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On Leland's strategy of option pricing with transactions costs

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Abstract. We compute the limiting hedging error of the Leland strategy for the approximate pricing of the European call option in a market with transactions costs. It is not equal to zero in the case when the level of transactions costs is a constant, in contradiction with the claim in Leland (1985).

Key words: Transactions costs, asymptotic hedging, call option, Black-Scholes formula

JEL classification: G13

Mathematics Subject Classification (1991): 90A09, 60H05

1. Introduction

In his paper devoted to the problem of option pricing in the presence of transactions costs, Heyne Leland (1985) suggested a trading strategy based on the nice idea of a periodic revision of a hedging portfolio using modified Black–Scholes betas. He assumed that the level k of transactions costs is a constant and claimed that the terminal value of the portfolio approximates the payoff as the length of a revision interval tends to zero. In a footnote remark he also mentioned that the same holds also when the level is $k_0 n^{-1/2}$, n being the number of revision intervals. Both of these results are considered very helpful for practitioners, and the paper is widely quoted in the literature. However, Leland's arguments were on a heuristic level and his conclusions have to be considered only as conjectures.

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Recently, Lott (1993) provided a rigorous mathematical proof of the footnote remark (together with a study of another approximating strategy). In the present note we show that, unfortunately, the main conjecture of Leland for the case of constant level of the transactions costs fails, and we calculate the hedging error. We also prove that the approximation result still holds in the case where the level is $k_0 n^{-\alpha}$, $\alpha \in]0, 1/2[$, $k_0 > 0$.

2. Description of the model

The stock price dynamics is given by the geometric Brownian motion

$$S_t = S_0 \exp\{(\mu - \sigma^2/2)t + \sigma W_t\},$$

where W is the Wiener process. The bond price is constant over time and equal to one (certainly, this is not a restriction). In the absence of transactions costs the "fair" price at time t of the European call option maturing at $T = 1$ with the striking price K, i.e. with the terminal payoff $H = h(S_1) = (S_1 - K)^+$, is given by the Black–Scholes formula $V_t = C(t, S_t)$ where

$$C(t, x) = C(t, x, \sigma) := x\Phi(d) - K\Phi(d - \sigma\sqrt{1-t}), \tag{1}$$

Φ is the standard normal distribution function with the density φ, and

$$d = d(x, \sigma) := \frac{\ln(x/K)}{\sigma\sqrt{1-t}} + \frac{1}{2}\sigma\sqrt{1-t}.$$

The terminal payoff is replicated by the value at maturity of the self-financing portfolio which has initial endowment $C(0, S_0)$ and at time t contains $\phi_t := C_x(t, S_t)$ units of the stock (and hence $V_t - C_x(t, S_t)S_t$ units of the bond).

Assume that in the stock market the cost of a single transaction is a fixed fraction of its trading volume and the corresponding coefficient is $k = k_n$ (our definition corresponds to one half of Leland's round trip coefficient). Let us consider the self-financing trading strategy with initial endowment $\widehat{C}(0, S_0)$ and the portfolio containing at time t a number ξ_t^n of shares of the stock given by the formula

$$\xi_t^n := \sum_{i=1}^{n} \widehat{\phi}_{t_{i-1}} I_{]t_{i-1}, t_i]}(t) = \sum_{i=1}^{n} \widehat{C}_x(t_{i-1}, S_{t_{i-1}}) I_{]t_{i-1}, t_i]}(t)$$

where $t_i := i/n$, $\widehat{C}(t, x) := C(t, x, \widehat{\sigma})$, $\widehat{\phi}_t := \widehat{C}_x(t, S_t)$,

$$\widehat{\sigma}^2 := \sigma^2\left(1 + \frac{\gamma}{\sigma}\right), \quad \gamma := 2\sqrt{\frac{2}{\pi}}k\sqrt{n} = 2\sqrt{\frac{2}{\pi}}k_0 n^{1/2-\alpha} \tag{2}$$

(to simplify formulae we omit the dependence on n in obvious cases). The value process now has the form

$$V_t(\xi^n) = \widehat{C}(0, S_0) + \int_0^t \xi_u^n dS_u - k \sum_{t_i \le t} S_{t_i} |\xi_{t_i}^n - \xi_{t_{i-1}}^n|. \tag{3}$$

Remarks. 1) We follow here the definition adopted by Lott (1993). Leland (1985) considered instead of self-financing an H-admissible strategy but the problem is essentially the same.
2) A reader may have some trouble with boundary effects. There is a transaction at time $t = 0$ when the investor enters the market. The last transaction at $t = 1$ is also special: the contract may admit different specification for the final settlement, e.g., to deliver or not to deliver the stock. We exclude these particular transactions from our considerations.

