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Ruin probabilities for a Sparre Andersen model with investments: the case of annuity payments

Yuri Kabanov[1,2] · **Platon Promyslov**[3,4]

Dedicated to the memory of Tomas Björk

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Abstract

This note is a complement to the paper (Stoch. Process. Appl. 144:72–84, 2022) by Eberlein, Kabanov and Schmidt on the asymptotics of the ruin probability in a Sparre Andersen non-life insurance model with investments into a risky asset whose price follows a geometric Lévy process. Using techniques from the theory of semi-Markov processes, we extend the result of (Eberlein, Kabanov and Schmidt in Stoch. Process. Appl. 144:72–84, 2022) to the case of annuities and models with two-sided jumps.

Keywords Ruin probabilities · Sparre Andersen model · Actuarial models with investments · Renewal processes · Annuities · Distributional equations

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1 Introduction

In the classical Sparre Andersen model of an insurance company, the numbers of claims form a renewal process; see Grandell [5, Chap. 3]. In recent studies, see

✉ Y. Kabanov
ykabanov@univ-fcomte.fr

P. Promyslov
platon.promyslov@gmail.com

1 Federal Research Center "Computer Science and Control" of the Russian Academy of Sciences Moscow, Lomonosov Moscow State University, Moscow, Russia

2 Laboratoire de Mathématiques, UMR CNRS 6623, Université de Franche-Comté, 16 Route de Gray, 25030 Besançon, France

3 Lomonosov Moscow State University, Moscow, Russia

4 Institute "Vega", Moscow, Russia

Albrecher et al. [1], Eberlein et al. [3] and references therein, this model was enriched by the assumption that the capital reserve of the insurance company is fully invested in a risky asset whose price evolves as a geometric Lévy process. In [3], Eberlein, Kabanov and Schmidt considered the non-life insurance version of such a model. It was shown that under rather mild hypotheses on the business process, the asymptotic behaviour of the (ultimate) ruin probability is essentially the same as in a Cramér–Lundberg model with risky investments. Namely, the ruin probability decays, up to a multiplicative constant, as the function $u^{-\beta}$ when the initial capital u tends to infinity. The decay rate β depends only on the characteristics of the price process. The method of analysis in [3] is based heavily on the assumption that the risk process has only downward jumps and therefore crosses the level zero only by a jump. This specific feature allows a straightforward reduction to a discrete-time Markovian framework.

The approach of [3] left as an open question whether similar results also hold in the case of upward jumps. This is a feature of the annuity model, where the risk process can cross the level zero in a continuous way. In a less popular model with two-sided jumps, the crossing may happen in both ways. Of course, a positive answer is expected also for the mentioned two models; this was already established for the Cramér–Lundberg models with investments analysed by Kabanov and Pergamenshchikov [6] and Kabanov and Pukhlyakov [8] as well as in very general Lévy Ornstein–Uhlenbeck models introduced and studied by Paulsen using the techniques based on the paper by Goldie [4]; see Paulsen [9, 10], Paulsen and Gjessing [11] and the more recent paper by Kabanov and Pergamenshchikov [7].

Our note, being a complement to the study [3], gives a positive answer for the renewal model in its annuity version, with positive jumps, as well as for the renewal model without restriction on the sign of the jumps. The key contribution lies in constructing a "bridge" between the ruin problem and implicit renewal theory to exploit results of the latter on the tail behaviour of solutions of affine distributional equations. Our technique uses an embedding of a scalar continuous-time semi-Markov process into a two-dimensional Markov process by adding a "clock" component measuring the time elapsed after the last arrival; see Sects. 3 and 4, and in particular Lemmas 3.1 and 4.5.

In the paper, we use standard notations of stochastic calculus and concepts discussed in detail in Kabanov and Pergamenshchikov [7] and Eberlein et al. [3].

2 The model

The Sparre Andersen model with risky investments considered here contains two ingredients. The first is the price process $S=(S_t)_{t\ge 0}$ of a risky financial asset, usually interpreted as a market index. We assume that it is of the form $S=\mathcal{E}(R)$, where $\mathcal{E}$ is the stochastic exponential, $R=(R_t)_{t\ge 0}$ with $R_0=0$ is a Lévy process with Lévy triplet (a,σ^2,Π) and such that $\Pi((-\infty,-1])=0$. The latter condition ensures for the jumps that $\Delta R>-1$, and hence for the price that $S>0$. In that case, $S=e^V$, where $V=\ln S$ is again a Lévy process which can be given by the formula

$$V_t=at-\frac{1}{2}\sigma^2 t+\sigma W_t+h*(\mu-\nu)_t+\big(\ln(1+x)-h\big)*\mu_t,$$

where $h(x) = xI_{\{|x|\le 1\}}$ for $x \in \mathbb{R}$. The Lévy triplet of V is (a_V, σ^2, Π_V) with

$$a_V = a - \frac{\sigma^2}{2} + \Pi\Big(h\big(\ln(1+x)\big) - h\Big)$$

and $\Pi_V = \Pi\varphi^{-1}$ with $\varphi(x) = \ln(1+x)$ for $x > -1$. We assume that R is non-deterministic, that is, at least one of the parameters σ^2 or Π is not zero.

