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Incompleteness of markets driven by a mixed diffusion

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Abstract. An incomplete market driven by a pair of Wiener and Poisson processes is considered. The range of European and American claim prices is determined.

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1 Introduction

It is well known that in an incomplete market there are several equivalent martingale measures and that perfect hedging is not always possible. In this situation, one way to price options is to choose a particular equivalent martingale measure, for example the Föllmer-Schweizer minimal probability [17], the canonical martingale measure which minimizes the relative entropy with respect to the original probability measure [23], or as in [9], the equivalent martingale measure associated with a utility function. In all these cases, the price is a viable price, i.e., it does not induce any arbitrage opportunities; however, there is no consensus on the choice of this martingale measure.

Another approach is to determine the range of prices compatible with no arbitrage or the minimal super-replication strategy. However, as shown by El Karoui and Quenez [15, 16], when the dynamics of the stock price are driven by a Wiener process or by Kramkov [20], in a general semi-martingale framework,

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these two approaches are closely related: the supremum of the possible prices is equal to the minimum initial value of an admissible self-financing strategy that super-replicates the contingent claim.

When the incompleteness arises from stochastic volatility and/or portfolio constraints, the minimal price needed to super-replicate a given contingent claim is studied in [3, 8, 18, 19] among others. Eberlein and Jacod [11] showed the absence of non-trivial bounds on European claim prices in a model where prices are driven by a purely discontinuous Levy process with unbounded jumps.

In this paper we address the problem of the range of viable prices for mixed diffusion dynamics. As in the above mentioned papers, the upper bound is proved to be a trivial one : for example, the minimal strategy to hedge a European call is a long position in the underlying asset. On the contrary, our work shows that the lower bound is not a trivial one, but the corresponding Black-Scholes function evaluated at the current stock price.

2 The model

Consider a financial market where a riskless asset, with deterministic return rate r, and a risky asset are traded up to a fixed horizon T.

Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $(\mathscr{F}_t)$ be a right-continuous filtration, which includes all P negligible sets in $\mathscr{F}$.

The dynamics of the riskless asset's price B are given by

$$dB(t) = B(t)r(t)\,dt,\; B(0) = 1\,;$$

whereas that of the risky asset's price S are

$$dS(t) = S(t_-)[b_t dt + \sigma(t, S_t)dW(t) + \phi(t, S_{t-})dM(t)]\,,\; S(0) = x > 0\,. \tag{2.1}$$

Here W is an $(\mathscr{F}_t)$-Brownian motion and M is the compensated $(\mathscr{F}_t)$-martingale associated with an $(\mathscr{F}_t)$-Poisson process N with deterministic intensity λ, i.e. $M_t = N_t - \int_0^t \lambda(s)ds$. It is well known that, in this setting, W and M are independent ([25], chap. V, ex. 4.25).

The following hypotheses will be upposed to hold through'out the paper:

Hypotheses H1

• The interest rate r and the jump intensity λ are assumed to be non-negative deterministic bounded functions, and the appreciation rate process b to be bounded and $(\mathscr{F}_t)$-predictable.

• The function $\sigma : [0,T] \times (0,\infty) \to \mathbb{R}$ is supposed to be continuous in (t,x), and Hölder continuous in $x \in [0,\infty[$ uniformly in $t \in [0,T]$, and bounded

$$0 < m \leq \sigma(t,x) \leq M\,.$$

• Assume moreover that $\rho(t,x) \overset{def}{=} \frac{\partial}{\partial x}(x\sigma(t,x))$ is continuous in (t,x) and Lipschitz in $x \in [0,\infty[$ uniformly in $t \in [0,T]$.

• The function $\phi : [0,T] \times (0,\infty) \to]-1,\infty[$ is supposed to be bounded and bounded away from 0 and -1 ($0 < m \leq |\phi(t,x)| \leq M$, $\phi(t,x) \geq \epsilon > -1$), continuous in (t,x), and Lipschitz in $x \in [0,\infty[$ uniformly in $t \in [0,T]$.
• We suppose that $\mathscr{F}_t$ is the usual augmentation of $\sigma(W_s, M_s\,;\, s \leq t)$. This implies that $(\mathscr{F}_t)$ is the augmented filtration of $\sigma(S_s\,;\, s \leq t)$ which is generated by the prices.

In particular, under these hypotheses, there exists a unique solution to the equation (2.1) which is strictly positive and can be written in the form

$$S(t) = x\,\exp\left(\int_0^t b_s\,ds\right)\,\mathscr{E}(\sigma W)(t)\,\mathscr{E}(\phi M)(t)\,.$$

Here the Doleans-Dade exponentials are the P-martingales defined by

$$\begin{cases} \mathscr{E}(\sigma W)(t) &= \exp\left(\int_0^t \sigma(s,S_s)dW(s) - \frac{1}{2}\int_0^t \sigma^2(s,S_s)ds\right) \\ \mathscr{E}(\phi M)(t) &= \exp\left(\int_0^t \ln(1+\phi(s,S_{s-}))\,dM(s)\right. \\ & \qquad \left. - \int_0^t \lambda(s)[\phi(s,S_s) - \ln(1+\phi(s,S_s))]\,ds\right) \end{cases}$$

