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Optimal time to invest when the price processes are geometric Brownian motions

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Abstract. Let $X_1(t), \cdots, X_n(t)$ be n geometric Brownian motions, possibly correlated. We study the optimal stopping problem: Find a stopping time $\tau^* < \infty$ such that

$$\sup_\tau \mathbb{E}^{\,x} \left\{ X_1(\tau) - X_2(\tau) - \cdots - X_n(\tau) \right\} = \mathbb{E}^{\,x} \left\{ X_1(\tau^*) - X_2(\tau^*) - \cdots - X_n(\tau^*) \right\},$$

the sup being taken all over all finite stopping times τ, and $\mathbb{E}^{\,x}$ denotes the expectation when $(X_1(0), \cdots, X_n(0)) = x = (x_1, \cdots, x_n)$. For $n = 2$ this problem was solved by McDonald and Siegel, but they did not state the precise conditions for their result. We give a new proof of their solution for $n = 2$ using variational inequalities and we solve the n-dimensional case when the parameters satisfy certain (additional) conditions.

Key words: Geometric Brownian motion, optimal stopping time, continuation region, stopping set

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1 Introduction

Let $X_1(t), X_2(t), \cdots, X_n(t)$ be stochastic processes modelling the prices of n assets, for example stocks. As is customary let us assume that the X_i's are geometric Brownian motions, possibly correlated, of the form

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$$\begin{cases} dX_1(t) = b_1X_1(t)dt + X_1(t)\big[q_{11}dB_1(t) + \cdots + q_{1n}dB_n(t)\big]; \quad X_1(0) = x_1 \\ \qquad\vdots \\ dX_n(t) = b_nX_n(t)dt + X_n(t)\big[q_{n1}dB_1(t) + \cdots + q_{nn}dB_n(t)\big]; \quad X_n(0) = x_n\,, \end{cases} \tag{1.1}$$

where $B(t) = (B_1(t), \cdots, B_n(t))$ is n-dimensional Brownian motion and b_i, q_{ij} are constants.

If we put

$$q_i = (q_{i1}, q_{i2}, \cdots, q_{in}) \in \mathbb{R}^n \tag{1.2}$$

then the solutions of these equations can be written

$$X_i(t) = x_i \exp\{(b_i - \frac{1}{2}a_{ii})t + q_i \cdot B(t)\}\,, \tag{1.3}$$

where

$$a_{ij} = q_{i1}q_{j1} + q_{i2}q_{j2} + \cdots + q_{in}q_{jn} = q_i \cdot q_j\,; \quad 1 \le i, j \le n \tag{1.4}$$

where $\cdot$ denotes the usual dot product in $\mathbb{R}^n$.

Suppose that $F(t) = X_2(t) + \cdots + X_n(t)$ represents the capital at time t that a person or a firm can use to make an irreversible investment in a project, whose value at time t is $X_1(t)$. The optimal time for making such a transaction will be the stopping time $\tau^* < \infty$ satisfying

$$\mathbb{E}^{\,x}\Big\{e^{-\rho\tau^*}\big[X_1(\tau^*) - X_2(\tau^*) - \cdots - X_n(\tau^*)\big]\Big\} = \sup_\tau \mathbb{E}^{\,x}\Big\{e^{-\rho\tau}\big[X_1(\tau) - X_2(\tau) - \cdots - X_n(\tau)\big]\Big\}\,, \tag{1.5}$$

where $\mathbb{E}^{\,x}$ denotes the expectation w.r.t the law Q^x of $X(t) = (X_1(t), \cdots, X_n(t))$ starting at x. Here $\rho > 0$ is a constant modelling the sum of the discounting exponent and the risk aversion of the investor. We may also regard ρ as the equilibrium expected rate of return on the investment opportunity. For a more detailed discussion of the application of this problem to financial decision making, we refer to McDonald and Siegel [1].

Note that $e^{-\rho t}X_i(t)$ is again a geometric Brownian motion, but with b_i replaced by

$$\hat{b}_i = b_i - \rho\,; \quad 1 \le i \le n\,. \tag{1.6}$$

Therefore we can, without loss of generality, assume that $\rho = 0$ in (1.5). In the case $n = 2$ this problem was solved by McDonald and Siegel [1]. The solution (for $n = 2$) is the following: The optimal time τ^* is given by

$$\tau^* = \inf\{t > 0; X_1(t) \ge \mu X_2(t)\}\,, \tag{1.7}$$

where

$$\mu = \frac{\lambda}{\lambda - 1} \tag{1.8}$$

and

$$\lambda = \frac{1}{2} - \frac{b_1 - b_2}{(q_1 - q_2)^2} + \sqrt{\left(\frac{1}{2} - \frac{b_1 - b_2}{(q_1 - q_2)^2}\right)^2 + \frac{2(\rho - b_2)}{(q_1 - q_2)^2}} \,. \tag{1.9}$$

In other words, if we define the *continuation region* D by

$$D = \{x; Q^x(\tau^* > 0) > 0\} \tag{1.10}$$

and the *stopping set* S by

$$S = \{x; \tau^* = 0 \quad a.s. \quad Q^x\} \tag{1.11}$$

then

$$D = \{(x_1, x_2) \in \mathbb{R}^2_+; \quad x_1 < \mu x_2\} \tag{1.12}$$

and

$$S = \{(x_1, x_2) \in \mathbb{R}^2_+; \quad x_1 \geq \mu x_2\}. \tag{1.13}$$

However, they did not state all the conditions needed for the validity of their result. As we shall see below, there are cases when the McDonald-Siegel solution does not hold.

