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Analysis of multipatch discontinuous Galerkin IgA approximations to elliptic boundary value problems

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Abstract In this work, we study the approximation properties of multipatch dG-IgA methods, that apply the multipatch Isogeometric Analysis discretization concept and the discontinuous Galerkin technique on the interfaces between the patches, for solving linear diffusion problems with diffusion coefficients that may be discontinuous across the patch interfaces. The computational domain is divided into non-overlapping subdomains, called patches in IgA, where B-splines, or NURBS approximations spaces are constructed. The solution of the problem is approximated in every subdomain without imposing any matching grid conditions and without any continuity requirements for the discrete solution across the interfaces. Numerical fluxes with interior penalty jump terms are applied in order to treat the discontinuities of the discrete solution on the interfaces. We provide a rigorous a priori discretization error analysis for diffusion problems in two- and three-dimensional domains, where solutions patchwise belong to $W^{l,p}$, with some $l \geq 2$ and $p \in (2d/(d+2(l-1)), 2]$. In any case, we show optimal convergence rates of the discretization with respect to the dG - norm.

Keywords Linear elliptic problems · Discontinuous coefficients · Discontinuous Galerkin discretization · Isogeometric analysis · Non-matching meshes · Low regularity solutions · A priori discretization error estimates

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1 Introduction

The finite element methods (FEM) and, in particular, discontinuous Galerkin (dG) finite element methods are very often used for solving elliptic boundary value problems which arise from engineering applications, see, e.g., [17] and [25]. For realistic problems in complicated geometries, the quality of the numerical results depends usually on the quality of the discretized geometry (triangulation of the domain), which is usually performed by a mesh generator. This is the case even if curved elements are used, see, e.g., [8,22,36] and [17]. In many practical situations, extremely fine meshes are required around fine-scale geometrical objects, singular corner points etc. in order to achieve numerical solutions with desired resolution. This fact leads to an increased number of degrees of freedom, and thus to an increased overall computational cost for solving the discrete problem, see, e.g., [32] for fluid dynamics applications.

Recently, the Isogeometric Analysis (IgA) concept has been applied for approximating solutions of elliptic problems [5,18]. IgA generalizes and improves the classical FE (even isoparametric FE) methodology in the following direction: complex technical computational domains can be exactly represented as images of some parameter domain, where the mappings are constructed by using superior classes of basis functions like B-spline, or Non-Uniform Rational B-spine(NURBS), see, e.g., [9] and [28]. The same class of functions is used to approximate the exact solution without increasing the computational cost for the computation of the resulting stiffness matrices [9], systematic *hpk* refinement procedures can easily be developed [33], and, last but not least, the method can be materialized in parallel environment incorporating fast domain decomposition solvers [20], [34], [2].

During the last two decades, there has been an increasing interest in discontinuous Galerkin (dG) finite element methods for the numerical solution of several types of partial differential equations, which is attributed to the advantages of the local approximation spaces without continuity requirements that dG methods offer, see, e.g., [3,10,27,29] and [31].

In this paper, we combine the best features of the two aforementioned methods, and develop a discretization method that we call multipatch discontinuous Galerkin Isogeometric Analysis (dG-IgA). We apply and analyze the proposed dG-IgA method to elliptic boundary value problems with discontinuous coefficients. It well known that the solutions of this type of problems are in general not smooth enough, see, e.g. [19,21], and the numerical method cannot produce an (optimal) accurate solution. The problem is set in a complex, bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, which is subdivided in a union of non-overlapping subdomains, say $\mathcal{S}(\Omega) := \{\Omega_i\}_{i=1}^N$, where we further assume that the discontinuity of the diffusion coefficients is only observed across the subdomain boundaries (interfaces). The weak solution of the problem is approximated in every subdomain applying IgA methodology, [5], without imposing continuity requirements for the approximation spaces across the interfaces. Having in our mind more general problems, where the use of independent subdomain meshes is more preferable, see, for example, [4] for the use of dG techniques in the case of rotating subdomains, we develop our numerical analysis for non-matching meshes across the interfaces. By construction, dG methods use discontinuous approximation spaces utilizing numerical fluxes on the interfaces, [3], and have been efficiently used for solving problems on non-matching grids in the past, [10,11,14]. Here, emulating the dG finite element methods, the numerical scheme is formulated by applying numerical fluxes with interior penalty coefficients on the interfaces of the subdomains (patches), and using IgA formulations in every patch independently. A crucial point in the presented work, is the expression of the numerical flux interface terms as a sum over the micro-elements edges taking note of the non-matching subdomain meshes. This gives the opportunity to proceed in the error analysis by applying the trace inequalities locally as in dG finite element methods. There are many papers, which present dG finite element approximations for elliptic problems, see, e.g., [3], the monographs [27,29], and, in particular, for the discontinuous coefficient case, [10,26]. However, there are only a few publications on the dG-IgA and their analysis. In [7], the author presented discretization error estimates for the dG-IgA of plane (2d) diffusion problems on meshes matching across the patch boundaries and under the assumption of sufficiently smooth solutions. This analysis obviously carries over to plane linear elasticity problems which have recently been studied numerically in [2]. In [12], the dG technology has been used to handle no-slip boundary conditions and multipatch geometries for IgA of Darcy-Stokes-Brinkman equations. DG-IgA discretizations of heterogeneous diffusion problems on open and closed surfaces, which are given by a multipatch NURBS representation, are constructed and rigorously analyzed in [24].

In the first part of this paper, we give a priori error estimates in the dG-norm $\|.\|_{dG}$ under the usual regularity assumption imposed on the exact solution, i.e. $u \in W^{1,2}(\Omega) \cap W^{l\geq 2,2}(\mathcal{S}(\Omega))$. Next, we consider the model problem with low regularity solution $u \in W^{1,2}(\Omega) \cap W^{l,p}(\mathcal{S}(\Omega))$, with $l \geq 2$ and $p \in (\frac{2d}{d+2(l-1)}, 2)$, and derive error estimates in the dG-norm $\|.\|_{dG}$. These estimates are optimal with respect to the space size discretization. We note that the error analysis in the case of low regularity solutions includes many ingredients of the dG FE error analysis presented in [35] and [26]. To the best of our knowledge, optimal error analysis for IgA discretizations combined with dG techniques for solving elliptic problems with discontinuous coefficients in general domains $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, have not been yet presented in the literature.

The paper is organized as follows. In Section 2, our model diffusion problem is described. Section 3 introduces some notations. The local approximation spaces $\mathbb{B}_h(\mathcal{S}(\Omega))$ and the numerical scheme are also presented in this section. Several auxiliary results and the analysis of the method for the case of sufficiently regular solutions are provided in Section 4. Section 5 is devoted to the analysis of the method for low regularity solutions. Section 6 includes several numerical examples that verify the theoretical convergence rates. Finally, we draw some conclusions.

2 The model problem

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d = 2, 3$, with the boundary $\partial\Omega$. For simplicity, we restrict our study to the model diffusion problem

$$-\mathrm{div}(\alpha\nabla u) = f \text{ in } \Omega, \text{ and } u = u_D \text{ on } \partial\Omega, \tag{2.1}$$

where f and u_D are given smooth data. In (2.1), α is the diffusion coefficient that is assumed to be bounded by strictly positive constants from above and below. Moreover, for the sake of simplicity, we will later assume that α is patchwise constant.

The weak formulation is to find a function $u \in W^{1,2}(\Omega)$ such that $u := u_D$ on $\partial\Omega$ and satisfies

$$a(u, \phi) = l(\phi), \ \forall \phi \in W_0^{1,2}(\Omega), \tag{2.2a}$$

where

$$a(u,\phi)=\int_{\Omega}\alpha\nabla u\nabla\phi\,dx,\quad \text{and}\quad l(\phi)=\int_{\Omega}f\phi\,dx. \tag{2.2b}$$

Results concerning the existence and uniqueness of the solution u of problem (2.2) can be derived by a simple application of Lax-Milgram Lemma, [13]. To avoid unnecessary long formulas below, we only considered in (2.1) non-homogeneous Dirichlet boundary conditions on $\partial\Omega$. However, the analysis can be easily generalized to Neumann and Robin type boundary conditions on a part of $\partial\Omega$, since they are naturally introduced in the dG formulation.

3 Preliminaries - dG notation

Throughout this work, we denote by $L^p(\Omega)$, $p>1$ *the Lebesgue spaces* for which $\int_\Omega |u(x)|^p\,dx<\infty$, endowed with the norm $\|u\|_{L^p(\Omega)}=\big(\int_\Omega |u(x)|^p\,dx\big)^{\frac{1}{p}}$. By $\mathcal{D}(\Omega)$, we define the space of C^∞ functions with compact support in Ω, and by $C^k(\Omega)$ the set of functions with $k-th$ order continues derivatives. In dealing with differential operators in Sobolev spaces, we use the following common conventions. For any (multi-index) $\alpha=(\alpha_1,\ldots,\alpha_d)$, $\alpha_j\geq 0$, $j=1,\ldots,d$, with degree $|\alpha|=\sum_{j=1}^d\alpha_j$, we define the *differential operator*

$$D^a=D_1^{\alpha_1}\cdots D_d^{\alpha_d},\ \text{with}\ D_j=\frac{\partial}{\partial x_j},\ D^{(0,\ldots,0)}u=u. \tag{3.1}$$

We also denote by $W^{l,p}(\Omega)$, l positive integer and $1\leq p\leq\infty$, the Sobolev space functions endowed with the norm

$$\|u\|_{W^{l,p}(\Omega)}=\Big(\sum_{0\leq|\alpha|\leq m}\|D^\alpha u\|^p_{L^p(\Omega)}\Big)^{\frac{1}{p}},\ \|u\|_{W^{l,\infty}(\Omega)}=\max_{0\leq|\alpha|\leq m}\|D^\alpha u\|_\infty. \tag{3.2a}$$

For more details for the above definitions, we refer [1]. In the sequel we write $a\sim b$ if $ca\leq b\leq Ca$, where c,C are positive constants independent of the mesh size.

In order to apply the IgA methodology for the problem (2.1), the domain Ω is subdivided into a union of subdomains $\mathcal{S}(\Omega):=\{\Omega_i\}_{i=1}^N$, such that

$$\bar{\Omega}=\bigcup_{i=1}^N\bar{\Omega}_i,\quad \text{with}\quad \Omega_i\cap\Omega_j=\emptyset,\ \text{if}\ j\neq i. \tag{3.3}$$

Throughout the paper we assume that the coefficient α in (2.1) is equal to some given positive constant $\alpha^{(i)}$ in each subdomain Ω_i for $i=1,\ldots,N$.

As it is common in the IgA analysis, we assume a parametric domain $\widehat{D}$ of unit length, (e.g. $\widehat{D}=[0,1]^d$). For any Ω_i, we associate $n=1,\ldots,d$ knot vectors $\Xi_n^{(i)}$ on $\widehat{D}$, which create a mesh $T^{(i)}_{h_i,\widehat{D}}=\{\hat{E}_m\}_{m=1}^{M_i}$, where $\hat{E}_m$ are the micro-elements, see details in [9]. We shall refer $T^{(i)}_{h_i,\widehat{D}}$ as the *parametric mesh of* Ω_i. For every $\hat{E}_m\in T^{(i)}_{h_i,\widehat{D}}$ we denote by $h_{\hat{E}_m}$ *its diameter* and by $h_i=\max\{h_{\hat{E}_m}\}$ *the meshsize* of $T^{(i)}_{h_i,\widehat{D}}$. We assume the following quasi-uniformity properties for every $T^{(i)}_{h_i,\widehat{D}}$: (i) for every $\hat{E}_m\in T^{(i)}_{h_i,\widehat{D}}$ holds $h_i\sim h_{\hat{E}_m}$, (ii) for the micro-element edges $e_{\hat{E}_m}\subset\partial\hat{E}_m$ holds $h_{\hat{E}_m}\sim e_{\hat{E}_m}$.

On every $T^{(i)}_{h_i,\widehat{D}}$, we construct the finite dimensional space $\hat{\mathbb{B}}^{(i)}_{h_i}$ spanned by B-spline basis functions of degree k, [9,30],

$$\hat{\mathbb{B}}^{(i)}_{h_i}=span\{\hat{B}^{(i)}_j(\hat{x})\}_{j=0}^{dim(\hat{\mathbb{B}}^{(i)}_{h_i})}, \tag{3.4a}$$

where every $\hat{B}^{(i)}_j(\hat{x})$ base function in (3.4a) is derived by means of tensor products of one-dimensional $\mathbb{B}$-spline basis functions, e.g.

$$\hat{B}^{(i)}_j(\hat{x})=\hat{B}^{(i)}_{j_1}(\hat{x}_1)\cdots\hat{B}^{(i)}_{j_d}(\hat{x}_d). \tag{3.4b}$$

For simplicity, we assume that the basis functions of every $\hat{\mathbb{B}}^{(i)}_{h_i}$, $i=1,\ldots,N$ are of the same degree k. We denote by $\tilde{D}^{(i)}_{\hat{E}}$ the support extension of $\hat{E}\in T^{(i)}_{h_i,\widehat{D}}$.

