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Vorticity–velocity–pressure formulation for Navier–Stokes equations

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Abstract. We analyze here the bidimensional boundary value problems, for both Stokes and Navier–Stokes equations, in the case where non standard boundary conditions are imposed. A well-posed vorticity–velocity–pressure formulation for the Stokes problem is introduced and its finite element discretization, which needs some stabilization, is then studied. We consider next the approximation of the Navier–Stokes equations, based on the previous approximation of the Stokes equations. For both problems, the convergence of the numerical approximation and optimal error estimates are obtained. Some numerical tests are also presented.

1 Introduction

We are interested in this paper in the stationary Navier–Stokes problem satisfying physical boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^2$, simply connected with a polygonal boundary $\Gamma = \partial\Omega$ such that Ω is on one side only of its boundary.

We begin by studying the linear Stokes equations with the same non-standard boundary conditions, for which we propose a three-fields variational formulation. After showing that this new vorticity–velocity–pressure formulation is well-posed, we discretize it by means of conforming low-order finite elements. The discrete *inf-sup* condition is then obvious, while the discrete coercivity is obtained by adding a stabilization term. The stabilization form is given by the jumps of the discrete vorticity and pressure on the internal edges of the triangulation. Optimal error bounds are deduced in a technical way. We thus obtain that the method is unconditionally convergent, i.e. without any particular hypothesis of regularity.

Next, we consider the Navier–Stokes equations and our goal is to propose a well-posed numerical approximation for this problem and to prove convergence as well as optimal error estimates. To do that, we first write the nonlinear term in an equivalent way, by means of a modified pressure. Thus, we can write the Navier–Stokes operator in terms of the previously introduced Stokes operator. To deal with the nonlinear aspects of the problem, we apply a variant of the implicit function theorem, which can be found for instance in Brezzi et al. 1980 or in Pousin and Rappaz 1994. The discretization is based on the approximation of the Stokes equations. Numerical tests illustrating the theoretical results are next presented, for both the Stokes and the Navier–Stokes cases.

For the sake of simplicity, we consider here homogeneous boundary conditions, but the method also applies to the non-homogeneous case.

The paper is organized as follows. In the next section we introduce the functional framework and the variational formulation corresponding to the Stokes problem. In Sect. 3 we describe the discretization method and we establish the error estimates. Section 4 deals with the continuous Navier–Stokes problem, while in Sect. 5 we present its approximation and we prove the well-posedness and the convergence results for the discrete nonlinear problem. Finally, in the last section some numerical tests are presented.

2 The Stokes problem

We begin by introducing some notations. For any $2D$ vector field $\boldsymbol{v} = (v_1, v_2)^t$, we denote $\boldsymbol{v}^\perp = (v_2, -v_1)^t$ and also :

$$div\, \boldsymbol{v} = \partial_1 v_1 + \partial_2 v_2, \ \ curl\, \boldsymbol{v} = \partial_1 v_2 - \partial_2 v_1$$

and, for any scalar field ϕ, $\mathbf{curl}\,\phi = (\partial_2\phi, -\partial_1\phi)^t$. We suppose that Γ is composed of three open and disjoint subsets $\Gamma_1, \Gamma_2, \Gamma_3$ such that $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$ and $\Gamma_2 \neq \emptyset$; we denote, as usually, by $\boldsymbol{n}$ the outward normal vector and by $\boldsymbol{t}$ the tangent vector to the boundary Γ.

We consider the incompressible Stokes equations

$$\begin{array}{ll} -\nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega \\ div\,\boldsymbol{u} = 0 & \text{in } \Omega \end{array}$$

and impose the following physical boundary conditions:

$$\begin{cases} \boldsymbol{u}\cdot\boldsymbol{n} = 0\,, & \boldsymbol{u}\cdot\boldsymbol{t} = 0 \quad \text{on } \Gamma_1 \\ \boldsymbol{u}\cdot\boldsymbol{t} = 0\,, & p = 0 \quad \text{on } \Gamma_2 \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0\,, & \omega = 0 \quad \text{on } \Gamma_3\,, \end{cases} \tag{1}$$

where $\omega = curl\,\boldsymbol{u}$ represents the scalar vorticity. The data $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ is given, as well as the kinematic viscosity $\nu > 0$.

Our approach also applies to non-homogeneous boundary conditions, see Amara et al. 2001.

Hence, we want to find a $2D$ velocity field $\boldsymbol{u}$ and two scalar fields ω and p satisfying

$$\begin{cases} \nu\,\mathbf{curl}\,\omega + \nabla p = \boldsymbol{f} & \text{in } \Omega \\ \omega = curl\,\boldsymbol{u} & \text{in } \Omega \\ div\,\boldsymbol{u} = 0 & \text{in } \Omega\,, \end{cases} \tag{2}$$

together with the boundary conditions (1).