In 1992 Olsen and Stensland [3] studied the general n-dimensional problem (1.5). They proved that, under certain conditions, the stopping set S *contains* a halfspace:

$$S \supseteq \{(x_1, \cdots, x_n) \in \mathbb{R}^n_+; \quad x_1 \geq \mu_{12} x_2 + \cdots + \mu_{1n} x_n\} \tag{1.14}$$

where μ_{1j} are the McDonald-Siegel barriers for the 2-dimensional problem

$$\sup_\tau \mathbb{E}^{\,x_1, x_j}\left[e^{-\rho}\big(X_1(\tau) - X_j(\tau)\big)\right]; \quad 2 \leq j \leq n\,. \tag{1.15}$$

They ask if we actually have equality in (1.14) and they perform some numerical calculations which support such a conjecture in some special cases. A discussion of the problem in some other cases can be found in [4].

One can assume that the capital $F(t)$ of the firm follows itself a geometric Brownian motion as McDonald and Siegel did [1]. But in today's world, many firms build one factory in one country with value $X_2(t)$ and another factory in another country with value $X_3(t)$. These factories are managed independently to certain extent. In this situation it should be more appropriate to assume that $X_2(t)$ and $X_3(t)$ follow geometric Brownian motions respectively than to assume that $F(t) = X_2(t) + X_3(t)$ follows geometric Brownian motion.

The purpose of this paper is twofold:

First, we give a short and rigorous mathematical proof of the McDonald-Siegel result, based on variational inequalities. Second, we consider the n-dimensional case and prove that, at least in some cases, the continuation region has the same simple form as McDonald and Siegel found for $n = 2$.

This paper is organized as follows: In Sect. 2 we state a sufficient variational inequality condition for our optimal stopping problem. In Sect. 3 we apply this to the 2-dimensional case and prove, under given conditions, the McDonald-Siegel result. We also give examples to show that the result can fail if our conditions are not satisfied. Then in Sect. 4 we discuss the general n-dimensional case. We prove a partial converse of the result of [3] mentioned above: The stopping region is always *contained* in some halfspace. Under additional assumptions we then show that the two halfspaces are the same, thereby obtaining an explicit description of the stopping region.

2 A variational inequality

In this section we formulate sufficient variational inequalities for our problem (1.5). In the following we let $X(t) = (X_1(t), \cdots, X_n(t))$ be the process defined in (1.1) and we let A denote the generator of the Itô diffusion $X(t)$. Then A coincides on $C_0^2(\mathbb{R}_+^n)$ with a second order semielliptic partial differential operator L (see (3.6) in the next section). Define

$$g(x_1, \cdots, x_n) = x_1 - x_2 - \cdots - x_n\,; \quad x = (x_1, \cdots, x_n) \in \mathbb{R}_+^n\,.$$

Then problem (1.5) has the form:
Find $\Phi(x)$ *and* $\tau^* = \tau^*(x, \omega) < \infty$ *such that*

$$\Phi(x) = \sup_\tau \mathbb{E}^x[g(X(\tau))] = \mathbb{E}^x[g(X(\tau^*))]\,. \tag{2.1}$$

The following result holds in a much more general context, but for simplicity we only state the version which is relevant to our problem. A proof can be found in [2, Theorem 10.18].

Theorem 2.1 *(Sufficient variational inequalities)*
a*) Suppose we can find a function* $\phi : \mathbb{R}_+^n \to \mathbb{R}$ *such that*

$$\phi \in C^1(\mathbb{R}_+^n) \tag{2.2}$$

and

$$\phi(x) \geq g(x) \quad \forall\, x \in \mathbb{R}_+^n\,. \tag{2.3}$$

Define

$$D = \{x \in \mathbb{R}_+^n\,; \phi(x) > g(x)\} \quad (\textit{the continuation region})\,. \tag{2.4}$$

Suppose

$$\mathbb{E}^x\Big[\int_0^\infty \chi_{\partial D}(X(t))dt\Big] = 0 \quad \forall\, x \in \mathbb{R}_+^n\,. \tag{2.5}$$

Moreover, suppose the following:

$$\phi \in C^2(\mathbb{R}_+^n \setminus \partial D) \text{ and the second order derivatives of } \phi \text{ are locally bounded near } \partial D\,, \tag{2.6}$$

$$L\phi \leq 0 \quad \textit{for} \quad x \in \mathbb{R}_+^n \setminus \bar{D}\,, \tag{2.7}$$

the family $\{\phi(X(\tau))\}_{\tau \in \mathcal{T}_D}$ *is uniformly integrable w.r.t.* Q^x *for all* $x \in D$, *where* $\mathcal{T}_D$ *is the set of all bounded stopping times* $\tau \leq \tau_D$ (2.8) *and*

$$\partial D \text{ is a Lipschitz surface.} \tag{2.9}$$

Then

$$\phi(x) \geq \mathbb{E}^{x}\left[g(X(\tau))\right] \quad \textit{for all stopping times} \quad \tau\,. \tag{2.10}$$

b) *If, in addition we also have*

$$L\phi = 0 \quad \textit{for} \quad x \in D \tag{2.11}$$

and

$$\tau_D := \inf\{t > 0\,; X(t) \notin D\} < \infty \quad \textit{a.s.} \quad Q^x \quad \textit{for} \quad x \in \mathbb{R}^n_+\,, \tag{2.12}$$

then

$$\phi(x) = \Phi(x) \quad \textit{and} \quad \tau^* = \tau_D \quad \textit{is optimal}\,. \tag{2.13}$$

3 The 2-dimensional case

We now apply the result of the previous section to discuss the case when there are only two types of stocks involved, $X_1(t)$ and $X_2(t)$. Even in this case there are many different types of values of the parameters involved. We will not try to cover all possibilities, but limit ourselves to these ranges of parameters values that appear reasonable from the point of view of our economic interpretation.