We shall often use the following expression for the Doléans-Dade martingale

$$\mathscr{E}(\phi M)(t) = \exp\left(\int_0^t \ln(1+\phi(s,S_{s-}))\,dN(s) - \int_0^t \lambda(s)\phi(s,S_s)\,ds\right)\,.$$

Moments of prices

For any $a \in \mathbb{R}$, an easy computation gives

$$\begin{aligned} [S(t)]^a &= x^a\,\mathscr{E}(a\sigma W)_t\,\mathscr{E}(\phi_a M)_t\,\exp\left[\frac{1}{2}\int_0^t a(a-1)\sigma^2(s,S_s)ds\right] \\ &\quad \times \exp\left[a\int_0^t b_s ds\right]\exp\left[\int_0^t \lambda(s)[\phi_a(s,S_s) - a\phi(s,S_s))]\,ds\right. \end{aligned} \quad (2.2)$$

where $\phi_a = (1+\phi)^a - 1$. From this we see that, for any $a > 0$, and any $t \in [0,T]$ $E(S^a(t))$ is finite.

3 Risk-neutral probability measures set

Recall that a probability Q on the space $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$ is called an equivalent martingale measure if it is equivalent to the original probability P and if, under Q the discounted price process $(R(t)S(t), t \geq 0)$, where $R(t) = \exp - \int_0^t r(s)\,ds$, is a martingale. We shall restrict our attention to martingale measures such that the Radon-Nikodym density ($L_t = \frac{dQ}{dP}|_{\mathscr{F}_t}, 0 \leq t \leq T$) is a P-square integrable martingale over the time interval $[0,T]$, i.e., $\sup_{t\in[0,T]} E[(L_t)^2] < \infty$.

Proposition 3.1. *The set $\mathcal{Q}$ of equivalent martingales measures with a P-square integrable density is the set of probability measures P^γ such that $\left.\frac{dP^\gamma}{dP}\right|_{\mathcal{F}_t} = L^\gamma(t)$ where $L^\gamma(t) \stackrel{def}{=} L^{\gamma W}(t) L^{\gamma N}(t)$ is the product of the Doleans-Dade martingales*

$$\begin{cases} L^{\gamma W}(t) &= \mathcal{E}(\psi W)(t) &= \exp[\int_0^t \psi_s dW(s) - \frac{1}{2}\int_0^t \psi_s^2 ds] \\ L^{\gamma N}(t) &= \mathcal{E}(\gamma M)(t) &= \exp[\int_0^t \ln(1+\gamma_s) dN(s) - \int_0^t \lambda(s)\gamma_s ds] \end{cases}$$

In these formulae, the two predictable processes ψ and γ are linked by

$$b_t - r(t) + \sigma(t, S_{t-})\psi_t + \lambda(t)\phi(t, S_{t-})\gamma_t = 0 \ \ , \ \ dP \otimes dt.\ a.s. \qquad (3.1)$$

and L^γ is assumed to be a P-square integrable strictly positive martingale. In particular, we assume that the process γ satisfies $(1+\gamma_t) > 0$, $dP \otimes dt\ a.s.$.

Proof. This result is now well known and can be found in [4, 6, 24] among others. The proof follows from the fact that the density L must be a strictly positive P-martingale. Then using the predictable representation theorem for the pair (W, M), it can be written in the form

$$dL(t) = L(t_-)[\psi_t dW(t) + \gamma_t dM(t)]$$

where (ψ, γ) are predictable processes. It remains to choose the pair (ψ, γ) so that the process RSL is a P-martingale; this leads, using Itô's lemma, to formula (3.1). In particular, if γ is a constant greater than -1, P^γ belongs to $\mathcal{Q}$. □

The terms ψ and γ are respectively the risk premium associated with the Brownian risk and the jump risk. On the contrary to the Brownian risk premium, the range of possible values for γ does not depend on the coefficients of the model and that is why we take it as a parameter. For example, $\psi = 0$ is not always a possible choice for the Brownian risk.

In order to price a European call in a mixed model with constant coefficients, Merton [22] choses the particular martingale measure associated with $\gamma = 0$ and $\psi = \frac{b-r}{\sigma}$. We shall therefore call Merton's measure the martingale measure P^0 associated with a zero jump risk premium. Under P^0 the intensity of N remains inchanged while $(W^0(t) = W(t) + \int_0^t \frac{b_s - r(s)}{\sigma(s, S_s)} ds, t \geq 0)$ becomes a standard Brownian motion independent of N.

In the sequel, we denote by Γ the set of the predictable processes γ such that L^γ is a P-square integrable strictly positive martingale. Under P^γ, the process W^γ defined as

$$W^\gamma(t) \stackrel{def}{=} W(t) - \int_0^t \psi_s\, ds$$

is a Brownian motion and $M^\gamma(t) \stackrel{def}{=} M(t) - \int_0^t \lambda(s)\gamma_s\, ds$ is a martingale.