In order to write a variational formulation of this problem, let us introduce the Hilbert spaces:

$$\boldsymbol{M} = \left\{ \boldsymbol{v} \in \boldsymbol{H}(div, curl; \Omega)\,;\ \boldsymbol{v}\cdot\boldsymbol{n}\,\big|_{\Gamma_1\cup\Gamma_3} = \boldsymbol{v}\cdot\boldsymbol{t}\big|_{\Gamma_1\cup\Gamma_2} = 0 \right\}$$
$$\boldsymbol{X} = \boldsymbol{L}^2(\Omega)\,.$$

The spaces $\boldsymbol{H}(div, curl; \Omega)$ and $\boldsymbol{M}$ are both normed by

$$\|\boldsymbol{v}\|_{\boldsymbol{M}} = \left(\|\boldsymbol{v}\|^2_{0,\Omega} + \|div\,\boldsymbol{v}\|^2_{0,\Omega} + \|curl\,\boldsymbol{v}\|^2_{0,\Omega}\right)^{1/2}$$

and we also introduce the seminorm

$$|\boldsymbol{v}|_{\boldsymbol{M}} = \left(\|div\,\boldsymbol{v}\|^2_{0,\Omega} + \|curl\,\boldsymbol{v}\|^2_{0,\Omega}\right)^{1/2}\,.$$

The following result will be used:

Lemma 1. *$\boldsymbol{M}$ is continuously embedded in $\boldsymbol{H}^s(\Omega)$, for some $s \in]1/2, 1]$.*

We assume in all this paper that one of the following situations hold : $|\Gamma_1| > 0$, or $|\Gamma_1| = |\Gamma_3| = 0$, or $|\Gamma_1| = 0$ and $|\Gamma_3| > 0$ with Γ_3 simply connected, where $|\cdot|$ denotes the Lebesgue measure.

Lemma 2. *Under the previous assumption, the seminorm $|\cdot|_{\boldsymbol{M}}$ is equivalent to the norm $\|\cdot\|_{\boldsymbol{M}}$ in $\boldsymbol{M}$.*

We will denote by $(\cdot,\cdot)$ the scalar product in $L^2(\Omega)$.

A variational formulation for the Stokes problem is given by:

$$\begin{cases} \text{Find } (\sigma, \boldsymbol{u}) \in \boldsymbol{X}\times\boldsymbol{M} \text{ such that} \\ a(\sigma,\tau) + b(\tau,\boldsymbol{u}) = 0 \ \ \forall\tau \in \boldsymbol{X}\,, \\ b(\sigma,\boldsymbol{v}) = -l(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{M}\,, \end{cases} \tag{3}$$

where, for all $\sigma = (\omega, p) \in \boldsymbol{X}$, $\tau = (\theta, q) \in \boldsymbol{X}$ and $\boldsymbol{v} \in \boldsymbol{M}$,

$$a(\sigma,\tau) = \nu(\omega,\theta)$$
$$b(\tau,\boldsymbol{v}) = -\nu(\theta, curl\,\boldsymbol{v}) + (q, div\,\boldsymbol{v})$$
$$l(\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})\,.$$

Then we can prove the next result:

Theorem 1. *The saddle point problem* (3) *has a unique solution $\sigma = (\omega, p) \in \boldsymbol{X}$ and $\boldsymbol{u} \in \boldsymbol{M}$, satisfying in $D'(\Omega)$ the equations* (2) *together with the boundary conditions* (1).

The proof is classical, based on the Babuska–Brezzi theorem for mixed formulations (see for instance Brezzi and Fortin 1991). One easily checks the *inf-sup* condition, as well as the coercivity of $a(\cdot,\cdot)$ on the kernel of $b(\cdot,\cdot)$.

From Lemma 1 and from the Sobolev theorem, it comes that $\boldsymbol{M}$ is also continuously embedded in $\boldsymbol{L}^4(\Omega)$.

Then one immediately gets that (3) is well-posed even for a less regular function $\boldsymbol{f} \in \boldsymbol{M}'$; in this case, $l(\boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle$ where $\langle\cdot,\cdot\rangle$ denotes the duality product between $\boldsymbol{M}'$ and $\boldsymbol{M}$. In particular, one can take $\boldsymbol{f} \in \boldsymbol{L}^{4/3}(\Omega) = (\boldsymbol{L}^4(\Omega))' \subset \boldsymbol{M}'$ and obtain a unique solution $(\sigma, \boldsymbol{u}) \in \boldsymbol{X}\times\boldsymbol{L}^4(\Omega)$.

Let us denote by S the previous Stokes operator

$$S : \boldsymbol{L}^{4/3}(\Omega) \longrightarrow \boldsymbol{X}\times\boldsymbol{L}^4(\Omega)\,,\quad S(\boldsymbol{f}) = (\sigma, \boldsymbol{u})\,,$$

which is clearly linear and continuous.

3 Finite element approximation

We are interested in the discretization of (3). Let $(T_h)_h$ be a regular family of triangulations of $\overline{\Omega}$, consisiting of triangles. We denote by h_K the diameter of the triangle K, by $h = \max_{K\in T_h} h_K$, by E_h the set of internal edges and by h_e the length of the edge e.