Every subdomain $\Omega_i\in\mathcal{S}(\Omega)$, $i=1,\ldots,N$, is exactly represented through a parametrization (one-to-one mapping), [9], having the form

$$\mathbf{\Phi}_i:\widehat{D}\rightarrow\Omega_i,\quad \mathbf{\Phi}_i(\hat{x})=\sum_j C^{(i)}_j\hat{B}^{(i)}_j(\hat{x}):=x\in\Omega_i, \tag{3.5a}$$

$$\text{with}\quad \hat{x}=\mathbf{\Psi}_i(x):=\mathbf{\Phi}_i^{-1}(x), \tag{3.5b}$$

where $C^{(i)}_j$ are the control points. For the purposes of this work, we assume that the components $\mathbf{\Psi}_i=(\Psi_{i,1},\ldots,\Psi_{i,d})$ and $\mathbf{\Phi}_i=(\Phi_{i,1},\ldots,\Phi_{i,d})$ are highly smooth functions.

Using $\mathbf{\Phi}_i$, we construct a mesh $T^{(i)}_{h_i,\Omega_i}=\{E_m\}_{m=1}^{M_i}$ for every Ω_i, whose vertices are the images of the vertices of the corresponding mesh $T^{(i)}_{h_i,\widehat{D}}$ through $\mathbf{\Phi}_i$. If $h_{\Omega_i}=\max\{h_{E_m}\}$, $E_m\in T^{(i)}_{h_i,\Omega_i}$ is the subdomain Ω_i mesh size, then based on Definition (3.5) of $\mathbf{\Phi}_i$, there is a constant $C:=C(\|\mathbf{\Phi}_i\|_\infty)$ such that $h_i\sim Ch_{\Omega_i}$. In what follows, we denote the subdomain mesh size by h_i without the constant $C:=C(\|\mathbf{\Phi}_i\|_\infty)$ explicitly appearing.

The mesh of Ω is considered to be $T_h(\Omega)=\bigcup_{i=1}^N T^{(i)}_{h_i,\Omega_i}$, where we note that there are no matching mesh requirements on the interior interfaces $\partial\Omega_i\bigcap\partial\Omega_j$, $i\neq j$.

For the sake of brevity in our notations, the interior faces of Ω_i which are common to the interior faces of Ω_j are denoted by F_{ij}, i.e., $F_{ij} = \partial\Omega_i \cap \partial\Omega_j$, $i \neq j$. We denote the collection of the faces that belong to $\partial\Omega_i \cap \partial\Omega$ by $\mathcal{F}_{i,B}$.

Lastly, we define on Ω the finite dimensional $\mathbb{B}-$spline space

$\mathbb{B}_h(\mathcal{S}(\Omega)) = \mathbb{B}^{(1)}_{h_1} \times \ldots \times \mathbb{B}^{(N)}_{h_N}$, where every $\mathbb{B}^{(i)}_{h_i}$ is defined on $T^{(i)}_{h_i,\Omega_i}$ as follows

$$\mathbb{B}^{(i)}_{h_i} := \{B^{(i)}_{h_i}|_{\Omega_i} : B^{(i)}_h(x) = \hat{B}^{(i)}_h \circ \boldsymbol{\Psi}_i(x),\ \forall \hat{B}^{(i)}_h \in \hat{\mathbb{B}}^{(i)}_{h_i}\}. \tag{3.6}$$

We define the union support in physical subdomain Ω_i as $D^{(i)}_E := \boldsymbol{\Phi}(\tilde{D}^{(i)}_{\hat{E}})$.

Assumption 1 We suppose that there exist constants $0 < c_m < c_M$ such that for all $\hat{x} \in \hat{D}$

$$c_m \leq |det(\boldsymbol{\Phi}^{'}_i(\hat{x}))| \leq c_M,\ i = 1, \ldots, N, \tag{3.7}$$

where $\boldsymbol{\Phi}^{'}_i(\hat{x})$ denotes the Jacobian matrix $\frac{\partial(x_1,\ldots,x_d)}{\partial(\hat{x}_1,\ldots,\hat{x}_d)}$.

Now, for any $\hat{u} \in W^{m,p}(\hat{D})$, $m \geq 0$, $p > 1$, we define the function

$$\mathcal{U}(x) = \hat{u}(\boldsymbol{\Psi}_i(x)),\ x \in \Omega_i, \tag{3.8}$$

and for the error analysis below, we need to show the relation

$$C_m \|\hat{u}\|_{W^{m,p}(\hat{D})} \leq \|\mathcal{U}\|_{W^{m,p}(\Omega_i)} \leq C_M \|\hat{u}\|_{W^{m,p}(\hat{D})}, \tag{3.9}$$

where C_m, C_M depending on $C_m := C_m (\max_{m_0\leq m}(\|D^{m_0}\boldsymbol{\Phi}_i\|_\infty), \|det(\boldsymbol{\Psi}^{'}_i)\|_\infty)$ and $C_M := C_M (\max_{m_0\leq m}(\|D^{m_0}\boldsymbol{\Psi}_i\|_\infty), \|det(\boldsymbol{\Phi}^{'}_i)\|_\infty)$, correspondingly.

Indeed, for any $\hat{u} \in W^{m,p}(\hat{D})$ we can find a sequence $\{\hat{u}_j\} \subset C^\infty(\bar{\hat{D}})$ converging to $\hat{u}$ in $\|.\|_{W^{m,p}(\hat{D})}$, by the chain rule in (3.8) we obtain

$$D_x(\boldsymbol{\Psi}_i(x))^{-1} D\mathcal{U}_j(x) = D\hat{u}_j(\boldsymbol{\Psi}_i(x)). \tag{3.10}$$

Then for any multi-index m we can get the following formula

$$D^m\mathcal{U}_j(x) = \sum_{m_0\leq m} P_{m,m_0}(x) D^{m_0}\mathcal{U}_j(x), \tag{3.11}$$

where $P_{m,m_0}(x)$ is a polynomial of degree less than k and includes the various derivatives of $\boldsymbol{\Psi}_i(x)$. Multiplying (3.11) by $\varphi(x) \in \mathcal{D}(\Omega)$, and integrating by parts we have

$$(-1)^{|m|} \int_{\Omega_i} \mathcal{U}_j(x) D^m\varphi(x)\, dx = \sum_{m_0\leq m} \int_{\Omega_i} P_{m,m_0}(x) D^{m_0}\mathcal{U}_j(x)\varphi(x)\, dx. \tag{3.12}$$

We transfer the integral in (3.12) to integrals over $\hat{D}$ and use the change of variable $x = \boldsymbol{\Phi}_i(\hat{x})$ to obtain

$$\begin{aligned}(-1)^{|m|} &\int_{\hat{D}} \hat{u}_j(\hat{x}) D^m\varphi(\boldsymbol{\Phi}_i(\hat{x}))|det(\boldsymbol{\Phi}^{'}_i(\hat{x}))|\, d\hat{x} \\ &= \sum_{m_0\leq m} \int_{\hat{D}} P_{m,m_0}(\boldsymbol{\Phi}_i(\hat{x})) D^{m_0}\hat{u}_j(\hat{x}) \\ &\quad \varphi(\boldsymbol{\Phi}_i(\hat{x}))|det(\boldsymbol{\Phi}^{'}_i(\hat{x}))|\, d\hat{x}.\end{aligned} \tag{3.13}$$

But it holds that $D^{m_0}\hat{u}_j \rightarrow D^{m_0}\hat{u}$ in $\|.\|_{L^p(\hat{D})}$, thus taking the limit $j \rightarrow \infty$ in (3.13) and transferring the integrals back to Ω_i, we can derive (3.12) with respect to $\mathcal{U}$. We conclude that (3.11) holds in the distributional sense, and therefore

$$\begin{aligned}\int_{\Omega_i} |D^m\mathcal{U}(x)|^p\, dx &\leq C_p \int_{\Omega_i} \sum_{m_0\leq m} \left|P_{m,m_0}(x) D^{m_0}\mathcal{U}(x)\right|^p dx \\ &\leq C_p \max_{m_0\leq m}\left(\max_{x\in\Omega_i}(P_{m,m_0}(x))\right) \sum_{m_0\leq m} \int_{\Omega_i} \left|D^{m_0}\mathcal{U}(x)\right|^p dx \\ &\leq C_p \max_{m_0\leq m}\left(\max_{x\in\Omega_i}(P_{m,m_0}(x))\right) \max_{\hat{x}\in\hat{D}}(|det(\boldsymbol{\Phi}^{'}_i(\hat{x}))| \\ &\quad \sum_{m_0\leq m} \int_{\hat{D}} \left|D^{m_0}\hat{u}(\hat{x})\right|^p d\hat{x} \\ &\leq C(\max_{m_0\leq m}(\|D^{m_0}\boldsymbol{\Psi}_i(x)\|_\infty, \|det(\boldsymbol{\Phi}^{'}_i(\hat{x}))\|_\infty) \\ &\quad \sum_{m_0\leq m} \left|D^{m_0}\hat{u}(\hat{x})\right|^p_{W^{m_0,p}(\hat{D})}.\end{aligned} \tag{3.14}$$

This proves the "right inequality" in (3.9). The "left inequality" in (3.9) can be shown following the same steps as above using the change of variable $\hat{x} = \boldsymbol{\Psi}_i(x)$.

3.1 The numerical scheme

We use the $\mathbb{B}-$spline spaces defined in (3.6) for approximating the solution of (2.2) in every subdomain Ω_i. Continuity requirements for $\mathbb{B}_h(\mathcal{S}(\Omega))$ are not imposed on the interfaces F_{ij} of the subdomains, and hence, the problem (2.2) is discretized by discontinuous Galerkin techniques on F_{ij}, [10]. Using the notation $\phi^{(i)}_h := \phi_h|_{\Omega_i}$, we define the average and the jump of ϕ_h

$$\{\phi_h\} := \frac{1}{2}(\phi^{(i)}_h + \phi^{(j)}_h), \quad [\![\phi_h]\!] := \phi^{(i)}_h - \phi^{(j)}_h, \quad \text{on} \quad F_{ij} = \partial\Omega_i \cap \partial\Omega_j, \tag{3.15a}$$

$$\{\phi_h\} := \phi^{(i)}_h, \quad [\![\phi_h]\!] := \phi^{(i)}_h, \quad \text{on} \quad F_{i\partial} = \partial\Omega_i \cap \partial\Omega. \tag{3.15b}$$

The dG-IgA method reads as follows: find $u_h \in \mathbb{B}_h(\mathcal{S}(\Omega))$ such that

$$a_h(u_h, \phi_h) = l(\phi_h) + p_D(u_D, \phi_h),\ \forall \phi_h \in \mathbb{B}_h(\mathcal{S}(\Omega)), \tag{3.16a}$$

where

$$\begin{aligned} a_h(u_h, \phi_h) = \sum_{i=1}^{N} a_i(u_h, \phi_h) - \sum_{i=1}^{N} \sum_{F_{ij} \subset \partial\Omega_i} \frac{1}{2} s_{F_{ij}}(u_h, \phi_h) \\ - p_{F_{ij}}(u_h, \phi_h) \\ - \sum_{i=1}^{N} \sum_{F_{i\partial} \in \mathcal{F}_{i,B}} s_{F_{i\partial}}(u_h, \phi_h) \\ - p_{F_{i\partial}}(u_h, \phi_h), \end{aligned} \tag{3.16b}$$

and the bilinear forms for the interior F_{ij} and the boundary faces $F_{i\partial}$ defined as

$$a_i(u_h, \phi_h) = \int_{\Omega_i} \alpha \nabla u_h \nabla \phi_h \, dx, \tag{3.16c}$$

$$s_{F_{ij}}(u_h, \phi_h) = \int_{F_{ij}} \{\alpha \nabla u_h\} \cdot \mathbf{n}_{F_{ij}} [\![\phi_h]\!] \, ds, \tag{3.16d}$$

$$p_{F_{ij}}(u_h, \phi_h) = \int_{F_{ij}} \left(\frac{\mu \alpha^{(j)}}{h_j} + \frac{\mu \alpha^{(i)}}{h_i}\right) [\![u_h]\!] [\![\phi_h]\!] \, ds, \tag{3.16e}$$

$$p_D(u_D, \phi_h) = \sum_{i=1}^{N} \sum_{F_{i\partial} \in \mathcal{F}_{i,B}} \int_{F_i} \frac{\mu \alpha^{(i)}}{h_i} u_D \phi_h \, ds, \tag{3.16f}$$

where $\alpha^{(i)}$ is the diffusion coefficient restricted on Ω_i, $\mathbf{n}_{F_{ij}}$ is the unit normal vector oriented from Ω_i towards the interior of Ω_j and the parameter $\mu > 0$ is large enough (will be specified later in the error analysis). For the faces $F_{i\partial} \in \mathcal{F}_{i,B}$, the forms in (3.16d) and (3.16e) are defined according to (3.15b). For notation convenience in what follows, we will use the same expression

$$\int_{F_{ij}} \{\alpha \nabla u_h\} \cdot \mathbf{n}_{F_{ij}} [\![\phi_h]\!] \, ds,$$

$$\int_{F_{ij}} \left(\frac{\mu \alpha^{(j)}}{h_j} + \frac{\mu \alpha^{(i)}}{h_i}\right) [\![u_h]\!] [\![\phi_h]\!] \, ds,$$

for both cases, boundary and interior faces. For the boundary jump terms, we will assume that $\alpha^{(j)} = 0$. If it is necessary to mention separately the integrals on $F_{i\partial} \in \mathcal{F}_{i,B}$, we will explicitly write this.

