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Optimal portfolio management rules in a non-Gaussian market with durability and intertemporal substitution

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Abstract. We consider an optimal portfolio-consumption problem which incorporates the notions of durability and intertemporal substitution. The logreturns of the uncertain assets are not necessarily normally distributed. The natural models then involve Lévy processes as driving noise instead of the more frequently used Brownian motion. The optimization problem is a singular stochastic control problem and the associated Hamilton-Jacobi-Bellman equation is a nonlinear second order degenerate elliptic integro-differential equation subject to gradient and state constraints. For utility functions of HARA type, we calculate the optimal investment and consumption policies together with an explicit expression for the value function when the Lévy process has only negative jumps. For the classical Merton problem, which is a special case of our optimization problem, we provide explicit policies for general Lévy processes having both positive and negative jumps. Instead of following the classical approach of using a verification theorem, we validate our solution candidates within a viscosity solution framework. To this end, the value function of our singular control problem is characterized as the unique constrained viscosity solution of the Hamilton-Jacobi-Bellman equation in the case of general utilities and general Lévy processes.

Key words: Portfolio choice, intertemporal substitution, singular stochastic control, dynamic programming method, integro-differential variational inequality, viscosity solution, closed form solution

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1 Introduction

The present paper continues our study in [4] of an optimal portfolio selection problem with consumption. The optimization problem captures the notions of durability and intertemporal substitution, and was first suggested and studied extensively by Hindy and Huang [14] for a market modeled by a geometric Brownian motion. In [4], we extended their model to exponential pure-jump Lévy processes and showed that the value function is the unique constrained viscosity solution of the associated Hamilton-Jacobi-Bellman equation, which is a first-order integro-differential equation subject to a gradient constraint (i.e., a first order integro-differential variational inequality).

The main topic here is to present explicit consumption and portfolio allocation rules in a market where the risky asset may have negative price shocks and the investor has a power utility function. We use a viscosity solution framework to validate our solution candidates, contrary to [14] which relies on a verification theorem. To this end, we extend the results on viscosity solutions in [4] to also account for Lévy processes having a continuous martingale part. We refer to Bank and Riedel [2], Framstad et al. [11], and Kallsen [16] for related results on portfolio optimization in Lévy markets.

Eberlein and Keller [8] and Barndorff-Nielsen [3] propose to model logreturns (i.e., the logarithmic price changes) of stock prices using distributions from the generalized hyperbolic family. Following their perspective, one is lead to an exponential stock price dynamics driven by *pure-jump* Lévy processes having paths of infinite variation. This was the main motivation in [4] for concentrating on Lévy models without a continuous martingale part.

In this paper our basic model for the asset price dynamics will be

$$S_t = S_0 e^{\sigma W_t + L_t}, \tag{1.1}$$

where L_t is a pure-jump Lévy process, W_t is a Wiener process independent of L_t and σ, S_0 are constants. There are several reasons for studying such a model. First of all, from the Lévy-Khintchine representation, we know that every Lévy process can be decomposed into a pure-jump process and a Wiener process where the Wiener process is the continuous martingale part. Hence, from a theoretical point of view, (1.1) is a generalization of the asset price dynamics considered in [4]. However, we can also view (1.1) as a model for the asset price where L_t is a pure-jump Lévy process accounting for sudden "big" changes in the price. The Brownian motion part, on the other hand, models the "small" or "normal" variations in the price movements. This is the modeling perspective of Honoré [13], although he considers a slightly different price process (see also Sect. 6). Rydberg [18] discusses an approximation procedure for numerical simulation of the normal inverse Gaussian Lévy process L_t which is a pure-jump Lévy process. She proposes to approximate L_t by a sum of a Brownian motion and a Lévy process of finite variation, i.e.,

$$L_t \approx \sigma W_t + \tilde{L}_t.$$

For a given ε, the jump process $\tilde{L}_t$ is assumed to be a Lévy process with Lévy measure

$$\tilde{\nu}(dz) = \mathbf{1}_{(-\varepsilon,\varepsilon)^C}\, \nu(dz),$$

where $\nu(dz)$ is the Lévy measure of L_t and

$$\sigma^2 = \int_{-\varepsilon}^{\varepsilon} z^2\, \nu(dz).$$

Since the support of $\tilde{\nu}$ is outside the interval $(-\varepsilon, \varepsilon)$, $\tilde{L}_t$ has paths of finite variation. We remark that this procedure is not restricted to the normal inverse Gaussian Lévy process alone. Such an approximation is highly relevant for a numerical treatment of the portfolio optimization problem using a Markov chain discretization (see [9]). In conclusion, generalizing the theory to asset price dynamics of the form (1.1) is of interest from both a practical and theoretical point of view.

The rest of this paper is organized as follows: In Sect. 2, we formulate the optimal portfolio-consumption problem and state the basic assumptions. In Sect. 3, the resulting singular control problem is analyzed via the dynamic programming method and the theory of viscosity solutions. In Sect. 4, we calculate explicit rules for portfolio allocation and consumption when the utility function is of HARA type and the Lévy process has only negative jumps. In Sect. 5, we treat the classical Merton problem (where utility is derived from present consumption only) for a general Lévy process. Finally, we discuss some related problems in Sect. 6.

2 The portfolio optimization problem and basic assumptions

Let $(\Omega, \mathscr{P}, \mathscr{F})$ be a probability space and $(\mathscr{F}_t)$ a given filtration satisfying the usual hypotheses. We consider a financial market consisting of a stock and a bond. Assume that the value of the stock follows the stochastic process

$$S_t = S_0 e^{L_t}, \tag{2.1}$$

where L_t is a Lévy process with Lévy-Khintchine decomposition

$$L_t = \mu t + \sigma W_t + \int_0^t \int_{|z|<1} z\, \tilde{N}(ds, dz) + \int_0^t \int_{|z|\geq 1} z\, N(ds, dz).$$

Here μ and σ are constants, W_t is a Wiener process, $N(dt, dz)$ is Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity measure $dt \times \nu(dz)$, $\nu(dz)$ is a σ-finite Borel measure on $\mathbb{R}\backslash\{0\}$ with the property

$$\int_{\mathbb{R}\backslash\{0\}} \min\left(1, z^2\right) \nu(dz) < \infty, \tag{2.2}$$

and $\tilde{N}(dt, dz) = N(dt, dz) - dt \times \nu(dz)$ is the compensated Poisson random measure. We assume W_t and $N(dt, dz)$ are independent stochastic processes. The measure $\nu(dz)$ is called the Lévy measure. We choose to work with the unique càdlàg version of L_t and denote this also by L_t. Under the additional integrability condition on the Lévy measure

$$\int_{|z|\geq 1} |e^z - 1|\,\nu(dz) < \infty, \tag{2.3}$$

we can write the differential of the stock price dynamics as (using Itô's Formula [15])

$$dS_t = \hat{\mu} S_t\,dt + \sigma S_t\,dW_t + S_{t-}\int_{\mathbb{R}\setminus\{0\}} \left(e^z - 1\right)\tilde{N}(dt, dz). \tag{2.4}$$

Here we have introduced the short-hand notation

$$\hat{\mu} = \mu + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^z - 1 - z\mathbf{1}_{|z|<1}\right)\nu(dz). \tag{2.5}$$

Note that condition (2.3) is effective only when $z \geq 1$ due to (2.2), and says essentially that e^z is $\nu(dz)$ - integrable on $\{z \geq 1\}$. Moreover, this condition implies that $\int_0^t \mathrm{E}[S_s]\,ds < \infty$ for all $t \geq 0$. Observe also that $e^z - 1 - z \geq 0$ for all $z \in \mathbb{R}$.

We let the bond have dynamics

$$dB_t = rB_t\,dt,$$

where $r > 0$ is the interest rate. Assume furthermore that $r < \hat{\mu}$, which means that the expected return from the stock is higher than the return of the bond.