We consider the following finite dimensional spaces

$$L_h = \left\{ q_h \in L^2(\Omega)\,;\ q_{h|K} \in P_0(K)\ \forall K \in \mathcal{T}_h \right\}\,,$$
$$\boldsymbol{X}_h = L_h \times L_h\,,$$
$$\boldsymbol{M}_h = \left\{ \boldsymbol{v}_h \in \left(C^0(\overline{\Omega})\right)^2\,;\ \boldsymbol{v}_{h|K} \in \boldsymbol{P}_1(K)\ \forall K \in \mathcal{T}_h \right\} \cap \boldsymbol{M}$$
$$= \left\{ v_h \in \boldsymbol{M}\,;\ \boldsymbol{v}_{h|K} \in \boldsymbol{P}_1(K)\ \forall K \in \mathcal{T}_h \right\} \tag{4}$$

and we write the discrete problem as follows:

$$\begin{cases} \text{Find } (\sigma_h, \boldsymbol{u}_h) \in \boldsymbol{X}_h\times\boldsymbol{M}_h \text{ such that} \\ a(\sigma_h,\tau_h) + b(\tau_h,\boldsymbol{u}_h) = 0 \ \ \forall\tau_h \in \boldsymbol{X}_h\,, \\ b(\sigma_h,\boldsymbol{v}_h) = -l(\boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{M}_h\,. \end{cases} \tag{5}$$

Then the *inf-sup* condition, which represents the main difficulty in the velocity-pressure formulation for Stokes problem, is satisfied. However, we lose the coercivity of $a(\cdot,\cdot)$ on the discrete kernel

$$\boldsymbol{V}_h = \{\tau_h \in \boldsymbol{X}_h\,;\ b(\tau_h, \boldsymbol{v}_h) = 0\,,\ \forall \boldsymbol{v}_h \in \boldsymbol{M}_h\}\,.$$

One can only prove the following inequality, for all $\tau_h = (\theta_h, q_h) \in \boldsymbol{V}_h$:

$$c\,\|\tau_h\|^2_{\boldsymbol{X}} \le a(\tau_h,\tau_h) + \|\nu\,\mathbf{curl}\,\theta_h + \nabla q_h\|^2_{-1,\Omega}\,. \tag{6}$$

In order to retrieve the coercivity, we change the bilinear form $a(\cdot,\cdot)$ in a consistent way, without changing the spaces. For that, let us first define the jump of $\tau = (\theta, q) \in \boldsymbol{X}_h$ across the edge $e \in E_h \cup \Gamma$ by:

$$[\tau]_e = \begin{cases} \nu[\theta]\boldsymbol{t}_e + [q]\boldsymbol{n}_e & \text{if } e \in E_h \\ \nu\theta\boldsymbol{t}_e & \text{if } e \in \Gamma_3 \\ q\boldsymbol{n}_e & \text{if } e \in \Gamma_2 \end{cases}\,.$$

Then we can establish:

Lemma3. *For $\tau_h = (\theta_h, q_h) \in \boldsymbol{V}_h$ there exists a positive constant C, independent of h, such that:*

$$\|\nu\,\mathbf{curl}\,\theta_h + \nabla q_h\|_{-1,\Omega} \le C\left(\sum_{e\in E_h\cup\Gamma} h_e^2[\tau]_e^2\right)^{1/2}\,.$$

We consider now the bilinear form $A_h : \boldsymbol{X}_h \times \boldsymbol{X}_h \to \mathbb{R}$ given by

$$A_h(\delta_h, \tau_h) = \sum_{e \in E_h \cup \Gamma} h_e \left([\delta_h]_e , [\tau_h]_e \right)_e$$

and we define

$$a_{h,\beta}(\cdot,\cdot) = a(\cdot,\cdot) + \beta\, A_h(\cdot,\cdot)\,,$$

where $\beta > 0$ is a stabilization parameter, which can be chosen eventually independently of the discretization parameter h.

Then $a_h(\cdot,\cdot)$ is clearly $\boldsymbol{V}_h$-coercive, thanks to (6) and to Lemma 3, while its continuity on $\boldsymbol{X}_h \times \boldsymbol{X}_h$ comes from the estimate:

$$\left(\sum_{e \in E_h \cup \Gamma} h_e^2 [\tau]_e^2 \right)^{1/2} \le C' \, \|\tau_h\|_{\boldsymbol{X}}$$

with a positive constant C' independent of h.

Therefore, we consider in the sequel the discrete problem (5) where the bilinear form $a(\cdot,\cdot)$ was replaced by $a_{h,\beta}(\cdot,\cdot)$. This new problem clearly fulfills the hypotheses of the Babuska–Brezzi theorem, uniformly with respect to h, so it is well-posed. Moreover, for any $\boldsymbol{f} \in \boldsymbol{L}^{4/3}(\Omega)$ one has, since $\boldsymbol{M} \subset \boldsymbol{L}^4(\Omega)$, that

$$|l(v)| = \left| \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \right| \le c\, \|\boldsymbol{f}\|_{\boldsymbol{L}^{4/3}(\Omega)}\, |\boldsymbol{v}|_{\boldsymbol{M}}\ .$$

Let us denote by $S_h : \boldsymbol{L}^{4/3}(\Omega) \to \boldsymbol{X} \times \boldsymbol{L}^4(\Omega)$ the discrete Stokes operator, defined by $S_h(\boldsymbol{f}) = (\sigma_h, \boldsymbol{u}_h) \in \boldsymbol{X}_h \times \boldsymbol{M}_h$, the unique solution of the previous mixed problem. Then it comes from the Babuska–Brezzi theorem that S_h satisfies

$$\forall \boldsymbol{f} \in \boldsymbol{L}^{4/3}(\Omega)\,,\ \ \|S_h(\boldsymbol{f})\|_{\boldsymbol{X} \times \boldsymbol{L}^4(\Omega)} \le c\, \|\boldsymbol{f}\|_{\boldsymbol{L}^{4/3}(\Omega)} \qquad (7)$$

with c a positive constant independent of h but depending on β.