Remark 1 We mention that, in [10], Symmetric Interior Penalty (SIP) dG formulations have been considered by introducing harmonic averages of the diffusion coefficients on the interface symmetric fluxes. Furthermore, harmonic averages of the two different grid sizes have been used to penalize the jumps. The possibility of using other averages for constructing the diffusion terms in front of the consistency and penalty terms has been analyzed in many other works as well, see, e.g. [16,26]. For simplicity of the presentation, we provide a rigorous analysis of the Incomplete Interior Penalty (IIP) forms (3.16d) and (3.16e). However, our analysis can easily be carried over to SIP dG-IgA that is preferred in practice for symmetric and positive definite (spd) variational problems due to the fact that the resulting systems of algebraic equations are spd and, therefore, can be solved by means of some preconditioned conjugate gradient method.

4 Auxiliary results

In order to proceed to error analysis, several auxiliary results must be shown for $u \in W^{l,p}(\mathcal{S}(\Omega))$ and $\phi_h \in \mathbb{B}_h(\mathcal{S}(\Omega))$. The general frame of the proofs consists of three steps: (i) the required relations are expressed-proved on a *parent element* D_p, see Fig. 1, (ii) the relations are "transformed" to $\hat{E} \in T^{(i)}_{h_i,\widehat{D}}$ using an affine-linear mapping and scaling arguments, (iii) by virtue of the mappings $\mathbf{\Phi}_i$ defined in (3.6) and relations (3.9), we express the results in every Ω_i.

Let D_p be the parent element e.g $[-x_b, x_b]^d \subset \mathbb{R}^d$, with diameter H_p, see Fig. 1. D_p is convex simply connected domain, thus for any $x \in \partial D_p$, $\exists x_0 \in D_p$ such that

$$(x - x_0) \cdot \mathbf{n}_{\partial D_p} \geq C_{H_p}. \tag{4.1}$$

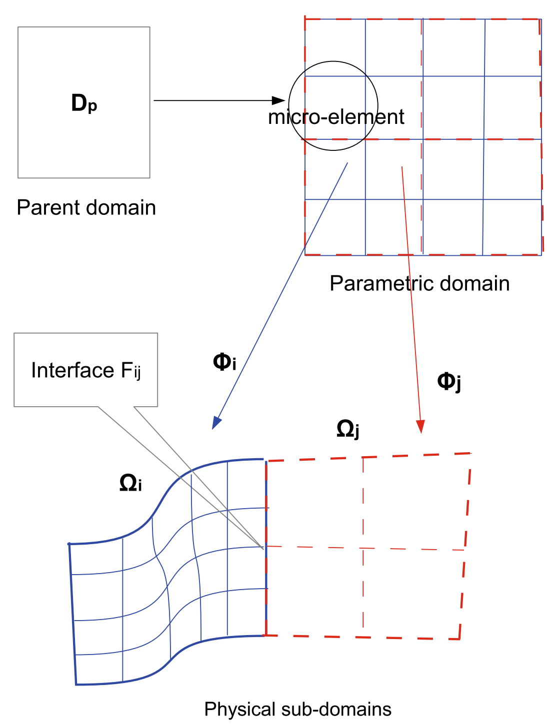

Fig. 1 The parent element, the parametric domain and two adjacent subdomains

Based on [15] and [13], we give the following trace inequality.

Lemma 1 *Let $u \in W^{l,p}(\Omega_i), l \geq 1, p > 1$, then there is a constant C depended on problem data and on the constants of (3.9) such that, the following trace inequality holds true for any $F_i \subset \partial\Omega_i$*

$$\int_{F_i} |u(s)|^p \, ds \leq C\Big(\frac{1}{h_i}\int_{\Omega_i} |u(x)|^p \, dx + h_i^{p-1}\int_{\Omega_i} |\nabla u(x)|^p \, dx\Big). \quad (4.2)$$

Proof Let $u \in W^{l,p}(D_p)$, then for $r = (x - x_0)$ we have

$$\int_{D_p} \nabla|u|^p \cdot r \, dx = \sum_{i=1}^{d}\int_{D_p} p|u|^{p-2}u\frac{\partial u}{\partial x_i}r_i \, dx = p\int_{D_p} |u|^{p-2}u\nabla u \cdot r \, dx. \quad (4.3)$$

The application of divergence theorem gives

$$\int_{D_p} \nabla|u|^p \cdot r \, dx = \int_{\partial D_p} |u|^p r \cdot \mathbf{n}_{\partial D_p} \, ds - \int_{D_p} |u|^p div(r) \, dx. \quad (4.4)$$

By (4.1), (4.3) and (4.4) it follows that

$$\int_{\partial D_p} |u|^p r \cdot \mathbf{n}_{\partial D_p} \, ds = p\int_{D_p} |u|^{p-2}u\nabla u \cdot r \, dx + \int_{D_p} |u|^p div(r) \, dx$$

and by (4.1), we get

$$C_{H_p}\int_{\partial D_p} |u|^p \, ds \leq p\int_{D_p} |u|^{p-2}u\nabla u \cdot r \, dx + \int_{D_p} |u|^p div(r) \, dx.$$

Applying Hölder and Youngs inequalities, we have

$$\begin{aligned}\int_{\partial D_p} |u|^p \, ds &\leq C_{H_p}\Big(C_{1,p}\big(\int_{D_p} |u|^p \, dx + |\nabla u|^p \, dx\big) + C_d\int_{D_p} |u|^p \, dx\Big)\\ &\leq C_{H_p,d,p}\Big(\int_{D_p} |u|^p \, dx + \int_{D_p} |\nabla u|^p \, dx\Big)\\ &= C_{H_p,d,p}\Big(\|u\|^p_{L^p(D_p)} + \|\nabla u\|^p_{L^p(D_p)}\Big). \quad (4.5)\end{aligned}$$

Now, D_p can be considered as a reference element of any micro-element $\hat{E} \in T^{(i)}_{h_i,\widehat{D}}$ with the linear affine map

$$\phi_{\hat{E}} : D_p \to \hat{E} \in T^{(i)}_{h_i,\widehat{D}}, \quad \phi_{\hat{E}}(x_{D_p}) = Bx_{D_p} + b, \quad (4.6)$$

where $|det(B)| = |\hat{E}|$, see [6]. By (4.6), we have that $|u|_{W^{l,p}(D_p)} = h_{\hat{E}}^{l-\frac{d}{p}}|\hat{u}|_{W^{l,p}(\hat{E})}$ and then for $e \subset \partial\hat{E}$ we deduce by (4.2) that

$$h_{\hat{E}}^{-(d-1)}\int_e |u|^p \, ds \leq C\Big(h_{\hat{E}}^{(0-\frac{d}{p})p}\int_{\hat{E}} |u|^p \, dx + h_{\hat{E}}^{p(1-\frac{d}{p})}\int_{\hat{E}} |\nabla u|^p \, dx\Big)$$

which directly gives

$$\int_e |u|^p \, ds \leq C\Big(\frac{1}{h_i}\int_{\hat{E}} |u|^p \, dx + h_i^{p-1}\int_{\hat{E}} |\nabla u|^p \, dx\Big). \quad (4.7)$$

Summing over all micro-elements $\hat{E} \in T^{(i)}_{h_i,\widehat{D}}$, we have for $\hat{F}_i \subset \partial\widehat{D}$

$$\int_{\hat{F}_i} |u|^p \, ds \leq C\Big(\frac{1}{h_i}\int_{\widehat{D}} |u|^p \, dx + h_i^{p-1}\int_{\widehat{D}} |\nabla u|^p \, dx\Big). \quad (4.8)$$

Finally, applying (3.9), we obtain the trace inequality on every subdomain

$$\int_{F_i} |u|^p \, ds \leq C\Big(\frac{1}{h_i}\int_{\Omega_i} |u|^p \, dx + h_i^{p-1}\int_{\Omega_i} |\nabla u|^p \, dx\Big). \quad (4.9)$$

□

We point out that similar proof has been given in [14] in case of $p = 2$.

Lemma 2 *For all $\phi_h \in \hat{\mathbb{B}}^{(i)}_{h_i}$ defined on $T^{(i)}_{h_i,\widehat{D}}$, there is a constant C depended on mesh quasi-uniformity parameters of the mesh but not on h_i, such that*

$$\|\nabla\phi_h\|^p_{L^p(\widehat{D})} \leq \frac{C}{h_i^p}\|\phi_h\|^p_{L^p(\widehat{D})}. \quad (4.10)$$

Proof The restriction of $\phi_h|_{\hat{E}}$ is a $B-$spline polynomial of the same order. Considering the same polynomial space on the D_p and by the equivalence of the norms on D_p we have, [6],

$$\|\nabla\phi_h\|^p_{L^p(D_p)} \leq C_{D_p}\|\phi_h\|^p_{L^p(D_p)}. \quad (4.11)$$

Applying scaling arguments and the mesh quasi-uniformity properties of $T^{(i)}_{h_i,\widehat{D}}$, the left and the right hand side of (4.11) can be expressed on every $\hat{E} \in T^{(i)}_{h_i,\widehat{D}}$ as

$$h_i^{p-\frac{d}{p}p}\|\nabla\phi_h\|^p_{L^p(\hat{E})} \leq Ch_i^{-\frac{d}{p}p}\|\phi_h\|^p_{L^p(\hat{E})}, \tag{4.12}$$

summing over all in (4.12) $\hat{E} \in T^{(i)}_{h_i,\widehat{D}}$, we can easily deduce (4.10). □

Lemma 3 *For all $\phi_h \in \hat{\mathbb{B}}^{(i)}_{h_i}$ defined on $T^{(i)}_{h_i,\widehat{D}}$ and for all $\hat{F}_i \in \partial\widehat{D}$, there is a constant C depended on mesh quasi-uniformity parameters of the mesh but not on h_i, such that*

$$\|\phi_h\|^p_{L^p(\hat{F}_i)} \leq \frac{C}{h_i}\|\phi_h\|^p_{L^p(\widehat{D})}. \tag{4.13}$$

Proof Applying the same scaling arguments as before and using the local quasi-uniformity of $T^{(i)}_{h_i,\widehat{D}}$, that is for every $\hat{e} \in \partial\hat{E}$ holds $|\hat{e}| \sim h_i$, we can show the following local trace inequality

$$\|\phi_h\|^p_{L^p(\hat{e})} \leq Ch_i^{-1}\|\phi_h\|^p_{L^p(\hat{E})}, \tag{4.14}$$

summing over all $\hat{E} \in T^{(i)}_{h_i,\widehat{D}}$ that have an edge on $\hat{F}_i$ we deduce (4.13). □

Next a Lemma for the relation among the $|\phi_h|_{W^{l,p}(\widehat{D})}$ and $|\phi_h|_{W^{m,p}(\widehat{D})}$.