The second ingredient is the "business process". It is a compound renewal process $P = (P_t)_{t\ge 0}$ independent of S. Classically, it can be written in the form

$$P_t = ct + \sum_{i=1}^{N_t} \xi_i,$$

where $N = (N_t)_{t\ge 0}$ is a counting renewal process with the interarrival times (lengths of the inter-jump intervals) $U_i := T_i - T_{i-1}$, $i \ge 1$, forming an i.i.d. sequence independent of the i.i.d. sequence of random variables $\xi_i = \Delta P_{T_i}$, $i \ge 1$, with the common law F_ξ with $F_\xi(\{0\}) = 0$. In the sequel, a "generic" random variable with that law is denoted by ξ. As usual, $T_0 := 0$. The common law of U_i, $i \ge 1$, is denoted by F and we use the same character for its distribution function.

The risk process $X = X^u$, $u > 0$, is defined as the solution of the non-homogeneous linear stochastic equation

$$X_t = u + \int_0^t X_{s-}\,dR_s + P_t.$$

The ruin probability is the function of the initial capital $\Psi(u) := \mathbf{P}[\tau^u < \infty]$, where $\tau^u := \inf\{t\colon X_t^u \le 0\}$.

The cases of major interest are $c > 0$ and $\xi_i < 0$, $i \ge 1$ (the non-life insurance model considered in [3]), and $c < 0$ and $\xi_i > 0$ (an annuities payments model). The latter case which is studied here is often interpreted as a model of a venture company paying salaries and selling innovations. The case where F_ξ charges both half-axes is also mathematically interesting, see e.g. Albrecher et al. [1], and we study it also.

If $c \ge 0$ and $\xi > 0$, ruin never happens. This case is excluded from our considerations as well as the trivial case $c < 0$ and $\xi \le 0$, where ruin happens for sure.

Standing assumption The cumulant-generating function $H(q) := \ln \mathbf{E}[e^{-qV_{T_1}}]$ of the random variable V_{T_1} has a root $\beta > 0$ not lying on the boundary of the effective domain of H. That is, if $\operatorname{int}(\operatorname{dom} H) = (\underline{q}, \bar{q})$, there is a unique root $\beta \in (0, \bar{q})$.

We are looking for conditions under which

$$0 < \liminf_{u\to\infty} u^\beta \Psi(u) \le \limsup_{u\to\infty} u^\beta \Psi(u) < \infty. \tag{2.1}$$

The paper [3] treats the case of non-life insurance. We formulate its main result in a more transparent form.

Theorem 2.1 **([3, Theorem 2.3])** *Suppose that the drift coefficient* $c \ge 0$, *the law* F_ξ *is concentrated on* $(-\infty, 0)$, $\mathbf{E}[|\xi|^\beta] < \infty$ *and* $\mathbf{E}[e^{\varepsilon T_1}] < \infty$ *for some* $\varepsilon > 0$. *Then* (2.1) *holds if at least one of the following conditions is fulfilled*:

1) $\sigma \ne 0$ *or* ξ *is unbounded from below.*

2a) $\Pi((-1, 0)) > 0$ *and* $\Pi((0, \infty)) > 0$.

2b) $\Pi((-1, 0)) = 0$ *and* $\Pi(h) = \infty$.

2c) $\Pi((0, \infty)) = 0$ *and* $\Pi(|h|) = \infty$.

2d) $\Pi((-\infty, 0)) = 0$, $0 < \Pi(h) < \infty$ *and* $F((0, t)) > 0$ *for every* $t > 0$.

2e) $\Pi((0, \infty)) = 0$, $0 < \Pi(|h|) < \infty$ *and* $F((0, t)) > 0$ *for every* $t > 0$.

In the above formulation, a function f in the argument of the measure Π means its integral with respect to this measure, i.e., $\Pi(f) = \int_{\mathbb{R}} f(x)\Pi(dx)$.

The proof in [3] used heavily the assumption that the business process has a positive drift and negative claims, corresponding to the non-life insurance setting. In that case, ruin may happen only at an instant of a jump, and therefore one needs to monitor the risk process only at T_1, T_2 and so on. Such a reduction to a discrete-time ruin model does not work if $\xi_i > 0$.

In our paper, we consider an annuity model of the Sparre Andersen type, where ruin occurs because of exhausting resources and the risk process reaches zero in a continuous way. The main result can be formulated as follows.

Theorem 2.2 *Suppose that the drift* $c < 0$, *the law* F_ξ *is concentrated on* $(0, \infty)$, $\mathbf{E}[\xi^\beta] < \infty$ *and* $\mathbf{E}[e^{\varepsilon T_1}] < \infty$ *for some* $\varepsilon > 0$. *Then* (2.1) *holds if at least one of the following conditions is fulfilled*:

1) $\sigma \ne 0$.

2a) $\Pi((-1, 0)) > 0$ *and* $\Pi((0, \infty)) > 0$.

2b) $\Pi((-1, 0)) = 0$ *and* $\Pi(h) = \infty$.

2c) $\Pi((0, \infty)) = 0$ *and* $\Pi(|h|) = \infty$.

2d) $\Pi((-1, 0)) = 0$, $0 < \Pi(h) < \infty$ *and* $F((t, \infty)) > 0$ *for every* $t > 0$.