First of all, we will assume, as in [1], that the combining discounting/risk aversion rate ρ is greater than the average relative growth rate b_i of each of the stock prices. This means that, if we replace b_i by $\hat{b}_i$ as in (1.6) we should have

$$\hat{b}_i = b_i - \rho < 0\,; \quad 1 \leq i \leq 2\,. \tag{3.1}$$

To make it easier to keep track of the signs of the quantities involved we put

$$p_i = -\hat{b}_i\,; \quad 1 \leq i \leq 2\,, \tag{3.2}$$

and then our 2-dimensional problem gets the form

Problem 3.1 *Find $\Phi(x_1, x_2)$ and τ^* such that*

$$\Phi(x_1, x_2) = \sup_{\tau} \mathbb{E}^{x_1,x_2}\left[X_1(\tau) - X_2(\tau)\right] = \mathbb{E}^{x_1,x_2}\left[X_1(\tau^*) - X_2(\tau^*)\right], \tag{3.3}$$

where

$$\begin{cases} dX_1(t) = -p_1 X_1(t)dt + X_1(t)\left[q_{11}dB_1(t) + q_{12}dB_2(t)\right]; \quad X_1(0) = x_1 \\ dX_2(t) = -p_2 X_2(t)dt + X_2(t)\left[q_{21}dB_1(t) + q_{22}dB_2(t)\right]; \quad X_2(0) = x_2 \end{cases} \tag{3.4}$$

with

$$p_1 > 0\,, \quad p_2 > 0\,. \tag{3.5}$$

Here $\mathbb{E}^{x_1,x_2}$ denotes the expectation with respect to the law Q^{x_1,x_2} of the process $X(t) = (X_1(t), X_2(t))$ starting at (x_1, x_2).

The Itô diffusion X_t has a generator A which on $C_0(\mathbb{R}^2)$ coincides with the differential operator L given by

$$\begin{aligned} Lf(x_1,x_2) \quad &= \quad \frac{1}{2}\big[a_{11}x_1^2\frac{\partial^2 f}{\partial x_1^2} + 2a_{12}x_1x_2\frac{\partial^2 f}{\partial x_1 x_2} + a_{22}x_1^2\frac{\partial^2 f}{\partial x_2^2}\big] \\ &\quad -p_1x_1\frac{\partial f}{\partial x_1} - p_2x_2\frac{\partial f}{\partial x_2}\,, \quad \text{for} \quad f \in C^2(\mathbb{R}^2)\,, \end{aligned} \tag{3.6}$$

where, with the notation of (1.2),

$$a_{11} = q_{11}^2 + q_{12}^2 = q_1^2 \tag{3.7}$$

$$a_{12} = q_{11}q_{21} + q_{12}q_{22} = q_1 \cdot q_2 \tag{3.8}$$

and

$$a_{22} = q_{21}^2 + q_{22}^2 = q_2^2\,. \tag{3.9}$$

In view of Theorem 2.1 we try to find a function ϕ of this form

$$\phi(x_1,x_2) = \begin{cases} \psi(x_1,x_2)\,; & x_1 < \mu x_2 \\ g(x_1,x_2)\,; & x_1 \geq \mu x_2, \end{cases} \tag{3.10}$$

where $g(x_1,x_2) = x_1 - x_2$. Here $\mu > 0$ is a constant and $\psi(x_1,x_2)$ is a function, both to be determined. By Theorem 2.1 we can conclude that $\phi = \Phi$ and that $D = \{(x_1,x_2)\,; x_1 < \mu x_2\}$ if we can find ψ and μ such that the following (3.11)-(3.16) hold:

$$\psi \in C^2(D) \quad \text{and} \quad L\psi(x_1,x_2) = 0 \quad \text{when} \quad x_1 < \mu x_2\,, \tag{3.11}$$

$$\psi(x_1,x_2) = g(x_1,x_2) \quad \text{when} \quad x_1 = \mu x_2\,, \tag{3.12}$$

$$\nabla\psi(x_1,x_2) = \nabla g(x_1,x_2) \quad \text{when} \quad x_1 = \mu x_2\,, \tag{3.13}$$

$$Lg(x_1,x_2) \leq 0 \quad \text{when} \quad x_1 > \mu x_2\,, \tag{3.14}$$

$$\psi(x_1,x_2) > g(x_1,x_2) \quad \text{when} \quad x_1 < \mu x_2\,, \tag{3.15}$$

the family

$$\{\psi(X_\tau)\}_{\tau\in\mathcal{T}_D} \quad \text{with} \quad \tau_D = \inf\{t > 0\,; X_1(t) \geq \mu X_2(t)\} \tag{3.16}$$

is uniformly integrable w.r.t. Q^x for all $x \in D$, where $\mathcal{T}_D$ is the set of all bounded stopping times $\tau \leq \tau_D$.

As a candidate for ψ we try

$$\psi(x_1,x_2) = Cx_1^\lambda x_2^{1-\lambda} \tag{3.17}$$

for suitable values of the constants $C > 0$ and $\lambda > 0$.

To find the possible values of λ (and subsequently C) we put

$$f(x_1, x_2) = x_1^{\lambda} x_2^{1-\lambda}$$

and compute, using (3.6)

$$\begin{aligned}
&Lf(x_1, x_2) \\
&= \frac{1}{2}\big[a_{11}x_1^2\lambda(\lambda-1)x_1^{\lambda-2}x_2^{1-\lambda} + 2a_{12}x_1x_2\lambda(1-\lambda)x_1^{\lambda-1}x_2^{-\lambda} \\
&\quad + a_{22}x_2^2(1-\lambda)(-\lambda)x_1^{\lambda}x_2^{-\lambda-1}\big] - p_1x_1\lambda x_1^{\lambda-1}x_2^{1-\lambda} - p_2x_2(1-\lambda)x_1^{\lambda}x_2^{-\lambda} \\
&= x_1^{\lambda}x_2^{1-\lambda}\big[\frac{1}{2}\gamma\lambda^2 + (p_2 - p_1 - \frac{1}{2}\gamma)\lambda - p_2\big]\,, \qquad (3.18)
\end{aligned}$$

where

$$\gamma = \gamma_{12} = a_{11} - 2a_{12} + a_{22} = (q_1 - q_2)^2 \geq 0\,. \qquad (3.19)$$

Note that

$$\begin{cases} \gamma \geq 0 \quad \text{for all} \quad q_{ij} \quad \text{and} \\ \gamma = 0 \Leftrightarrow q_{11} = q_{21} \quad \& \quad q_{12} = q_{22}\,. \end{cases} \qquad (3.20)$$

