Omega)$ for all n and

$$\sum_{n=0}^{N-1}\delta t^2 \|s^n - v_c^n\|^2 \leq \epsilon \sum_{n=0}^{N-1}\delta t^2 \|s^n\|^2 . \tag{A.2}$$

where $s^n \equiv -\nu \Delta u^n + \boldsymbol{a} \cdot \nabla u^n$. It is easy to prove that the term in the right-hand-side is bounded. From (A.2) one can easily show that

$$B(U, \delta t V_c) \geq -\sum_{n=0}^{N-1}\frac{\delta t}{T}\left\|u^{n+1} - u^n\right\|\left\|v_c^{n+1}\right\| + \delta t \sum_{n=0}^{N-1}\delta t \left\|s^{n+1}\right\|^2 (1 - \epsilon) - \sigma \sum_{n=0}^{N-1}\delta t^2 \left\|u^{n+1}\right\|\left\|v_c^{n+1}\right\| .$$

This inequality, together with (A.1), allows us to obtain a stability estimate of the form $B(U, U + c\delta t V_c(U)) \geq C |\!|\!| U |\!|\!|^2$ for all U and for c small enough. The next step is to obtain a consistency estimate. If U_{ex} is the exact solution in Theorem 1 and U the semidiscrete solution, we have that

$$B\left(U - U_{\text{ex}}, V\right) = \frac{1}{T}\sum_{n=0}^{N-1}\left(e_c^{n+1}, v^{n+1}\right) ,$$

where $e_c^{n+1} := u\left(t^{n+1}\right) - u(t^n) - \delta t\, \partial_t u|_{t^{n+1}}$ is the consistency error at time $t^{n+1} = (n+1)\delta t$. It can be expressed in terms of the Taylor residual, whose $\ell^2(L^2(\Omega))$-norm can be shown to be bounded by $C\delta t$ provided $\partial_{tt}^2 u \in L^2(0, T; L^2(\Omega))$, as stated in Theorem 1.

Once stability and consistency has been proven, a result similar to Theorem 3 (which in the present setting can be considered Lax's Theorem) yields convergence.

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