The price process may be written in terms of W^γ and M^γ in the form

$$dS(t) = S(t_-)[r(t)dt + \sigma(t, S_t)dW^\gamma(t) + \phi(t, S_{t-})dM^\gamma(t)]$$

and it satisfies

$$R(t)S(t) = x\,\mathscr{E}(\sigma W^\gamma)(t)\,\mathscr{E}(\phi M^\gamma)(t)\,.$$

The moments of S_t under P^γ may be computed using the same method as in (2.2) where $\lambda(s)$ is replaced by $\lambda(s)(1+\gamma_s)$ and b by r.

In the case where γ is a deterministic function, the Poisson process N has a deterministic P^γ intensity equal to $\lambda(t)(1+\gamma(t))$, the martingale M^γ has the predictable representation property and is independent of W^γ. This is no longer the case when γ depends on W and the pair (W^γ, M^γ) can fail to be independent as it is easily seen in the case $\gamma_t = 1\!\!1_{W_t>0}$ where $E^\gamma(W_t^\gamma(M_t^\gamma)^2) \neq 0$.

In what follows, we denote by $\varUpsilon$ the subset of $\mathscr{Q}$ consisting of the probability measures P^γ associated with a constant jump-risk premium γ. For any $\gamma \in \varUpsilon$ the martingales W^γ and M^γ are P^γ-independent and the process S is a P^γ-Markov process.

4 Range of prices

Let ζ be a European contingent claim, i.e. a non-negative random variable in $L^2(\Omega, \mathscr{F}_T, P)$. A t-time viable price $V^\gamma(t)$ for the contingent claim ζ is defined as the conditional expectation (with respect to the information $\mathscr{F}_t$) of the discounted contingent claim under the martingale-measure P^γ, i.e., $R(t)V^\gamma(t) \stackrel{def}{=} E^\gamma(R(T)\zeta|\mathscr{F}_t)$.

The determination of the range of viable prices, i.e., the explicit form of the interval $]\inf_{\gamma\in\Gamma} V^\gamma(t)\,,\ \sup_{\gamma\in\Gamma} V^\gamma(t)[$ is studied in [11] for European claims, in the case where the underlying asset is a purely discontinuous process.

We give here, in a more general framework, a study based on the convexity of the Black-Scholes price using the simple arguments of El Karoui et al. [13, 14]. The crux of the matter is the convexity, with respect to the stock price, of the Black-Scholes function. This assumption is satisfied under some hypotheses on the pay-off function.

In order to obtain this convexity property, we restrict our attention to the case where $\zeta = h(S_T)$ for some convex function h having bounded one sided derivatives.

4.1 The "call" case

Definition. For a given convex function h, we define the corresponding Black-Scholes function $\mathscr{H}(t, x)$ by

$$R(t)\mathscr{H}(t,x) = E(R(T)\,h(X_T)\,|X_t = x)\,,\quad \mathscr{H}(T,x) = h(x)$$

when the dynamics of X are given by

$$dX_t = X_t(r(t)dt + \sigma(t, X_t)\, dW_t)\, , X_0 = x\, . \tag{4.1}$$

Here $\sigma(t,x)$ is the same function as in (2.1). In the case where $\sigma(t,x) = \sigma(t)$ is a deterministic function of time only, the Black-Scholes function is known to be given by

$$\mathscr{H}(t,x) = \frac{R(T)}{R(t)}\, E\left[h(\frac{xR(T)}{R(t)}\, \exp[\Sigma(t)U - \frac{1}{2}\Sigma^2(t)]\,)\right]$$

where U is a standard normal random variable and $\Sigma^2(t) = \int_t^T \sigma^2(s)ds$.

Under the hypotheses on σ and h, the Black-Scholes function $\mathscr{H}$ is convex w.r.t. x, belongs to $C^{1,2}$ [13] and its “Delta” is bounded : $\left|\frac{\partial \mathscr{H}}{\partial x}(t,x)\right| \leq C$.

Consider the operators $\mathscr{L}$ and Λ defined on $C^{1,2}$ functions by

$$\begin{aligned} \mathscr{L}(f)(t,x) &= \frac{\partial f}{\partial t}(t,x) + rx\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}x^2\sigma^2(t,x)\frac{\partial^2 f}{\partial x^2}(t,x) \\ \Lambda f(t,x) &= f(t,(1+\phi(t,x))x) - f(t,x) - \phi(t,x)x\,\frac{\partial f}{\partial x}(t,x)\, . \end{aligned}$$

Here, the Black-Scholes function $\mathscr{H}$ satisfies $\mathscr{L}(R\mathscr{H})(t,x) = 0$.

Hypothesis H2
A convex function h satisfies the hypothesis **H2** if $0 \leq h(x) \leq x$, $h(0) = 0$ and if the function g defined by $g(x) = x - h(x)$ is bounded.

As an example, we have the European call, where $h(x) = (x - K)^+$.