Theorem 1 *Assume that* $k = k_n = k_0 n^{-\alpha}$ *where* $\alpha \in]0, 1/2]$, $k_0 > 0$. *Then*

$$P - \lim_{n\to\infty} V_1(\xi^n) = H. \tag{4}$$

Theorem 2 *Let* $k = k_0 > 0$ *be a constant. Then*

$$P - \lim_{n\to\infty} V_1(\xi^n) = H + J_1 - J_2 \tag{5}$$

where

$$J_1 := \min\{K, S_1\}, \tag{6}$$

$$J_2 = J_2(k_0) := \frac{1}{4}\int_0^\infty \frac{S_1}{\sqrt{v}} G(S_1, v, k_0) \exp\left\{-\frac{v}{2}\left(\frac{\ln(S_1/K)}{v} + \frac{1}{2}\right)^2\right\} dv, \tag{7}$$

$$G(S_1, v, k_0) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \left| x - \frac{2k_0 \ln(S_1/K)}{\sqrt{2\pi v}} + \frac{k_0}{\sqrt{2\pi}}\right| e^{-x^2/2} dx \tag{8}$$

Remark. The integral in (8) can be calculated explicitly, in particular, $G(S_1, v, 0) = 2/\sqrt{2\pi}$. From the other hand, it is easy to check that

$$J_1 := \min\{K, S_1\} = \frac{1}{2\sqrt{2\pi}}\int_0^\infty \frac{S_1}{\sqrt{v}} \exp\left\{-\frac{v}{2}\left(\frac{\ln(S_1/K)}{v} + \frac{1}{2}\right)^2\right\} dv = J_2(0).$$

It follows that $0 \le J_2 - J_1 \le Bk_0$ where the constant B depends on S_1 and K. Thus, the option is always underpriced in the limit though the hedging error is small for small values of k_0 (see Fig. 1).

3. Conclusion

We have shown that the limiting error in Leland's hedging strategy for the approximate pricing of the European call is equal to zero only when the level of transaction costs decreases to zero as the revision interval tends to zero. In the case when the level of transaction costs is a constant the limiting hedging error is given in Theorem 2 and, in general, is not equal to zero.

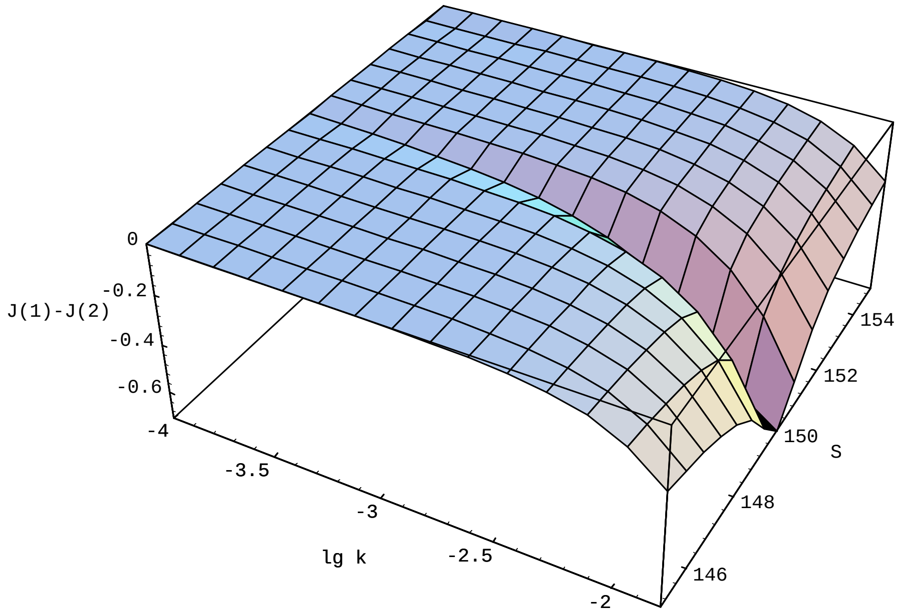

Fig. 1. Dependence of $J_1 - J_2$ on $k = k_0$ and $S = S_1$ for $K = 150$

Appendix A. Proof of Theorem 1

We start with calculations that are common for both theorems, assuming also without loss of generality that P is the risk-neutral measure (i.e., $\mu = 0$).

The case of $\alpha = 1/2$ (when $\widehat{\sigma}$ and hence $\widehat{C}$ do not depend on n) has been considered in Lott (1993). So we suppose from now on that $\alpha \in [0, 1/2[$. This implies that $\widehat{\sigma}^2 = O(n^{1/2-\alpha}) \to \infty$ as $n \to \infty$). However, some ideas from Lott's study work well here and we use them in several places below, e.g., in Lemma 4.