2e) $\Pi((0, \infty)) = 0$, $0 < \Pi(|h|) < \infty$ *and* $F((t, \infty)) > 0$ *for every* $t > 0$.

For the mixed case, we have the following result.

Theorem 2.3 *Suppose that the drift* $c \in \mathbb{R}$, *the law* F_ξ *charges both half-lines* $(-\infty, 0)$ *and* $(0, \infty)$, $\mathbf{E}[|\xi|^\beta] < \infty$ *and* $\mathbf{E}[e^{\varepsilon T_1}] < \infty$ *for some* $\varepsilon > 0$. *Then* (2.1) *holds if at least one of the following conditions is fulfilled*:

1) $\sigma \ne 0$ *or* $|\xi|$ *is unbounded.*

2a) $\Pi((-1, 0)) > 0$ *and* $\Pi((0, \infty)) > 0$.

2b) $\Pi((-1, 0)) = 0$ *and* $\Pi(h) = \infty$.

2c) $\Pi((0, \infty)) = 0$ *and* $\Pi(|h|) = \infty$.

2d) $\Pi((-1, 0)) = 0$, $0 < \Pi(h) < \infty$, *and* $F((t, \infty)) > 0$ *for every* $t > 0$ *in the case* $c < 0$, *or* $F_\xi((0, \varepsilon)) > 0$ *for every* $\varepsilon > 0$ *in the case* $c \ge 0$.

2e) $\Pi((0, \infty)) = 0$, $0 < \Pi(|h|) < \infty$, *and* $F((t, \infty)) > 0$ *for every* $t > 0$ *in the case* $c < 0$, *or* $F_\xi((0, \varepsilon)) > 0$ *for every* $\varepsilon > 0$ *in the case* $c \ge 0$.

3 The clock process

The annuity and the mixed setting require a different approach inspired by the theory of semi-Markov processes. Namely, we consider the business process P as one component of the two-dimensional Markov process (P, D), where the second component $D = D^r$ is a "clock", i.e., a process measuring the time elapsed after the instant of the last claim. We assume that for $r > 0$, the law of $U_1 = T_1$ may be different from the common law of the further interarrival times; so at the instant zero, a portion $r > 0$ of the interarrival time is already elapsed. This feature admits obvious justifications, e.g. the venture company may change the governance when a project is still in progress.

Here and **throughout the paper**, we use the superscript r to emphasize that the law of a random variable or a process depends on $r \geq 0$, skipping usually $r = 0$. The reserve process is denoted by $X^{u,r}$, the ruin time by $\tau^{u,r} := \inf\{t \geq 0\colon X_t^{u,r}\}$ and the ruin probability by $\Psi^r(u) := \mathbf{P}[\tau^{u,r} < \infty]$.

Formally, the "clock" $D^r = (D_t^r)$ is a process with the initial value $D_0^r = r$, $D_t^r = r + t$ on the interval $[0, T_1)$, and $D_t^r := t - T_n^r$ on all other interarrival intervals $[T_n^r, T_{n+1}^r)$, $n \geq 1$. That is, the "clock" restarts from zero at each instant T_n^r. We denote by F^r the law of the first interarrival time $T_1^r = T_1^r - T_0$. In accordance with our convention, $F^0 = F$. Alternatively, D^r can be represented as the solution of the linear equation

$$D_t^r = r + t - \int_{[0,t]} D_{s-}^r dN_s.$$

We assume that the distribution functions satisfy $F^r(t) \geq F(t)$ for all t to reflect the fact that a part of the interarrival time already elapsed.

Recall that the assumed independence of P^r and R implies that the joint quadratic characteristic $[P^r, R]$ is zero and that the reserve process $X^{u,r}$ can be written in the form, resembling the Cauchy formula for solutions of linear differential equations,

$$X_t^{u,r} = e^{V_t}(u - Y_t^r),$$

where

$$Y_t^r := -\int_{(0,t]} \mathcal{E}_{s-}^{-1}(R) dP_s^r = -\int_{(0,t]} e^{-V_{s-}} dP_s^r.$$

The strict positivity of the process $\mathcal{E}(R) = e^V$ implies that the ruin time is

$$\tau^{u,r} := \inf\{t \geq 0\colon X_t^{u,r} \leq 0\} = \inf\{t \geq 0\colon Y_t^r \geq u\}.$$

The crucial element of our study is the following result.

Lemma 3.1 *Suppose that $Y_t^r \to Y_\infty^r$ almost surely as $t \to \infty$, where Y_∞^r is a finite random variable such that $\bar{G}(u, r) := \mathbf{P}[Y_\infty^r > u] > 0$ for every $u \in \mathbb{R}$ and $r \geq 0$. If $\bar{G}_* := \inf_{q \geq 0} \bar{G}(0, q) > 0$, then*

$$\bar{G}(u, r) \leq \Psi^r(u) = \frac{\bar{G}(u, r)}{\mathbf{E}[\bar{G}(X_{\tau^{u,r}}^{u,r}, D_{\tau^{u,r}}^r) | \tau^{u,r} < \infty]} \leq \frac{1}{\bar{G}_*} \bar{G}(u, r).$$