We conclude that $Lf(x_1, x_2) = 0$ for some, and then for all, (x_1, x_2) if and only if λ satisfies the equation

$$\frac{1}{2}\gamma\lambda^2 + (p_2 - p_1 - \frac{1}{2}\gamma)\lambda - p_2 = 0\,. \qquad (3.21)$$

The solutions of this equation are

$$\lambda = \begin{cases} \frac{1}{\gamma}\big[\frac{1}{2}\gamma + p_1 - p_2 \pm \sqrt{(\frac{1}{2}\gamma + p_1 - p_2)^2 + 2\gamma p_2}\big] & \text{if } \gamma > 0 \\ \frac{p_2}{p_2 - p_1} & \text{if } \gamma = 0. \end{cases} \qquad (3.22)$$

Since we need to have $\lambda > 0$ we must require

$$p_2 > p_1 \quad \text{if} \quad \gamma = 0\,. \qquad (3.23)$$

From now on we choose the plus sign in (3.22) and let

$$\lambda = \begin{cases} \frac{1}{\gamma}\big[\frac{1}{2}\gamma + p_1 - p_2 + \sqrt{(\frac{1}{2}\gamma + p_1 - p_2)^2 + 2\gamma p_2}\big] & \text{if } \gamma > 0 \\ \frac{p_2}{p_2 - p_1} & \text{if } \gamma = 0 \text{ and } p_2 > p_1. \end{cases} \qquad (3.24)$$

For this value of λ put

$$\psi(x_1, x_2) = Cx_1^{\lambda}x_2^{1-\lambda} \quad \text{for some constant} \quad C\,. \qquad (3.25)$$

The requirement (3.12) then gives

$$C(\mu x_2)^{\lambda}x_2^{1-\lambda} = \mu x_2 - x_2 \quad \text{for all} \quad x_2 > 0\,,$$

or

$$C\mu^{\lambda} = \mu - 1\,. \qquad (3.26)$$

The requirement (3.13) gives

$$C\lambda(\mu x_2)^{\lambda-1}x_2^{1-\lambda} = 1 \quad \text{and} \quad C(1-\lambda)(\mu x_2)^{\lambda}x_2^{-\lambda} = -1$$

i.e.

$$C\lambda\mu^{\lambda-1} = 1 \quad \text{and} \quad C(1-\lambda)\mu^{\lambda} = -1\,. \tag{3.27}$$

The three equations in (3.26) and (3.27) have the unique solution

$$\mu = \frac{\lambda}{\lambda-1} \quad \text{and} \quad C = \frac{1}{\lambda}\Big(\frac{\lambda}{\lambda-1}\Big)^{1-\lambda}. \tag{3.28}$$

Since we need to have $\mu > 0$ and $C > 0$ it is necessary to check that $\lambda > 1$:
(i) If $\gamma = 0$ and $p_2 > p_1$, this is clear from (3.24).
(ii) If $\gamma > 0$ we see that

$$\begin{aligned}
\lambda > 1 \quad &\Leftrightarrow \quad p_1 + \sqrt{(\frac{1}{2}\gamma + p_1 - p_2)^2 + 2\gamma p_2} > \frac{1}{2}\gamma + p_2 \\
&\Leftrightarrow \quad p_1^2 + 2p_1\sqrt{(\frac{1}{2}\gamma + p_1 - p_2)^2 + 2\gamma p_2} \\
&\qquad + (\frac{1}{2}\gamma + p_1 - p_2)^2 + 2\gamma p_2 > \frac{1}{4}\gamma^2 + \gamma p_2 + p_2^2 \\
&\Leftrightarrow \quad 2p_1 + \gamma + 2\sqrt{(\frac{1}{2}\gamma + p_1 - p_2)^2 + 2\gamma p_2} > 2p_2
\end{aligned} \tag{3.29}$$

Now $\sqrt{(\frac{1}{2}\gamma + p_1 - p_2)^2 + 2\gamma p_2} > |\frac{1}{2}\gamma + p_1 - p_2|$, so by checking the two cases
a) $\frac{1}{2}\gamma + p_1 - p_2 \geq 0$
b) $\frac{1}{2}\gamma + p_1 - p_2 < 0$
separately, we verify that (3.29) always holds. We conclude that we always have

$$\lambda > 1\,, \tag{3.30}$$

when λ is defined by (3.24).

We proceed to check the requirement (3.14): Since

$$Lg(x_1, x_2) = -p_1x_1 + p_2x_2\,,$$

we see that

$$Lg(x_1, x_2) \leq 0 \quad \text{iff} \quad x_1 \geq \frac{p_2}{p_1}x_2\,.$$

Therefore (3.14) leads to the condition that

$$\frac{p_2}{p_1} \leq \mu\,. \tag{3.31}$$

To verify this inequality, note that from (3.18) we have

$$\frac{1}{2}a_{11}\lambda(\lambda-1) - a_{12}\lambda(\lambda-1) + \frac{1}{2}a_{22}\lambda(\lambda-1) - p_1\lambda + p_2(\lambda-1) = 0$$

or

$$\big(\frac{1}{2}a_{11} - a_{12} + \frac{1}{2}a_{22}\big)\lambda - p_1\frac{\lambda}{\lambda - 1} + p_2 = 0$$

or

$$\frac{\lambda}{\lambda - 1} \cdot \frac{p_1}{p_2} = 1 + \frac{\gamma\lambda}{2p_2} \geq 1\,.$$

Since $\mu = \frac{\lambda}{\lambda-1}$ this proves that (3.31), and hence (3.14) holds.