Lemma 4 *Let $\phi_h \in \hat{\mathbb{B}}^{(i)}_{h_i}$ such that $\phi_h \in W^{l,p}(\hat{E}) \cap W^{m,q}(\hat{E})$, $\hat{E} \in T^{(i)}_{h_i,\widehat{D}}$, and $0 \leq m \leq l$, $1 \leq p, q \leq \infty$. Then there is a constant $C := C(l,p,m,q)$ depended on mesh quasi-uniformity parameters of the mesh but not on h_i, such that*

$$|\phi_h|_{W^{l,p}(\hat{E})} \leq Ch_i^{m-l-\frac{d}{q}+\frac{d}{p}}|\phi_h|_{W^{m,q}(\hat{E})}. \tag{4.15}$$

Proof We mimic the analysis of Chp 4 in [6]. For any $\phi_h \in \hat{\mathbb{B}}^{(i)}_{h_i}|_{D_p}$, we have that

$$|\phi_h|_{W^{l,p}(D_p)} \leq C|\phi_h|_{W^{m,q}(D_p)},\ \phi_h \in \hat{\mathbb{B}}^{(i)}_{h_i}|_{D_p}. \tag{4.16}$$

Using the scaling arguments as in proof of (4.7),

$$h_{\hat{E}}^{l-\frac{d}{p}}|\phi_h|_{W^{l,p}(\hat{E})} \leq Ch_{\hat{E}}^{m-\frac{d}{q}}|\phi_h|_{W^{m,q}(\hat{E})},$$

which directly implies

$$|\phi_h|_{W^{l,p}(\hat{E})} \leq C\,h_i^{m-l-\frac{d}{q}+\frac{d}{p}}|\phi_h|_{W^{m,q}(\hat{E})},\ \phi_h \in \hat{\mathbb{B}}^{(i)}_{h_i}. \tag{4.17}$$

For the particular case of $m = l = 0$ in (4.15), we have that

$$\|\phi_h\|_{L^p(\hat{E})} \leq Ch_i^{d(\frac{1}{p}-\frac{1}{q})}\|\phi_h\|_{L^q(\hat{E})}. \tag{4.18}$$

□

Remark 2 The foregoing results can be expressed on every $\Omega_i \in \mathcal{S}(\Omega)$ using (3.9).

4.1 Analysis of the dG-IgA discretization

We next study the convergence properties of the method (3.16) under the following regularity assumption for the solution u.

Assumption 2 We assume for u that $u \in W^{l,2}_{\mathcal{S}} := W^{1,2}(\Omega) \cap W^{l,2}(\mathcal{S}(\Omega))$, $l \geq 2$. Under this assumption, we consider the B-spline degree k to be $k \geq l-1$.

We consider the enlarged space $W^{l,2}_h := W^{l,2}_{\mathcal{S}} + \mathbb{B}_h(\mathcal{S}(\Omega))$, equipped with the broken dG-norm

$$\|u\|^2_{dG} = \sum_{i=1}^N\Big(\alpha^{(i)}\|\nabla u^{(i)}\|^2_{L^2(\Omega_i)} + p_i(u^{(i)}, u^{(i)})\Big),\ u \in W^{l,2}_h, \tag{4.19}$$

where denote $p_i(u^{(i)}, u^{(i)}) = \sum_{F_{ij}\subset\partial\Omega_i} p_{F_{ij}}(u^{(i)}, u^{(i)})$. For the error analysis is necessary to show the continuity and coercivity properties of the bilinear form $a_h(.,.)$ of (3.16). Initially, we give a bound for the consistency terms.

Lemma 5 *For $(u, \phi_h) \in W^{l,2}_h \times \mathbb{B}_h(\mathcal{S}(\Omega))$, there are $C_{1,\varepsilon}, C_{2,\varepsilon} > 0$ such that for $F_{ij} \subset \partial\Omega_i$*

$$\begin{aligned}|s_{F_{ij}}| &= \Big|\int_{F_{ij}}\{\alpha\nabla u\}\cdot\mathbf{n}_{F_{ij}}(\phi_h^{(i)} - \phi_h^{(j)})\,ds\Big| \\ &\leq C_{1,\varepsilon}\Big(h_i\alpha^{(i)}\|\nabla u^{(i)}\|^2_{L^2(F_{ij})} \\ &\quad + h_j\alpha^{(j)}\|\nabla u^{(j)}\|^2_{L^2(F_{ij})}\Big) \\ &\quad + \frac{1}{C_{2,\varepsilon}}\Big(\frac{\alpha^{(i)}}{h_i} + \frac{\alpha^{(j)}}{h_j}\Big)\|\phi_h^{(i)} - \phi_h^{(j)}\|^2_{L^2(F_{ij})}.\end{aligned} \tag{4.20}$$

Proof Expanding the terms and applying Cauchy-Schwartz inequality yields

$$\begin{aligned}|s_{F_{ij}}| &\leq C\Big|\int_{F_{ij}}\{\alpha\nabla u\}\cdot\mathbf{n}_{F_{ij}}(\phi_h^{(i)} \\ &\quad -\phi_h^{(j)})\,ds\Big| \leq \\ &C\Big(\alpha^{(i)}\|\nabla u^{(i)}\|_{L^2(F_{ij})} + \alpha^{(j)}\|\nabla u^{(j)}\|_{L^2(F_{ij})}\Big)\|\phi_h^{(i)} \\ &\quad -\phi_h^{(j)}\|_{L^2(F_{ij})}.\end{aligned}$$

Applying Young's inequality:

$$\alpha^{(i)}\|\nabla u^{(i)}\|_{L^2(F_{ij})}\|\phi_h^{(i)}-\phi_h^{(j)}\|_{L^2(F_{ij})}$$
$$\leq C_{1,\varepsilon}h_i\alpha^{(i)}\|\nabla u^{(i)}\|^2_{L^2(F_{ij})}$$
$$+\frac{\alpha^{(i)}}{C_{2,\varepsilon}h_i}\|\phi_h^{(i)}-\phi_h^{(j)}\|^2_{L^2(F_{ij})}$$

we obtain

$$\begin{aligned}|s_{F_{ij}}| \leq\ & C_{1,\varepsilon}h_i\alpha^{(i)}\|\nabla u^{(i)}\|^2_{L^2(F_{ij})} \\ &+ C_{1,\varepsilon}h_j\alpha^{(j)}\|\nabla u^{(j)}\|^2_{L^2(F_{ij})} \\ &+ \frac{\alpha^{(i)}}{C_{2,\varepsilon}h_i}\|\phi_h^{(i)}-\phi_h^{(j)}\|^2_{L^2(F_{ij})} \\ &+ \frac{\alpha^{(j)}}{C_{2,\varepsilon}h_j}\|\phi_h^{(i)}-\phi_h^{(j)}\|^2_{L^2(F_{ij})} \\ =\ & C_{1,\varepsilon}\Big(h_i\alpha^{(i)}\|\nabla u^{(i)}\|^2_{L^2(F_{ij})}+h_j\alpha^{(j)}\|\nabla u^{(j)}\|^2_{L^2(F_{ij})}\Big) \\ &+ \frac{1}{C_{2,\varepsilon}}\Big(\frac{\alpha^{(i)}}{h_i}+\frac{\alpha^{(j)}}{h_j}\Big)\|\phi_h^{(i)}-\phi_h^{(j)}\|^2_{L^2(F_{ij})}.\end{aligned}$$

□

Lemma 6 *Suppose $u_h \in \mathbb{B}_h(\mathcal{S}(\Omega))$ is the dG-IgA solution derived by (3.16). There exist a $C>0$ independent of α and h_i but depended on μ such that*

$$a_h(u_h,u_h) \geq C\|u_h\|^2_{dG},\ \ u_h \in \mathbb{B}_h(\mathcal{S}(\Omega)) \tag{4.21}$$

Proof By (3.16a), we have that

$$\begin{aligned}a_h(u_h,u_h) =\ & \sum_{i=1}^N\Big(a_i(u_h,u_h)-\sum_{F_{ij}\subset\partial\Omega_i}\frac{1}{2}s_{F_{ij}}(u_h,u_h) \\ &+ p_{F_{ij}}(u_h,u_h)\Big) \\ =\ & \sum_{i=1}^N\Big(\alpha_i\|\nabla u_h\|^2_{L^2(\Omega_i)} \\ &- \sum_{F_{ij}\subset\partial\Omega_i}\frac{1}{2}\int_{F_{ij}}\{\alpha\nabla u_h\}\cdot\mathbf{n}_{F_{ij}}[\![u_h]\!]\,ds \\ &+ \sum_{F_{ij}\subset\partial\Omega_i}\mu\Big(\frac{\alpha^{(i)}}{h_i}+\frac{\alpha^{(j)}}{h_j}\Big)\|[\![u_h]\!]\|^2_{L^2(F_{ij})}\Big).\end{aligned} \tag{4.22}$$

For the second term on the right hand side, Lemma 5 and the trace inequality (4.13) expressed on $F_{ij}\in\mathcal{F}$ yield the bound

$$\begin{aligned}&-\sum_{F_{ij}\subset\partial\Omega_i}\frac{1}{2}\int_{F_{ij}}\{\alpha\nabla u_h\}\cdot\mathbf{n}_{F_{ij}}[\![u_h]\!]\,ds \geq \\ &-C_{1,\varepsilon}\sum_{i=1}^N\Big(\alpha_i\|\nabla u_h\|^2_{L^2(\Omega_i)} \\ &-\sum_{F_{ij}\subset\partial\Omega_i}\frac{1}{C_{2,\varepsilon}}\Big(\frac{\alpha^{(i)}}{h_i}+\frac{\alpha^{(j)}}{h_j}\Big)\|[\![u_h]\!]\|^2_{L^2(F_{ij})}\Big).\end{aligned} \tag{4.23}$$

Inserting (4.23) into (4.22) and choosing $C_{1,\varepsilon}<\frac{1}{2}$ and $\mu>\frac{2}{C_{2,\varepsilon}}$ we obtain (4.21). □

Lemma 7 *There are $C_1, C_2>0$ independent of h_i such that for all $(u,\phi_h)\in W_h^{l,2}\times\mathbb{B}_h(\mathcal{S}(\Omega))$*

$$\begin{aligned}a_h(u,\phi_h) \leq\ & C_1\Big(\|u\|^2_{dG}+\sum_{i=1}^N\sum_{F_{ij}\subset\partial\Omega_i}\alpha^{(i)}h_i\|\nabla u^{(i)}\|^2_{L^2(F_{ij})}\Big) \\ &+C_2\|\phi_h\|^2_{dG}.\end{aligned} \tag{4.24}$$

Proof We have by (3.16a) that

$$\begin{aligned}a_h(u,\phi_h) =\ & \sum_{i=1}^N\Big(\int_{\Omega_i}\alpha\nabla u\nabla\phi_h\,dx \\ &+ \sum_{F_{ij}\subset\partial\Omega_i}\frac{1}{2}\int_{F_{ij}}\{\alpha\nabla u\}\cdot\mathbf{n}_{F_{ij}}[\![\phi_h]\!]\,ds \\ &+ \sum_{F_{ij}\subset\partial\Omega_i}\int_{F_{ij}}\Big(\frac{\mu\alpha^{(j)}}{h_j} \\ &+ \frac{\mu\alpha^{(i)}}{h_i}\Big)[\![u]\!][\![\phi_h]\!]\,ds\Big) = T_1+T_2+T_3.\end{aligned} \tag{4.25}$$

Applying Cauchy-Schwartz inequality and consequently Young's inequality on every term in (4.25) yield the bounds

$$T_1 \leq C_1\|u\|^2_{dG}+C_2\|\phi_h\|^2_{dG}.$$

For the term T_2, owing to the Lemma 5, we have

$$\begin{aligned}T_2 \leq\ & \sum_{i=1}^N\sum_{F_{ij}\subset\partial\Omega_i}\Big(C_1\alpha^{(i)}h_i\|\nabla u^{(i)}\|^2_{L^2(F_{ij})}+C_2\Big(\frac{\mu\alpha^{(j)}}{h_j} \\ &+ \frac{\mu\alpha^{(i)}}{h_i}\Big)\|[\![\phi_h]\!]\|^2_{L^2(F_{ij})}\Big) \\ \leq\ & C_1\sum_{i=1}^N\sum_{F_{ij}\subset\partial\Omega_i}\alpha^{(i)}h_i\|\nabla u^{(i)}\|^2_{L^2(F_{ij})}+C_2\|\phi_h\|^2_{dG},\end{aligned}$$

$$T_3 \leq \sum_{i=1}^N\sum_{F_{ij}\subset\partial\Omega_i}\Big(\frac{\mu\alpha^{(j)}}{h_j}+\frac{\mu\alpha^{(i)}}{h_i}\Big)$$

$$\times \Big(C_1\|[\![u]\!]\|^2_{L^2(F_{ij})} + C_2\|[\![\phi_h]\!]\|^2_{L^2(F_{ij})}\Big) \leq C_1\|u\|^2_{dG} + C_2\|\phi_h\|^2_{dG}.$$

Substituting the bounds of T_1, T_2, T_3 into (4.25), we can derive (4.24). □

In Chp 12 in [30], B-spline quasi-intrpolants, say Π_h, are defined for $u \in W^{l,p}$ functions. Next, we consider the same quasi-interpolant and give an estimate on how well $\Pi_h u$ approximates functions $u \in W^{l,2}(\Omega_i)$ in $\|.\|_{dG}$-norm.