The next technical estimate was established in Amara et al. 2001:

Theorem 2. *Let $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and let $\bar{\sigma}_h$ denote the $\boldsymbol{L}^2(\Omega)$-projection of σ on $\boldsymbol{X}_h$. Then one has:*

$$\|(S - S_h)(\boldsymbol{f})\|_{\boldsymbol{X} \times \boldsymbol{M}} \le C \left\{ E_c + \|\sigma - \overline{\sigma}_h\|_{\boldsymbol{X}} + \inf_{\boldsymbol{v}_h \in \boldsymbol{M}_h} |\boldsymbol{u} - \boldsymbol{v}_h|_{\boldsymbol{M}} \right\},$$

where C is a constant independent of h (but depending on β) and E_c represents the consistency error:

$$E_c = \|\sigma - \overline{\sigma}_h\|_{\boldsymbol{X}} + h\, \|\boldsymbol{f}\|_{0,\Omega}\ .$$

Therefore, with no regularity assumption, the method is unconditionally convergent. Supposing moreover that the exact solution $(\sigma, \boldsymbol{u})$ belongs to $\boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^2(\Omega)$, one gets:

$$\|(S - S_h)(\boldsymbol{f})\|_{\boldsymbol{X} \times \boldsymbol{M}} \le ch\, \|\boldsymbol{f}\|_{0,\Omega}\ ,$$

i.e. the method has an optimal convergence rate $O(h)$.

4 The Navier–Stokes problem

We want to study now the stationary incompressible Navier–Stokes equations

$$\begin{array}{ll} \boldsymbol{u}\nabla\boldsymbol{u} - \nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega \\ div\, \boldsymbol{u} = 0 & \text{in } \Omega \end{array}$$

with the same boundary conditions (1) as in the Stokes problem.

For the analysis of this problem, it is useful to write it in a different form. By introducing the kinematic pressure $\widetilde{p} = p + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}$, one has the relation

$$\boldsymbol{u}\nabla\boldsymbol{u} + \nabla p = \omega\boldsymbol{u}^\perp + \nabla\widetilde{p}\,.$$

Therefore we obtain the equivalent equations:

$$\begin{cases} \nu\, \mathbf{curl}\, \omega + \nabla\widetilde{p} + \omega\boldsymbol{u}^\perp = \boldsymbol{f} & \text{in } \Omega \\ \omega = curl\, \boldsymbol{u} & \text{in } \Omega \\ div\, \boldsymbol{u} = 0 & \text{in } \Omega\,. \end{cases} \qquad (8)$$

By introducing the nonlinear operator

$$G : \boldsymbol{X} \times \boldsymbol{L}^4(\Omega) \longrightarrow \boldsymbol{L}^{4/3}(\Omega)\,, \qquad G(\sigma, \boldsymbol{u}) = \omega\boldsymbol{u}^\perp\,,$$

the Navier–Stokes equations (8) can be put in the general setting of a nonlinear problem as follows:

$$F(\sigma, \boldsymbol{u}) = (\mathbf{0}, \mathbf{0}) \qquad (9)$$

where the mapping F is defined by:

$$\begin{array}{l} F : \boldsymbol{X} \times \boldsymbol{L}^4(\Omega) \longrightarrow \boldsymbol{X} \times \boldsymbol{L}^4(\Omega)\,, \\ F(\tau, \boldsymbol{v}) = (\tau, \boldsymbol{v}) - S(\boldsymbol{f} - G(\tau, \boldsymbol{v}))\,. \end{array}$$

S represents the previous Stokes operator and $\boldsymbol{f}$ belongs *a priori* to $\boldsymbol{L}^{4/3}(\Omega)$.

In order to simplify the writing, we agree to put $\boldsymbol{Y} = \boldsymbol{X} \times \boldsymbol{L}^4(\Omega)$. It is obvious that $\boldsymbol{Y}$ is a Banach space and that we have $\boldsymbol{X} \times \boldsymbol{M} \subset \boldsymbol{Y}$.

We assume in the sequel that there exists a solution $(\sigma, \boldsymbol{u})$ such that $F(\sigma, \boldsymbol{u}) = (\mathbf{0}, \mathbf{0})$ and $DF(\sigma, \boldsymbol{u})$ is an isomorphism on $\boldsymbol{X} \times \boldsymbol{L}^4(\Omega)$ and we are interested in the numerical approximation of (9).

5 Discrete Navier–Stokes problem

We consider the discrete version of (9):

$$F_h(\sigma_h, \boldsymbol{u}_h) = 0\,, \qquad (10)$$

for which we want to prove existence and uniqueness of the solution, convergence of the numerical approximation as well as optimal error estimates.

We define the mapping $F_h : \boldsymbol{Y} \to \boldsymbol{Y}$ by putting

$$F_h(\tau, \boldsymbol{v}) = (\tau, \boldsymbol{v}) - S_h(\boldsymbol{f} - G(\tau, \boldsymbol{v}))$$

where $S_h : \boldsymbol{L}^{4/3}(\Omega) \to \boldsymbol{Y}$ is the discrete Stokes operator previously introduced.