Proof Let τ be an arbitrary stopping time with respect to the filtration $(\mathcal{F}_t^{P,D,R})$. As we assume that the finite limit Y_∞^r exists, the random variable

$$Y_{\tau,\infty}^r := \begin{cases} -\lim_{N\to\infty} \int_{(\tau,\tau+N]} e^{-(V_{s-}-V_\tau)} dP_s^r, & \tau < \infty, \\ 0, & \tau = \infty, \end{cases}$$

is well defined. On the set $\{\tau < \infty\}$,

$$Y_{\tau,\infty}^r = e^{V_\tau}(Y_\infty^r - Y_\tau^r) = X_\tau^{u,r} + e^{V_\tau}(Y_\infty^r - u). \tag{3.1}$$

Let ζ be an $\mathcal{F}_\tau^{P,D,R}$-measurable random variable. Using the strong Markov property, we get that

$$\mathbf{P}[Y_{\tau,\infty}^r > \zeta, \tau < \infty] = \mathbf{E}[\bar{G}(\zeta, D_\tau^r) I_{\{\tau<\infty\}}]. \tag{3.2}$$

Noting that $\Psi^r(u) = \mathbf{P}[\tau^{u,r} < \infty] \geq \mathbf{P}[Y_\infty^r > u] > 0$, we deduce from (3.1) and (3.2) that

$$\begin{aligned} \bar{G}(u,r) &= \mathbf{P}[Y_\infty^r > u, \tau^{u,r} < \infty] \\ &= \mathbf{P}[Y_{\tau^{u,r},\infty}^r > X_{\tau^{u,r}}^{u,r}, \tau^{u,r} < \infty] \\ &= \mathbf{P}[\tau^{u,r} < \infty]\mathbf{E}[\bar{G}(X_{\tau^{u,r}}^{u,r}, D_{\tau^{u,r}}^r) | \tau^{u,r} < \infty] \\ &\geq \mathbf{P}[\tau^{u,r} < \infty]\mathbf{E}[\bar{G}(0, D_{\tau^{u,r}}^r) | \tau^{u,r} < \infty] \\ &\geq \mathbf{P}[\tau^{u,r} < \infty] \inf_{q\geq 0} \bar{G}(0,q) \end{aligned}$$

and get the result. □

In view of Lemma 3.1, the proof of the main theorems is reduced to establishing the existence of finite limits Y_∞^r and finding the asymptotics of the tail of their distributions. One needs also to check that $\inf_{q\geq 0} \bar{G}(0,q) > 0$.

4 The existence of the limit Y_∞^r

Let us introduce the notations

$$Q_k^r := -\int_{(T_{k-1}^r, T_k^r]} e^{-(V_{s-}-V_{T_{k-1}^r})} dP_s^r, \qquad M_k^r := e^{-(V_{T_k^r}-V_{T_{k-1}^r})}. \tag{4.1}$$

First, we recall several results from [3]. In the following result, H is the cumulant-generating function of the random variable V_{T_1} satisfying the standing assumption.

Lemma 4.1 (**[3, Lemma 2.1]**) *Let $T > 0$ be a random variable independent of R. Suppose that $\mathbf{E}[e^{\varepsilon T}] < \infty$ for some $\varepsilon > 0$. Let $\beta \in (0, \bar{q})$ be the (unique) root of the equation $H(q) = 0$. If $q \in [\beta, \bar{q})$ is such that $H(q) \leq \varepsilon/2$, then*

$$\mathbf{E}\Big[\sup_{s\leq T} e^{-qV_s}\Big] < \infty.$$

Corollary 4.2 *Suppose that* $\mathbf{E}[e^{\varepsilon T_1}] < \infty$ *for some* $\varepsilon > 0$. *Let*

$$\widehat{Q}_1 := \sup_{t \le T_1} \left| \int_{[0,t]} e^{-V_{s-}} dP_s \right|.$$

If $\mathbf{E}[|\xi_1|^\beta] < \infty$, *then* $\mathbf{E}[\widehat{Q}_1^\beta] < \infty$.

Although the assertion in Corollary 4.2 is a bit more general than Eberlein et al. [3, Corollary 2.2], the proof is exactly the same. Note also that it does not depend on the signs of c or ξ_1 and needs only the integrability of $|\xi_1|^\beta$. It implies in particular that $\mathbf{E}[|Q_1|^\beta] < \infty$.

Lemma 4.3 *Suppose that* $\mathbf{E}[e^{\varepsilon T_1}] < \infty$ *and* $\mathbf{E}[|\xi_1|^{\beta \wedge \varepsilon \wedge 1}] < \infty$ *for some* $\varepsilon > 0$. *Then* $Y_t \to Y_\infty$ *almost surely as* $t \to \infty$, *where* Y_∞ *is a finite random variable.*

Proof The convergence a.s. of the sequence (Y_{T_n}) to a finite random variable Y_∞ has been proved in [3, Lemma 4.1], as has the fact that $\rho := \mathbf{E}[M_1^p] < 1$ for any $p \in (0, \beta \wedge \varepsilon \wedge 1)$.