Consider an investor who wants to put her money in the stock and the bond so as to maximize her utility. Let $\pi_t \in [0, 1]$ be the fraction of her wealth invested in the stock at time t and assume that there are no transaction costs in the market. If we denote the cumulative consumption up to time t by C_t, we have the wealth process $X_t^{\pi,C}$ given as

$$\begin{aligned} X_t^{\pi,C} &= x - C_t + \int_0^t \left(r + (\hat{\mu} - r)\pi_s\right)X_s^{\pi,C}\,ds + \int_0^t \sigma\pi_s X_s^{\pi,C}\,dW_s \\ &\quad + \int_0^t \pi_{s-}X_{s-}^{\pi,C}\int_{\mathbb{R}\setminus\{0\}} \left(e^z - 1\right)\tilde{N}(ds, dz), \end{aligned}$$

where x is the initial wealth of the investor. To incorporate the idea of intertemporal substitution, Hindy and Huang [14] introduce the process $Y_t^{\pi,C}$ modeling the average past consumption. The process has dynamics

$$Y_t^{\pi,C} = ye^{-\beta t} + \beta e^{-\beta t}\int_{[0,t]} e^{\beta s}\,dC_s, \tag{2.6}$$

where $y > 0$ and β is a positive weighting factor. We shall frequently use the notation Y_t for $Y_t^{\pi,C}$ and X_t for $X_t^{\pi,C}$. The integral is interpreted pathwise in a Lebesgue-Stieltjes sense. The differential form of Y_t is

$$dY_t = -\beta Y_t\,dt + \beta\,dC_t.$$

The objective of the investor is to find an allocation process π_t^* and a consumption pattern C_t^* which optimizes the expected discounted utility over an investment horizon. We shall here focus on an investor with an infinite investment horizon. We define the value function as

$$V(x,y) = \sup_{\pi,C\in\mathscr{A}_{x,y}} \mathbb{E}\Big[\int_0^\infty e^{-\delta t}U(Y_t^{\pi,C})\,dt\Big], \tag{2.7}$$

where $\delta > 0$ is the discount factor and $\mathscr{A}_{x,y}$ is a set of admissible controls. Let

$$\mathscr{D} = \Big\{(x,y)\in\mathbb{R}^2 : x > 0, y > 0\Big\}.$$

We say that a pair of controls (π, C) is admissible for $x, y \in \overline{\mathscr{D}}$ and write $\pi, C \in \mathscr{A}_{x,y}$ if:

(c_i) C_t is an adapted process that is right continuous with left-hand limits (càdlàg), nondecreasing, with initial value $C_{0-} = 0$ (to allow an initial jump when $C_0 > 0$), and satisfies $\mathbb{E}[C_t] < \infty$ for all $t \geq 0$.

(c_{ii}) π_t is an adapted càdlàg process with values in $[0,1]$.

(c_{iii}) $X_t^{\pi,C}, Y_t^{\pi,C} \geq 0$ almost everywhere for all $t \geq 0$.

Note that condition (c_{iii}) introduces a state space constraint into our control problem. The utility function $U : [0,\infty) \to [0,\infty)$ is assumed to have the following properties:

(u_i) $U \in C([0,\infty))$ is nondecreasing and concave.

(u_{ii}) There exist constants $K > 0$ and $\gamma \in (0,1)$ such that $\delta > k(\gamma)$ and

$$U(z) \leq K(1+z)^\gamma,$$

for all nonnegative z, where

$$\begin{aligned} k(\gamma) \quad = \quad & \max_{\pi\in[0,1]}\Big[\gamma(r + (\hat{\mu} - r)\pi) - \frac{1}{2}\sigma^2\pi^2\gamma(1-\gamma) \\ & + \int_{\mathbb{R}\setminus\{0\}}\Big(\big(1+\pi(e^z-1)\big)^\gamma - 1 - \gamma\pi(e^z-1)\Big)\,\nu(dz)\Big]. \end{aligned} \tag{2.8}$$

By a Taylor expansion we see that the integral term of $k(\gamma)$ is well-defined in a neighborhood of zero. Condition (2.3) ensures that the integral is finite outside this neighborhood, which shows that (2.8) is finite for $\gamma \in (0,1]$. Note that condition (u_{ii}) guarantees that the value function of the related Merton problem is well-defined, see Sect. 5. In the case when the integral term is absent, i.e., when $\nu(dz) = 0$, we denote $k(\gamma)$ by $k_0(\gamma)$.

In this paper we will assume that the dynamic programming principle holds, that is, for any stopping time τ and $t \geq 0$,

$$V(x,y) = \sup_{\pi,C\in\mathscr{A}_{x,y}} \mathbb{E}\Big[\int_0^{t\wedge\tau} e^{-\delta s} U(Y_s^{\pi,C})\,ds + e^{-\delta(t\wedge\tau)} V(X_{t\wedge\tau}^{\pi,C}, Y_{t\wedge\tau}^{\pi,C})\Big], \quad (2.9)$$

where $a \wedge b = \min(a,b)$.

Straightforward modifications (which we omit) of the proofs of Lemma 3.1 and Theorem 3.1 in [4] (see also Alvarez [1]) yield the next theorem concerning the regularity properties of the value function.

Theorem 2.1 *The value function defined in (2.7) is non-decreasing, concave and uniformly continuous in $\overline{\mathscr{D}}$. Furthermore, V is non-negative and has the same sublinear growth as the utility function, i.e., $0 \le V(x,y) \le K(1+x+y)^\gamma$ for $x,y \in \overline{\mathscr{D}}$. If for some $\alpha \in (0,1]$, we have $\delta > k(\alpha)$ and $U \in C^{0,\alpha}([0,\infty))$, then $V \in C^{0,\alpha}(\overline{\mathscr{D}})$. If $\delta > k(1+\alpha)$ and $U \in C^{1,\alpha}([0,\infty))$, then $V \in C^{1,\alpha}(\overline{\mathscr{D}})$.*

To our optimization problem we can associate a Hamilton-Jacobi-Bellman equation, which is a degenerate elliptic integro-differential equation subject to a gradient constraint:

$$\begin{aligned}\max\Big\{&\beta v_y - v_x; U(y) - \delta v - \beta y v_y + \max_{\pi\in[0,1]}\Big[(r + (\hat{\mu} - r)\pi) x v_x + \frac{1}{2}\sigma^2\pi^2 x^2 v_{xx} \\ &+ \int_{\mathbb{R}\setminus\{0\}} \Big(v(x + \pi x(e^z - 1), y) - v(x,y) - \pi x v_x(x,y)(e^z - 1)\Big) \\ &\times \nu(dz)\Big]\Big\} = 0 \text{ in } \mathscr{D}. \end{aligned} \quad (2.10)$$

In other words, the Hamilton-Jacobi-Bellman equation is an integro-differential variational inequality. Note that $x + \pi x(e^z - 1) \ge 0$ for all $x \ge 0$ and $z \in \mathbb{R}$. If v is C^2 and sublinearly growing, then a straightforward Taylor expansion shows that (2.10) is well-defined (see [4]).

It will be convenient to write the Hamilton-Jacobi-Bellman equation in a more compact and simplified form. To this end, we introduce the following notations: $X = (x_1, x_2) \in \overline{\mathscr{D}}$, $D_X = (\partial_{x_1}, \partial_{x_2})$, $D_X^2 = (\partial^2_{x_i x_j})_{i,j=1,2}$ and $G(D_X v) = \beta v_{x_2} - v_{x_1}$. Furthermore, let $\mathscr{B}^\pi$ be the integral operator

$$\mathscr{B}^\pi(X,v) = \int_{\mathbb{R}\setminus\{0\}} \Big(v(x_1 + \pi x_1(e^z - 1), x_2) - v(X) - \pi x_1 v_{x_1}(X)(e^z - 1)\Big)\nu(dz), \quad (2.11)$$

and let

$$\begin{aligned}(X, v, D_X v, D_X^2 v, \mathscr{B}^\pi(X,v)) &= U(x_2) - \delta v - \beta x_2 v_{x_2} \\ &+ \max_{\pi\in[0,1]}\Big[(r + (\hat{\mu} - r)\pi) x_1 v_{x_1} + \frac{\sigma^2}{2}\pi^2 x_1^2 v_{x_1 x_1} + \mathscr{B}^\pi(X,v)\Big].\end{aligned}$$

The Hamilton-Jacobi-Bellman Eq. (2.10) can now be written as

$$\max\Big(G(D_X v); F(X, v, D_X v, D_X^2 v, \mathscr{B}^\pi(X,v))\Big) = 0. \quad (2.12)$$

This is the form that we will employ in Sect. 3. Finally, we define the set

$$C_\ell(\overline{\mathscr{D}}) = \Big\{\phi \in C(\overline{\mathscr{D}}) \,:\, \sup_{\overline{\mathscr{D}}} \frac{|\phi(X)|}{(1 + x_1 + x_2)^\ell} < \infty\Big\}, \qquad \ell \ge 0. \quad (2.13)$$

3 Viscosity solutions

We shall rely on a viscosity solution framework to verify the closed form solutions derived in Sections 4 and 5. The (constrained) viscosity solution framework presented below is an adaption (to the second order case) of the framework developed in [4] for first order integro-differential variational inequalities. Because of the strong similarities with [4], we will be very brief in this section and instead refer to [4] for details not found herein. Also, we refer to [4] for an overview of the existing literature on viscosity solutions of integro-differential equations. For a general overview of the viscosity solution theory, we refer to the survey paper by Crandall et al. [7] and the book by Fleming and Soner [10].