By the Ito formula we have that

$$\widehat{C}_x(t, S_t) = \widehat{C}_x(0, S_0) + M_t^n + A_t^n$$

where

$$\begin{aligned} M_t^n &:= \int_0^t \widehat{C}_{xx}(u, S_u) dS_u = \int_0^t \sigma S_u \widehat{C}_{xx}(u, S_u) d w_u, \\ A_t^n &:= \int_0^t \left[\widehat{C}_{xt}(u, S_u) + \frac{1}{2} \sigma^2 S_u^2 \widehat{C}_{xxx}(u, S_u) \right] du. \end{aligned}$$

The process M^n is a square integrable martingale on $[0, 1]$ with

$$\langle M^n \rangle_t = \frac{1}{2\pi} \int_0^t \frac{\sigma^2}{\widehat{\sigma}^2(1-s)} \exp\left\{ - \left(\frac{\ln(S_s/K)}{\widehat{\sigma}\sqrt{1-s}} + \frac{1}{2}\widehat{\sigma}\sqrt{1-s} \right)^2 \right\} ds.$$

Following Lott we represent the difference $V_1 - H$ in the form convenient for a further study.

Lemma 1 *We have* $V_1 - H = F_1(\xi^n) + F_2(\xi^n)$ *where*

$$F_1(\xi^n) := \int_0^1 (\xi_t^n - \widehat{\phi}_t) dS_t, \tag{9}$$

$$F_2(\xi^n) := \frac{1}{2}\gamma\sigma \int_0^1 S_t^2 \widehat{C}_{xx}(t, S_t) dt - k \sum_{i=1}^n |\xi_{t_i}^n - \xi_{t_{i-1}}^n| S_{t_i}. \tag{10}$$

Proof. According to the Black–Scholes theorem the claim H admits the representation

$$H = C(0, S_0) + \int_0^1 \phi_u dS_u. \tag{11}$$

Comparing (3) and (11) we get that

$$\begin{aligned} V_1 - H &= \int_0^1 (\xi_t^n - \widehat{\phi}_t) dS_t + \int_0^1 (\widehat{\phi}_t - \phi_t) dS_t + \widehat{C}(0, S_0) - C(0, S_0) \\ &\quad - k \sum_{i=1}^n |\xi_{t_i}^n - \xi_{t_{i-1}}^n| S_{t_i}. \end{aligned}$$

We have left to check that

$$\widehat{C}(0, S_0) - C(0, S_0) = \frac{1}{2}\gamma\sigma \int_0^1 S_t^2 \widehat{C}_{xx}(t, S_t) dt - \int_0^1 (\widehat{\phi}_t - \phi_t) dS_t.$$

This identity follows easily from the Ito formula and the observation that $C(t, x)$ is the solution of the parabolic equation $(1/2)\sigma^2 x^2 C_{xx} + C_t = 0$ with the boundary condition $C(1, x) = h(x)$ while $\widehat{C}(t, x)$ is the solution of $(1/2)\widehat{\sigma}^2 x^2 \widehat{C}_{xx} + \widehat{C}_t = 0$ with the same boundary condition. □

Lemma 2 *For any* $\alpha \in [0, 1/2[$

$$P - \lim_{n\to\infty} F_1(\xi^n) = 0. \tag{12}$$

Proof. By the Lenglart inequality (see, e.g., Jacod and Shiryayev (1987)) it is sufficient to show that

$$P - \lim_{n\to\infty} \int_0^1 (\xi_t^n - \widehat{\phi}_t)^2 \sigma^2 S_t^2 dt = 0. \tag{13}$$

Here the integrand is bounded (by $\sigma^2 \sup_{t\le 1} S_t^2$). But for all $t < 1$ we have that $\xi_t^n \to 1$ and $\widehat{\phi}_t \to 1$ as $n \to \infty$. □

The study of $F_2(\xi^n)$ is more delicate.

Put $\Delta t := 1/n$. It is easily seen that $F_2(\xi^n) = \sum_{i=1}^5 L_i^n$ where