Next we check the requirement (3.15): Define

$$h(x_1, x_2) = \psi(x_1, x_2) - g(x_1, x_2) = Cx_1^{\lambda}x_2^{1-\lambda} - x_1 + x_2\,; \quad x_1, x_2 \geq 0\,.$$

Then we have from (3.12) that

$$h(x_1, x_2) = 0 \quad \text{if} \quad x_1 = \mu x_2\,.$$

Moreover

$$\frac{\partial h}{\partial x_1} = C\lambda x_1^{\lambda-1}x_2^{1-\lambda} - 1 = 0 \quad \text{iff} \quad \frac{x_1}{x_2} = \mu\,.$$

Since $\frac{\partial h}{\partial x_1} \to -1$ as $x_1 \to 0$, we must have $\frac{\partial h}{\partial x_1} < 0$ for all $x_1 < \mu x_2$ and therefore $h(x_1, x_2) > 0$ for $x_1 < \mu x_2$. This proves that (3.15) holds.

The requirement (3.16) now follows easily from (3.17) and the definition of D:

Note that if $\tau \leq \tau_D$ then $X(\tau) \in \bar{D}$ and therefore

$$\psi(X(\tau)) = C\big(\frac{X_1(\tau)}{X_2(\tau)}\big)^{\lambda}X_2(\tau) \leq C\mu^{\lambda}X_2(\tau)\,. \tag{3.32}$$

So if $\tau \in \mathcal{T}_D$ and $r > 1$ satisfies $-p_2 + \frac{1}{2}a_{22}(r-1) < 0$ then

$$\begin{aligned}
\mathbb{E}^x[\psi(X_\tau)^r] &\leq C^r\mu^{r\lambda}\mathbb{E}^x[X_2(\tau)^r] \\
&\leq C^r\mu^{r\lambda}x_2^r\mathbb{E}\left[\{r[(-p_2 - \frac{1}{2}a_{12})\tau + q_{21}B_1(\tau) + q_{22}B_2(\tau)]\}\right] \\
&\leq C^r\mu^{r\lambda}x_2^r\mathbb{E}\left[\{r[(-p_2 + \frac{1}{2}a_{22}(r-1))\tau\}\right. \\
&\qquad\qquad \left.\cdot\ \exp\{-\frac{1}{2}r^2a_{22}\tau + rq_{21}B_1(\tau) + rq_{22}B_2(\tau)\}\right] \\
&\leq C^r\mu^{r\lambda}x_2^r\mathbb{E}\left[\exp\{-\frac{1}{2}r^2a_{22}\tau + rq_{21}B_1(\tau) + rq_{22}B_2(\tau)\}\right] \\
&\leq C^r\mu^{r\lambda}x_2^r\,,
\end{aligned}$$

where we have used that

$$M_t := \exp\{-\frac{1}{2}r^2a_{22}t + rq_{21}B_1(t) + rq_{22}B_2(t)]\}$$

is a martingale. Therefore, if r is chosen such that

$$1 < r < 1 + \frac{2p_2}{a_{22}} \quad \text{if} \quad a_{22} > 0 \quad (1 < r \text{ if } a_{22} = 0)\,,$$

then

$$\sup_{\tau \in \mathcal{T}_D} \mathbb{E}^{\,x}[\psi(X(\tau))^r] < \infty$$

and hence $\{\psi(X(\tau))\}_{\tau \in \mathcal{T}_D}$ is uniformly integrable. Thus (3.16) is verified.

Therefore, from Theorem 2.1a we conclude that if (3.23) holds, then with ϕ given by (3.10), (3.17), (3.22) and (3.28) we have

$$\phi(x_1, x_2) \geq \mathbb{E}^{\,x_1,x_2}\big[X_1(\tau) - X_2(\tau)\big] \quad \text{for all stopping times } \tau\,. \tag{3.33}$$

In order to apply Theorem 2.1b the last requirement we have to check is that

$$\tau_D < \infty \quad \text{a.s.} \quad Q^{x_1,x_2} \quad \text{for all} \quad (x_1, x_2)\,. \tag{3.34}$$

Since the solution of (3.4) is

$$X_i(t) = x_i \exp\{(-p_i - \frac{1}{2}a_{ii}t + q_{i1}B_1(t) + q_{i2}B_2(t)\}\,;\, i = 1, 2 \tag{3.35}$$

we have that, with a_{ij} as in (3.7)-(3.9),

$$\frac{X_1(t)}{X_2(t)} = \frac{x_1}{x_2}\exp\{(p_2 + \frac{1}{2}a_{22} - p_1 - \frac{1}{2}a_{11})t + (q_{11} - q_{21})B_1(t) + (q_{12} - q_{22})B_2(t)\}\,. \tag{3.36}$$

Using the law of iterated logarithm for Brownian motion, we see that

$$\overline{\lim_{t\to\infty}}\ \frac{X_1(t)}{X_2(t)} = \infty \quad \text{a.s.} \quad Q^{x_1,x_2}$$

when

$$p_2 + \frac{1}{2}a_{22} \geq p_1 + \frac{1}{2}a_{11}\,. \tag{3.37}$$

We have now completed the proof of the following result, which is the main result of [1]. Moreover, we have obtained precise conditions for its validity:

Theorem 3.2 *([1]) Let* $q_i = (q_{i1}, q_{i2})$*;* $1 \leq i \leq 2$ *as in (1.2).*
a) *Assume that*

$$q_2 \neq q_1 \quad \textit{or} \quad p_2 > p_1\,. \tag{3.38}$$

Define λ *as in (3.24). Then* $\lambda > 1$*. Put*

$$\mu = \frac{\lambda}{\lambda - 1} \quad \textit{and} \quad C = \frac{1}{\lambda}\Big(\frac{\lambda - 1}{\lambda}\Big)^{\lambda - 1} \tag{3.39}$$

and let

$$\phi(x_1, x_2) = \begin{cases} Cx_1^{\lambda}x_2^{1-\lambda}\,; & x_1 < \mu x_2 \\ x_1 - x_2\,; & x_1 \geq \mu x_2 \end{cases}. \tag{3.40}$$

Then

$$\phi(x_1, x_2) \geq \mathbb{E}^{\,x_1,x_2}\big[X_1(\tau) - X_2(\tau)\big] \quad \textit{for all stopping times} \quad \tau\,. \tag{3.41}$$

In particular,

$$S \supseteq \{(x_1, x_2); x_1 \geq \mu x_2\} \tag{3.42}$$

where S is the stopping set (see (1.11)).
b) *Assume, in addition to (3.38), that*

$$q_1 = q_2 \quad or \quad p_2 + \frac{1}{2}q_2^2 \geq p_1 + \frac{1}{2}q_1^2 \,. \tag{3.43}$$

Then ϕ is the solution of Φ of Problem 3.1 and the optimal stopping time τ^* is

$$\tau^* = \inf\{t > 0\,; X_1(t) \geq \mu X_2(t)\} < \infty \quad a.s. \tag{3.44}$$

Hence

$$S = \{(x_1, x_2);\ x_1 \geq \mu x_2\} \,. \tag{3.45}$$

We give some examples to show that if one of our conditions (3.38), (3.43) fails, then the optimal stopping time τ^* may not be of form (3.44).