A constrained viscosity solution of (2.12) is defined as follows:

Definition 3.1 *(i) Let $\mathscr{O} \subset \overline{\mathscr{D}}$. Any $v \in C(\overline{\mathscr{D}})$ is a viscosity subsolution (supersolution) of (2.12) in $\mathscr{O}$ if and only if we have, for every $X \in \mathscr{O}$ and $\phi \in C^2(\overline{\mathscr{D}}) \cap C_1(\overline{\mathscr{D}})$ such that X is a global maximum (minimum) relative to $\mathscr{O}$ of $v - \phi$,*

$$\max\Big(G(D_X\phi); F(X, v, D_X\phi, D_X^2\phi, \mathscr{B}^\pi(X,\phi))\Big) \geq 0\,(\leq 0).$$

(ii) Any $v \in C(\overline{\mathscr{D}})$ is a constrained viscosity solution of (2.12) if and only if and v is a viscosity subsolution of (2.12) in $\overline{\mathscr{D}}$ and v is a viscosity supersolution of (2.12) in $\mathscr{D}$.

Exactly the same argumentation (which we omit) as in the proof of Theorem 4.1 in [4] leads to the constrained viscosity property of the value function.

Theorem 3.1 *The value function $V(x,y)$ defined in (2.7) is a constrained viscosity solution of the integro-differential variational inequality (2.12).*

To prove that the value function is the *only* solution of (2.12), we need a comparison principle similar to Theorem 4.2 in [4]. We outline below how we can extend the proof of Theorem 4.2 in [4] to the second order integro-differential variational inequality (2.12).

First, note that to distinguish the singularities at zero and infinity it is advantageous to split the integral operator into two parts. For any $\kappa \in (0,1)$, $X \in \overline{\mathscr{D}}$, $\phi \in C^1(\overline{\mathscr{D}}) \cap C_1(\overline{\mathscr{D}})$, and $P = (p_1, p_2) \in \mathbb{R}^2$, define

$$\mathscr{B}^{\pi,\kappa}(X,\phi,P) = \int_{|z|>\kappa} \Big(\phi(x_1 + \pi x_1(e^z - 1), x_2) - \phi(X) - \pi x_1 p_1 (e^z - 1)\Big)\nu(dz).$$

For any $\kappa \in (0,1)$, $X \in \overline{\mathscr{D}}$, $\phi \in C^2(\overline{\mathscr{D}})$, and $P = (p_1, p_2) \in \mathbb{R}^2$, define

$$\mathscr{B}^{\pi}_{\kappa}(X,\phi) = \int_{|z|\leq\kappa} \Big(\phi(x_1 + \pi x_1(e^z - 1), x_2) - \phi(X) - \pi x_1 \phi_{x_1}(X)(e^z - 1)\Big)\nu(dz).$$

Observe that for $\phi \in C^2(\overline{\mathscr{D}}) \cap C_1(\overline{\mathscr{D}})$, we can write (see [4])

$$\mathscr{B}^\pi(X,\phi) = \mathscr{B}^{\pi,\kappa}(X,\phi,D_X\phi) + \mathscr{B}^\pi_\kappa(X,\phi). \qquad (3.1)$$

Equipped with this decomposition, we introduce the (slightly shorter) notation

$$F(X, v, P, A, \mathscr{B}^{\pi,\kappa}(X, v, P), \mathscr{B}_\kappa^\pi(X, \phi)) := F(X, v, P, A, \mathscr{B}^{\pi,\kappa}(X, v, P) + \mathscr{B}_\kappa^\pi(X, \phi)),$$

for $v \in C_1(\overline{\mathscr{D}})$ and $\phi \in C^2(\overline{\mathscr{D}})$.

When proving comparison results for second order equations, it is more convenient to use a formulation of viscosity solutions based on the notions of subjet and superjet.

Definition 3.2 *Let $\mathscr{S}^N$ denotes the set of $N \times N$ symmetric matrices, $\mathscr{O} \subset \overline{\mathscr{D}}$, $v \in C(\mathscr{O})$, and $X \in \mathscr{O}$. The second order superjet (subjet) $J_{\mathscr{O}}^{2,+(-)}v(X)$ is the set of $(P, A) \in \mathbb{R}^2 \times \mathscr{S}^2$ such that*

$$v(Y) \le (\ge 0)\, v(X) + \langle P, Y - X\rangle + \frac{1}{2}\langle A(Y - X), Y - X\rangle + o(|X - Y|^2) \text{ as } \mathscr{O} \ni Y \to X.$$

The closure $\overline{J}_{\mathscr{O}}^{2,+(-)}v(X)$ is the set of (P, A) for which there exists a sequence $(P_n, A_n) \in J_{\mathscr{O}}^{2,+(-)}v(X_n)$ such that $(X_n, v(X_n), P_n, A_n) \to (X, v(X), P, A)$ as $n \to \infty$.

Before we can give a suitable definition of viscosity solutions based on sub- and superjets, we need an equivalent formulation of viscosity solutions in $C_1(\overline{\mathscr{D}})$ based on test functions (which takes into account the decomposition (3.1).)

Lemma 3.1 *Let $v \in C_1(\overline{\mathscr{D}})$ and $\mathscr{O} \subset \overline{\mathscr{D}}$. Then v is a viscosity subsolution (supersolution) of (2.12) in $\mathscr{O}$ if and only if we have, for every $\phi \in C^2(\overline{\mathscr{D}})$ and $\kappa > 0$,*

$$\max\Big(G(D_X\phi); F(X, v, D_X\phi, D_X^2\phi, \mathscr{B}^{\pi,\kappa}(X, v, D_X\phi), \mathscr{B}_\kappa^\pi(X, \phi))\Big) \ge 0$$

whenever $X \in \mathscr{O}$ is a global maximum (minimum) relative to $\mathscr{O}$ of $v - \phi$.

This lemma is a straightforward extension of [4, Lemmma 4.1] and the proof is therefore omitted. Let $v \in C(\overline{\mathscr{D}})$ and $\mathscr{O} \subset \overline{\mathscr{D}}$. Then using the arguments in, e.g., [10] one can easily prove that $(P, A) \in J_{\mathscr{O}}^{2,+(-)}v(X)$ if and only if there exists $\phi \in C^2(\overline{\mathscr{D}})$ such that $\phi(x) = v(x)$, $D_X\phi(X) = P$, $D_X^2\phi(X) = A$, and $v - \phi$ has a global maximum (minimum) relative to $\mathscr{O}$ at X. In view of Lemma 3.1 and continuity of the governing equation, the following formulation of viscosity solutions in C_1 based on sub- and superjets is now immediate.

Lemma 3.2 *([7]) Let $v \in C_1(\overline{\mathscr{D}})$ be a subsolution (supersolution) of (2.12) in $\mathscr{O} \subset \overline{\mathscr{D}}$. Then, for all $\kappa > 0$, $X \in \mathscr{O}$, $(P, A) \in \overline{J}_{\mathscr{O}}^{2,+(-)}v(X)$, there exists $\phi \in C^2(\overline{\mathscr{D}})$ such that*

$$\max\Big(G(P); F(X, v, P, A, \mathscr{B}^{\pi,\kappa}(X, v, P), \mathscr{B}_\kappa^\pi(X, \phi))\Big) \ge 0\,(\le 0).$$

The test function ϕ is such that $v - \phi$ has a global maximum (minimum) relative to $\mathscr{O}$ at X_n with $X_n \to X$ as $n \to \infty$.