$$\begin{aligned}
L_1^n &:= \sigma\frac{\gamma}{2}\int_0^1 S_t^2\widehat{C}_{xx}(t,S_t)dt \\
&\quad -\sigma\frac{\gamma}{2}\int_0^1\sum_{i=1}^n S_{t_{i-1}}^2\widehat{C}_{xx}(t_{i-1},S_{t_{i-1}})I_{]t_{i-1},t_i]}(t)dt, \\
L_2^n &:= \sigma\frac{\gamma}{2}\sum_{i=1}^n S_{t_{i-1}}^2\widehat{C}_{xx}(t_{i-1},S_{t_{i-1}})\Delta t \\
&\quad -k\sigma\sum_{i=1}^n S_{t_{i-1}}^2\widehat{C}_{xx}(t_{i-1},S_{t_{i-1}})|w_{t_i}-w_{t_{i-1}}|, \\
L_3^n &:= k\sigma\sum_{i=1}^n S_{t_{i-1}}^2\widehat{C}_{xx}(t_{i-1},S_{t_{i-1}})|w_{t_i}-w_{t_{i-1}}| \\
&\quad -k\sum_{i=1}^n S_{t_{i-1}}|M_{t_i}^n-M_{t_{i-1}}^n|, \\
L_4^n &:= k\sum_{i=1}^n S_{t_{i-1}}|M_{t_i}^n-M_{t_{i-1}}^n| - k\sum_{i=1}^n S_{t_{i-1}}|\xi_{t_i}^n-\xi_{t_{i-1}}^n|, \\
L_5^n &:= k\sum_{i=1}^n (S_{t_{i-1}}-S_{t_i})|\xi_{t_i}^n-\xi_{t_{i-1}}^n|.
\end{aligned}$$

Lemma 3 *For any $\alpha\in[0,1/2[$ we have*

$$\sigma\frac{\gamma}{2}\int_0^1 S_t^2\widehat{C}_{xx}(t,S_t)dt\to J_1\quad a.s.,$$

$$\sigma\frac{\gamma}{2}\int_0^1\sum_{i=1}^n S_{t_{i-1}}^2\widehat{C}_{xx}(t_{i-1},S_{t_{i-1}})I_{]t_{i-1},t_i]}(t)dt\to J_1\quad a.s.,$$

and, hence, $L_1^n\to 0$ a.s. when $n\to\infty$.

Proof. After the substitution $v=\widehat{\sigma}^2(1-t)$ the first integral can be written as

$$\frac{\sigma}{2}\frac{\gamma}{\widehat{\sigma}^2}\frac{1}{\sqrt{2\pi}}\int_0^{\widehat{\sigma}^2}\frac{S_{1-v/\widehat{\sigma}^2}}{\sqrt{v}}\exp\left\{-\frac{v}{2}\left(\frac{\ln(S_{1-v/\widehat{\sigma}^2}/K)}{v}+\frac{1}{2}\right)^2\right\}dv$$

and the second one as

$$\begin{aligned}
&\frac{\sigma}{2}\frac{\gamma}{\widehat{\sigma}^2}\frac{1}{\sqrt{2\pi}}\int_0^{\widehat{\sigma}^2}\sum_{i=1}^n\frac{S_{1-v_{i-1}/\widehat{\sigma}^2}}{\sqrt{v_{i-1}}} \\
&\times\exp\left\{-\frac{v_{i-1}}{2}\left(\frac{\ln(S_{1-v_{i-1}/\widehat{\sigma}^2}/K)}{v_{i-1}}+\frac{1}{2}\right)^2\right\}I_{]v_i,v_{i-1}]}(v)dv
\end{aligned}$$

where $v_i:=\widehat{\sigma}^2(1-t_i)$. Clearly, both expressions tends a.s. to the integral

$$\frac{1}{2\sqrt{2\pi}}\int_0^\infty \frac{S_1}{\sqrt{v}} \exp\left\{-\frac{v}{2}\left(\frac{\ln(S_1/K)}{v}+\frac{1}{2}\right)^2\right\} dv$$

(which is equal to $J_1 = \min\{K, S_1\}$) since $\gamma/\widehat{\sigma}^2 \to 1/\sigma$ and the integrands above (when $S_1(\omega) \neq K$) are dominated by the function of the form

$$c_1(\omega)[v^{-1/2}e^{-c_2/v}I_{[0,\varepsilon]}(v) + I_{]\varepsilon,N]}(v) + e^{-c_3/v}I_{[N,\infty[}(v)] \tag{14}$$

and we get the result (see Remark after Theorem 2). □

Lemma 4 *For any* $\alpha \in [0, 1/2[$ *we have* $L_2^n \to 0$ *in probability.*

Proof. The sequence of independent random variables

$$|w_{t_i} - w_{t_{i-1}}| - n^{-1/2}\sqrt{2/\pi}$$

is a martingale difference with respect to the discrete filtration $(\mathscr{F}_{t_i})$,

$$E\left(|w_{t_i} - w_{t_{i-1}}| - n^{-1/2}\sqrt{2/\pi}\right)^2 = (1-2/\pi)n^{-1} = (1-2/\pi)\Delta t.$$

By the Lenglart inequality we need to check only the convergence to zero in probability of the sequence

$$\sigma^2(1-2/\pi)k^2\sum_{i=1}^n S_{t_{i-1}}^4 \widehat{C}_{xx}^2(t_{i-1}, S_{t_{i-1}})\Delta t$$

which for almost all ω is of the order $k^2/\widehat{\sigma}^2 = O(n^{-1/2-\alpha})$. □