Put $I_n := (T_{n-1}, T_n]$ and

$$\Delta_n := \sup_{v \in I_n} \left| \int_{(T_{n-1},v]} e^{-V_{s-}} dP_s \right| = \prod_{i=1}^{n-1} M_i \sup_{v \in I_n} \left| \int_{(T_{n-1},v]} e^{-(V_{s-} - V_{T_{n-1}})} dP_s \right|.$$

By virtue of the Borel–Cantelli lemma, to get the announced result, it is sufficient to show that for every $\delta > 0$,

$$\sum_{n=1}^{\infty} \mathbf{P}[\Delta_n \ge \delta] < \infty.$$

But this is true because the Chebyshev inequality and Corollary 4.2 imply that we have $\mathbf{P}[\Delta_n \ge \delta] \le \delta^{-p} \rho^n \mathbf{E}[\widehat{Q}_1^p]$. □

Lemma 4.4 *For each* $r \ge 0$, *the finite limit* Y_∞^r *exists a.s. and admits the representation*

$$Y_\infty^r = Q_1^r + M_1^r \tilde{Y}_\infty^r,$$

where

$$Q_1^r := -\int_{[0,T_1^r]} e^{-V_{s-}} dP_s^r, \qquad M_1^r := e^{-V_{T_1^r}},$$

(Q_1^r, M_1^r) *and* $\tilde{Y}_\infty^r$ *are independent, and the laws of* $\tilde{Y}_\infty^r$ *and* Y_∞ *coincide.*

Proof Note that

$$Y^r_{T^r_n} = -\int_{[0,T^r_1]} e^{-V_{s-}} dP^r_s - \sum_{k=2}^{n} e^{V_{T^r_{k-1}}} \int_{(T^r_{k-1},T^r_k]} e^{-(V_{s-}-V_{T^r_{k-1}})} dP^r_s$$
$$= Q^r_1 + M^r_1 \Big(Q^r_2 + \sum_{k=3}^{n} M^r_2 \cdots M^r_{k-1} Q^r_k \Big),$$

where the random variable in the parentheses is independent of (Q^r_1, M^r_1) and has the same distribution as $Y_{T_{n-1}} := Y^0_{T_{n-1}}$. By Lemma 4.3, the sequence $(Y^r_{T^r_n})$ converges a.s. to Y^r_∞ as $n \to \infty$, and the same arguments as above allow us to conclude that Y^r_t also converges a.s. as $t \to \infty$. □

Lemma 4.5 *Suppose that Y_∞ is unbounded from above. If $c < 0$, then*

$$\inf_{q \ge 0} \bar G(0,q) \ge \mathbf{E}[\bar G(\xi, 0)] > 0.$$

If $c \in \mathbb{R}$ and the distribution functions F^r and F satisfy $F^r \ge F$, then

$$\inf_{q \ge 0} \bar G(0,q) > 0.$$

Proof Using Lemma 4.4, we have

$$\begin{aligned}
\bar G(0,r) &= \mathbf{P}[Y^r_\infty > 0] \\
&= \mathbf{P}[Q^r_1/M^r_1 + \tilde Y^r_\infty > 0] \\
&= \int_{\mathbb{R}_+} \mathbf{P}\Big[|c| e^{V_t} \int_{[0,t]} e^{-V_s} ds - \xi_1 + \tilde Y^r_\infty > 0 \Big] F_{T^r_1}(dt) \\
&\ge \mathbf{P}[\tilde Y^r_\infty > \xi_1] \\
&= \int_{\mathbb{R}_+} \mathbf{P}[\tilde Y^r_\infty > x] F_\xi(dx) \\
&= \int_{\mathbb{R}_+} \bar G(x,0) F_\xi(dx) > 0
\end{aligned}$$

since Y_∞ is unbounded. An inspection of the above proof reveals that it still works with minor changes also for the case where c has an arbitrary sign and the law F_ξ is arbitrary.

Put $f_t := |\xi_1| + |c| t e^{2V^*_t}$, where $V^*_t := \sup_{s \le t} |V_s|$. Then

$$|Q^r_1|/M^r_1 \le |\xi_1| + |c| e^{V_{T^r_1}} \int_{[0,T^r_1]} e^{-V_s} ds \le f_{T^r_1}.$$

It follows that

$$\begin{aligned}
\bar{G}(0,r) &= \mathbf{P}[\tilde{Y}^r_\infty > -Q^r_1/M^r_1] \\
&= \mathbf{E}[\bar{G}(-Q^r_1/M^r_1, 0)] \\
&\ge \mathbf{E}[\bar{G}(|Q^r_1|/M^r_1, 0)] \\
&\ge \mathbf{E}\bigg[\int_{\mathbb{R}_+} \bar{G}(f_t, 0) F^r(dt)\bigg] \\
&= -\mathbf{E}\bigg[\int_{\mathbb{R}_+} F^r(t) d\bar{G}(f_t, 0)\bigg] \\
&\ge -\mathbf{E}\bigg[\int_{\mathbb{R}_+} F(t) d\bar{G}(f_t, 0)\bigg] \\
&\ge \mathbf{E}\bigg[\int_{\mathbb{R}_+} \bar{G}(f_t, 0) F(dt)\bigg] > 0,
\end{aligned}$$

where we use the property $F^r \ge F$. Thus $\inf_{r\ge 0} \bar{G}(0,r) > 0$. □

5 Tails of solutions of distributional equations

Like a number of results on the ruin with investments, the proof is based on implicit renewal theory. As in [3], we use the following formulation combining several useful facts; see Kabanov and Pergamenshchikov [7, Theorem A.6].