Lemma 8 *Let $m, l \geq 2$ be positive integers with $0 \leq m \leq l \leq k+1$ and let $E = \mathbf{\Phi}_i(\hat{E})$, $\hat{E} \in T^{(i)}_{h_i,\hat{D}}$. For $u \in W^{l,2}(\Omega_i)$ there exist a quasi-interpolant $\Pi_h u \in \mathbb{B}^{(i)}_h$ and a constant C_i depended on C_m and C_M of (3.9) such that*

$$\sum_{E\in T^{(i)}_{h_i,\Omega_i}} |u - \Pi_h u|^2_{W^{m,2}(E)} \leq C_i h_i^{2(l-m)} \|u\|^2_{W^{l,2}(\Omega_i)}. \tag{4.26}$$

Further, for any $F_{ij} \subset \partial\Omega_i$ the following estimates are true

$$h_i\alpha^{(i)}\|(\nabla u^{(i)} - \nabla \Pi_h u^{(i)})\cdot \mathbf{n}_{F_{ij}}\|^2_{L^2(F_{ij})} \leq C_i h_i^{2l-2}, \tag{4.27a}$$

$$\left(\frac{\alpha^{(j)}}{h_j} + \frac{\alpha^{(i)}}{h_i}\right)\|u^{(i)} - \Pi_h u^{(i)}\|^2_{L^2(F_{ij})} \leq C_i\left(\alpha^{(i)}h_i^{2l-2} + \frac{\alpha^{(j)}h_i^{2l-1}}{h_j}\right), \tag{4.27b}$$

$$\|u - \Pi_h u\|^2_{dG} \leq \sum_{i=1}^{N} C_i\left(h_i^{2l-2} + \sum_{F_{ij}\subset\partial\Omega_i} \alpha^{(j)}\frac{h_i}{h_j}h_i^{2l-2}\right). \tag{4.27c}$$

Proof The proof of (4.26) is included in Lemma 10 (see below) if we set $p = 2$.

Applying the trace inequality (4.9) for $u := u^{(i)} - \Pi_h u^{(i)}$ and consequently using the approximation estimate (4.26) the result (4.27a) easily follows.

To prove (4.27b), we apply again (4.9) and obtain

$$\Big(\frac{\alpha^{(j)}}{h_j} + \frac{\alpha^{(i)}}{h_i}\Big)\|u^{(i)} - \Pi_h u^{(i)}\|^2_{L^2(F_{ij})} \leq C_i\Big(\frac{\alpha^{(j)}}{h_j} + \frac{\alpha^{(i)}}{h_i}\Big)\Big(\frac{1}{h_i}\|u^{(i)} - \Pi_h u^{(i)}\|^2_{L^2(\Omega_i)} + h_i\|\nabla u^{(i)} - \nabla\Pi_h u^{(i)}\|^2_{L^2(\Omega_i)} \leq C_i\Big(\frac{\alpha^{(j)}}{h_j} + \frac{\alpha^{(i)}}{h_i}\Big)h_i^{2l-1} \leq C_i\Big(\alpha^{(i)}h_i^{2l-2} + \frac{\alpha^{(j)}h_i^{2l-1}}{h_j}\Big)$$

Recalling the approximation result (4.26) and using (4.27b) we can deduce (4.27c). □

In order to proceed and to give an estimate for the error $\|u - u_h\|_{dG}$, we need to show that the weak solution satisfies the form (3.16a).

Lemma 9 *Under the Assumption 2, the weak solution u of the variational formulation (2.2) satisfies the dG-IgA variational identity (3.16), that is for all $\phi_h \in \mathbb{B}_h(\mathcal{S}(\Omega))$, we have*

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega_i}\alpha\nabla u\cdot\nabla\phi_h\,dx \\
&\quad - \sum_{i=1}^{N}\sum_{F_{ij}\subset\partial\Omega_i}\Big(\int_{F_{ij}}\{\alpha\nabla u\}\cdot\mathbf{n}_{F_{ij}}[\![\phi_h]\!]\,ds \\
&\quad + \big(\frac{\mu\alpha^{(i)}}{h_i} + \frac{\mu\alpha^{(j)}}{h_j}\big)\int_{F_{ij}}[\![u]\!][\![\phi_h]\!]\,ds\Big) \\
&\quad + \sum_{i=1}^{N}\sum_{F_{i\partial}\in\mathfrak{F}_{i,B}}\Big(\int_{F_{i\partial}}\alpha^{(i)}\nabla u\cdot\mathbf{n}_{F_{i\partial}}\phi_h\,ds \\
&\quad + \frac{\mu\alpha^{(i)}}{h_i}\int_{F_{i\partial}}u\phi_h\,ds\Big) \\
&= \sum_{i=1}^{N}\Big(\int_{\Omega_i}f\phi_h\,dx + \sum_{F_{i\partial}\in\mathfrak{F}_{i,B}}\frac{\mu\alpha^{(i)}}{h_i}\int_{F_{i\partial}}u_D\phi_h\,ds\Big).
\end{aligned} \tag{4.28}$$

Proof We multiply (2.1) by $\phi_h \in \mathbb{B}_h(\mathcal{S}(\Omega))$ and integrating by parts on each subdomain Ω_i we get

$$\int_{\Omega_i}\alpha\nabla u\cdot\nabla\phi_h\,dx - \int_{\partial\Omega_i}\alpha\nabla u\cdot\mathbf{n}_{\partial\Omega_i}\phi_h\,ds = \int_{\Omega_i}f\phi_h\,dx.$$

Summing over all subdomains

$$\sum_{i=1}^{N}\Big(\int_{\Omega_i}\alpha\nabla u\cdot\nabla\phi_h\,dx - \sum_{F_{ij}\subset\partial\Omega_i}\int_{F_{ij}}[\![\alpha\nabla u\phi_h]\!]\cdot\mathbf{n}_{F_{ij}}\,ds\Big) = \sum_{i=1}^{N}\int_{\Omega_i}f\phi_h\,dx. \tag{4.29}$$

The regularity Assumption 2 implies that $[\![\alpha\nabla u]\!]\cdot\mathbf{n}_{F_{ij}} = 0$. Making use of the identity

$$[\![ab]\!] = a_1b_1 - a_2b_2 = \{a\}[\![b]\!] + [\![a]\!]\{b\},$$

the relation (4.29) can be reformulated as

$$\sum_{i=1}^{N}\Big(\int_{\Omega_i}\alpha\nabla u\cdot\nabla\phi_h\,dx - \sum_{F_{ij}\subset\partial\Omega_i}\frac{1}{2}\int_{F_{ij}}\{\alpha\nabla u\}\cdot\mathbf{n}_{F_{ij}}[\![\phi_h]\!]\,ds$$

$$+ \sum_{F_{i\partial}\in\mathcal{F}_{i,B}} \int_{F_{i\partial}} \alpha\nabla u\cdot \mathbf{n}_{F_{i\partial}}\phi_h\,ds\Big)$$
$$= \sum_{i=1}^{N} \int_{\Omega_i} f\phi_h\,dx. \tag{4.30}$$

The continuity of u implies further that

$$\sum_{i=1}^{N}\Big(\sum_{F_{ij}\subset\partial\Omega_i}\Big(\frac{\mu\alpha^{(i)}}{h_i}+\frac{\mu\alpha^{(j)}}{h_j}\Big)\int_{F_{ij}}[\![u]\!][\![\phi_h]\!]\,ds$$
$$+\sum_{F_i\in\mathcal{F}_{i,B}}\frac{\mu\alpha^{(i)}}{h_i}\int_{F_i}u\phi_h\,ds\Big)$$
$$=\sum_{i=1}^{N}\sum_{F_i\in\mathcal{F}_{i,B}}\frac{\mu\alpha^{(i)}}{h_i}\int_{F_{i\partial}}u_D\phi_h\,ds. \tag{4.31}$$

Adding (4.31) and (4.30) we obtain (4.28). □

We can now give an error estimate in $\|.\|_{dG}$-norm.

Theorem 1 *Let $u\in W^{l,2}_{\mathcal{S}}$ be the solution of (2.2) and let $u_h\in\mathbb{B}_h(\mathcal{S}(\Omega))$ be the solution of the discrete problem (3.16). Then the error $u-u_h$ satisfies*

$$\|u-u_h\|^2_{dG}<\sum_{i=1}^{N}C_i\Big(h_i^{2l-2}+\sum_{F_{ij}\subset\partial\Omega_i}\alpha^{(j)}\frac{h_i}{h_j}h_i^{2l-2}\Big), \tag{4.32}$$

where the positive constant C_i is depended on C_m and C_M of (3.9) and $|u|_{W^{l,2}(\Omega_i)}$.

Proof Let $\Pi_h u\in\mathbb{B}_h(\mathcal{S}(\Omega))$ as in Lemma 8, by subtracting (4.28) from (3.16a) we get

$$a_h(u_h,\phi_h)=a_h(u,\phi_h),$$

and adding $-a_h(\Pi_h u,\phi_h)$ on both sides

$$a_h(u_h-\Pi_h u,\phi_h)=a_h(u-\Pi_h u,\phi_h). \tag{4.33}$$

Note that $u_h-\Pi_h u\in\mathbb{B}_h(\mathcal{S}(\Omega))$. Therefore we may set $\phi_h=u_h-\Pi_h u$ in (4.33), and consequently applying Lemma 6 and Lemma 7, we find

$$\|u_h-\Pi_h u\|^2_{dG}\le C\Big(\|u-\Pi_h u\|^2_{dG}$$
$$+\sum_{i=1}^{N}\sum_{F_{ij}\subset\partial\Omega_i}\alpha^{(i)}h_i\|\nabla(u^{(i)}-\Pi_h u^{(i)})\|^2_{L^2(F_{ij})}\Big). \tag{4.34}$$

Using the triangle inequality in (4.34) and consequently applying the estimates of (4.27) we can obtain (4.32). □

5 Low-regularity solutions

In this section, we investigate the convergence of the u_h produced by the dG-IgA method (3.16), under the assumption that the weak solution u of the model problem (2.1) has less regularity, that is $u\in W^{l,p}_{\mathcal{S}}:=W^{1,2}(\Omega)\cap W^{l,p}(\mathcal{S}(\Omega))$, $l{\ge}2$, $p\in(\frac{2d}{d+2(l-1)},2]$. Problems with low regularity solutions can be found in several cases, as for example, when the domain has singular boundary points, points with changing boundary conditions, see e.g. [15], even in particular choices of the discontinuous diffusion coefficient, [19]. We use the enlarged space $W^{l,p}_h=W^{l,p}_{\mathcal{S}}+\mathbb{B}_h(\mathcal{S}(\Omega))$ and will show that the dG-IgA method converges in optimal rate with respect to $\|.\|_{dG}$ norm defined in (4.19). We develop our analysis inspired by the techniques used in [35], [27]. A basic tool that we will use is the Sobolev embeddings theorems, see [1,13]. Let $l=j+m\ge 2$, then for $j=0$ or $j=1$ it holds that

$$\|u\|_{W^{j,2}(\Omega_i)}\le C(l,p,2,\Omega_i)\|u\|_{W^{l,p}(\Omega_i)},\ \text{for } p>\frac{2d}{d+2m}. \tag{5.1}$$

We start by proving estimates on how well the quasi-interpolant $\Pi_h u$ defined in Lemma 8 approximates $u\in W^{l,p}(\Omega_i)$. We consider always for the B-spline degree k that $k\ge l-1$.

The constants appear below are depended on the constants of (3.9) and (4.9) and are not explicitly specified.

Lemma 10 *Let $u\in W^{l,p}(\Omega_i)$ with $l\ge 2$, $p\in(\max\{1,\frac{2d}{d+2(l-1)}\},2]$ and let $E\in T^{(i)}_{h_i,\Omega_i}$. Then for $0\le m\le l\le k+1$, there exist constants C_i such that*

$$\sum_{E\in T^{(i)}_{h_i,\Omega_i}}|u-\Pi_h u|^p_{W^{m,p}(E)}\le C_i h_i^{p(l-m)}\|u\|_{W^{l,p}(\Omega_i)}. \tag{5.2}$$

Moreover, we have the following estimates

- $h_i^{\beta}\|\nabla u^{(i)}-\nabla\Pi_h u^{(i)}\|^p_{L^p(F_{ij})}$
$$\le Ch_i^{p(l-1)-1+\beta}, \tag{5.3a}$$
- $\big(\frac{\alpha^{(j)}}{h_j}+\frac{\alpha^{(i)}}{h_i}\big)\|[\![u-\Pi_h u]\!]\|^2_{L^2(F_{ij})}$
$$\le\ C_i\alpha^{(j)}\frac{h_i}{h_j}\Big(h_i^{\delta(p,d)}|u|^p_{W^{l,p}(\Omega_i)}\Big)^2$$
$$+C_j\alpha^{(i)}\frac{h_j}{h_i}\Big(h_j^{\delta(p,d)}\|u\|_{W^{l,p}(\Omega_j)}\Big)^2$$
$$+C_j\Big(h_j^{\delta(p,d)}|u|_{W^{l,p}(\Omega_j)}\Big)^2+C_i\Big(h_i^{\delta(p,d)}\|u\|_{W^{l,p}(\Omega_i)}\Big)^2, \tag{5.3b}$$

$$\bullet\|u-\Pi_h u\|_{dG}^2 \leq \sum_{i=1}^{N} C_i\Big(h_i^{\delta(p,d)}\|u\|_{W^{l,p}(\Omega_i)}\Big)^2 + \sum_{i=1}^{N}\sum_{F_{ij}\subset\partial\Omega_i} C_i\alpha^{(j)}\frac{h_i}{h_j}\Big(h_i^{\delta(p,d)}\|u\|_{W^{l,p}(\Omega_i)}\Big)^2, \quad (5.3c)$$

where $\delta(p,d)=l+(\frac{d}{2}-\frac{d}{p}-1)$.