Lemma 5 *For* $t \in [0, 1[$ *we have*

$$ES_t^2\widehat{C}_{xx}^2(t, S_t) = \frac{1}{2\pi\widehat{\sigma}^2(1-t)}\frac{1}{\sqrt{2a^2+1}}\exp\left\{-\frac{b^2}{2a^2+1}\right\}$$

where

$$a := \frac{\sigma\sqrt{t}}{\widehat{\sigma}\sqrt{1-t}}, \qquad b := \frac{\ln(S_0/K) - \sigma^2 t/2}{\widehat{\sigma}\sqrt{1-t}} + \frac{1}{2}\widehat{\sigma}\sqrt{1-t}.$$

Proof. Let η be a standard normal random variable. Then for any a and b

$$E\exp\{-(a\eta+b)^2\} = \frac{1}{\sqrt{2a^2+1}}\exp\left\{-\frac{b^2}{2a^2+1}\right\}.$$

Since

$$S_t^2\widehat{C}_{xx}^2(t, S_t) = \frac{1}{2\pi}\frac{1}{\widehat{\sigma}^2(1-t)}$$
$$\exp\left\{-\left(\frac{\sigma w_t}{\widehat{\sigma}\sqrt{1-t}} + \frac{\ln(S_0/K) - \sigma^2 t/2}{\widehat{\sigma}\sqrt{1-t}} + \frac{1}{2}\widehat{\sigma}\sqrt{1-t}\right)^2\right\}$$

the result follows. □

As a corollary we get that for $t \in [1/2, 1[$

$$ES_t^2 \widehat{C}_{xx}^2(t, S_t) \le \frac{c}{\widehat{\sigma}\sqrt{1-t}}. \tag{15}$$

We shall use below some simple bounds for higher order derivatives of the function $\widehat{C}_x(t, x) = \Phi(\widehat{d})$ where $\widehat{d} := d(x, \widehat{\sigma})$. We have:

$$\begin{aligned}
\widehat{C}_{xx}(t,x) &= \frac{1}{x\widehat{\sigma}\sqrt{1-t}}\varphi(\widehat{d}),\\
\widehat{C}_{xxx}(t,x) &= \frac{-1}{x^2\widehat{\sigma}\sqrt{1-t}}\left(1+\frac{\widehat{d}}{\widehat{\sigma}\sqrt{1-t}}\right)\varphi(\widehat{d}),\\
\widehat{C}_{xxxx}(t,x) &= \frac{1}{x^3}\left[\frac{2}{\widehat{\sigma}\sqrt{1-t}}\left(1+\frac{\widehat{d}}{\widehat{\sigma}\sqrt{1-t}}\right)\right.\\
&\quad \left.+\frac{\widehat{d}}{\widehat{\sigma}^2(1-t)}\left(1+\frac{\widehat{d}}{\widehat{\sigma}\sqrt{1-t}}\right)\right]\varphi(\widehat{d}),\\
\widehat{C}_{xt}(t,x) &= \left(-\frac{1}{2}\frac{\ln(x/K)}{\widehat{\sigma}(1-t)^{3/2}}+\frac{1}{4}\frac{\widehat{\sigma}}{\sqrt{1-t}}\right)\varphi(\widehat{d}),\\
\widehat{C}_{xxt}(t,x) &= \left[-\frac{1}{2}\frac{1}{x\widehat{\sigma}(1-t)^{3/2}}\right.\\
&\quad \left.+\frac{1}{x\widehat{\sigma}\sqrt{1-t}}\left(-\frac{1}{2}\frac{\ln(x/K)}{\widehat{\sigma}(1-t)^{3/2}}+\frac{1}{4}\frac{\widehat{\sigma}}{\sqrt{1-t}}\right)\right]\varphi(\widehat{d}).
\end{aligned}$$

It follows that for $t < 1$ we have

$$\widehat{C}_{xxx}^2(t,x) \le \frac{c}{x^4}\left(\frac{1}{\widehat{\sigma}^2(1-t)}+\frac{1}{\widehat{\sigma}^4(1-t)^2}\right), \tag{16}$$

$$\widehat{C}_{xxxx}^2(t,x) \le \frac{c}{x^4}\left(\frac{1}{\widehat{\sigma}^2(1-t)}+\frac{1}{\widehat{\sigma}^4(1-t)^2}+\frac{1}{\widehat{\sigma}^6(1-t)^3}\right), \tag{17}$$

$$|\widehat{C}_{xt}(t,x)| = c\left(1+\frac{\widehat{\sigma}}{\sqrt{1-t}}\right), \tag{18}$$

$$|\widehat{C}_{xxt}(t,x)| = \frac{c}{x}\left(\frac{1}{\widehat{\sigma}(1-t)^{3/2}}+\frac{1}{1-t}\right). \tag{19}$$