A similar formulation is also used in Pham [17]. To prove a comparison principle for (2.12), we shall need the following maximum principle for semicontinuous function taken from Crandall et al. [7]:

Lemma 3.3 ([7]) *Let $\mathscr{O} \subset \mathbb{R}^N$ be locally compact. Let $u_1, -u_2$ be upper semi-continuous and φ twice continuously differentiable in a neighborhood of $\mathscr{O} \times \mathscr{O}$. Suppose $(\hat{X}, \hat{Y}) \in \mathscr{O} \times \mathscr{O}$ is a local maximum of $u_1(X) - u_2(Y) - \varphi(X, Y)$ relative to $\mathscr{O} \times \mathscr{O}$. Then for every $\varsigma > 0$ there exist two matrices $A, B \in S^N$ such that*

$$\left(D_X\varphi(\hat{X}, \hat{Y}), A\right) \in \overline{J}^{2,+}_{\mathscr{O}} u_1(\hat{X}), \qquad \left(-D_Y\varphi(\hat{X}, \hat{Y}), B\right) \in \overline{J}^{2,-}_{\mathscr{O}} u_2(\hat{Y}),$$

and

$$-\left(\frac{1}{\varsigma} + \|D^2\varphi(\hat{X}, \hat{Y})\|\right) I \leq \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \leq D^2\varphi(\hat{X}, \hat{Y}) + \varsigma\left(D^2\varphi(\hat{X}, \hat{Y})\right)^2. \quad (3.2)$$

Let $\underline{v} \in C(\overline{\mathscr{D}})$ be a subsolution of (2.12) in $\overline{\mathscr{D}}$ and $\overline{v} \in C(\overline{\mathscr{D}})$ a supersolution of (2.12) in $\mathscr{D}$. Choosing $\tilde{K}$ and $\overline{\gamma} \in (0, 1)$ properly, one can show (following closely the proof of Lemma 4.3 in [4]) that $w = \tilde{K} + \left(1 + x_1 + \frac{x_2}{2\beta}\right)^{\overline{\gamma}}$ and

$$\overline{v}^\theta = (1 - \theta)\overline{v} + \theta w, \qquad \theta \in (0, 1],$$

are strict supersolutions of (2.12) in any bounded subset of $\mathscr{D}$. We claim that

$$\underline{v} \leq \overline{v}^\theta \text{ in } \overline{\mathscr{D}},$$

which immediately implies that the comparison principle holds between $\underline{v}$ and $\overline{v}$.

Except for the treatment of the second order term, which relies in an essential way on Lemma 3.3, the proof of our comparison principle is very similar to the proof of Theorem 4.2 in [4], which the reader is referred to for details not found below.

As in the first-order case [4], we utilize our choice of a strict supersolution $\overline{v}^\theta$ to "localize" the proof to the following bounded domain

$$\mathscr{K} := \left\{(x_1, x_2) : 0 < x_1 < R(1 + e^1), 0 < x_2 < R\right\}, \quad (3.3)$$

where R is some positive constant chosen such that $\underline{v} \leq \overline{v}^\theta$ in $\{x_1, x_2 \geq R\}$. To prove the comparison result it is now sufficient to show that $\underline{v} \leq \overline{v}^\theta$ in $\overline{\mathscr{K}}$.

Assume that the contrary is true, i.e., we have

$$M := \max_{\overline{\mathscr{K}}}(\underline{v} - \overline{v}^\theta) = (\underline{v} - \overline{v}^\theta)(Z) > 0 \quad (3.4)$$

for some $Z \in \overline{\mathscr{K}}$. Then either $Z \in (0, R) \times (0, R)$ or $Z \in \Gamma_{\text{SC}}$, where

$$\Gamma_{\text{SC}} = \left\{(x_1, x_2) : x_1 = 0, 0 \leq x_2 < R \text{ or } 0 \leq x_1 < R, x_2 = 0\right\}.$$

Here we consider only the latter case, the case $Z \in \mathscr{K}$ is treated similarly (consult Case II in the proof of [4, Theorem 4.2]).

Let (X_α, Y_α) be a maximizer of the function $\Phi(X,Y) : \overline{\mathscr{K}} \times \overline{\mathscr{K}} \to \mathbb{R}$, defined for any $\alpha > 1$ and $0 < \varepsilon < 1$ as

$$\Phi(X,Y) = \underline{v}(X) - \overline{v}^\theta(Y) - |\alpha(X-Y) + \varepsilon\eta(Z)|^2 - \varepsilon|X-Z|^2. \tag{3.5}$$

The uniformly continuous function $\eta : \overline{\mathscr{K}} \to \mathbb{R}^2$ satisfies

$$B(X + t\eta(Z), td) \subset \mathscr{K} \text{ for all } X \in \overline{\mathscr{K}} \text{ and } t \in (0, t_0],$$

for positive constants d, t_0 and $B(X,r)$ denotes the open ball in $\mathbb{R}^2$ centered at X and with radius r. The construction (3.5) is ultimately due to Soner [19].

It is standard to see that the penalized maxima (X_α, Y_α) satisfy as $\alpha \to \infty$ (see, e.g., [4]): (i) $X_\alpha, Y_\alpha \to Z$, (ii) $\alpha(X_\alpha - Y_\alpha) + \varepsilon\zeta(Z) \to 0$, (iii) $\big(\underline{v}(X_\alpha) - \overline{v}^\theta(Y_\alpha)\big) \to M$, (iv) $M_\alpha \to M$. In view of (ii) and (3.4), we conclude that $Y_\alpha \in (0,R) \times (0,R)$ and $X_\alpha \in [0,R) \times [0,R)$. Using the maximum principle for semicontinuous functions (Lemma 3.3) with

$$\varphi(X,Y) = |\alpha(X-Y) + \varepsilon\eta(Z)|^2 + \varepsilon|X-Z|^2, \quad u_1 = \underline{v}, \quad u_2 = \overline{v}^\theta, \quad \mathscr{O} = \overline{\mathscr{K}},$$

we conclude that there exist matrices $A = (a_{ij})_{i,j=1,2}, B = (b_{ij})_{i,j=1,2} \in \mathscr{S}^2$ such that

$$(P,A) \in \overline{J}^{2,+}_{\overline{\mathscr{K}}}\underline{v}(X_\alpha), P = D_X\varphi(X_\alpha, Y_\alpha) = 2\alpha[\alpha(X_\alpha - Y_\alpha) + \varepsilon\eta(Z)] + 2\varepsilon(X_\alpha - Z),$$
$$(Q,B) \in \overline{J}^{2,-}_{\overline{\mathscr{K}}}\overline{v}^\theta(Y_\alpha), Q = -D_Y\varphi(X_\alpha, Y_\alpha) = 2\alpha[\alpha(X_\alpha - Y_\alpha) + \varepsilon\eta(Z)].$$

Following, e.g., [7] it is not difficult to show that (3.2) implies

$$\lim_{\varepsilon\to 0}\lim_{\alpha\to\infty}\left(\frac{\sigma^2}{2}\pi x_{\alpha 1}^2 a_{11} - \frac{\sigma^2}{2}\pi y_{\alpha 1}^2 b_{11}\right) \le 0. \tag{3.6}$$

Since $\overline{v}^\theta$ is a strict supersolution of (2.12) in $\mathscr{D}$ there exists, thanks to Lemma 3.2, $\psi \in C^2(\overline{\mathscr{D}})$ such that

$$F(Y_\alpha, \overline{v}^\theta, Q, B, \mathscr{B}^{\pi,\kappa}(Y_\alpha, \overline{v}^\theta, Q), \mathscr{B}^\pi_\kappa(Y_\alpha, \psi)) < -\vartheta, \tag{3.7}$$

for some constant $\vartheta > 0$. Similarly, since $\underline{v}$ is a subsolution of (2.12) in $\overline{\mathscr{D}}$, there exists $\phi \in C^2(\overline{\mathscr{D}})$ such that

$$F(X_\alpha, \underline{v}, P, A, \mathscr{B}^{\pi,\kappa}(X_\alpha, \underline{v}, P), \mathscr{B}^\pi_\kappa(X_\alpha, \phi)) \ge 0. \tag{3.8}$$

Having (3.6) in mind, we now subtract (3.7) from (3.8) and send (in that order) $\alpha \to \infty$, $\varepsilon \to 0$, and $\kappa \to 0$. These limit operations lead (after some tedious work) to the contradiction $(\underline{v} - \overline{v}^\theta)(Z) < 0$ (consult Case I in the proof of [4, Theorem 4.2]).