Then we can establish the next result:

Lemma 4. *For $\boldsymbol{f} \in \boldsymbol{L}^{4/3}(\Omega)$, the following error bound is true:*

$$\|(S-S_h)(\boldsymbol{f})\|_{\boldsymbol{Y}} \leq C\left\{\sqrt{h}\,\|\boldsymbol{f}\|_{L^{4/3}(\Omega)} + \|\sigma - \overline{\sigma}_h\|_{\boldsymbol{X}} + \inf_{\boldsymbol{v}_h\in\boldsymbol{M}_h} |\boldsymbol{u}-\boldsymbol{v}_h|_{\boldsymbol{M}}\right\},$$

where the constant C is independent of h.

Proof. The proof is completely similar to the one of the estimate given in Theorem 2, which holds in the case where $\boldsymbol{f}$ belongs to $\boldsymbol{L}^2(\Omega)$ instead of $\boldsymbol{L}^{4/3}(\Omega)$. The only difference appears when estimating , by means of an inverse inequality (cf. Ciarlet 1978 for instance), the term:

$$\left|\int_\Omega \boldsymbol{f}\cdot\phi_h\right| \leq \|\boldsymbol{f}\|_{L^{4/3}(\Omega)}\,\|\phi_h\|_{L^4(\Omega)} \leq c\sqrt{h}\,\|\boldsymbol{f}\|_{L^{4/3}(\Omega)}\,|\phi_h|_{1,\Omega}\ ,$$

where ϕ_h belongs to a finite dimensional space. Then we just use that $\boldsymbol{X}\times\boldsymbol{M}$ is continuously embedded in $\boldsymbol{Y}$.

So, from (7) and Lemma 4 it comes that the discrete operator $S_h : \boldsymbol{L}^{4/3}(\Omega) \to \boldsymbol{Y}$ satisfies the following conditions, with c independent of h:

(A1) $\|S_h(\boldsymbol{f})\|_{\boldsymbol{Y}} \leq c\,\|\boldsymbol{f}\|_{L^{4/3}(\Omega)}$
(A2) $\lim_{h\to 0}\|(S-S_h)(\boldsymbol{f})\|_{\boldsymbol{Y}} = 0$. Moreover, for a regular data $\boldsymbol{f}\in\boldsymbol{L}^2(\Omega)$ and assuming that the exact solution $(\sigma,\boldsymbol{u})$ belongs to $\boldsymbol{H}^1(\Omega)\times\boldsymbol{H}^2(\Omega)$, one gets:

$$\|(S-S_h)(\boldsymbol{f})\|_{\boldsymbol{Y}} \leq ch\,\|\boldsymbol{f}\|_{0,\Omega}\ .$$

These results concern only the Stokes problem and are the key-point of our next proofs.

Let us now come back to the numerical approximation (10) of the initial Navier–Stokes problem. For its analysis we use a result established in Pousin and Rappaz 1994, which is mainly based on the implicit function theorem. Some variants can be found in Brezzi et al. 1980 or in Caloz and Rappaz 1994. To apply this general result, we first show:

Theorem 3. *The following conditions are fulfilled:*

(H1) *for all $(\tau,\boldsymbol{v})\in\boldsymbol{Y}$, one has*

$$\|DF_h(\sigma,\boldsymbol{u}) - DF_h(\tau,\boldsymbol{v})\|_{\boldsymbol{Y}} \leq c\,\|(\sigma,\boldsymbol{u})-(\tau,\boldsymbol{v})\|_{\boldsymbol{Y}}$$

(H2) $\lim_{h\to 0}\|F_h(\sigma,u)\|_{\boldsymbol{Y}} = 0$ *(consistency)*
(H3) *for $h\leq 1$, $DF_h(\sigma,\boldsymbol{u})$ is an isomorphism of $\boldsymbol{Y}$ and there exists a constant $M>0$ such that*

$$\left\|DF_h(\sigma,\boldsymbol{u})^{-1}\right\|_{\boldsymbol{Y}} \leq M \quad (stability)\,.$$

Proof. One has that: $DF_h(\sigma,\boldsymbol{u}) = I + S_h(DG(\sigma,\boldsymbol{u}))$, where $DG(\sigma,\boldsymbol{u}) : \boldsymbol{Y}\to L^{4/3}(\Omega)$ is given, for any $\delta=(\varrho,r)\in\boldsymbol{X}$ and $\boldsymbol{w}\in\boldsymbol{L}^4(\Omega)$ by:

$$DG(\sigma,\boldsymbol{u})(\delta,\boldsymbol{w}) = (\omega\boldsymbol{w}^\perp + \varrho\boldsymbol{u}^\perp)\,.$$