Example 3.3 Consider the deterministic case

$$X_1(t) = x_1 e^{-\alpha t} \,, \quad X_2(t) = x_2 e^{-\beta t} \,,$$

where α, β are constants, $\alpha > \beta > 0$. In this case

$$p_1 = \alpha \,, \quad q_1 = (0, 0)$$
$$p_2 = \beta \,, \quad q_2 = (0, 0)$$

and hence $q_1 = q_2$ and yet

$$p_1 > p_2 \,.$$

So (3.38) does not hold. In this case we find by direct computation

$$\sup_{t \geq 0} \{X_1(t) - X_2(t)\} = \begin{cases} 0 & ; \quad x_1 < x_2 \\ x_1 - x_2 & ; \quad x_1 \geq x_2 \end{cases}$$

which is different from the conclusion in Theorem 3.2.

Example 3.4 Let

$$dX_1(t) = -\alpha X_1(t)dt \,, \quad dX_2(t) = -\beta X_2(t)dt + X_2(t)dB_t$$

where $\alpha > 0$, $0 < \beta < \alpha - \frac{1}{2}$ and B_t is 1-dimensional Brownian motion. Here we have

$$p_1 = \alpha \,, \quad q_1 = (0, 0)$$
$$p_2 = \beta \,, \quad q_2 = (0, 1) \,.$$

Hence

$$p_2 + \frac{1}{2}q_2^2 = \beta + \frac{1}{2} < \alpha = p_1 + \frac{1}{2}q_1^2 \,.$$

Hence (3.43) does not hold. In this case

$$\frac{X_1(t)}{X_2(t)} = \frac{X_1(0)}{X_2(0)} \exp\left((-\alpha + \beta + \frac{1}{2})t - B_t\right) .$$

Therefore, if τ^* is defined by (3.44), then

$$Q^{x_1, x_2}\left[\tau^* = \infty\right] > 0$$

if $x_1 < \mu x_2$. So the conclusion of Theorem 3.2 does not hold.

4 The general case

In this section first we introduce an order relation among the geometric Brownian motions and then use it to obtain some new results for the general problem (1.5).

Definition 4.1 *Let $X(t)$ and $Y(t)$, $t \geq 0$ be two geometric Brownian motions. Let $A, C \in \mathbb{R}$ be constants. We write*

$$X \leq AY + C \quad or \quad AY + C \geq X \quad (w.r.t. \quad x_1, x_2) \tag{4.1}$$

iff for any stopping time $\tau < \infty$ a.s. we have

$$\mathbb{E}^{x_1,x_2}\left[X(\tau)\right] \leq A\mathbb{E}^{x_1,x_2}\left[Y(\tau)\right] + C . \tag{4.2}$$

As in Sect. 2, we put

$$p_i = -(b_i - \rho) > 0 , \quad 1 \leq i \leq n . \tag{4.3}$$

Consider the following problem:

Problem 4.2 *Find $\Phi(x_1, x_2, \cdots, x_n)$ and $\tau^* < \infty$ such that*

$$\begin{aligned} \Phi(x_1, x_2, \cdots, x_n) &= \sup_\tau \mathbb{E}^x\left[X_1(\tau) - X_2(\tau) - \cdots - X_n(\tau)\right] \\ &= \mathbb{E}^x\left[X_1(\tau^*) - X_2(\tau^*) \cdots - X_n(\tau^*)\right] , \end{aligned} \tag{4.4}$$

where $x = (x_1, x_2, \cdots, x_n)$ and

$$\begin{cases} dX_1(t) = X_1(t)\left[-p_1 dt + q_1 dB(t)\right] ; \quad X_1(0) = x_1 \\ \qquad\vdots \\ dX_n(t) = X_n(t)\left[-p_n dt + q_n dB(t)\right] ; \quad X_n(0) = x_n \end{cases} \tag{4.5}$$

with

$$p_1 > 0 , \cdots , \quad p_n > 0 . \tag{4.6}$$

Here $\mathbb{E}^x$ denotes the expectation with respect to the law Q^x of the process $X(t) = (X_1(t), \cdots, X_n(t))$ starting at $x = (x_1, \cdots, x_n) \in [0, \infty)^n$.

In this general case it is hard to find $\Phi(x_1, x_2, \cdots, x_n)$. But it is possible to deduce some information about the optimal stopping time τ^*. From Theorem 2.1 we know that τ^* is typically given by the hitting time of some set S of $\mathbb{R}^n$ (called *the stopping set*). We will prove that S is contained in some halfspace. Under some further conditions on the parameters, we identify S explicitly. We shall need the following

Remark 4.3 *It is easy to see that one can extend Theorem 3.2 to the case with two geometric Brownian motions of the form*

$$\begin{aligned} dX_i(t) &= -p_i X_i(t)dt + X_i(t)\big[q_{i1}dB_1(t) + \cdots + q_{in}dB_n(t)\big]; \\ & \quad X_i(0) = x_i \end{aligned} \tag{4.7}$$

$$\begin{aligned} dX_j(t) &= -p_j X_j(t)dt + X_j(t)\big[q_{j1}dB_1(t) + \cdots + q_{jn}dB_n(t)\big]; \\ & \quad X_j(0) = x_j \, . \end{aligned} \tag{4.8}$$