Proof We give the proof of (5.2) based on the results of Chap 12 in [30]. Given $f\in W^{l,p}(\hat D)$, there exists a tensor-product polynomial $T^m f$ of order m, such that, for every $\hat E\in T^{(i)}_{h_i,\hat D}$ the estimate

$$|f-T^m f|_{W^{m,p}(\hat E)} \leq C_{d,l,m} h_i^{l-m}|f|_{W^{l,p}(D^{(i)}_{\hat E})}, \quad (5.4)$$

holds, cf. [6] and [30]. Because of $m\leq k$ holds $\Pi_h(T^m f)=T^m f$ and $\|\Pi_h f\|_{L^p(\hat E)}\leq C\|f\|_{L^p(D^{(i)}_{\hat E})}$. Hence, we have that

$$\begin{aligned}
|u-\Pi_h u|_{W^{m,p}(\hat E)} &\leq |u-T^m u|_{W^{m,p}(\hat E)} + |\Pi_h u - T^m u|_{W^{m,p}(\hat E)}\\
&\leq |u-T^m u|_{W^{m,p}(\hat E)} + |\Pi_h(u-T^m u)|_{W^{m,p}(\hat E)}\\
&\leq C_1 h_i^{l-m}|u|_{W^{l,p}(D^{(i)}_{\hat E})} + C_2 h_i^{-m+\frac{d}{p}-\frac{d}{p}}|\Pi_h(u-T^m u)|_{L^p(\hat E)} \text{ (by (4.10))}\\
&\leq C_1 h_i^{l-m}|u|_{W^{l,p}(D^{(i)}_{\hat E})} + C_2 h_i^{-m}|u-T^m u|_{L^p(\hat E)} \text{ (by (5.4))}\\
&\leq C h_i^{l-m}|u|_{W^{l,p}(D^{(i)}_{\hat E})}. \quad (5.5)
\end{aligned}$$

Recalling (3.9), the above inequality is expressed on every $E\in T^{(i)}_{h_i,\Omega_i}$. Then, taking the $p-th$ power and summing over the elements we obtain the estimate (5.2).

We consider now the interface $F_{ij}=\partial\Omega_i\cap\Omega_j$. Applying (4.9) and using the uniformity of the mesh we get

$$\begin{aligned}
&h_i^\beta\|\nabla u^{(i)}-\nabla\Pi_h u^{(i)}\|^p_{L^p(F_{ij})}\\
&\leq C h_i^\beta\big(\frac{1}{h_i}\|\nabla u^{(i)}-\nabla\Pi_h u^{(i)}\|^p_{L^p(\Omega_i)} + h_i^{p-1}\|\nabla^2 u^{(i)}-\nabla^2\Pi_h u^{(i)}\|^p_{L^p(\Omega_i)}\big)\\
&\stackrel{\text{by (5.2)}}{\leq} C h_i^{p(l-1)-1+\beta}. \quad (5.6)
\end{aligned}$$

To prove(5.3b), we again make use of the trace inequality (4.9)

$$\begin{aligned}
&\frac{\alpha^{(i)}}{h_i}\|u^{(i)}-\Pi_h u^{(i)}\|^2_{L^2(F_{ij})}\\
&\leq C\alpha^{(i)}\big(\frac{1}{h_i^2}\int_{\Omega_i}|u^{(i)}-\Pi_h u^{(i)}|^2\,dx + \int_{\Omega_i}|\nabla(u^{(i)}-\Pi_h u^{(i)})|^2\,dx\big)\\
&= C\alpha^{(i)}\Big(\frac{1}{h_i^2}\sum_{E\in T^{(i)}_{h_i,\Omega_i}}\int_E|u^{(i)}-\Pi_h u^{(i)}|^2\,dx + \sum_{E\in T^{(i)}_{h_i,\Omega_i}}\int_E|\nabla(u^{(i)}-\Pi_h u^{(i)})|^2\,dx\Big). \quad (5.7)
\end{aligned}$$

The Sobolev embedding (5.1) gives

$$\|u\|_{L^2(D_p)} \leq C(p,2,D_p)\big(\|u\|^p_{L^p(D_p)}+|u|^p_{W^{1,p}(D_p)}\big)^{\frac{1}{p}}. \quad (5.8)$$

Using the scaling arguments, see (4.6), and the bounds (3.9) we can derive the coresponding expression of (5.8) on every $E\in T^{(i)}_{h_i,\Omega_i}$,

$$h_i^{\frac{-d}{2}}\|u\|_{L^2(E)} \leq C_i h_i^{\frac{-d}{p}}\big(\|u\|^p_{L^p(E)}+h_i^p|u|^p_{W^{1,p}(E)}\big)^{\frac{1}{p}},$$

where a straight forward computation gives

$$h_i^{-2}\|u\|^2_{L^2(E)} \leq C h_i^{2(\frac{d}{2}-\frac{d}{p}-1)}\big(\|u\|^p_{L^p(E)}+h_i^p|u|^p_{W^{1,p}(E)}\big)^{\frac{2}{p}}, \quad (5.9)$$

$$\|u\|^2_{W^{1,2}(E)} \leq C h_i^{2(\frac{d}{2}-\frac{d}{p}-1)}\big(\|u\|^p_{L^p(E)} + h_i^p|u|^p_{W^{1,p}(E)}+h_i^{2p}|u|^p_{W^{2,p}(E)}\big)^{\frac{2}{p}}. \quad (5.10)$$

Setting in (5.9) and (5.10) $u:=u^{(i)}-\Pi_h u^{(i)}$ and applying (5.2), we obtain that

$$\begin{aligned}
&\sum_{E\in T^{(i)}_{h_i,\Omega_i}}\alpha^{(i)}\big(h_i^{-2}\|u^{(i)}-\Pi_h u^{(i)}\|^2_{L^2(E)} + \|u^{(i)}-\Pi_h u^{(i)}\|^2_{W^{1,2}(E)}\big)\\
&\leq \sum_{E\in T^{(i)}_{h_i,\Omega_i}}\big(\alpha^{(i)}C_i h_i^{l+(\frac{d}{2}-\frac{d}{p}-1)}\|u\|_{W^{l,p}(D^{(i)}_E)}\big)^2\\
&\leq \text{(note that } f(x)=(a^x+b^x)^{\frac{1}{x}}\downarrow)\\
&\alpha^{(i)}C_i\Big(\sum_{E\in T^{(i)}_{h_i,\Omega_i}}\big(h_i^{lp+p(\frac{d}{2}-\frac{d}{p}-1)}\|u\|^p_{W^{l,p}(D^{(i)}_E)}\big)\Big)^{\frac{2}{p}}
\end{aligned}$$

$$\leq \alpha^{(i)} C_i \Big(h_i^{l+(\frac{d}{2}-\frac{d}{p}-1)} \|u\|_{W^{l,p}(\Omega_i)} \Big)^2. \tag{5.11}$$

Moreover, by (5.11) we can deduce that

$$\frac{\alpha^{(j)} h_i}{h_j} \frac{1}{h_i} \|u^{(i)} - \Pi_h u^{(i)}\|^2_{L^2(F_{ij})} \leq C_i \frac{\alpha^{(j)} h_i}{h_j} \Big(h_i^{l+(\frac{d}{2}-\frac{d}{p}-1)} \|u\|_{W^{l,p}(\Omega_i)} \Big)^2, \tag{5.12}$$

similarly

$$\frac{\alpha^{(i)} h_j}{h_i} \frac{1}{h_j} \|u^{(j)} - \Pi_h u^{(j)}\|^2_{L^2(F_{ji})} \leq C_i \frac{\alpha^{(i)} h_j}{h_i} \Big(h_j^{l+(\frac{d}{2}-\frac{d}{p}-1)} \|u\|_{W^{l,p}(\Omega_j)} \Big)^2. \tag{5.13}$$

Now, we return to the left hand side of (5.3b) and use (5.11), (5.12) and (5.13), to obtain

$$\begin{aligned}
\Big(\frac{\alpha^{(j)}}{h_j} + \frac{\alpha^{(i)}}{h_i} \Big) \|[\![u - \Pi_h u]\!]\|^2_{L^2(F_{ij})}
&\leq \frac{\alpha^{(j)} h_i}{h_j} \frac{1}{h_i} \|u^{(i)} - \Pi_h u^{(i)}\|^2_{L^2(F_{ij})} \\
&\quad + \frac{\alpha^{(i)} h_j}{h_i} \frac{1}{h_j} \|u^{(j)} - \Pi_h u^{(j)}\|^2_{L^2(F_{ji})} \\
&\quad + \frac{\alpha^{(j)}}{h_j} \|u^{(j)} - \Pi_h u^{(j)}\|^2_{L^2(F_{ji})} \\
&\quad + \frac{\alpha^{(i)}}{h_i} \|u^{(i)} - \Pi_h u^{(i)}\|^2_{L^2(F_{ij})} \\
&\leq C_i \frac{\alpha^{(j)} h_i}{h_j} \Big(h_i^{l+(\frac{d}{2}-\frac{d}{p}-1)} |u|_{W^{l,p}(\Omega_i)} \Big)^2 \\
&\quad + C_j \frac{\alpha^{(i)} h_j}{h_i} \Big(h_j^{l+(\frac{d}{2}-\frac{d}{p}-1)} \|u\|_{W^{l,p}(\Omega_j)} \Big)^2 \\
&\quad + C_j \Big(h_j^{l+(\frac{d}{2}-\frac{d}{p}-1)} \|u\|_{W^{l,p}(\Omega_j)} \Big)^2 \\
&\quad + C_i \Big(h_i^{l+(\frac{d}{2}-\frac{d}{p}-1)} \|u\|_{W^{l,p}(\Omega_i)} \Big)^2.
\end{aligned} \tag{5.14}$$

For the proof (5.3c), we recall the definition (4.19) for $u - \Pi_h u$ and have

$$\|u - \Pi_h u\|^2_{dG} = \sum_{i=1}^{N} \Big(\alpha^{(i)} \|\nabla(u^{(i)} - \Pi_h u^{(i)})\|^2_{L^2(\Omega_i)} + \sum_{i=1}^{N} \sum_{F_{ij} \subset \partial\Omega_i} \Big(\frac{\mu\alpha^{(i)}}{h_i} + \frac{\mu\alpha^{(j)}}{h_j} \Big) \|[\![u - \Pi_h u]\!]\|^2_{L^2(F_{ij})} \Big). \tag{5.15}$$

Estimating the first term on the right hand side in (5.15) by (5.2) and the second term by (5.3b), the approximation estimate (5.3c) follows. □

We need further discrete coercivity, consistency and boundedness. The discrete coercivity (Lemma 6) can also be applied here. Using the same arguments as in Lemma 9, we can prove the consistency for u. Due to assumed regularity of the solution, the normal interface flux $(\alpha\nabla u)|_{\Omega_i} \cdot \mathbf{n}_{F_{ij}}$ belongs (in general) to $L^p(F_{ij})$. Thus, we need to prove the boundedness for $a_h(.,.)$ by estimating the flux terms (3.16d) in different way than this in Lemma 7. We work in a similar way as in [26] and show bounds for the interface fluxes in $\|.\|_{L^p}$ setting.