Lemma 6 *For any $\alpha \in [0, 1/2[$ the sequence $L_3^n \to 0$ in probability.*

Proof. Using the inequality $||a| - |b|| \le |a - b|$ we get that

$$|L_3^n| \le k\sum_{i=1}^n S_{t_{i-1}}\left|\int_{t_{i-1}}^{t_i}\sigma[S_{t_{i-1}}\widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) - S_t\widehat{C}_{xx}(t, S_t)]dw_t\right|$$

and it sufficient to show that the sequence

$$\sum_{i=1}^n\left|\int_{t_{i-1}}^{t_i}[S_{t_{i-1}}\widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) - S_t\widehat{C}_{xx}^2(t, S_t)]dw_t\right|$$

tends to zero in probability. But by the Burkholder and Jensen inequalities

$$\sum_{i=1}^{n} E\Big|\int_{t_{i-1}}^{t_i} [S_{t_{i-1}}\widehat{C}_{xx}(t_{i-1},S_{t_{i-1}}) - S_t\widehat{C}_{xx}(t,S_t)]dw_t\Big| \le$$

$$\le c\sum_{i=1}^{n} E\left(\int_{t_{i-1}}^{t_i} [S_{t_{i-1}}\widehat{C}_{xx}(t_{i-1},S_{t_{i-1}}) - S_t\widehat{C}_{xx}(t,S_t)]^2dt\right)^{1/2} \le$$

$$\le c\sum_{i=1}^{n} \left(\int_{t_{i-1}}^{t_i} E[S_{t_{i-1}}\widehat{C}_{xx}(t_{i-1},S_{t_{i-1}}) - S_t\widehat{C}_{xx}(t,S_t)]^2dt\right)^{1/2}.$$

It follows from (15) that the last summand in the right-hand side of the above inequality is finite and tends to zero. By the Ito formula

$$d[S_t\widehat{C}_{xx}(t,S_t)] = f_t dw_t + g_t dt$$

where

$$\begin{aligned} f_t &:= \sigma S_t\widehat{C}_{xx}(t,S_t) + \sigma^2 S_t^2\widehat{C}_{xxx}(t,S_t),\\ g_t &:= S_t\widehat{C}_{xxt}(t,S_t) + \frac{1}{2}\sigma^2 S_t^3\widehat{C}_{xxxx}(t,S_t) + \sigma^2 S_t^2\widehat{C}_{xxx}(t,S_t) \end{aligned}$$

and we can estimate all other summands as follows:

$$\begin{aligned} &\left(\int_{t_{i-1}}^{t_i} E[S_{t_{i-1}}\widehat{C}_{xx}(t_{i-1},S_{t_{i-1}}) - S_t\widehat{C}_{xx}(t,S_t)]^2dt\right)^{1/2}\\ &\quad\le \left(\int_{t_{i-1}}^{t_i} E\Big[\int_{t_{i-1}}^{t} f_u dw_u + \int_{t_{i-1}}^{t} g_u du\Big]^2 dt\right)^{1/2}\\ &\quad\le \sqrt{2}(\Delta t)^{1/2}\left(\int_{t_{i-1}}^{t_i} Ef_u^2 du + \Delta t\int_{t_{i-1}}^{t_i} E g_u^2 du\right)^{1/2}\\ &\quad\le c\Delta t\left(\frac{1}{\widehat{\sigma}^2(1-t_i)} + \frac{1}{\widehat{\sigma}^4(1-t_i)^2}\right)^{1/2}\\ &\qquad + c(\Delta t)^{3/2}\left(\frac{1}{\widehat{\sigma}(1-t_i)^{3/2}} + \frac{1}{1-t_i} + \frac{1}{\widehat{\sigma}^2(1-t_i)}\right.\\ &\qquad \left. + \frac{1}{\widehat{\sigma}^4(1-t_i)^2} + \frac{1}{\widehat{\sigma}^6(1-t_i)^3}\right)^{1/2}\\ &\quad\le c\Delta t\left(\frac{1}{\widehat{\sigma}(1-t_i)^{1/2}} + \frac{1}{\widehat{\sigma}^2(1-t_i)}\right)\\ &\qquad + c(\Delta t)^{3/2}\left(\frac{1}{\widehat{\sigma}^{1/2}(1-t_i)^{3/4}}\right.\\ &\qquad \left. + \frac{1}{(1-t_i)^{1/2}} + \frac{1}{\widehat{\sigma}(1-t_i)^{1/2}} + \frac{1}{\widehat{\sigma}^2(1-t_i)} + \frac{1}{\widehat{\sigma}^3(1-t_i)^{3/2}}\right). \end{aligned}$$