Theorem 5.1 *Let $M > 0$ and Q be random variables such that for some $\beta > 0$,*

$$\mathbf{E}[M^\beta] = 1, \qquad \mathbf{E}[M^\beta (\ln M)^+] < \infty, \qquad \mathbf{E}[|Q|^\beta] < \infty. \tag{5.1}$$

Let Y_∞ be the solution of the distributional equation $Y_\infty \overset{d}{=} Q + MY_\infty$ and define $\bar{G}(u) := \mathbf{P}[Y_\infty > u]$. Then $\limsup_{u\to\infty} u^\beta \bar{G}(u) < \infty$. If the random variable Y_∞ is unbounded from above, then $\liminf_{u\to\infty} u^\beta \bar{G}(u) > 0$.

In Sect. 3, we introduced a process $Y = (Y_t)_{t\ge 0}$. We proved that under the integrability assumptions $\mathbf{E}[|\xi|^\beta] < \infty$ and $\mathbf{E}[e^{\varepsilon T_1}] < \infty$ for some $\varepsilon > 0$, the process Y has at infinity a finite limit Y_∞. The random variable Y_∞ solves the required distributional equation with $M = M_1$ and $Q = Q_1$ given by (4.1) having the properties (5.1). To apply Theorem 5.1 and get the claimed lower and upper bounds for the ruin probabilities, it remains to check that the remaining hypotheses of Theorems 2.2 and 2.3 ensure that Y_∞ is unbounded from above. We do this in the next section.

6 When is Y_∞ unbounded from above?

The above question is studied in [3] for the non-life insurance case, i.e., when $c < 0$ and $F_\xi((0,\infty)) = 1$. In the present paper, we provide sufficient conditions for the unboundedness from above for all new cases by using the techniques developed in [3]. This is based on the following elementary observation: If $f : X \times Y \to \mathbb{R}$ is a measurable function and the random variables η and ζ are independent and have the laws F_η and F_ζ, then the random variable $f(\eta, \zeta)$ is unbounded from above provided that there exists a measurable set $X_0 \subseteq X$ with $F_\eta(X_0) > 0$ such that the random variable $f(x, \zeta)$ is unbounded from above for every $x \in X_0$.

Let $A_n := M_1 \cdots M_n$ for $n \geq 1$ with $A_0 := 1$. A tractable sufficient condition is given by the following result.

Lemma 6.1 **([3, Lemma 5.1])** *If there exists $n \geq 1$ such that the random variables Q_1 and $(Q_1 + \cdots + A_{n-1} Q_n)/A_n$ are unbounded from above, then Y_∞ is unbounded from above.*

Lemma 6.1 usually works already with $n = 1$, but sometimes we need it with $n = 2$. A short look at the expressions

$$Q_1 = -c \int_0^{T_1} e^{-V_s} ds - e^{-V_{T_1}} \xi_1, \tag{6.1}$$

$$Q_1/A_1 = -c e^{V_{T_1}} \int_0^{T_1} e^{-V_s} ds - \xi_1, \tag{6.2}$$

$$Q_1/A_2 + Q_2/M_2 = -c e^{V_{T_2}} \int_0^{T_2} e^{-V_s} ds - \xi_1 e^{V_{T_2} - V_{T_1}} - \xi_2$$

shows that Y_∞ is unbounded from above when ξ is unbounded from below (of course, the latter property is not fulfilled for the annuity model).

Using the above sufficient condition for unboundedness, we examine different cases.

1) Let $\sigma \neq 0$. In this case, the following lemma is helpful.

Lemma 6.2 *Let W be a Wiener process and let $K > 0$, $\sigma \neq 0$ and $0 \leq s < t$. Then the random variables*

$$\zeta := K e^{\sigma W_t} - \int_0^t e^{\sigma W_s} ds, \qquad \tilde{\zeta} := K e^{\sigma (W_t - W_s)} - e^{\sigma W_t} \int_0^t e^{\sigma W_s} ds$$

are unbounded from below and from above.

Proof The property that ζ and $\tilde{\zeta}$ are unbounded from above has been proved in Eberlein et al. [3, Lemma 5.2]. The unboundedness from below can be established by similar arguments. It is also clear that if $K = 0$, then ζ and $\tilde{\zeta}$ are unbounded from below. □

The process $\bar{V} := V - \sigma W$ is independent of the Wiener process W. If $c < 0$, then

$$Q_1 \geq |c| \inf_{s \leq T_1} e^{-\bar{V}_s} \int_0^{T_1} e^{-\sigma W_s} ds - \xi_1 e^{-\bar{V}_{T_1}} e^{-\sigma W_{T_1}}.$$

By conditioning with respect to $\bar{V}$, ξ_1, T_1 and using Lemma 6.2, we get that Q_1 is unbounded from above. Since

$$Q_1/A_1 \geq |c| e^{\bar{V}_{T_1}} \inf_{s\leq T_1} e^{-\bar{V}_s} e^{\sigma W_{T_1}} \int_0^{T_1} e^{-\sigma W_s} ds - \xi_1,$$

we conclude in the same way that Q_1/A_1 is unbounded from above. If $c \geq 0$, then necessarily $F_\xi(-\infty, 0)) > 0$ (recall that we exclude the case $c \geq 0$, $\xi > 0$ where ruin is impossible). Lemma 6.2 implies that the random variables Q_1 and $Q_1/A_2 + Q_2/M_2$ are unbounded from above.