Theorem 4.1. *Let $P^\gamma \in \mathscr{Q}$ and $V^\gamma(t)$ be the associated viable price process defined by $R(t)V^\gamma(t) = R(T)E^\gamma(h(S_T)|\mathscr{F}_t)$. Then,*
1. The hedging error caused by jumps is given by $\Lambda\mathscr{H}(s,x)$. More precisely,

$$R(t)V^\gamma(t) = R(t)\,\mathscr{H}(t,S_t) + \mathscr{R}_t^\gamma$$

where $\mathscr{R}_t^\gamma = E^\gamma\left[\int_t^T R(s)(1+\gamma_s)\lambda(s)\Lambda\mathscr{H}(s,S_s)ds\,|\mathscr{F}_t\right]$.
2. Any viable price is bounded below by the Black-Scholes function, evaluated at the underlying asset value i.e.,

$$\mathscr{H}(t,S_t) \leq V^\gamma(t)\ ,\ \forall \gamma \in \Gamma.$$

3. If moreover h satisfies the hypotheses **H2**, *any viable price is bounded above by the underlying asset value*

$$\mathscr{H}(t,S_t) \leq V^\gamma(t) \leq\ S_t\ ,\ \forall\gamma \in \Gamma.$$

Proof. Let S be the solution of (2.1) and suppose that $\mathscr{H}$ is $C^{1,2}$. (Otherwise, apply Itô's lemma for convex functions, see e.g. [25]). Itô's formula for mixed processes gives

$$\begin{aligned} R(T).\mathscr{H}(T,S_T) \\ = \quad & R(t).\mathscr{H}(t,S_t) \\ & + \int_t^T \left[\mathscr{L}(R.\mathscr{H})(s,S_s) + R(s)\lambda(s)(\gamma_s+1)\Lambda.\mathscr{H}(s,S_s)\right]\, ds \\ & + \int_t^T R(s)\frac{\partial \mathscr{H}}{\partial x}(s,S_{s_-})\,S_{s_-}\,(\sigma(s,S_s)dW^\gamma(s) + \phi(s,S_{s_-})dM^\gamma(s)) \\ & + \int_t^T R(s)\Lambda.\mathscr{H}(s,S_{s_-})\,dM^\gamma(s) \end{aligned} \tag{4.2}$$

From $E^\gamma(S_t^2) \leq \sqrt{E[(L_t^\gamma)^2]\,E(S_t^4)}$, the boundedness of the Delta and the existence of the moments of price, the stochastic integrals on the right-hand side of (4.2) are P^γ-martingales. The Black-Scholes equation gives $\mathscr{L}[R.\mathscr{H}](s,x) = 0$. Taking the P^γ conditional expectation with respect to $\mathscr{F}_t$ then leads to

$$\begin{aligned} E^\gamma(R(T).\mathscr{H}(T,S_T)|\mathscr{F}_t) \quad = \quad & E^\gamma(R(T)h(S_T)|\mathscr{F}_t) \\ = \quad & R(t).\mathscr{H}(t,S_t) \\ & + E^\gamma\left(\int_t^T R(s)\lambda(s)(\gamma_s+1)\Lambda.\mathscr{H}(s,S_s)\,ds\,|\mathscr{F}_t\right) . \end{aligned}$$

The lower bound of the interval of viable prices follows from the convexity of $\mathscr{H}(t,\cdot)$ which implies that $\Lambda.\mathscr{H}(t,x) \geq 0$. The upper bound is a trivial one from the hypothesis that $h(x) \leq x$ and the P^γ martingale property of the process RS. □

The convexity of the set of equivalent martingale measure implies that the map $\gamma \to E^\gamma(BR(T))$ is convex and, therefore, continuous. Consequently, the range of prices is an interval.

To establish that the viable prices of the claim $h(S_T)$ span the whole interval $]\mathscr{H}(t,S_t), /, S_t[$, we may (and will) restrict our attention to the case of a constant jump risk premium.

Theorem 4.2. *1. The lower bound of the range of prices is the Black and Scholes function evaluated at the unerlying asset's price :* $\lim_{\gamma\to-1} V^\gamma(t) = \mathscr{H}(t,S_t)$.
2. If moreover h satisfies hypothesis **H2***, the upper bound is the trivial one:* $\lim_{\gamma\to+\infty} V^\gamma(t) = S_t$.

Proof. If γ is a constant, the price process S is a Markov process under P^γ. Therefore, it suffices to prove the lemma for $t = 0$.

When γ goes to -1, the intensity of the Poisson process goes to zero, therefore there are, at least intuitively, no more jumps and the viable price converges to the Black-Scholes function. This argument can be made precise as follows. From the inequality $|\Lambda\mathscr{H}(t,x)| \le 2xMC$ where M is the bound for the size of the jumps ϕ, it follows that

$$\begin{aligned} 0 &\le E^{\gamma}[\int_0^T R(s)\lambda(s)(\gamma+1)\Lambda\mathscr{H}(s,S_s)\,ds] \\ &\le 2C(\gamma+1)M\int_0^T \lambda(s)E^{\gamma}(R(s)S_s)\,ds \\ &= 2xC(\gamma+1)M\int_0^T \lambda(s)\,ds \end{aligned}$$

Thus, the proof is complete because the right-hand side converges a.s. to 0 when γ goes to -1.