So it comes, thanks to **(A1)** and to Holder's inequality, that for any $\tau=(\theta,q)\in\boldsymbol{X}$ and $\boldsymbol{v}\in\boldsymbol{L}^4(\Omega)$,

$$\begin{aligned}\|DF_h(\sigma,\boldsymbol{u})(\delta,\boldsymbol{w}) - DF_h(\tau,\boldsymbol{v})(\delta,\boldsymbol{w})\|_{\boldsymbol{Y}} &\leq c\,\|DG(\sigma,\boldsymbol{u})(\delta,\boldsymbol{w}) - DG(\tau,\boldsymbol{v})(\delta,\boldsymbol{w})\|_{L^{4/3}(\Omega)}\\ &\leq c\Big(\|\omega-\theta\|_{L^2}\left\|\boldsymbol{w}^\perp\right\|_{L^4} + \|\varrho\|_{L^2}\left\|\boldsymbol{u}^\perp - \boldsymbol{v}^\perp\right\|_{L^4}\Big)\\ &\leq c\,\|(\sigma,\boldsymbol{u})-(\tau,\boldsymbol{v})\|_{\boldsymbol{Y}}\,\|(\delta,\boldsymbol{w})\|_{\boldsymbol{Y}}\ ,\end{aligned}$$

which leads to **(H1).**

Condition **(H2)** is satisfied thanks to **(A2)** and to the fact that

$$\|F_h(\sigma,\boldsymbol{u})\|_{\boldsymbol{Y}} = \|F(\sigma,\boldsymbol{u}) - F_h(\sigma,\boldsymbol{u})\|_{\boldsymbol{Y}} = \|(S-S_h)(\boldsymbol{f}-G(\sigma,\boldsymbol{u}))\|_{\boldsymbol{Y}}\ .$$

We now want to show that the linear operator $DF_h(\sigma,\boldsymbol{u})$ is an isomorphism of $\boldsymbol{Y}$. To do that, we write it in a different form:

$$DF_h(\sigma,\boldsymbol{u}) = DF(\sigma,\boldsymbol{u})\,(I+B_h)\ ,$$

where $B_h = DF(\sigma,\boldsymbol{u})^{-1}(DF_h(\sigma,\boldsymbol{u}) - DF(\sigma,\boldsymbol{u}))$. Since by hypothesis $DF(\sigma,\boldsymbol{u})$ is invertible, one has that $DF_h(\sigma,\boldsymbol{u})$ is an isomorphism if $\|B_h\|_{\boldsymbol{Y}} < 1$. Let us put $\left\|DF(\sigma,\boldsymbol{u})^{-1}\right\|_{\boldsymbol{Y}} = K$ and estimate:

$$\|B_h\|_{\boldsymbol{Y}} \leq K\,\|(S-S_h)(DG(\sigma,\boldsymbol{u}))\|_{\boldsymbol{Y}}\ .$$

We already know from **(A2)** that $S_h(\boldsymbol{f})$ converges in $\boldsymbol{Y}$ towards $S(\boldsymbol{f})$, for any $\boldsymbol{f}\in\boldsymbol{L}^{4/3}(\Omega)$. We show next that $DG(\sigma,\boldsymbol{u})$ is compact from $\boldsymbol{Y}$ to $L^{4/3}(\Omega)$. Then

$$\lim_{h\to 0}\|(S-S_h)(DG(\sigma,\boldsymbol{u}))\|_{\boldsymbol{Y}} = 0$$

so for h sufficiently small one has $\|B_h\|_{\boldsymbol{Y}} < \frac{1}{2}$. This finally gives:

$$\left\|DF_h(\sigma,\boldsymbol{u})^{-1}\right\|_{\boldsymbol{Y}} \leq \frac{\left\|DF(\sigma,\boldsymbol{u})^{-1}\right\|_{\boldsymbol{Y}}}{1-\|B_h\|_{\boldsymbol{Y}}} \leq 2K\,,$$

which ends the proof of **(H3)**.

Concerning the compactness of $DG(\sigma,\boldsymbol{u})$, we assume that $\omega\in L^{2+\alpha}(\Omega)$ with $\alpha>0$. One deduces that $\omega\boldsymbol{w}^\perp\in\boldsymbol{L}^r(\Omega)$ where $r>4/3$ since $\frac{1}{r}=\frac{1}{2+\alpha}+\frac{1}{4}$. On the other hand, one has that $\boldsymbol{u}\in\boldsymbol{M}\subset\boldsymbol{H}^s(\Omega)$ with $s>1/2$. Since by the Kondrasov theorem $\boldsymbol{H}^s(\Omega)$ is compactly embedded in $\boldsymbol{L}^q(\Omega)$ for some $q>4$, it comes that the term $\varrho\boldsymbol{u}^\perp$ belongs to $\boldsymbol{L}^{r'}(\Omega)$, with $r'>4/3$ too $\left(\frac{1}{r'}=\frac{1}{2}+\frac{1}{q}\right)$. Finally, the fact that $\boldsymbol{L}^r(\Omega)$ is compactly embedded in $\boldsymbol{L}^{4/3}(\Omega)$ for any $r>4/3$ achieves the proof.