2) Now we study the case where $\sigma = 0$ and ξ is bounded from below. We treat separately several subcases.

2a) Suppose first that $\Pi((-1, 0)) > 0$ and $\Pi((0, \infty)) > 0$. Fix $\varepsilon > 0$ such that $\Pi((-1, -\varepsilon)) > 0$ and $\Pi((\varepsilon, \infty)) > 0$ and put

$$V^{(1)} := \big(I_{\{-1<x<-\varepsilon\}} \ln(1+x)\big) * \mu + \big(I_{\{x>\varepsilon\}} \ln(1+x)\big) * \mu.$$

Then the processes $V^{(1)}$ and $V^{(2)} := V - V^{(1)}$ are independent.

Note that $V^{(1)}$ is the sum of two independent compound Poisson processes with negative and positive jumps, respectively, and the absolute values of the jumps are larger than some constant $c_\varepsilon > 0$.

Lemma 6.3 *Let $K > 0$, $t > 0$. Then the random variable*

$$\zeta := K e^{-V_t^{(1)}} - \int_0^t e^{-V_s^{(1)}} ds$$

is unbounded from above and from below, and the random variable

$$\widehat{\zeta} := e^{-V_t^{(1)}} \int_0^t e^{-V_s^{(1)}} ds$$

is unbounded from above.

Proof The arguments are simple and we explain only the idea. One can consider trajectories where $V^{(1)}$ has a lot of negative jumps in a neighbourhood of zero while all positive jumps are concentrated in a neighbourhood of t. Choosing suitable parameters and using the independence of the processes with positive and negative jumps, we obtain that with a strictly positive probability, the first term in the definition of ζ is arbitrarily close to zero while the integral is arbitrarily large. Thus ζ is unbounded from below. Symmetric arguments lead to the conclusion that ζ is unbounded from above.

For $\widehat{\zeta}$, fix an arbitrary $N > 0$. On a set of strictly positive probability, the process $V^{(1)}$ has no positive jumps on $[0, t]$, but $V_1^{(1)} \leq -N$. Thus $\widehat{\zeta} \geq e^N t$. □

Let $c<0$. Then we have from (6.1) and (6.2) the obvious bounds

$$Q_1 \ge |c| \inf_{s\le T_1} e^{-V_s^{(2)}} \int_0^{T_1} e^{-V_s^{(1)}} ds - \xi_1 e^{-\bar V_{T_1}^{(2)}} e^{-V_{T_1}^{(1)}},$$

$$Q_1/A_1 \ge |c| e^{V_{T_1}^{(2)}} \inf_{s\le T_1} e^{-V_s^{(2)}} e^{V_{T_1}^{(1)}} \int_0^{T_1} e^{-V_s^{(1)}} ds - \xi_1.$$

By conditioning with respect to the random variables $V^{(2)}$, T_1, ξ_1 which are independent of $V^{(1)}$ and by using Lemma 6.3, we easily obtain that the random variables Q_1 and Q_1/A_1 are unbounded from above. By Lemma 6.1, so is then Y_∞.

Let $c\ge 0$. The same arguments as in [3] show that Q_1 and $Q_1/A_2+Q_2/M_2$ are unbounded from above on the non-null set $\{\xi_1<0,\ \xi_2<0\}$.

2b) We next consider the case where $\Pi((-1,0))=0$ and $\Pi(h)=\infty$. Here, we use a decomposition of V depending on the choice of $\varepsilon\in(0,1)$. Namely, we put

$$V^\varepsilon := (I_{\{x\le\varepsilon\}}h)*(\mu-\nu) + \Big(I_{\{x\le\varepsilon\}}\big(\ln(1+x)-h\big)\Big)*\mu, \tag{6.3}$$

$$\tilde V^\varepsilon := (I_{\{x>\varepsilon\}}h)*(\mu-\nu) + \Big(I_{\{x>\varepsilon\}}\big(\ln(1+x)-h\big)\Big)*\mu. \tag{6.4}$$

Note that $V_t = at + V_t^\varepsilon + \tilde V_t^\varepsilon$ and

$$\tilde V^\varepsilon = \big(I_{\{x>\varepsilon\}}\ln(1+x)\big)*\mu - (I_{\{x>\varepsilon\}}h)*\nu.$$

Lemma 6.4 *Let $K>0$, $t>0$. Then the random variables*

$$\eta := \int_0^t e^{-V_s} ds - Ke^{-V_t}, \qquad \eta' := e^{V_t}\int_0^t e^{-V_s} ds$$

are unbounded from above.