To handle such cases we define

$$a_{kl} = \sum_{m=1}^{n} q_{km} q_{lm} = q_k \cdot q_l \, ; \quad 1 \leq k, l \leq n \, . \tag{4.9}$$

For the case with the processes (X_i, X_j) *of (4.7) we then put*

$$\gamma_{ij} = a_{ii} - 2a_{ij} + a_{jj} = (q_i - q_j)^2 \tag{4.10}$$

and similarly we let $\lambda = \lambda_{ij}$, $\mu = \mu_{ij}$, $C = C_{ij}$ *be defined by the same formulas as before, but with these new values of* a_{ij}*. Then the conclusions of Theorem 3.2 a and b hold under the conditions*

$$q_i \neq q_j \quad \text{or} \quad p_j > p_i \tag{4.11}$$

for Theorem 3.2 a and the additional condition

$$q_i = q_j \quad \text{or} \quad p_j + \frac{1}{2}q_j^2 > p_i + \frac{1}{2}q_i^2 \tag{4.12}$$

for Theorem 3.2 b.

Lemma 4.4 *Fix* $1 \leq i, j \leq n$*. Let* X_i *and* X_j *be as above and* $x_i \geq \mu_{ij} K x_j$ *(*$K > 0$ *is constant). If* X_i *and* X_j *satisfy (4.11), then*

$$X_i \leq K X_j + x_i - K x_j \, . \tag{4.13}$$

Proof. Let $g(x_i, x_j) := x_i - x_j$ and

$$\phi(x_i, x_j) = \begin{cases} \psi(x_i, x_j)\, ; & x_i < \mu_{ij} x_j \\ \\ g(x_1, x_2)\, ; & x_i \geq \mu_{ij} x_j \, , \end{cases} \tag{4.14}$$

where $\psi(x_i, x_j)$ is obtained exactly the same way as in Sect. 3 for the pair X_i and X_j. If X_i and X_j satisfy (4.11), then by Theorem 3.2 a) we have

$$\begin{aligned} \mathbb{E}^{\,x_i, x_j}\{X_i(\tau) - K X_j(\tau)\} &\leq \phi(x_i, K x_j) \\ &= g(x_i, K x_j) = x_i - K x_j \end{aligned}$$

for all stopping times τ, since $x_i \geq \mu_{ij} K x_j$. This proves the lemma. □

Let X_0 be given by

$$dX_0(t) = X_0(t)\{-p_0 dt + q_0 dB(t)\} \quad \text{with} \quad X_0(0) = 1 \tag{4.15}$$

where $p_0 > 0$ and $q_0 = (q_{01}, \cdots, q_{0n}) \in \mathbb{R}^n$ will be determined later.

We now extend our notation to include our auxiliary process X_0. Denote by μ_{ij} the μ computed for the pair (X_i, X_j); for $0 \le i \le n$, $0 \le j \le n$. Namely, define

$$\lambda_{ij} = \begin{cases} \frac{1}{\gamma_{ij}}\left[\frac{1}{2}\gamma_{ij} + p_i - p_j + \sqrt{(\frac{1}{2}\gamma_{ij} + p_i - p_j)^2 + 2\gamma_{ij}p_j}\ \right] & \text{if } \gamma_{ij} > 0 \\ \frac{p_j}{p_j - p_i} & \text{if } \gamma_{ij} = 0 \text{ and } p_j > p_i, \end{cases} \tag{4.16}$$

where

$$\gamma_{ij} = a_{ii} + a_{jj} - 2a_{ij} = (q_i - q_j)^2; \quad 0 \le i, j \le n. \tag{4.17}$$

Then put

$$\mu_{ij} := \frac{\lambda_{ij}}{\lambda_{ij} - 1}; \quad 0 \le i, j \le n. \tag{4.18}$$

If for $i = 2, \cdots, n$, (X_i, X_0) satisfies (4.11), *i.e.* $q_i \not\equiv q_0$ or $p_0 > p_i$, then by Lemma 4.4 we have

$$X_i \le K_i x_i X_0 + x_i - K_i x_i, \quad i = 2, \cdots, n \tag{4.19}$$

for any K_i satisfying

$$1 \ge \mu_{i0} K_i. \tag{4.20}$$

Thus

$$X_1 - X_2 - \cdots - X_n \ge X_1 - [K_2 x_2 + \cdots + K_n x_n] X_0 - \sum_{i=2}^n x_i + \sum_{i=2}^n K_i x_i. \tag{4.21}$$

On the other hand, if both conditions (4.11) and (4.12) are satisfied for (X_1, X_0) and if

$$X_1(0) = x_1 < \mu_{10}[K_2 x_2 + \cdots + K_n x_n], \tag{4.22}$$

then by Theorem 3.2 there is a stopping time $\tilde{\tau} < \infty$ a.s. such that

$$\mathbb{E}^{x_1,1}\left[X_1(\tilde{\tau}) - (K_2 x_2 + \cdots + K_n x_n) X_0(\tilde{\tau})\right] > x_1 - K_2 x_2 - \cdots - K_n x_n. \tag{4.23}$$

Combining (4.19) and (4.23), we have

$$\mathbb{E}^x\left\{X_1(\tilde{\tau}) - X_2(\tilde{\tau}) - \cdots - X_n(\tilde{\tau})\right\} > x_1 - x_2 - \cdots - x_n.$$

This means that the set

$$\left\{x_1 \ge \mu_{10}\left[K_2 x_2 + \cdots + K_n x_n\right]\right\}$$

contains the stopping set S of Problem 4.2. Choosing $K_i = \frac{1}{\mu_{i0}}$ in (4.20) we obtain

Theorem 4.5 *Assume that*

$$q_i \neq q_0 \quad \text{or} \quad p_0 > p_i \quad \text{for} \quad 2 \leq i \leq n \tag{4.24}$$

and

$$q_1 = q_0 \quad \text{or} \quad p_0 + \frac{1}{2}q_0^2 \geq p_1 + \frac{1}{2}q_1^2 \,. \tag{4.25}$$