Remark 1. One may propose a more direct proof for **(H3)**, using the regularity of the exact solution of the associated Stokes problem with data $\boldsymbol{f}\in\boldsymbol{L}^{4/3}(\Omega)$. Indeed, if we assume that $(\sigma,\boldsymbol{u})\in\boldsymbol{H}^s(\Omega)\times\boldsymbol{H}^{1+s}(\Omega)$ for $s>0$ and moreover, that

$$|\boldsymbol{u}|_{1+s} + |\sigma|_s \leq c\,\|\boldsymbol{f}\|_{L^{4/3}(\Omega)}\ ,$$

then using Lemma 4 it comes that:

$$\|(S-S_h)(\boldsymbol{f})\|_{\boldsymbol{Y}} \leq \varepsilon_1(h)\,\|\boldsymbol{f}\|_{L^{4/3}(\Omega)}\ .$$

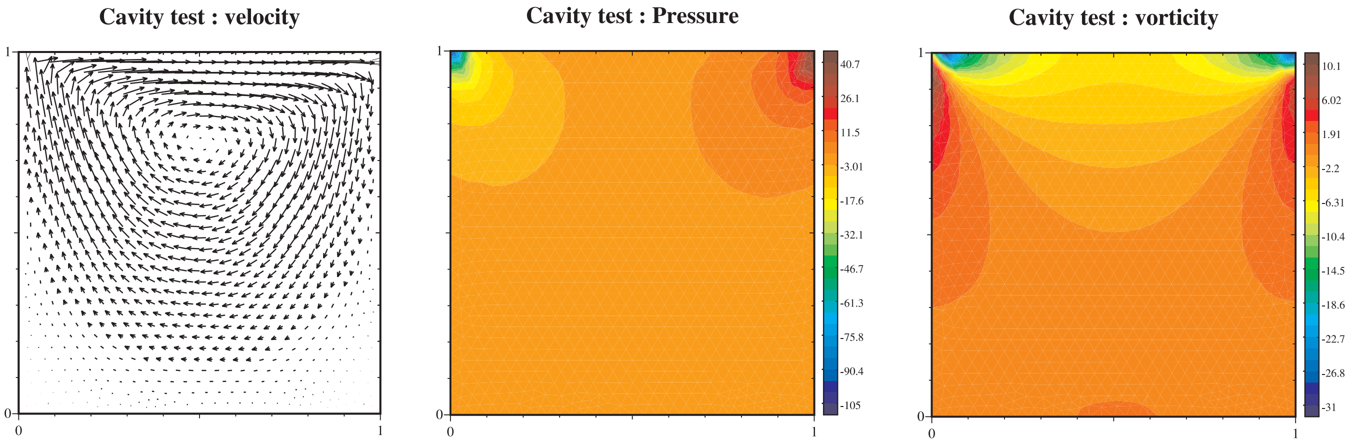

Fig. 1. Calculated velocity, pressure and vorticity for the cavity test

So, for h small enough one has:

$$\|B_h\|_Y \leq K \, \|(S - S_h)(DG(\sigma, \boldsymbol{u}))\|_Y < \frac{1}{2} .$$

Then the next statement is true, according to Pousin and Rappaz 1994:

Theorem 4. *There exists $h_0 > 0$ such that, for all $h < h_0$, problem* (10) *has a unique solution. Moreover, the following error estimate holds:*

$$\|(\sigma, \boldsymbol{u}) - (\sigma_h, \boldsymbol{u}_h)\|_Y \leq 2 \left\| DF_h(\sigma, \boldsymbol{u})^{-1} \right\|_Y \|F_h(\sigma, \boldsymbol{u})\|_Y$$
$$\leq c \, \|F_h(\sigma, \boldsymbol{u})\|_Y \ .$$

So, in conclusion, the approximation method for the Navier–Stokes problem is unconditionally convergent and its convergence rate is given by an upper bound for $\|F_h(\sigma, \boldsymbol{u})\|_Y$. If we consider smooth data $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and we admit that the exact solution $(\sigma, \boldsymbol{u})$ of the initial Navier–Stokes problem satisfies $\omega \in L^4(\Omega)$, then $G(\sigma, \boldsymbol{u}) \in L^2(\Omega)$ since $\boldsymbol{u} \in \boldsymbol{L}^4(\Omega)$. Then the second part of **(A2)** leads to the same convergence rate $O(h)$ as for the Stokes problem, that is our method is optimal in terms of finite elements.

6 Numerical results

6.1 Stokes problem

We present first the cavity test: the domain Ω is the unit square, the righthand side functions are equal to zero and we impose $\boldsymbol{u} = (1, 0)$ on the upper boundary and $\boldsymbol{u} = \boldsymbol{0}$ on the other three boundaries. We represent in Fig. 1 the calculated solution. Remark that the method used here can rigorously take into account boundaries conditions on $\boldsymbol{u}$ which belong only to $L^2(\partial\Omega)$.

We present now the Bercovier–Engelman test which allow us to compute the error between the exact solution and its numerical approximation. For this test, we have $\Omega =]0, 1[^2$ and the boundary condition: $\boldsymbol{u} = \boldsymbol{0}$ on Γ. The right-hand sides f_1 and f_2 of equations are given here such that the exact solution is:

Table 1. Error for the Bercovier–Engelman test

	$\beta = 1$	$\beta = 0.1$	$\beta = 0.05$	$\beta = 0.01$
$\|\omega - \omega_h\|_{L^2(\Omega)}$	0.57	0.2	0.2	0.32
$\|p - p_h\|_{L^2(\Omega)}$	1.78	0.55	1.07	4.97
$\|u_1 - u_{1h}\|_{1,\Omega}$	1.06	0.24	0.23	0.27
$\|u_2 - u_{2h}\|_{1,\Omega}$	1.06	0.24	0.23	0.28

$$u_1(x, y) = -256y(y-1)(2y-1)x(x-1)^2 ,$$
$$u_2(x, y) = -u_1(y, x) , \quad p(x, y) = (x - 0.5)(y - 0.5)$$

Besides, in Table 1 we present the absolute error in L^2-norm for the unknowns ω and p and in H^1-norm for (u_1, u_2). These errors are calculated on a unstructured mesh for different values of β. The results show that β has to be chosen correctly, not too big but not too small neither.