The proof is somewhat more difficult for the upper bound. When the intensity goes to infinity, there are more and more jumps and the Brownian motion has no effect. Furthermore, the process M is a martingale and jumps very quickly above and below M_0. It is then nearly impossible to see these jumps and it seems that the process stays at the initial point. This point of view is made precise in the following lemma.

Lemma 4.3. *For any $\eta > 0$, $P^{\gamma}[\mathscr{E}(\phi M^{\gamma})_T \ge \eta]$ tends to 0 when γ goes to infinity.*

Proof. From the Markov inequality, it suffices to check that $E^{\gamma}\left[\mathscr{E}(\phi M^{\gamma})_T)^a\right]$ converges to zero, for $a > 0$. Using the same computation as in (2.2), it can be shown that, for γ constant

$$[\mathscr{E}(\phi M^{\gamma})_T]^a \le \mathscr{E}(\phi_a M^{\gamma})_T \ \exp(1+\gamma)\int_0^T \lambda(s)F(s,S_s)ds$$

where $F(s,x) = (1+\phi(s,x))^a - a\phi(s,x) - 1$.

For $1-a > 0$, from the hypothesis $|\phi| > m$, some elementary computations on the function F lead to $F(s,x) \le k$ where

$$k = [(1+m)^a - (1+am)] \vee [(1-m)^a - (1-am)] < 0\,.$$

Therefore $E^{\gamma}\left([\mathscr{E}(\phi M^{\gamma})_T]^a\right) \le \exp[(1+\gamma)k\int_0^T \lambda(s)ds]$ goes to 0 when γ goes to infinity and the result follows . □

We now return to the proof of theorem (4.2); from hypotheses **H2** we get

$$\begin{aligned} E^{\gamma}[R(T)h(S_T)] &= E^{\gamma}[R(T)S_T] - E^{\gamma}[R(T)g(S_T)] \\ &= x - E^{\gamma}[R(T)g(S_T)] = x - R(T)E^{\gamma}[G(x\mathscr{E}(\phi M^{\gamma})_T] \end{aligned}$$

where $G(x) = E[g(xR(T)^{-1}\mathscr{E}(\sigma W)_T)]$ is a continuous bounded function. The convergence of $E^{\gamma}(G(x\mathscr{E}(\phi M^{\gamma})_T)$ towards 0 as γ goes to infinity follows from the boundness and continuity of G.

4.2 The put case

Hypothesis H3:
A function g satisfies the hypothesis **H3** if g is a convex bounded function having bounded one sided derivatives and if $0 \leq g(x) \leq g(0)$.

Under this hypothesis the Black-Scholes function $\mathscr{G}(t,x)$ associated with the contingent claim $g(S_T)$ is convex w.r.t. x and, following our method, we obtain

$$R(t)\mathscr{G}(t,S_t) \leq E^\gamma(R(T)g(S_T)|\mathscr{F}_t) \leq R(T)g(0)$$

As above, the viable prices span the whole interval.

4.3 Substrategy

Our proof provides an explicit substrategy. Suppose that a self-financing portfolio is built on the "Delta" of Black-Scholes, i.e. by investing $\frac{\partial \mathscr{H}}{\partial x}(t,S_t)$ in the risky asset. The value $\Pi(t)$ of this portfolio satisfies

$$d\Pi(t) = (\Pi(t) - S_t\frac{\partial \mathscr{H}}{\partial x}(t,S_t))r(t)dt + \frac{\partial \mathscr{H}}{\partial x}(t,S_t)dS_t$$

The tracking error is defined as $e(t) \stackrel{def}{=} \Pi(t) - \mathscr{H}(t,S_t)$.

Theorem 4.4. *The discounted tracking error* $(R(t)e(t), t \geq 0)$ *is a non-increasing process*

Proof. The tracking error satisfies

$$de(t) = -\Lambda\mathscr{H}(t,S_t)\,dN_t + r(t)e(t)dt\,.$$

Consequently $R(t)e(t)$ is a decreasing process. □

It seems interesting to obtain some estimates for the expected value and the variance of the error from the Black-Scholes hedging. Some easy computation lead to $\mathrm{Var}(R(t)e(t)) \leq Cx^2K(t)$ where $K(t) = \alpha t^2 e^{\beta t}$, $\alpha, \beta > 0$.

4.4 Remarks and comments

1. Eberlein and Jacod [12] consider the case where the dynamics of the prices is a general Lévy process. While restricting their attention to equivalent martingale measures under which the underlying asset remains a Lévy process, they show that the range of prices is included in the interval $]\mathscr{H}(t,S_t), S_t[$ using a different method to ours.
2. Note that if $\mathscr{E}(\sigma W^\gamma)$ and $\mathscr{E}(\phi M^\gamma)$ are independent processes, the convexity of the Black-Scholes function and Jensen's inequality lead easily to a comparison of the viable P^γ price and the Black-Scholes one. Indeed, let

$\mathscr{H}(x) \stackrel{def}{=} R(T)E[h(xR(T)^{-1}\mathscr{E}(\sigma W)_T)]$ be the price of the claim in a Black-Scholes framework. Then, assuming that $\mathscr{H}$ is a convex function, (see the needed assumptions above)

$$\mathscr{H}(x) = \mathscr{H}[E^\gamma(x\mathscr{E}(\phi M^\gamma)_T)] \leq E^\gamma[\mathscr{H}(x\mathscr{E}(\phi M^\gamma)_T)] = R(T)E^\gamma(h(S_T))$$

where the last equality follows from the assumed independence between $\mathscr{E}(\sigma W^\gamma)$ and $\mathscr{E}(\phi M^\gamma)$ and the definition of $\mathscr{H}$.