Then

$$S \subseteq \Big\{ x_1 \geq \mu_{10} \big[\frac{1}{\mu_{20}} x_2 + \cdots + \frac{1}{\mu_{n0}} x_n \big] \Big\} \,. \tag{4.26}$$

Conversely we have, as mentioned in the Introduction:

Theorem 4.6 *[3] Suppose*

$$q_1 \neq q_j \quad \text{or} \quad p_j > p_1 \quad \text{for} \quad 2 \leq j \leq n \,. \tag{4.27}$$

Then we have

$$\Big\{ x_1 \geq \mu_{12} x_2 + \cdots + \mu_{1n} x_n \Big\} \subseteq S \,. \tag{4.28}$$

Proof. Under the above condition (4.27) the argument in [3, Proposition 2] works. □

Thus if we can find X_0 such that for $i = 2, \cdots, n$ we have $\mu_{1i}\mu_{i0} \leq \mu_{10}$, then we can conclude that the stopping set is (4.28). Namely, we obtain the following theorem.

Theorem 4.7 *Suppose (4.27) holds. Suppose there exist* $p_0 > 0, q_{01}, \cdots, q_{0n} \in \mathbb{R}$ *such that (4.24) and (4.25) hold and such that*

$$\mu_{1i}\mu_{i0} \leq \mu_{10} \,, \quad i.e. \quad \lambda_{10} \leq \frac{\lambda_{1i}\lambda_{i0}}{\lambda_{1i} + \lambda_{i0} - 1} \,; \quad i = 2, \cdots, n \,. \tag{4.29}$$

Then

$$S = \Big\{ x_1 \geq \mu_{12} x_2 + \cdots + \mu_{1n} x_n \Big\} \,. \tag{4.30}$$

The condition (4.29) seems a bit difficult to check in general. We illustrate the condition by looking at some special cases.

First note that λ_{ij} defined in (4.16) can be written

$$\lambda_{ij} = \begin{cases} \frac{1}{2} + \frac{p_i - p_j}{\gamma_{ij}} + \sqrt{(\frac{1}{2} + \frac{p_i - p_j}{\gamma_{ij}})^2 + \frac{2p_j}{\gamma_{ij}}} & \text{if } \gamma_{ij} > 0 \\ \frac{p_j}{p_j - p_i} & \text{if } \gamma_{ij} = 0 \text{ and } p_j > p_i \end{cases} \tag{4.31}$$

where

$$\gamma_{ij} = (q_i - q_j)^2 \,; \quad 0 \leq i, j \leq n \,. \tag{4.32}$$

Suppose p_0 is chosen to be large compared to p_i and γ_{ij} for $i \neq 0$. Then if $\gamma_{i0} > 0$ we have

$$\begin{aligned}
\lambda_{i0} &= \frac{1}{2} + \frac{p_i - p_0}{\gamma_{i0}} + \left|\frac{1}{2} + \frac{p_i - p_0}{\gamma_{i0}}\right| \cdot \sqrt{1 + \frac{2p_0}{\gamma_{i0}(\frac{1}{2} + \frac{p_i - p_0}{\gamma_{i0}})^2}} \\
&\approx \frac{1}{2} + \frac{p_i - p_0}{\gamma_{i0}} + \Big(- \frac{1}{2} - \frac{p_i - p_0}{\gamma_{i0}}\Big) \cdot \Big(1 + \frac{p_0}{\gamma_{i0}(\frac{1}{2} + \frac{p_i - p_0}{\gamma_{i0}})^2}\Big) \\
&= \frac{p_0}{\gamma_{i0}\big(\frac{p_0 - p_i}{\gamma_{i0}} - \frac{1}{2}\big)} = \frac{p_0}{p_0 - p_i - \frac{1}{2}\gamma_{i0}} \\
&\approx 1 + \frac{p_i + \frac{1}{2}\gamma_{i0}}{p_0} . \qquad (4.33)
\end{aligned}$$

Substituting this in (4.29) we see that it suffices to have

$$1 + \frac{p_1 + \frac{1}{2}\gamma_{10}}{p_0} < \frac{(1 + \frac{p_i + \frac{1}{2}\gamma_{i0}}{p_0})\lambda_{1i}}{\frac{p_i + \frac{1}{2}\gamma_{i0}}{p_0} + \lambda_{1i}} \quad \text{for} \quad i = 2, \cdots, n \qquad (4.34)$$

or

$$p_1 + \frac{1}{2}\gamma_{10} < \frac{\lambda_{1i} - 1}{\lambda_{1i}}\Big(p_i + \frac{1}{2}\gamma_{i0}\Big) \quad \textit{for} \quad i = 2, \cdots, n . \qquad (4.35)$$

Choosing q_0 arbitrarily close to q_1 we get $\gamma_{10} = (q_1 - q_0)^2$ arbitrarily close to 0. Therefore we get from (4.35) that it suffices to have

$$p_1 < \frac{\lambda_{1i} - 1}{\lambda_{1i}}\Big(p_i + \frac{1}{2}(q_i - q_1)^2\Big) ; \quad i = 2, \cdots, n . \qquad (4.36)$$

From the expression for λ_{1i} we see that this inequality is satisfied if p_1 is small enough.

We have proved :

Corollary 4.8 *Suppose (4.27) and (4.36) hold. Then*

$$S = \{x_1 \geq \mu_{12}x_2 + \cdots + \mu_{1n}x_n\} .$$

We have proved that, under the conditions (4.27) and (4.29), the stopping set S of the optimal stopping problem (1.5) has the simple form (4.30). It is not clear to us how restrictive the condition (4.29) is, although Corollary 4.8 shows that it *is* satisfied in some parameter domains. Nor is it clear how necessary the condition (4.29) is. It seems natural to conjecture that the stopping set has the form (4.30) in a wide generality of cases.

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