6.2 Navier–Stokes problem

We first present an academic test where the solution is given by the next expressions:

$$u_1 = -\sin(\pi x)\cos(\pi y) ; \quad u_2 = \cos(\pi x)\sin(\pi y) ,$$
$$p = \sin(\pi x)\sin(\pi y)$$

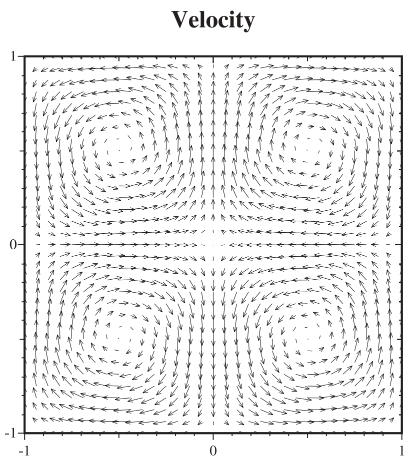

Fig. 2. Calculated velocity for the academic test boxes

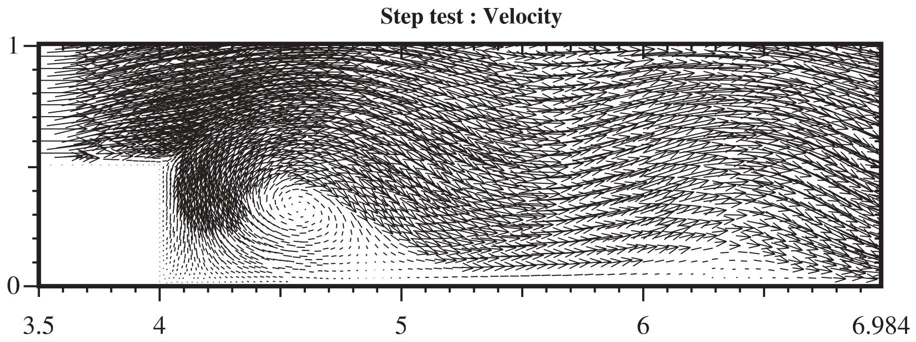

Fig. 3. Calculated solutions for the Step Test boxes

Table 2. Errors for Reynolds number equal to 1

Reynolds = 1	8 × 8	16 × 16	32 × 32	64×64
$\|\omega-\omega_h\|_{L^2(\Omega)}$	0.4	0.2	0.09	0.046
$\|p-p_h\|_{L^2(\Omega)}$	2.2	1.07	0.54	0.26
$\|u_1-u_{1h}\|_{1,\Omega}$	0.17	0.08	0.04	0.02
$\|u_2-u_{2h}\|_{1,\Omega}$	0.13	0.06	0.03	0.016

Table 3. Errors for Reynolds number equal to 100

Reynolds = 100	8 × 8	16 × 16	32 × 32	64×64
$\|\omega-\omega_h\|_{L^2(\Omega)}$	2.15	1.1	0.59	0.3
$\|p-p_h\|_{L^2(\Omega)}$	1.82	0.88	0.49	0.26
$\|u_1-u_{1h}\|_{1,\Omega}$	0.9	0.47	0.24	0.13
$\|u_2-u_{2h}\|_{1,\Omega}$	0.88	0.44	0.23	0.11

Table 4. Errors for Reynolds number equal to 1000

Reynolds = 1000	16 × 16	32 × 32	64×64
$\|\omega-\omega_h\|_{L^2(\Omega)}$	3.6	1.9	0.92
$\|p-p_h\|_{L^2(\Omega)}$	2.04	0.9	0.43
$\|u_1-u_{1h}\|_{1,\Omega}$	1.12	0.6	0.31
$\|u_2-u_{2h}\|_{1,\Omega}$	1.01	0.57	0.26

and the domain Ω is the square $]-1,1[^2$. We impose the exact value of ω and of the normal component of the velocity on the left and right boundaries, the pressure and the tangential component of the velocity on the upper boundary and finally the velocity on the rest of the boundary. We represent in Fig. 2 the exact velocity:

We give then in Tables 2, 3 and 4 the error on u_1, u_2, p and ω for different Reynolds numbers and different meshes.

In order to improve the stability in the case of large Reynolds numbers, we proceed to an upwinding on the convection term $\int_\Omega \omega \boldsymbol{u}^\perp \cdot \boldsymbol{v} dx$.

Finally, we represent in Fig. 3 the velocity obtained in the case of the step test. Note that in this example, we impose the pressure on the inlet and oulet boundaries. Assuming that the domain Ω is the half part of the real domain, we impose the condition $\omega = 0$ on the upper boundary. The Reynolds number for this test is taken equal to 1000. Note that, generally, for this test the two components of the velocity are imposed on the upperwall boundary. We obtain in this case a vortex closed to the step and a more laminar flow.

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