Summing up, we have proven the following comparison (uniqueness) theorem:

Theorem 3.2 *Let $\gamma' > 0$ be such that $\delta > k(\gamma')$. Assume $\underline{v} \in C_{\gamma'}(\overline{\mathscr{D}})$ is a subsolution of (2.12) in $\overline{\mathscr{D}}$ and $\overline{v} \in C_{\gamma'}(\overline{\mathscr{D}})$ is a supersolution of (2.12) in $\mathscr{D}$. Then $\underline{v} \le \overline{v}$ in $\overline{\mathscr{D}}$. Consequently, in the class of sublinearly growing solutions, the Hamilton-Jacobi-Bellman Eq. (2.12) admits at most one constrained viscosity solution.*

4 Explicit consumption and portfolio allocation rules

In this section we study a case where we can construct an explicit solution to the control problem. The case is taken from Hindy and Huang [14], who construct an explicit solution to the optimization problem when the utility function is of HARA (Hyperbolic Absolute Risk Aversion) type and the price of the stock follows a geometric Brownian motion. We show in this section that a more realistic price model with a Lévy process instead of Brownian motion leads to a similar solution. To obtain explicit results we need to restrict our attention to Lévy processes having only negative jumps. Our optimization problem leads to the second-order integro-differential variational inequality (2.10) where $\nu(dz)$ has mass on the negative part of the real line. We are able to solve this equation, and construct optimal consumption and portfolio allocation strategies by closely following the arguments in [14]. Note, however, that our results are not as explicit as those in [14]. For instance, the optimal allocation strategy π^* is the solution of an integral equation involving the Lévy measure of the noise process. We mention that the results presented below also hold in the limiting case $\sigma = 0$, which corresponds to the pure-jump case [4].

For $\gamma \in (0, 1)$, consider the utility function

$$U(z) = \frac{z^\gamma}{\gamma}.$$

We recall that $1 - \gamma$ is the risk aversion coefficient. Motivated by Hindy and Huang [14], we guess that the optimization problem has a constrained viscosity solution of the form

$$V(x,y) = \{ k_1 y^\gamma + k_2 y^\gamma \left[\frac{x}{ky}\right]^\rho, 0 \le x < ky, k_3\left(\frac{y+\beta x}{1+\beta k}\right)^\gamma, x \ge ky > 0, \quad (4.1)$$

for some constants k_1, k_2, k_3, k, and $\rho > \gamma$. This solution is constructed from the assumption that we can split the state space into two parts, on which each of the terms in the variational inequality (2.10) is effective. Hence, for $0 \le x < ky$, we construct the solution from the assumption that

$$\frac{y^\gamma}{\gamma} - \delta V - \beta y V_y + \max_{\pi\in[0,1]}\Big[(r + (\hat{\mu} - r)\pi)xV_x + \frac{1}{2}\sigma^2\pi^2x^2V_{xx} + \quad (4.2)$$
$$\int_{-\infty}^{0-}\Big(V(x+\pi x(e^z - 1), y) - V(x,y) - \pi x V_x(x,y)(e^z - 1)\Big)\,\nu(dz)\Big] = 0$$

and, when $x \ge ky > 0$,

$$\beta V_y - V_x = 0. \quad (4.3)$$

We see that the integral in (4.2) is well defined by the condition in (2.3). In what follows, all the displayed integrals are convergent by the same condition. In the rest of this section we derive expressions for the different constants in the solution, and find the optimal allocation and consumption processes. Optimize the kernel of (4.2) with respect to π to find the first order condition for an optimum

$$(\hat{\mu} - r)xV_x + \sigma^2 \pi x^2 V_{xx} + \int_{-\infty}^{0-} \Big(V_x(x + \pi x(e^z - 1), y) x(e^z - 1) - xV_x(x,y) \\ \times (e^z - 1) \Big) \nu(dz) = 0.$$

Inserting the guessed solution (4.1) for $x < ky$, we get the expression

$$(\hat{\mu}-r)-(1-\rho)\sigma^2\pi+\int_{-\infty}^{0-}\Big(\big(1+\pi(e^z-1)\big)^{\rho-1}(e^z-1)-(e^z-1)\Big)\,\nu(dz)=0. \quad (4.4)$$

Assume from now on that π^* is a solution of (4.4) that lies in $(0,1)$, i.e., we assume that the parameters of the problem are so that we can find an interior optimum. Note that π^* is constant with respect to time which gives that the optimal investment rule is to hold a constant fraction of the wealth in the stock. With this π^*, we can find equations for the unknown constants k_1 and ρ. Inserting (4.1) into (4.2), we obtain

$$\begin{aligned} y^\gamma \Big(\frac{1}{\gamma} - \delta k_1 &- \beta\gamma k_1 \Big) + k_2 y^\gamma \left[\frac{x}{ky} \right]^\rho \Big\{ -\delta - \beta(\gamma - \rho) + (r + (\hat{\mu} - r)\pi^*)\rho \\ &- \frac{1}{2}\sigma^2\pi^2\rho(1-\rho) + \int_{-\infty}^{0-} \Big(\big(1 + \pi^*(e^z - 1)\big)^\rho \\ &- 1 - \rho\pi^*(e^z - 1) \Big) \nu(dz) \Big\} = 0. \end{aligned}$$

The only way the left-hand side can be zero is when

$$\begin{aligned} &\Big(r + (\hat{\mu} - r)\pi^* + \beta - \frac{1}{2}\sigma^2\pi^{*2}(1-\rho) \Big)\rho \\ &\quad = \delta + \beta\gamma - \int_{-\infty}^{0-} \Big(\big(1 + \pi^*(e^z - 1)\big)^\rho - 1 - \rho\pi^*(e^z - 1) \Big) \nu(dz) \quad (4.5) \end{aligned}$$

and

$$k_1 = \frac{1}{\gamma(\delta + \beta\gamma)}.$$

The first equation is an expression for ρ.

From now on we assume that (4.4) and (4.5) have a solution $(\pi^*, \rho) \in (0,1) \times (\gamma, 1)$. We can find expressions for k_2 and k_3 by imposing a *smooth fit* condition along the boundary $x = ky$. From continuity we easily get

$$k_1 + k_2 = k_3.$$

Moreover, if the derivatives of V are to be continuous as well, we need to have $V_x = \beta V_y$ when $x = ky$ for the solution (4.1) ($x < ky$). But differentiating and equating give

$$k_2 = \frac{\beta k_1 \gamma}{\rho/k - \beta(\gamma - \rho)} = \frac{\beta k}{(\delta + \beta\gamma)(\rho(1 + \beta k) - \beta k\gamma)}.$$

For $x < ky$, we need to show that $\beta V_y - V_x \leq 0$. Direct differentiation gives

$$V_x = k_2 y^\gamma \left[\frac{x}{ky}\right]^{\rho-1} \frac{\rho}{ky} = k_2 \frac{\rho}{k} y^{\gamma-1} \left[\frac{x}{ky}\right]^{\rho-1},$$
$$V_y = k_1\gamma y^{\gamma-1} + k_2(\gamma-\rho) y^{\gamma-\rho-1} \left[\frac{x}{k}\right]^{\rho} = k_1\gamma y^{\gamma-1} + k_2(\gamma-\rho) y^{\gamma-1} \left[\frac{x}{ky}\right]^{\rho}.$$

Hence

$$\beta V_y - V_x = y^{\gamma-1}\Big(k_1\beta\gamma + \beta k_2(\gamma-\rho)\left[\frac{x}{ky}\right]^{\rho} - k_2\frac{\rho}{k}\left[\frac{x}{ky}\right]^{\rho-1}\Big).$$

Inserting the expressions for k_1 and k_2 yields

$$\beta V_y - V_x = \frac{\beta y^{\gamma-1}}{\delta+\beta\gamma}\Big(1 - (1-\rho)\left[\frac{x}{ky}\right]^{\rho} - \rho\left[\frac{x}{ky}\right]^{\rho-1}\Big).$$

We see that $\beta V_y - V_x \le 0$ if and only if

$$h(z) := 1 - (1-\rho)z^{\rho} - \rho z^{\rho-1} \le 0, \qquad \text{for all } z \in [0,1].$$

But $h(1) = 0$ and

$$h'(z) = \rho(1-\rho)z^{\rho-2}(1-z) \ge 0.$$

Hence $h(z)$ is an increasing function on $[0,1]$ with maximum $h(1) = 0$, which implies $h(z) \le 0$. This completes the proof of $\beta V_y - V_x \le 0$ for $x < ky$.