It is clear that

$$\begin{aligned}
\sum_{i=1}^{n-1}\frac{\Delta t}{\widehat{\sigma}(1-t_i)^{1/2}} &\asymp \widehat{\sigma}^{-1}\int_0^1\frac{dt}{(1-t)^{1/2}}\to 0,\\
\sum_{i=1}^{n-1}\frac{\Delta t}{\widehat{\sigma}^2(1-t_i)} &\asymp \widehat{\sigma}^{-2}\ln n\to 0,\\
(\Delta t)^{1/2}\sum_{i=1}^{n-1}\frac{\Delta t}{\widehat{\sigma}^{1/2}(1-t_i)^{3/4}} &\asymp n^{-1/2}\widehat{\sigma}^{-1/2}\int_0^1\frac{dt}{(1-t)^{3/4}}\to 0,\\
(\Delta t)^{1/2}\sum_{i=1}^{n-1}\frac{\Delta t}{(1-t_i)^{1/4}} &\asymp n^{-1/2}\int_0^1\frac{dt}{(1-t)^{1/4}}\to 0,\\
(\Delta t)^{1/2}\sum_{i=1}^{n-1}\frac{\Delta t}{\widehat{\sigma}(1-t_i)} &\asymp n^{-1/2}\ln n\to 0,\\
(\Delta t)^{1/2}\sum_{i=1}^{n-1}\frac{\Delta t}{\widehat{\sigma}^3(1-t_i)^{3/2}} &\asymp n^{-1/2}\widehat{\sigma}^{-3}n^{1/2}\to 0,
\end{aligned}$$

and the result follows. □

Lemma 7 *For any $\alpha\in]0,1/2[$ the sequence $L_4^n\to 0$ in probability and bounded in probability for $\alpha=0$.*

Proof. Using again the inequality $||a|-|b||\le|a-b|$ we get that

$$\begin{aligned}
|L_4^n| &\le k\sum_{i=1}^{n}S_{t_{i-1}}|A_{t_i}-A_{t_{i-1}}|\\
&\le kc(\omega)\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}|\widehat{C}_{xt}(u,S_u)|du\\
&\quad +kc(\omega)\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\sigma^2S_u^2|\widehat{C}_{xxx}(u,S_u)|du. \qquad (20)
\end{aligned}$$

It follows from (16) that

$$\begin{aligned}
\sum_{i=1}^{n-1}\int_{t_{i-1}}^{t_i}\sigma^2S_u^2|\widehat{C}_{xxx}(u,S_u)|du &\le c\sum_{i=1}^{n-1}\left(\frac{\Delta t}{\widehat{\sigma}(1-t_i)^{1/2}}+\frac{\Delta t}{\widehat{\sigma}^2(1-t_i)}\right)\\
&\le c(\widehat{\sigma}^{-1}+\widehat{\sigma}^{-2}\ln n)\to 0.
\end{aligned}$$

But the first sum for any $\alpha\in[0,1/2[$ converges to a finite limit

$$c(\omega)k_\infty\frac{1}{\sqrt{2\pi}}\int_0^\infty\left|-\frac{1}{2}\frac{\ln(S_1/K)}{v}+\frac{1}{4\sqrt{v}}\right|\exp\left\{-\frac{v}{2}\left(\frac{\ln(S_1/K)}{v^{3/2}}+\frac{1}{2}\right)^2\right\}dv$$

where $k_\infty:=\lim k_n$ (which is to zero when $\alpha>0$ and to k_0 when $\alpha=0$). The convergence to zero of the last summand in (20) follows from (15). □

Lemma 8 *For any* $\alpha \in [0, 1/2[$ *the sequence* $L_5^n \to 0$ *in probability.*

Proof. It is sufficient to show that the sequence

$$k \sum_{i=1}^{n} |\xi_{t_i}^n - \xi_{t_{i-1}}^n| = k \sum_{i=1}^{n} |\widehat{C}_x(t_i, S_{t_i}) - \widehat{C}_x(t_{i-1}, S_{t_{i-1}})|$$

is bounded in probability. But this fact follows easily from Lemma 7 since we proved above that $k \sum_{i=1}^{n} S_{t_{i-1}} |M_{t_i}^n - M_{t_{i-1}}^n|$ converges in probability to J_1. □

Thus, we established that for $\alpha \in]0, 1/2[$ the sequence $F_2(\xi^n)$ converges to zero in probability and Theorem 1 is proved. □