Lemma 11 *There is a constant C such that the following inequality for $(u, \phi_h) \in W_h^{l,p} \times \mathbb{B}_h(\mathcal{S}(\Omega))$ holds true*

$$\begin{aligned}
&\sum_{i=1}^{N} \sum_{F_{ij} \subset \partial\Omega_i} \frac{1}{2} \int_{F_{ij}} \{\alpha \nabla u\} \cdot \mathbf{n}_{F_{ij}} [\![\phi_h]\!] \, ds \leq \\
&C \Big(\sum_{i=1}^{N} \sum_{F_{ij} \subset \partial\Omega_i} \alpha^{(i)} h_i^{1+\gamma_{p,d}} \|\nabla u^{(i)}\|^p_{L^p(F_{ij})} \\
&+ \alpha^{(j)} h_j^{1+\gamma_{p,d}} \|\nabla u^{(j)}\|^p_{L^p(F_{ij})} \Big)^{\frac{1}{p}} \|\phi_h\|_{dG},
\end{aligned}$$

where $\gamma_{p,d} = \frac{1}{2} d(p-2)$. (5.16)

Proof For the interface edge $e_{ij} \subset F_{ij}$ Hölder inequality yield

$$\begin{aligned}
&\frac{1}{2} \int_{e_{ij}} \frac{1}{2} |\alpha^{(i)} \nabla u^{(i)} + \alpha^{(j)} \nabla u^{(j)}| |[\![\phi_h]\!]| \, ds \\
&\leq C \int_{e_{ij}} (\alpha^{(i)} h_i^{1+\gamma_{p,d}})^{\frac{1}{p}} |\nabla u^{(i)}| \frac{{\alpha^{(i)}}^{\frac{1}{q}}}{h_i^{\frac{1+\gamma_{p,d}}{p}}} |[\![\phi_h]\!]| \\
&\quad + (\alpha^{(j)} h_j^{1+\gamma_{p,d}})^{\frac{1}{p}} |\nabla u^{(j)}| \frac{{\alpha^{(j)}}^{\frac{1}{q}}}{h_j^{\frac{1+\gamma_{p,d}}{p}}} |[\![\phi_h]\!]| \, ds \\
&\leq C (\alpha^{(i)} h_i^{1+\gamma_{p,d}})^{\frac{1}{p}} \|\nabla u^{(i)}\|_{L^p(e_{ij})} \frac{{\alpha^{(i)}}^{\frac{1}{q}}}{h_i^{\frac{1+\gamma_{p,d}}{p}}} \|[\![\phi_h]\!]\|_{L^q(e_{ij})} \\
&\quad + C (\alpha^{(j)} h_j^{1+\gamma_{p,d}})^{\frac{1}{p}} \|\nabla u^{(j)}\|_{L^p(e_{ij})} \frac{{\alpha^{(j)}}^{\frac{1}{q}}}{h_j^{\frac{1+\gamma_{p,d}}{p}}} \|[\![\phi_h]\!]\|_{L^q(e_{ij})}.
\end{aligned} \tag{5.17}$$

We employ the inverse inequality (4.18) with $p = q > 2$, $q = 2$ and use the analytical form $\frac{1+\gamma_{p,d}}{p} = \frac{2+d(p-2)}{2p}$ to express the jump terms in (5.17) in the *convenient L^2 form* as follows

$$\frac{\alpha^{(i)\frac{1}{q}}}{h_i^{\frac{2+d(p-2)}{2p}}}\|[\![\phi_h]\!]\|_{L^q(e_{ij})} \le C_{inv,p,2}\alpha^{(i)\frac{1}{q}} h_i^{(d-1)(\frac{1}{q}-\frac{1}{2})-\frac{2+d(p-2)}{2p}} \|[\![\phi_h]\!]\|_{L^2(e_{ij})} \le C_{inv,p,2}\alpha^{(i)\frac{1}{q}} h_i^{\frac{-1}{2}} \|[\![\phi_h]\!]\|_{L^2(e_{ij})}. \tag{5.18}$$

Inserting the result (5.18) into (5.17) and summing over all $e_{ij} \in F_{ij}$ we obtain for $q > 2$,

$$\begin{aligned}\frac{1}{2}\int_{F_{ij}} \{\alpha\nabla u\}\cdot \mathbf{n}_{F_{ij}}[\![\phi_h]\!]\,ds &\le C\sum_{e_{ij}\in F_{ij}}\int_{e_{ij}} |\alpha^{(i)}\nabla u^{(i)} +\alpha^{(j)}\nabla u^{(j)}||[\![\phi_h]\!]|\,ds\\ &\le C\Big(\sum_{e_{ij}\in F_{ij}} \alpha^{(i)} h_i^{1+\gamma_{p,d}}\|\nabla u^{(i)}\|^p_{L^p(e_{ij})}\Big)^{\frac{1}{p}} \Big(\sum_{e_{ij}\in F_{ij}} \alpha^{(i)}\big(\frac{1}{h_i^{\frac{1}{2}}}\|[\![\phi_h]\!]\|_{L^2(e_{ij})}\big)^q\Big)^{\frac{1}{q}}\\ &+C\Big(\sum_{e_{ij}\in F_{ij}} \alpha^{(j)} h_j^{1+\gamma_{p,d}}\|\nabla u^{(j)}\|^p_{L^p(e_{ij})}\Big)^{\frac{1}{p}} \Big(\sum_{e_{ij}\in F_{ij}} \alpha^{(j)}\big(\frac{1}{h_j^{\frac{1}{2}}}\|[\![\phi_h]\!]\|_{L^2(e_{ij})}\big)^q\Big)^{\frac{1}{q}}.\end{aligned} \tag{5.19}$$

Now, using that the function $f(x) = (\lambda\alpha^x + \lambda\beta^x)^{\frac{1}{x}}$, $\lambda > 0, x > 2$ is decreasing, we estimate the "q-power terms" in the sum of the right hand side in (5.19) as follows

$$\begin{aligned}\Big(\sum_{e_{ij}\in F_{ij}} \alpha^{(j)}\big(\frac{1}{h_j^{\frac{1}{2}}}\|[\![\phi_h]\!]\|_{L^2(e_{ij})}\big)^q\Big)^{\frac{1}{q}} &\le \Big(\sum_{e_{ij}\in F_{ij}} \alpha^{(j)}\big(\frac{1}{h_j^{\frac{1}{2}}}\|[\![\phi_h]\!]\|_{L^2(e_{ij})}\big)^2\Big)^{\frac{1}{2}}\\ &\le \Big(\big(\frac{\mu\alpha^{(i)}}{h_i}+\frac{\mu\alpha^{(j)}}{h_j}\big)\|[\![\phi_h]\!]\|^2_{L^2(F_{ij})}\Big)^{\frac{1}{2}}.\end{aligned} \tag{5.20}$$

Applying (5.20) into (5.19) we get

$$\begin{aligned}\frac{1}{2}\int_{F_{ij}} \{\alpha\nabla u\}\cdot \mathbf{n}_{F_{ij}}[\![\phi_h]\!]\,ds &\le 2C\Big(\alpha^{(i)} h_i^{1+\gamma_{p,d}}\|\nabla u^{(i)}\|^p_{L^p(F_{ij})} +\alpha^{(j)} h_j^{1+\gamma_{p,d}}\|\nabla u^{(j)}\|^p_{L^p(F_{ij})}\Big)^{\frac{1}{p}}\\ &\Big(\big(\frac{\mu\alpha^{(i)}}{h_i}+\frac{\mu\alpha^{(j)}}{h_j}\big)\|[\![\phi_h]\!]\|^2_{L^2(F_{ij})}\Big)^{\frac{1}{2}}.\end{aligned} \tag{5.21}$$

In (5.21), we sum over all $F_{ij} \subset \partial\Omega_i$ for all $i = 1, \ldots, N$ and consequently we apply Hölder inequality

$$\begin{aligned}\frac{1}{2}\sum_{F_{ij}}\int_{F_{ij}} \{\alpha\nabla u\}[\![\phi_h]\!]\,ds &\le 2C\Big(\sum_{F_{ij}}\alpha^{(i)} h_i^{1+\gamma_{p,d}}\|\nabla u^{(i)}\|^p_{L^p(F_{ij})} +\alpha^{(j)} h_j^{1+\gamma_{p,d}}\|\nabla u^{(j)}\|^p_{L^p(F_{ij})}\Big)^{\frac{1}{p}}\\ &\Big(\sum_{F_{ij}}\Big(\big(\frac{\mu\alpha^{(i)}}{h_i} +\frac{\mu\alpha^{(j)}}{h_j}\big)\|[\![\phi_h]\!]\|^2_{L^2(F_{ij})}\Big)^{\frac{q}{2}}\Big)^{\frac{1}{q}}.\end{aligned} \tag{5.22}$$

Following in much the same arguments as in proof of (5.20), we can bound the last $\sum_{F_{ij}}$ in (5.22) as

$$\begin{aligned}\Big(\sum_{F_{ij}}\Big(\big(\frac{\mu\alpha^{(i)}}{h_i}+\frac{\mu\alpha^{(j)}}{h_j}\big)\|[\![\phi_h]\!]\|^2_{L^2(F_{ij})}\Big)^{\frac{q}{2}}\Big)^{\frac{1}{q}} &\le \Big(\sum_{F_{ij}}\big(\frac{\mu\alpha^{(i)}}{h_i}+\frac{\mu\alpha^{(j)}}{h_j}\big)\|[\![\phi_h]\!]\|^2_{L^2(F_{ij})}\Big)^{\frac{1}{2}}\\ &\le \|\phi_h\|_{dG}.\end{aligned} \tag{5.23}$$

Employing (5.23) in (5.22), we can easily obtain (5.16). □

Lemma 12 *There is a C independent of h_i such that $\forall(u,\phi_h) \in W_h^{l,p} \times \mathbb{B}_h(\mathcal{S}(\Omega))$*

$$a_h(u,\phi_h) \le C\Big(\|u\|^p_{dG} + \sum_{i=1}^N\sum_{F_{ij}\subset\partial\Omega_i} h_i^{1+\gamma_{p,d}}\alpha^{(i)}\|\nabla u^{(i)}\|^p_{L^p(F_{ij})} + h_j^{1+\gamma_{p,d}}\alpha^{(j)}\|\nabla u^{(j)}\|^p_{L^p(F_{ij})}\Big)^{\frac{1}{p}}\|\phi_h\|_{dG}. \tag{5.24}$$

Proof We estimate the terms of $a_h(u,\phi_h)$ in (3.16b) separately. Applying Cauchy-Schwartz for the terms (3.16c) and (3.16e) we have

$$\sum_{i=1}^N a_i(u,\phi_h) \le C\|u\|_{dG}\|\phi_h\|_{dG} \tag{5.25a}$$

$$\sum_{i=1}^N\sum_{F_{ij}\subset\partial\Omega_i} p_{F_{ij}}(u,\phi_h) \le C\|u\|_{dG}\|\phi_h\|_{dG}. \tag{5.25b}$$

For the term (3.16d) we use Lemma 11

$$\sum_{i=1}^N s_i(u,\phi_h) \le C\Big(\sum_{i=1}^N\sum_{F_{ij}\subset\partial\Omega_i} \alpha^{(i)} h_i^{1+\gamma_{p,d}}\|\nabla u^{(i)}\|^p_{L^p(F_{ij})}$$

$$+\alpha^{(j)}h_j^{1+\gamma_{p,d}}\|\nabla u^{(j)}\|^p_{L^p(F_{ij})}\Big)^{\frac{1}{p}}\|\phi_h\|_{dG}, \tag{5.26}$$

Combining (5.25) with (5.26) we can derive (5.24). □

Next, we prove the main convergence result of this section.

Theorem 2 *Let $u \in W^{l,p}_{\mathcal{S}}$, $l \geq 2$, $p \in (\max\{1, \frac{2d}{d+2(l-1)}\}, 2]$ be the solution of (2.2a). Let $u_h \in \mathbb{B}_h(\mathcal{S}(\Omega))$ be the dG-IgA solution of (3.16a) and $\Pi_h u \in \mathbb{B}_h(\mathcal{S}(\Omega))$ is the interpolant of Lemma 10. Then there are constants C_i specified by the constants of (5.3c), (5.16) and (5.24), such that*

$$\|u-u_h\|_{dG} \leq \sum_{i=1}^{N}\Big(C_i\Big(h_i^{\delta(p,d)} + \sum_{F_{ij}\subset\partial\Omega_i}\alpha^{(j)}\big(\frac{h_i}{h_j}\big)^{\frac{1}{2}}h_i^{\delta(p,d)}\Big)\|u\|_{W^{l,p}(\Omega_i)}\Big), \tag{5.27}$$

where $\delta(p,d) = l + (\frac{d}{2} - \frac{d}{p} - 1)$.