For the second case we specify the value of k to be the same as in [14] and show that this gives the desired inequality under an additional condition on the parameters in the problem. Let

$$k = \frac{1-\rho}{\beta(\rho-\gamma)}.$$

This gives

$$k_3 = \frac{\rho(1-\gamma)}{\gamma(\rho-\gamma)(\delta+\beta\gamma)}$$

and thus

$$V(x,y) = c(y+\beta x)^{\gamma}, \qquad \text{for } x \ge ky, \qquad c = \frac{\rho}{\gamma(\delta+\beta\gamma)}\left(\frac{1-\gamma}{\rho-\gamma}\right)^{1-\gamma}.$$

We show next that

$$\frac{y^\gamma}{\gamma} - \delta V - \beta y V_y + \max_{\pi\in[0,1]}\Big[(r + (\hat{\mu}-r)\pi)xV_x + \frac{1}{2}\sigma^2\pi^2x^2V_{xx}$$
$$+ \int_{-\infty}^{0-}\Big(V(x+\pi x(e^z-1),y) - V(x,y) - \pi x V_x(x,y)(e^z-1)\Big)\,\nu(dz)\Big] \le 0,$$

whenever $x \ge ky > 0$. Note now that since V is concave, the integral term in this expression is non-positive. Inserting the expression for $V(x,y)$ in the left-hand side of the above inequality and using $\frac{\beta x}{y+\beta x} \in (0,1)$, we get

$$\frac{y^\gamma}{\gamma} - \delta c(y+\beta x)^\gamma - \beta\gamma\frac{y}{y+\beta x}c(y+\beta x)^\gamma$$
$$+c(y+\beta x)^\gamma \max_{\pi\in[0,1]}\Big[(r+(\hat\mu-r)\pi)\gamma\frac{\beta x}{y+\beta x}+\frac{1}{2}\sigma^2\pi^2\Big(\frac{\beta x}{y+\beta x}\Big)^2\gamma(\gamma-1)\Big]$$
$$\leq \frac{y^\gamma}{\gamma} - c(y+\beta x)^\gamma\big(\delta - k_0(\gamma)\big),$$

where $k_0(\gamma)$ equals $k(\gamma)$ with $\nu(dz)=0$, see (2.8). But since $x \geq ky$ and $\delta - k_0(\gamma)$ and c are both positive, we have

$$\frac{y^\gamma}{\gamma} - c(\delta - k_0(\gamma))(y+\beta x)^\gamma$$
$$\leq \frac{y^\gamma}{\gamma} - c(\delta-k_0(\gamma))(1+\beta k)^\gamma y^\gamma = y^\gamma\Big(\frac{1}{\gamma} - c(\delta-k_0(\gamma))(1+\beta k)^\gamma\Big),$$

which is less than or equal to zero if and only if

$$\frac{1}{\gamma} - c(\delta-k_0(\gamma))(1+\beta k)^\gamma \leq 0.$$

But this happens if and only if

$$\frac{\rho(1-\gamma)}{\rho-\gamma} \geq \frac{\delta+\beta\gamma}{\delta-k_0(\gamma)}. \tag{4.6}$$

By construction V is a constrained viscosity solution in $\{x \geq 0, y > 0\}$. Note that a subsolution in $\{x \geq 0, y > 0\}$ is also a subsolution in $\overline{\mathscr{D}}$. We refer to the first remark in Sect. 3 in [1] for a proof of this. Thanks to the Theorems 3.1 and 3.2, V is thus the unique constrained viscosity solution of (2.10) and hence coincides with the value function (2.7). Summing up, we have proven the following theorem:

Theorem 4.1 *For $\gamma \in (0,1)$, let $U(y) = \frac{y^\gamma}{\gamma}$ and assume that $\nu\big((0,\infty)\big) = 0$ and (4.6) holds. Then the value function $V(x,y)$ associated with our optimization problem is explicitly given by (4.1), where*

$$k_1 = \frac{1}{\gamma(\delta+\beta\gamma)},\ k_2 = \frac{1-\rho}{(\rho-\gamma)(\delta+\beta\gamma)},\ k_3 = \frac{\rho(1-\gamma)}{\gamma(\rho-\gamma)(\delta+\beta\gamma)},\ k = \frac{1-\rho}{\beta(\rho-\gamma)}.$$

The optimal allocation of money in the stock is given by π^*, where $\pi^* \in (0,1)$ and $\rho \in (\gamma, 1]$ are solutions (when such exist) to the system of equations

$$(\hat\mu - r) - (1-\rho)\sigma^2\pi + \int_{-\infty}^{0-}\big(1+\pi(e^z-1)\big)^{\rho-1}(e^z-1) - (e^z-1)\,\nu(dz) = 0,$$
$$\Big(r+(\hat\mu-r)\pi+\beta-\frac{1}{2}\sigma^2\pi^2(1-\rho)\Big)\rho$$
$$= \delta+\beta\gamma - \int_{-\infty}^{0-}\Big(\big(1+\pi(e^z-1)\big)^\rho - 1 - \rho\pi(e^z-1)\Big)\,\nu(dz).$$

Note that the constants k_1, k_2, and k_3 are equal to the constants found by Hindy and Huang [14]. However, our expressions for ρ and π^* are quite different. Furthermore, π^* is independent of time and thus gives a constant fraction of wealth to be invested in the stock. It is easily seen that in the case of geometric Brownian motion, Theorem 4.1 coincides with the results of Hindy and Huang [14]. Note that we have implicitly assumed π^* to be an interior optimum in $[0, 1]$.

Example 4.1 To include the possibility of a sudden price drop (a “crack”) in a stock, a natural model could be a geometric Brownian motion with a Poisson component:

$$S_t = S_0 e^{\mu t + \sigma W_t - \xi N_t},$$

where μ, σ, ξ, S_0 are constants and N_t is a Poisson process with intensity $\lambda > 0$. The Lévy measure is easily seen to be

$$\nu(dz) = \lambda \delta_{-\xi}(dz),$$

where δ_a is the Dirac measure located at a. Assume now that $0 < \xi < 1$. The expected rate of return for this stock is

$$\hat{\mu} = \mu + \frac{1}{2}\sigma^2 - \lambda(1 - e^{-\xi}).$$

Moreover, the equations for π^* and ρ become

$$\hat{\mu} - r - \lambda(1 - e^{-\xi})\Big(\big(1 - \pi(1 - e^{-\xi})\big)^{\rho - 1} - 1\Big) - \sigma^2(1 - \gamma)\pi = 0\,,$$

$$\Big(r + (\hat{\mu} - r)\pi + \beta - \frac{1}{2}\sigma^2\pi^2(1 - \rho)\Big)\rho = \delta + \beta\gamma - \lambda\Big(\big(1 - \pi(1 - e^{-\xi})\big)^{\rho} - 1 + \rho\pi(1 - e^{-\xi})\Big).$$

If $\sigma = 0$ and the conditions

$$\mu > r \text{ and } (\mu - r)e^{-(1-\rho)\xi} < \lambda(1 - e^{-\xi}) < \mu - r$$

hold, we have the following explicit expression for $\pi^* \in (0, 1)$ in terms of ρ:

$$\pi^* = \frac{1}{1 - e^{-\xi}}\left(1 - \left[\frac{\lambda(1 - e^{-\xi})}{\mu - r}\right]^{\frac{1}{1-\rho}}\right).$$