3. As we have recalled, the convexity of the Black-Scholes function holds when h is convex, under some assumptions on the dynamics of the prices. The reader may refer to the thesis of Martini [21] for an exhaustive study of the convexity of the Black-Scholes function.

4. The impact of jump size and jump risk can be made precise. Let γ be a fixed constant parameter. Then, the viable price is a function of the underlying asset, say $V^\gamma(t) = V^\gamma(t, S_t)$ where $V^\gamma(t,x)$ depends on ϕ and on the other parameters of the model. Pham [24] proves that the function $\gamma \to V^\gamma(t,x)$ is non-decreasing. Bellamy [5] establishes that the function $\phi \to V^\gamma(t,x)$ is non-increasing on the interval $]-1,0[$ and non-decreasing on $\mathbb{R}^+$.

5. If h satisfies **H2**, and if the upper bound belongs to the range of prices, i.e., there exists P^γ such that $R(t)S_t = E^\gamma(R(T)h(S_T)|\mathscr{F}_t)$, then $g(S_T) = S_T - h(S_T)$ is hedgeable and bounded, the range of prices is a singleton and $\mathscr{H}(t, S_t) = h(S_t)$ which implies that $h(x) = x$. It is tempting, but not true to extend this property to any contingent claim. If a contingent claim is hedgeable, the upperbound is reached, however, except in the case when this claim is bounded, this does not imply that the range is a singleton, see e.g. Ansel and Stricker [2].

4.5 Generalisations

1. The result in theorem 4.2 holds under a more general dynamics for the jump part. Suppose for example that the dynamics of S are given by

$$dS(t) = S(t_-)[b_t dt + \sigma(t, S_t)dW(t) + \int_{\mathbb{R}} \phi(t,y)\widetilde{v}(dt,dy)]$$

where $\widetilde{v}(dt,dy) = v(dt,dy) - \lambda(t)m(dx)dt$ is the compensated jump martingale of a homogeneous Poisson random measure, and σ, ϕ and h satisfy the hypotheses **H1** and **H2** respectively.

The proof follows by analogy with the previous proof.

The set $\mathscr{Q}$ of equivalent martingales measures is the set of probability measures P^γ such that $dP^\gamma = L_T^\gamma dP$ where the process L_t^γ satisfies the stochastic differential equation

$$dL^\gamma(t) = L^\gamma(t_-)\left(\psi(t)dW(t) + \int_{\mathbb{R}} \gamma(t,y)\widetilde{v}(dt,dy)\right)$$

and where the processes ψ and γ are related by the equation

$$b_t - r(t) + \sigma(t, S_t)\psi(t) + \int_{\mathbb{R}} \lambda(t)\varphi(t, y)\,(1 + \gamma(t, y))\,m(dx) = 0\,.$$

It suffices to check that

$$E^{\gamma}[R(T)h(S_T)|\mathcal{F}_t] = R(t)\mathcal{H}(t, S_t) + E^{\gamma}\left[\int_t^T R(s)\lambda(s)\widetilde{\Lambda}\mathcal{H}(s, S_s)ds\,|\mathcal{F}_t\right]\,.$$

Here

$$\widetilde{\Lambda}f(t,x) = \int_{\mathbb{R}}\left[f(t,(1+\varphi(t,y))x) - f(t,x) - x\varphi(t,y)\frac{\partial f}{\partial x}(t,x)\right](1+\gamma(t,x))\,m(dx)$$

and it is clear that $\widetilde{\Lambda}\mathcal{H}(t,x) \geq 0$ as soon as $\mathcal{H}$ is a convex function of x.

2. The same result extends also to the case where

$$dS_t = S_{t-}\left(b_t dt + \sigma(t, S_t)dW_t + \sum_{i=1}^{i=k}\varphi_i(t, S_{t-})dM_{i,t}\right)$$

We shall not provide the proof here, since it uses the same ideas as in the previous one.

3. Our results extend also to the case where the process N has a stochastic bounded intensity under the historical probability measure. Actually, we used the deterministic intensity assumption in order to establish that the local martingales in (4.2) are martingales, which can be done in a general setting. It remains, in the proof of the corresponding theorem and lemma, to restrict our attention to the set of jump risk premium γ for which the processes W^{γ} and M^{γ} remain independent under P^{γ}. This can be done with $1+\gamma_t = \dfrac{\alpha}{\lambda_t}$ with constant coefficient α varying in $]0,\infty[$.

4. The same method applies for average contingent claims of the form $\zeta = \int_0^T h(s, S_s)ds$ for some function $h(s,\cdot)$ satisfying the hypothesis **H2** or **H3**, see Sect. 5.2 for the Asian case.