Proof Since $(u_h - \Pi_h u) \in \mathbb{B}_h(\mathcal{S}(\Omega))$ by the discrete coercivity (4.21) we have

$$\|u_h - \Pi_h u\|^2_{dG} \leq a_h(u_h - \Pi_h u, u_h - \Pi_h u). \tag{5.28}$$

By orthogonality we have

$$\begin{aligned}\|u_h - \Pi_h u\|^2_{dG} &\leq a_h(u_h - \Pi_h u, u_h - \Pi_h u)\\ &= a_h\big((u_h - u) + (u - \Pi_h u), u_h - \Pi_h u\big)\\ &= a_h\big(u - \Pi_h u, u_h - \Pi_h u\big)\\ &\leq C\Big(\|u - \Pi_h u\|_{dG} + \Big(\sum_{F_{ij}\subset\partial\Omega_i} h_i^{1+\gamma_{p,d}}\alpha^{(i)}\|\nabla u^{(i)}\\ &-\Pi_h u^{(i)}\|^p_{L^p(F_{ij})}\\ &+h_j^{1+\gamma_{p,d}}\alpha^{(j)}\|\nabla u^{(j)}\\ &-\Pi_h u^{(j)}\|^p_{L^p(F_{ij})}\Big)^{\frac{1}{p}}\Big)\|u_h - \Pi_h u\|_{dG},\end{aligned}$$

where immediately we get

$$\begin{aligned}\|u_h - \Pi_h u\|_{dG} &\leq \|u - \Pi_h u\|_{dG}\\ &+\Big(\sum_{F_{ij}\subset\partial\Omega_i} h_i^{1+\gamma_{p,d}}\alpha^{(i)}\|\nabla u^{(i)} - \Pi_h u^{(i)}\|^p_{L^p(F_{ij})}\\ &+h_j^{1+\gamma_{p,d}}\alpha^{(j)}\|\nabla u^{(j)}\\ &-\Pi_h u^{(j)}\|^p_{L^p(F_{ij})}\Big)^{\frac{1}{p}}.\end{aligned} \tag{5.29}$$

Now, using triangle inequality, the estimates (5.3) and the bound (5.16) in (5.29), we obtain

$$\|u_h - u\|_{dG} \leq \sum_{i=1}^{N} C_i\Big(h_i^{\delta(p,d)} + \sum_{F_{ij}\subset\partial\Omega_i}\alpha^{(j)}\big(\frac{h_i}{h_j}\big)^{\frac{1}{2}}h_i^{\delta(p,d)}\Big)\|u\|_{W^{l,p}(\Omega_i)}, \tag{5.30}$$

which is the required error estimate (5.27). □

6 Numerical examples

In this section, we present a series of numerical examples to validate numerically the theoretical results of the previous Sections. We first validate the estimates by considering the model problem with uniform diffusion coefficients for both cases regular and low regular exact solutions. The second example comes from [19], where an appropriate choice of highly heterogeneous coefficients on the interfaces produces a low regularity solution. In the third example, we solve the problem utilizing non-matching meshes and study the influence of the term $\frac{h_i}{h_j}$ on the convergence rates. We have chosen the domain to be a circular sector where the parametrization mapping, see (3.5), has a singular point. All the numerical tests have been performed in G+SMO[1].

1 Regular solution and solution with an interior point singularity
We consider the problem in $\Omega = (\frac{-1}{2}, \frac{1}{2})^{d=3}$, with $\Gamma_D = \partial\Omega$ and $\alpha = 1$ uniformly in Ω. The domain Ω is subdivided in four equal subdomains Ω_i, $i = 1,\ldots,4$, where for simplicity every Ω_i is initially partitioned into a mesh $T^{(i)}_{h_i,\Omega_i}$ with $h := h_i = h_j, i \neq j, i,j = 1,\ldots,4$. Successive uniform refinements are performed on every $T^{(i)}_{h_i,\Omega_i}$ in order to compute numerically the convergence rates. In the first test, the data u_D and f in (2.1) are determined so that the exact solution is given by $u(x) = \sin(2.5\pi x)\sin(2.5\pi y)\sin(2.5\pi z)$ (smooth test case). The first two columns of Table 1 display the convergence rates. As it was expected, the convergence rates are optimal. In the second case, the exact solution is $u(x) = |x|^\lambda$. The parameter λ is chosen such that $u \in W^{l,p=1.4}(\Omega)$. In the last columns of Table 1, we display the convergence rates for degree $k = 2$, $k = 3$ and $l = 2$, $l = 3$. We observe that, for each of the two tests, the error in the dG-norm behaves according to the main error estimate given by (5.27).

[1] G+SMO: http://www.gs.jku.at/gs-gismo.shtml

Table 1 Regular/interior point singularity test: the numerical convergence rates

$h/2^s$	Highly smooth		$k=2$		$k=3$	
	$k=2$	$k=3$	$l=2$	$l=3$	$l=2$	$l=3$
–	Convergence rates					
$s=0$	–	–	–	–	–	–
$s=1$	0.15	2.91	0.62	0.76	0.24	1.64
$s=2$	2.34	2.42	0.29	1.10	0.28	0.89
$s=3$	2.08	3.14	0.35	1.32	0.47	1.25
$s=4$	2.02	3.04	0.35	1.36	0.36	1.37

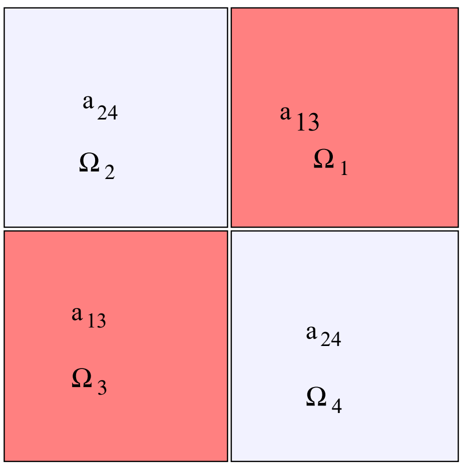

Fig. 2 Kellog's test: Ω_i, $i=1,\ldots,4$

2 Kellogg's test problem, highly heterogeneous diffusion coefficients
The solutions of problem (2.1) with rough diffusion coefficients may not be very smooth. We examine such a case by solving the so-called Kellogg test problem [19]. We consider the computational domain $\Omega=(-1,1)^2$, which is subdivided into four subdomains Ω_i, $i=1,\ldots,4$, see Fig. 2. We choose a piecewise constant diffusion coefficient α in (2.1), having the same value, say $\alpha:=\alpha_{13}$, in Ω_1 and Ω_3, similarly, $\alpha:=\alpha_{24}$ in the Ω_2 and Ω_4, see Fig. 2. The exact solution of the problem for $f=0$ is given in polar coordinates by $u(r,\theta)=r^\lambda\varphi(\theta)$, where

$$\varphi(\theta)=\begin{cases}\cos((\pi/2-\sigma)\lambda)\cos((\theta-\pi/2+\rho)\lambda), & \text{if } 0\le\theta<\pi/2,\\ \cos(\rho\lambda)\cos((\theta-\pi+\sigma)\lambda), & \text{if } \pi/2\le\theta<\pi,\\ \cos(\sigma\lambda)\cos((\theta-\pi-\rho)\lambda), & \text{if } \pi\le\theta<3\pi/2,\\ \cos((\pi/2-\rho)\lambda)\cos((\theta-3\pi/2-\sigma)\lambda), & \text{if } 3\pi/2\le\theta\le 2\pi,\end{cases}$$

and the numbers λ,ρ,σ satisfy the nonlinear relations

Table 2 Kellog's test: the convergence rates

$h/2^s$	$k=2,\ \lambda=0.3$ Rates
$s=0$	–
$s=1$	0.318
$s=2$	0.312
$s=3$	0.308
$s=4$	0.305
$s=5$	0.303

$$\begin{cases}-\tan((\pi/2-\sigma)\lambda)\cot(\rho\lambda)=\mathfrak{R},\\ -\tan(\rho\lambda)\cot(\sigma\lambda)=1/\mathfrak{R},\\ -\tan(\sigma\lambda)\cot((\pi/2-\rho)\lambda)=\mathfrak{R},\\ 0<\lambda<2,\\ \max\{0,\pi\lambda-\pi\}<2\lambda\rho<\min\{\pi\lambda,\pi\},\\ \max\{0,\pi-\pi\lambda\}<-2\lambda\sigma<\min\{\pi,2\pi-\lambda\pi\},\end{cases} \tag{6.1}$$

where $\mathfrak{R}:=\alpha_{13}/\alpha_{24}$. The above system of equations admits several solutions. For this example, we set $\lambda=0.3$, and $\rho=\pi/4$. A Newton iteration recovers one solution for the rest of the parameters, namely $\sigma=-4.4505895927$ and $\mathfrak{R}=17.34972217$. Note that in this case $u\in W^{1.3,2}(\Omega)$. We solved the problem using B-spline spaces with degree $k=2$. In Table 2, we display the convergence rates of the error. We can see that, the experimental convergence rates approach the value 0.3, which is in agreement with the regularity of the solution and (5.27).

3 Circular sector domain, non-matching meshes
Consider the problem on a circular sector domain described in polar coordinates as $\Omega=\{(r,\phi):0\le r\le 3, 0\le\phi\le\frac{\pi}{2}\}$. The diffusion coefficient is set to be uniformly $\alpha=1$ in Ω, the function f and the boundary condition u_D in (2.1) are determined to have exact solution $u(x,y)=\sin(2\pi x)\sin(2\pi y)$. The domain Ω is divided into two subdomains Ω_1 (circular sector) and Ω_2 (annulus) and non-matching meshes are considered, as it is presented in Fig. 3. We performed three numerical tests, where the mesh size h_1 of the circular sector and the mesh size h_2 of the annulus are such that $\frac{h_1}{h_2}=\kappa$ with $\kappa=2,4$ and 8 correspondingly. The results for the three test cases are displayed in Table 3. The second, fourth and sixth column include the error $\|u_h-u\|_{dG(\Omega)}$ values. The third, the fifth and the seventh column include the convergence rates of the error for the different ratio κ. We observe that the convergence rates for all cases of the non-matching meshes, tend to get the optimal value with respect to the B-spline degree $k=2$, (see also estimate (4.27c)). Thus, for this problem the different grid

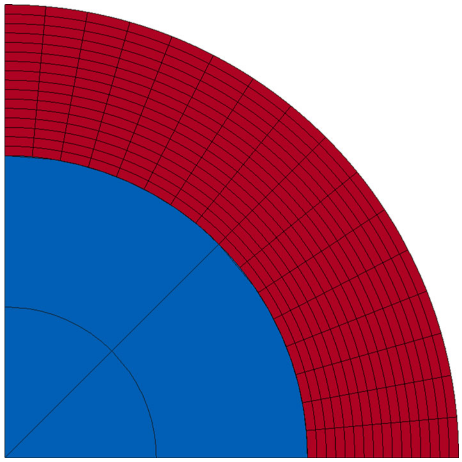

Fig. 3 Circular sector: the two subdomains

Table 3 Circular sector: the $\|u - u_h\|_{dG(\Omega)}$ values and the convergence rates for $h_1/h_2 = \kappa$

$h/2^s$	$k = 2,\ h_1/h_2 = 2$		$k = 2,\ h_1/h_2 = 4$		$k = 2,\ h_1/h_2 = 8$	
	$\|u_h - u\|_{dG}$	rates	$\|u_h - u\|_{dG}$	rates	$\|u_h - u\|_{dG}$	rates
$s = 1$	1.65054	–	1.72928	–	1.73149	–
$s = 2$	0.446972	1.88	0.371505	2.21	0.355404	2.28
$s = 3$	0.0313984	3.83	0.0301854	3.62	0.0298388	3.57
$s = 4$	0.00660832	2.24	0.00656384	2.20	0.00655856	2.18
$s = 5$	0.00161788	2.03	0.00161349	2.02	0.00161317	2.02
$s = 6$	0.000403005	2.00	0.000402154	2.00	0.000402099	2.00

sizes of the subdomains do not influence the behavior of the convergence rate. We point out that for the present example the point (0, 0) is a singular point and the Jacobian determinant is vanished. As we have seen in the results in Table 3, this singularity of the mapping did not affect the convergence rates. However, if we continue the refinement steps reaching the limits of our code, we would have numerical problems. This occurs, because in that case, the quadrature points would be located too close to the singularity of the mapping, and thus we might be lead to division by (almost) zero while performing the numerical integration. We mention that, similar type problems, where the parametrization mappings have singular points, have been solved and discussed more thoroughly by the authors in [23].

7 Conclusions

In this paper, we presented theoretical error estimates of the dG-IgA method applied to a model elliptic problem with discontinuous coefficients. The problem was discretized according to IgA methodology using discontinuous $\mathbb{B}$-spline spaces. Due to global discontinuity of the approximate solution on the subdomain interfaces, dG discretizations techniques were utilized. In the first part, we assumed higher regularity for the exact solution, that is $u \in W^{l\geq 2,2}$, and we showed optimal error estimates with respect to $\|.\|_{dG}$. In the second part, we assumed low regularity for the exact solution, that is $u \in W^{l,p}$ for $l \geq 2$ and $p \in (\frac{2d}{d+2(l-1)}, 2)$, applying the Sobolev embedding theorem we proved optimal convergence rates with respect to $\|.\|_{dG}$. The theoretical error estimates were validated by numerical tests. The results can obviously be carried over to diffusion problems on open and closed surfaces as studied in [24], and to more general second-order boundary value problems like linear elasticity problems as studied in [2].

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