An optimal consumption process is provided by the following theorem:

Theorem 4.2 *An optimal consumption process C_t^* is given as*

$$C_t^* = \Delta C_0^* + \int_0^t \frac{X_t^*}{1 + \beta k}\, dZ_s, \qquad k = \frac{1 - \rho}{\beta(\rho - \gamma)},$$

$$\Delta C_0^* = \left[\frac{x - kY_{0-}}{1 + \beta k}\right]^+, \quad Z_t = \sup_{0 \le s \le t}\left[\ln\frac{\hat{X}_t}{\hat{Y}_t} - \ln k\right]^+, \qquad \hat{Y}_t = (Y_0 + \beta\Delta C_0^*)e^{-\beta t},$$

and

$$\hat{X}_t = (x - \Delta C_0^*) + \int_0^t \big(r + (\hat{\mu} - r)\pi^*\big)\hat{X}_s\, ds + \int_0^t \sigma\pi^*\hat{X}_s\, dB_s$$
$$+ \int_0^t \pi^*\hat{X}_{s-} \int_{-\infty}^{0-} \big(e^z - 1\big)\, \tilde{N}(ds, dz).$$

The processes X^* and Y^* are the state variables associated with C^*.

Proof. This argument follows closely the proof in [14, Proposition 5]. From the results in [14], we need to find a k ratio barrier policy which ensures that $X_t^*/Y_t^* \le k$, $P-$a.s. at every t. This leads to an initial jump of C_t^* if $x/Y_{0-} > k$, from where we get the expression of ΔC_0^*. Now define

$$Z_t = \sup_{0\le s\le t}\left[\ln\frac{\hat{X}_t}{\hat{Y}_t} - \ln k\right]^+$$

and let $\ln(X_t^*/Y_t^*)$ be the "regulated" process defined by

$$\ln\frac{X_t^*}{Y_t^*} = \ln\frac{\hat{X}_t}{\hat{Y}_t} - Z_t. \tag{4.7}$$

Note that the processes $\hat{X}_t$ and $\hat{Y}_t$ are unregulated in the sense that we do not apply any consumption process except for the initial jump. The process Z_t is easily seen to be nondecreasing, $Z_0(\omega) = 0$, and increasing only when $\ln(X_t^*/Y_t^*) = \ln k$. Applying Itô's formula, we find that

$$\begin{aligned} d\ln\frac{X_t^*}{Y_t^*} &= d\ln X_t^* - d\ln Y_t^* - \left(\frac{1}{X_t^*} + \frac{\beta}{Y_t^*}\right) dC_t^* \\ &= (r + \beta + (\hat{\mu} - r)\pi^* - \frac{1}{2}\sigma^2\pi^{*2})\, dt - \left(\frac{1}{X_t^*} + \frac{\beta}{Y_t^*}\right) dC_t^* \\ &\quad + \int_{-\infty}^{0-} \ln\Big(1 + \pi^*(e^z - 1)\Big)\, \tilde{N}(dt, dz) + \sigma\pi^*\, dB_t \\ &\quad + \int_{-\infty}^{0-} \Big(\ln\Big(1 + \pi^*(e^z - 1)\Big) - \pi^*(e^z - 1)\Big)\, \nu(dz) \end{aligned}$$

and

$$\begin{aligned} d\ln\frac{\hat{X}_t}{\hat{Y}_t} = d\ln\hat{X}_t \quad &- \quad d\ln\hat{Y}_t = (r + \beta + (\hat{\mu} - r)\pi^* - \frac{1}{2}\sigma^2\pi^{*2})\, dt \\ &+ \quad \int_{-\infty}^{0-} \ln\Big(1 + \pi^*(e^z - 1)\Big)\, \tilde{N}(dt, dz) + \sigma\pi^*\, dB_t \\ &+ \quad \int_{-\infty}^{0-} \Big(\ln\Big(1 + \pi^*(e^z - 1)\Big) - \pi^*(e^z - 1)\Big)\, \nu(dz). \end{aligned}$$

Thus, relation (4.7) is fulfilled exactly when

$$Z_t = \int_0^t \left(\frac{Y_s^* + \beta X_s^*}{X_s^* Y_s^*}\right) dC_s^* \quad \text{or} \quad C_t^* = \int_0^t \frac{X_s^* Y_s^*}{Y_s^* + \beta X_s^*}\, dZ_s = \int_0^t \frac{X_s^*}{1 + \beta k}\, dZ_s.$$

Here the relation for C_t^* follows since Z_t only increases when $X_t^*/Y_t^* = k$. This completes the proof of the theorem.

5 Merton's problem with consumption and HARA utility

In this section we consider Merton's problem with consumption when the stock price is modeled as (2.1) for a general Lévy process, i.e., here we allow both positive and negative jumps as opposed to Sect. 4. Merton's problem can be thought of as the limiting case when $\beta \to \infty$ in the particular model considered in Sect. 2. In this problem we thus optimize the expected utility of the consumption directly. The consumption process is assumed to be absolutely continuous with respect to the Lebesgue measure on the real positive half-line, and can thus be specified on the form $C_t = \int_0^t c_s\, ds$, where c_s is the consumption rate at time s. The value function will only be dependent on one variable, namely the initial fortune x. We note that this problem has been treated by Framstad et al. [11] when the price process S_t is modeled as the solution of a stochastic differential equation with jumps, see also [12] which takes into account transaction costs. However, they have a more restrictive condition on the Lévy measure in a neighborhood of zero. For example, the normal inverse Gaussian Lévy process of Barndorff-Nielsen [3] does not fit into the framework of [11, 12].

In the present context, the wealth process is given as

$$dX_t = \big(r + (\hat{\mu} - r)\pi_t\big)X_t\, dt - c_t\, dt + \sigma X_t \pi_t\, dB_t + X_{t-}\pi_{t-}\int_{\mathbb{R}\setminus\{0\}} \big(e^z - 1\big)\, \tilde{N}(dt, dz)$$

with initial wealth $X_0 = x$ and $\hat{\mu}$ as defined in (2.5). We consider the optimal control problem

$$V(x) = \sup_{c,\pi\in\mathscr{A}_x} \mathrm{E}^x\left[\int_0^\infty e^{-\delta t}\left[\frac{c_t^\gamma}{\gamma}\right] dt\right], \qquad \text{for } \gamma \in (0,1),$$

where the set of admissible controls $\mathscr{A}_x$ is defined as follows: $\pi, c \in \mathscr{A}_x$ if

(cm_i) c_t is a positive and adapted process such that $\int_0^t \mathrm{E}[c_s]\, ds < \infty$ for all $t \geq 0$.

(cm_{ii}) π_t is an adapted cádlág process with values in $[0,1]$.

(cm_{iii}) c_t is such that $X_t^{\pi,c} \geq 0$ almost everywhere for all $t \geq 0$.