5 American and Asian cases

5.1 American case

The comparison result still holds for American claims.

If h satisfies the hypothesis **H2**, it can be proved that the American and the European prices agree under any P^{γ}, see e.g. [13].

Let us now study the “put” case and assume that the pay-off function g satisfies **H3**.

Let $\mathcal{G}^{Am,\gamma}$ defined by

$$R(t)\,\mathscr{G}^{Am,\gamma}(t) = \text{ess sup}_{\tau\in\mathscr{T}(t,T)} E^{\gamma}(R(\tau)g(S_{\tau})\,|\mathscr{F}_t)$$

be the American viable price corresponding to the martingale measure P^{γ}. Here, $\mathscr{T}(t,T)$ is the class of stopping times with values in the interval $[t,T]$. Let $\mathscr{G}^{Am}$ be defined as the American- Black-Scholes function for an underlying asset following (4.1), i.e.,

$$R(t)\mathscr{G}^{Am}(t,X_t) \stackrel{def}{=} \text{ess sup}_{\tau\in\mathscr{T}(t,T)} E(R(\tau)g(X_{\tau})\,|X_t)$$

Then, $\mathscr{G}^{Am}(t,.)$ is a convex function [13]. Moreover, this function is smooth before the exercice time τ^* and satisfies $\mathscr{L}(R\mathscr{G}^{Am}) = 0$ on this region.

Proposition 5.1. *Let $P^{\gamma} \in \mathscr{Q}$. Then,*

$$\mathscr{G}^{Am}(t,S_t) \le \mathscr{G}^{Am,\gamma}(t) \le g(0)$$

The viable prices span the whole interval.

Proof. We prove the result for $t = 0$. Applying Itô's formula to $\mathscr{G}^{Am}(t,S_t)$ yields

$$\begin{aligned} R(t)\mathscr{G}^{Am}(t,S_t) &= \mathscr{G}^{Am}(0,S_0) \\ &+ \int_0^t \left[\mathscr{L}(R\mathscr{G}^{Am})(s,S_s) + R(s)S_s\lambda(s)(1+\gamma_s)\Lambda\mathscr{G}^{Am}(s,S_s)\right] ds + Z_t \end{aligned}$$

where Z is a martingale. Using the convexity of $\mathscr{G}^{Am}$ and introducing the stopping time

$$D \stackrel{def}{=} \inf\{t \mid \mathscr{G}^{Am}(t,S_t) = g(S_t)\}$$

it follows that

$$\mathscr{G}^{\gamma}(0) \ge E^{\gamma}(R(D)g(S_D)) \ge \mathscr{G}^{Am}(0,x)$$

where we have used that $\mathscr{G}^{\gamma}(0)$ is the supremum over stopping times of $E^{\gamma}(R(\tau)g(S_{\tau}))$. The general result (i.e. at time t) is now routine, see [13]. The upper bound is obvious since $R(t) \le 1$.

When γ goes to -1, the viable prices converge to the Black-Scholes function evaluated at S_t, a fact that has already been pointed out by Pham [24]. When γ goes to infinity, the exercise boundary tends to $g(0)$. From $\mathscr{G}^{Am,\gamma}(t) \ge E^{\gamma}(R(T)g(S_T))$ and using the same arguments as in the previous section, we obtain the required limit. □

Pham [24] has proved this result for a put option, under the restriction that γ is a deterministic parameter, using the maximum principle. American options in a mixed model are intensively studied by Zhang [26], and Chesney [7] among others. In these papers, the authors compute the value of the American option under a particular martingale measure, such as Merton's measure or the Föllmer-Schweizer martingale measure.

5.2 Asian options

In the Black-Scholes framework (4.1), the value of a fixed-strike Asian call is defined to be

$$C_t^{As} \stackrel{def}{=} E\left[\frac{R(T)}{R(t)}\left(\frac{1}{T}\int_0^T X_u\,du - K\right)^+ |\mathscr{F}_t\right]$$

Décamps and Koehl [1] have established that the value of an Asian claim is given by $C_t^{As} = X(t)\mathscr{A}(t, Y_t^X)$ where Y^X is determined in terms of X as

$$Y_t^X \stackrel{def}{=} \frac{1}{X(t)}\left(\frac{1}{T}\int_0^t X(u)\,du - K\right),$$

and $\mathscr{A}$ satisfies the partial derivative equation

$$\frac{\partial\mathscr{A}}{\partial t} + (\frac{1}{T} - ry)\frac{\partial\mathscr{A}}{\partial y} + \frac{1}{2}\sigma^2 y^2\frac{\partial^2\mathscr{A}}{\partial y^2} = 0$$

together with the terminal condition $\mathscr{A}(T, y) = y^+$. Further, they proved the convexity of the value of the Asian claim with respect to the underlying asset X. Following our method, the convexity of the Asian price and Itô's lemma lead to the inequality

$$E^\gamma\left(e^{-rT}(\frac{1}{T}\int_0^T S(u)\,du - K)^+|\mathscr{F}_t\right) \geq S_t\,\mathscr{A}(t, Y_t^S)$$

where Y^S is defined in terms of S as above.