Appendix B. Proof of Theorem 2

In view of Lemmas 1, 2, 8, and 3, 4, 6 it remains to show only that

$$k_0 \sum_{i=1}^{n} S_{t_{i-1}} |\xi_{t_i}^n - \xi_{t_{i-1}}^n| \to J_2.$$

Put

$$\begin{aligned} Z_i^n &:= \left| \sqrt{n}(w_{t_i} - w_{t_{i-1}}) - \frac{\ln(S_{t_{i-1}}/K)}{2\sigma(1 - t_{i-1})\sqrt{n}} + \frac{\widehat{\sigma}^2}{4\sigma\sqrt{n}} \right|, \\ \tilde{Z}_i^n &:= Z_i^n - E(Z_i^n \mid \mathscr{F}_{t_{i-1}}) \end{aligned}$$

Evidently,

$$\sum_{i=1}^{n} S_{t_{i-1}} |\xi_{t_i}^n - \xi_{t_{i-1}}^n| - k_0^{-1} J_2 = I_1^n + I_2^n + I_3^n$$

where

$$\begin{aligned} I_1^n &:= \sum_{i=1}^{n} S_{t_{i-1}} |\xi_{t_i}^n - \xi_{t_{i-1}}^n| - \sum_{i=1}^{n} \sigma S_{t_{i-1}}^2 C_{xx}(t_{i-1}, S_{t_{i-1}}) \Big| w_{t_i} - w_{t_{i-1}} \\ &\quad - \frac{\ln(S_{t_{i-1}}/K)}{2\sigma(1 - t_{i-1})} \Delta t + \frac{1}{4\sigma} \widehat{\sigma}^2 \Delta t \Big|, \end{aligned}$$

$$\begin{aligned} I_2^n &:= \sum_{i=1}^{n} \sigma S_{t_{i-1}}^2 \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) \tilde{Z}_i^n n^{-1/2}, \\ I_3^n &:= \sum_{i=1}^{n} \sigma S_{t_{i-1}}^2 \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) E(Z_i^n \mid \mathscr{F}_{t_{i-1}}) n^{-1/2} - k_0^{-1} J_2. \end{aligned}$$

Since $\widehat{\sigma}^2/\sqrt{n} \to 2\sqrt{2/\pi} k_0 \sigma$ we get using the definition (8) that

$$\begin{aligned} I_3^n &= \frac{1}{4} \sum_{i=1}^{n} \frac{S_{t_{i-1}}}{\widehat{\sigma}\sqrt{1 - t_{i-1}}} G(S_{t_{i-1}}, \widehat{\sigma}^2(1 - t_{i-1})) \varphi(\widehat{d}(S_{t_{i-1}}, \widehat{\sigma})) \widehat{\sigma}^2 \Delta t \\ &\quad + o(1) - k_0^{-1} J_2 \to 0 \quad \text{a.s.} \end{aligned}$$

By the same considerations as in Lemma 4 we can show that $I_2^n \to 0$ in probability. Indeed, for any n the sequence $\tilde{Z}_i^n$ is a martingale difference (with respect to the discrete filtration $(\mathscr{F}_{t_{i-1}})$) and for a certain easily calculated function R we have

$$\sum_{i=1}^{n} \sigma^2 S_{t_{i-1}}^4 \widehat{C}_{xx}^2(t_{i-1}, S_{t_{i-1}}) E((\tilde{Z}_i^n)^2 \mid \mathscr{F}_{t_{i-1}}) n^{-1}$$
$$\leq c(\omega) n^{-1/2} \sum_{i=1}^{n} R(S_{t_{i-1}}, \widehat{\sigma}^2(1-t_{i-1})) \to 0 \quad \text{a.s.}$$

implying by the Lenglart inequality the convergence $I_2^n \to 0$ in probability.

Finally,

$$\begin{aligned} |I_1^n| \quad \leq \quad & \sum_{i=1}^{n} S_{t_{i-1}} |M_{t_i} - M_{t_{i-1}} - \sigma S_{t_{i-1}} \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}})(w_{t_i} - w_{t_{i-1}})| \\ & + \sum_{i=1}^{n} S_{t_{i-1}} \Big| A_{t_i} - A_{t_{i-1}} - \sigma S_{t_{i-1}} \widehat{C}_{xx}(t_{i-1}, S_{t_{i-1}}) \\ & \quad \times \left(-\frac{\ln(S_{t_{i-1}}/K)}{2\sigma(1-t_{i-1})} + \frac{1}{4\sigma} \widehat{\sigma}^2 \right) \Delta t \Big|. \end{aligned}$$

It follows from Lemma 6 that the first sum in the right-hand side of the above inequality converges to zero. The second sum is equivalent to the sum

$$\sum_{i=1}^{n} S_{t_{i-1}} \left| \int_{t_{i-1}}^{t_i} \widehat{C}_{xt}(t, S_t) dt - \widehat{C}_{xt}(t_{i-1}, S_{t_{i-1}}) \Delta t \right|$$

which tends to zero as $n \to 0$. □

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