Note that condition (cm_{iii}) introduces a state space constraint into our control problem. The Hamilton-Jacobi-Bellman equation for this problem is

$$\begin{aligned}\max_{c\geq 0,\pi\in[0,1]}\Big[&(r + (\hat{\mu} - r)\pi)xv'(x) - cv'(x) - \delta v(x) + \frac{c^\gamma}{\gamma} + \frac{1}{2}\sigma^2\pi^2x^2v''(x)\\ &+ \int_{\mathbb{R}\setminus\{0\}} \Big(v(x + \pi x(e^z - 1)) - v(x) - \pi x v'(x)(e^z - 1)\Big)\, \nu(dz)\Big]\\ &= 0 \text{ in } \{x > 0\}. \end{aligned} \tag{5.1}$$

Note that the integral in (5.1) as well as the other integrals displayed in this section are convergent by the condition in (2.3). We now construct an explicit (unique) constrained viscosity solution to this problem. First maximize with respect to c to obtain

$$-V'(x) + c^{\gamma-1} = 0 \implies c = \left[V'(x)\right]^{\frac{1}{\gamma-1}}.$$

Maximizing with respect to π, assuming that the optimal π lies in $(0,1)$, gives the expression

$$\begin{aligned}(\hat{\mu}-r)xV'(x) + \sigma^2\pi x^2 V''(x) \quad &+ \quad \int_{\mathbb{R}\setminus\{0\}} \Big(V'(x+\pi x(e^z-1))x(e^z-1) \\ &- \quad xV'(x)(e^z-1)\Big)\,\nu(dz) = 0.\end{aligned}$$

We guess a solution of the form $V(x) = Kx^\gamma$. Then a straightforward calculation gives the following integral equation for π:

$$(\hat{\mu}-r) - (1-\gamma)\sigma^2\pi + \int_{\mathbb{R}\setminus\{0\}} \Big(\big(1+\pi(e^z-1)\big)^{\gamma-1} - 1\Big)(e^z-1)\,\nu(dz) = 0. \quad (5.2)$$

Note that a π solving this equation will be independent on t. Using the guessed solution, we can obtain an expression for c as well:

$$c = (K\gamma)^{\frac{1}{\gamma-1}} x. \quad (5.3)$$

This expression gives us an explicit consumption rule, that is, consume the fraction $(K\gamma)^{1/\gamma-1}$ of the present wealth. We now set out to find the constant K. Inserting (5.3) into the Hamilton-Jacobi-Bellman Eq. (5.1), we get

$$\begin{aligned}\max_{\pi\in[0,1]}\Big[(r \quad &+ \quad (\hat{\mu}-r)\pi)\gamma - (K\gamma)^{\frac{1}{\gamma-1}}\gamma - \delta + (K\gamma)^{\frac{\gamma}{\gamma-1}-1} - \frac{1}{2}\sigma^2\gamma(1-\gamma)\pi^2 \\ &+ \quad \int_{\mathbb{R}\setminus\{0\}} \Big(\big(1+\pi(e^z-1)\big)^\gamma - 1 - \gamma\pi(e^z-1)\,\nu(dz)\Big)\Big]Kx^\gamma = 0.\end{aligned}$$

We thus conclude that

$$K = \frac{1}{\gamma}\left[\frac{1-\gamma}{\delta - k(\gamma)}\right]^{1-\gamma},$$

where $k(\gamma)$ is defined in (2.8). Note that the condition $\delta > k(\gamma)$ imposed in Sect. 2 implies that K is positive.

We state a condition ensuring the existence of a unique solution $\pi \in (0,1)$ to (5.2). To this end, define the function

$$f(\pi) = (\hat{\mu}-r) - (1-\gamma)\sigma^2\pi + \int_{\mathbb{R}\setminus\{0\}} \Big(\big(1+\pi(e^z-1)\big)^{\gamma-1}(e^z-1) - (e^z-1)\Big)\,\nu(dz).$$

Inserting $\pi = 0$ and $\pi = 1$, we obtain

$$f(0) = \hat{\mu} - r > 0$$

and

$$\begin{aligned} f(1) &= (\hat{\mu} - r) - (1-\gamma)\sigma^2 + \int_{\mathbb{R}\setminus\{0\}} \Big(e^{(\gamma-1)z}(e^z - 1) - (e^z - 1) \Big)\, \nu(dz) \\ &= (\hat{\mu} - r) - (1-\gamma)\sigma^2 - \int_{\mathbb{R}\setminus\{0\}} \big(1 - e^{-(1-\gamma)z}\big)(e^z - 1)\, \nu(dz). \end{aligned}$$

In order to have a solution in $(0,1)$, we need $f(1) < 0$, i.e.,

$$\int_{\mathbb{R}\setminus\{0\}} \big(1 - e^{-(1-\gamma)z}\big)(e^z - 1)\, \nu(dz) > (\hat{\mu} - r) - (1-\gamma)\sigma^2. \qquad (5.4)$$

This solution is unique since

$$f'(\pi) = -(1-\gamma)\Big\{ \sigma^2 + \int_{\mathbb{R}\setminus\{0\}} \big(1 + \pi(e^z - 1)\big)^{\gamma-2}(e^z-1)^2\, \nu(dz) \Big\} < 0.$$

It is well known that in the case of a geometric Brownian motion, $S_t = S_0 \exp(\mu t + \sigma B_t)$, the optimal allocation of money in the portfolio is independent of time:

$$\pi^*_{\mathrm{GBM}} = \frac{\mu + \sigma^2/2 - r}{(1-\gamma)\sigma^2}.$$

On the other hand, we have seen that S_t given as in (2.1) also gives a constant fraction, denoted by π^*_J, which solves (5.2).

$$\begin{aligned} f(\pi^*_{\mathrm{GBM}}) &= \int_{\mathbb{R}\setminus\{0\}} \big(e^z - 1 - z\mathbf{1}_{|z|<1}\big)\nu(dz) \\ &\quad + \int_{\mathbb{R}\setminus\{0\}} \Big(\big(1 + \pi^*_{\mathrm{GBM}}(e^z - 1)\big)^{\gamma-1} - 1 \Big) \big(e^z - 1\big)\nu(dz). \end{aligned}$$

Thus, $\pi^*_J < \pi^*_{\mathrm{GBM}}$ if $f(\pi^*_{\mathrm{GBM}}) < 0$ and $\pi^*_J > \pi^*_{\mathrm{GBM}}$ if $f(\pi^*_{\mathrm{GBM}}) > 0$. Note that the first integral in the expression of $f(\pi^*_{\mathrm{GBM}})$ is positive, while the second is negative. Where to put the most of your fortune depends on the parameters of the specific model in question. In Benth et al. [6] we have compared numerically geometric Brownian motion with the normal inverse Gaussian model proposed by Barndorff-Nielsen [3].

6 Other models and concluding remarks

Instead of modeling the price process S_t directly as in (2.1) or (1.1), one can let S_t be the solution of a stochastic differential equation with jumps

$$dS_t = \mu S_t\, dt + \sigma S_t\, dB_t + S_{t-} \int_{-1}^{\infty} z\, \tilde{N}(dt, dz). \qquad (6.1)$$

Note that S_t is positive due to the restriction of the jump size to be greater than -1. As noted by Eberlein and Keller [8], it is the large jumps that are responsible for the empirically observed heavy tails of the logreturn data. Therefore, (6.1) may not be a good model if heavy tails are to be accounted for in the model.

Assuming a price dynamics defined by (6.1), condition (2.3) must be substituted by

$$\int_1^\infty z\,\nu(dz) < \infty. \tag{6.2}$$

Under this restriction on the Lévy measure we can show, by arguing as before, that the value function $V(x,y)$ is the unique constrained viscosity solution of the Hamilton-Jacobi-Bellman equation

$$\max\Big\{\beta v_y - v_x; U(y) - \delta v - \beta y v_y + \max_{\pi\in[0,1]}\Big[(r + (\mu - r)\pi)x v_x + \frac{1}{2}\sigma^2\pi^2x^2 v_{xx} + \int_{-1}^\infty \Big(v(x + \pi xz, y) - v(x,y) - \pi xz\, v_x(x,y)\Big)\,\nu(dz)\Big]\Big\} = 0 \text{ in } \mathscr{D}. \tag{6.3}$$

The condition (6.2), which ensures that (6.3) is well defined for all sublinearly growing $v \in C^2$, is satisfied for the normal inverse Gaussian Lévy process discussed in Sect. 2 and for all α-stable Lévy processes with $\alpha > 1$.

In Framstad et al. [11], the price model (6.1) is chosen for the analysis of Merton's problem with consumption. Using a verification theorem, they show that the value function in Merton's problem with consumption (see Sect. 5) is a unique classical solution of (6.3) under condition (6.2) and $\nu(\{(-1,\infty)\}) < \infty$. Honoré [13] has developed estimation techniques for price processes of the type (6.1). This opens for a numerical comparison of the different stock price models for financial data.

Except for a few special cases such as those considered in Sects. 4 and 5, the Hamilton-Jacobi-Bellman Eq. (2.10) cannot be solved explicitly and one has to consider numerical approximations. The construction and analysis of numerical schemes for (first and second order) integro-differential variational inequalities will be reported in future work (see [9]).

Finally, we mention that the portfolio model studied in this paper is generalized to account for transaction costs in [5].

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