6 Conclusion

In incomplete markets driven by mixed diffusion, the range of prices is too large. The non existence of non-trivial super hedging strategy seems to be a constant fact. However, the lower bound proves that, one more time, the Black-Scholes function satisfies some robustness condition. In recent works, many authors are interested with modelling prices including jumps. More precise studies have to be conducted, either to price the jump risk in an economically satisfying manner, or to construct, as Dritschel and Protter [10], a complete model allowing for jumps in the stock price dynamics.

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Note added in proof: It is possible to show that $S_t\mathscr{A}(t, Y_t^S)$ is the lower bound of the viable prices. See [27].

References

1. Alziary, B., Decamps, J.P., Koehl, P.F.: A P.D.E. approach to Asian options: analytical and numerical evidence. J. Banking and Finance **21**, 613–640 (1997)
2. Ansel, J.P., Stricker, C.: Couverture des actifs contingents. Ann. Inst. H. Poincaré. Prob. Stat. **30**, 303–315 (1994)
3. Avellaneda, M., Levy, A., Paras, A.: Pricing and hedging derivative securities in markets with uncertain volatilities. Appl. Math. Finance **2**, 73–88 (1995)
4. Bardhan, I., Chao, X.: On martingale measures when asset returns have unpredictable jumps. Stoc. Proc. and Appl. **63**, 35–54 (1996)
5. Bellamy, N.: Evaluation et couverture dans un marché dirigé par des processus discontinus. Thèse de doctorat, Université d'Evry, May 1999
6. Buhlmann, H., Delbaen, F., Embrecht, P., Shiryaev, A.: No-arbitrage, change of measure and conditional Esscher transform in a semi-martingale model of stock price. CWI Quaterly **9**, 291–317 (1996)
7. Chesney, M.: A simple method for the valuation of american options in a jump diffusion setting. Preprint (1996)
8. Cvitanic, J., Pham, H., Touzi, N.: Super-replication in stochastic volatility models under portfolio constraints. J. Appl. Probab. (To appear) (1999)
9. Davis, M.H.A.: Option pricing in Incomplete markets . In: Demtser, M.H.A., Pliska, S.R. (eds.), Mathematics of Derivative Securities. Publication of the Newton Institute. Cambridge: Cambridge University Press 1997, pp. 216–227
10. Dritschel, M., Protter, P.: Complete markets with discontinuous security price. Finance and Stochast. **3**, 203–214 (1999)
11. Eberlein, E., Jacod, J.: On the range of option pricing. Finance and Stochast. **1**, 131–140 (1997)
12. Eberlein, E., Jacod, J., Shiryaev, A.: On the range of option pricing. Unpublished manuscript, 1997
13. El Karoui, N., Jeanblanc-Picqué, M., Shreve, S.: Robustness of Black and Scholes formula. Math. Finance **8**, 93–126 (1998)
14. El Karoui, N., Jeanblanc-Picqué, M., Viswanathan, R.: Bounds for the price of options. Proceedings of a US-French Workshop, Rutgers University. 1991. Lecture notes in Control and Information sciences. Berlin Heidelberg New York: Springer 1992, **177**, pp. 234–237
15. El Karoui, N., Quenez, M-C.: Programmation dynamique et évaluation des actifs contingents en marchés incomplets. CRAS, Paris **331**, 851–854 (1991)
16. El Karoui, N., Quenez, M-C.: Dynamic programming and pricing of contingent claims in an incomplete market. SIAM J. Control and Optim. **33**, 29–66 (1995)
17. Föllmer, H., Schweizer, M.: Hedging of contingent claims under incomplete information. In: Davis, M.H.A., Elliott, R.J. (eds.), Applied Stochastic Analysis. London: Gordon and Breach 1990
18. Frey, R.: Superreplication and optimal stopping. (forthcoming) Finance and Stochast. (1998)
19. Frey, R., Sin, C.: Bounds on european option prices under stochastic volatility. Mathemat. Finance **9**, 97–116 (1999)
20. Kramkov, D.: Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets. Prob. Theo. and related fields **105**, 459–479 (1996)
21. Martini, C.: Propagation de la convexité. Thèse de doctorat, Université d'Evry, Mars 1996
22. Merton, R.: Option pricing when underlying stock returns are discontinuous. J. Financial Econ. **3**, 125–144 (1976)
23. Miyahara, Y.: Canonical martingale measures of incomplete assets markets. In: Watanabe, S., Prohorov, Yu.V., Fukushima, M., Shiryaev, A.N. (eds.), Proceedings of the seventh Japan Russia Symposium. Probability theory and mathematical statistics. World Scientific 1997, pp. 343–352
24. Pham, H.: Optimal stopping, free boundary and American option in a jump diffusion model. Applied Math. and Optim **35**, 145–164 (1997)
25. Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion, Second edition. Berlin: Springer 1994
26. Zhang, X.: Numerical analysis of American option pricing in a jump-diffusion model. Math. Operations Res. **22**, 668–690 (1997)
27. Bellamy, N.: Asian Options in a Market driven by a discontinuous Process. Preprint (1999)