Proof Without loss of generality, we assume for the Lévy triplet of R that $a=0$. Fix $N>0$ and choose $\varepsilon>0$ small enough to ensure that $\Pi(I_{\{x>\varepsilon\}}h)\ge N$. Let $\Gamma_\varepsilon := \{\sup_{s\le t}|V_s^\varepsilon|\le 1\}$. Denote by J^ε and $\bar J^\varepsilon$ the processes on the right-hand side of (6.3). The first is a martingale with bounded jumps, the second is a decreasing process. Using the Doob inequality and the elementary bound $x-\ln(1+x)\le x^2/2$ for $x>0$, as well as $h(x)=x$ for $|x|\le 1$, we get that

$$\begin{aligned}\mathbf{P}\Big[\sup_{s\le t}|V_s^\varepsilon|>1\Big] &\le \mathbf{P}\Big[\sup_{s\le t}|J_s^\varepsilon|>1/2\Big] + \mathbf{P}[|\bar J_t^\varepsilon|>1/2]\\ &\le 2\mathbf{E}\Big[\sup_{s\le t}|J_s^\varepsilon|\Big] + 2\mathbf{E}[|\bar J_t^\varepsilon|]\\ &\le 2\big((I_{\{x\le\varepsilon\}}h^2)*\nu_t\big)^{1/2} + (I_{\{x\le\varepsilon\}}h^2)*\nu_t \longrightarrow 0 \qquad \text{as } \varepsilon\to 0.\end{aligned}$$

Thus the set Γ_ε is non-null for sufficiently small ε, and on this set

$$\eta \ge \frac{1}{e}\int_0^t e^{-\tilde V_s^\varepsilon} ds - Ke^{-\tilde V_t^\varepsilon+1}.$$

On the intersection $\Gamma_\varepsilon \cap \{(I_{\{x>\varepsilon\}}h)*\mu_{t/2}=0, \ln(1+\varepsilon)\mu((t/2,t]\times(\varepsilon,1])\geq Nt+1\}$, we have

$$\eta \geq \frac{1}{e}\int_0^{t/2} e^{Ns}ds - Ke = \frac{1}{eN}(e^{Nt/2}-1) - Ke.$$

Due to the independence of V^ε and $\tilde{V}^\varepsilon$, this intersection is a non-null set. Since N is arbitrary large, the required property of η holds.

The analysis of η' follows along the same lines. At the first stage, we replace V by $\tilde{V}$ and compensate the linear decrease of V by a large number of positive jumps on the second half of the interval $[0,t]$. □

Let $c<0$. Then the random variables from (6.1) and (6.2),

$$Q_1 = |c|\int_0^{T_1} e^{-V_s}ds - e^{-V_{T_1}}\xi_1,$$

$$Q_1/A_1 = |c|e^{V_{T_1}}\int_0^{T_1} e^{-V_s}ds - \xi_1,$$

are unbounded from above by Lemma 6.4.

Let $c\geq 0$. The same arguments as in [3] lead to the conclusion that Q_1 and $Q_1/A_2+Q_2/M_2$ are unbounded from above on the non-null set $\{\xi_1<0,\ \xi_2<0\}$.

2c) The next case we consider is when $\Pi((0,\infty))=0$ and $\Pi(|h|)=\infty$. First, let $c<0$. We use again a decomposition of V depending on the choice of $\varepsilon\in(0,1)$. Put

$$V^\varepsilon := (I_{\{x\geq-\varepsilon\}}h)*(\mu-\nu) + \Big(I_{\{x\geq-\varepsilon\}}\big(\ln(1+x)-h\big)\Big)*\mu,$$

$$\tilde{V}^\varepsilon := (I_{\{x<-\varepsilon\}}h)*(\mu-\nu) + \Big(I_{\{x<-\varepsilon\}}\big(\ln(1+x)-h\big)\Big)*\mu,$$

$L := (I_{\{x<-\varepsilon\}}\ln(1+x))*\mu$ and $\pi_\varepsilon := \Pi(I_{\{x<-\varepsilon\}}|h|)\uparrow\infty$ as $\varepsilon\to 0$. Then

$$\tilde{V}_t^\varepsilon = \big(I_{\{x<-\varepsilon\}}\ln(1+x)\big)*\mu_t - (I_{\{x<-\varepsilon\}}h)*\nu = L_t + \pi_\varepsilon t.$$

To prove that Q_1 from (6.1) is unbounded from above, we argue as follows. As in the previous subcase 2b), we reduce the problem to checking that the random variable

$$\tilde{\eta} = \int_0^t e^{-\tilde{V}_s^\varepsilon}ds - Ke^{-\tilde{V}_t^\varepsilon}$$

is unbounded from above. Let $t_1 := t_2/2$, where $t_2 := 1/\pi_\varepsilon$. Note that $t_2\leq t$ when $\varepsilon>0$ is sufficiently small. On the set $\{L_t = L_{t_1}\}$, we have that

$$\begin{aligned}\tilde{\eta} &\geq (t_2-t_1)e^{|L_{t_1}|}e^{-\pi_\varepsilon t_2} - Ke^{L_{t_1}}e^{-\pi_\varepsilon t}\\ &= e^{|L_{t_1}|}\big(1/(2e\pi_\varepsilon) - Kee^{-\pi_\varepsilon t}\big)\\ &\geq e^{|L_{t_1}|}/(4e\pi_\varepsilon)\end{aligned}$$

for sufficiently small ε. Since the random variable $|L_{t_1}|$ is unbounded from above, so is Q_1.

On the set $\{L_t = 0, \xi_1 \le K\}$ which is non-null for any $\varepsilon > 0$ and sufficiently large K, we have the bound

$$Q_1/A_1 \ge (|c|/\pi_\varepsilon)(e^{\pi_\varepsilon t} - 1) - K.$$

It follows that Q_1/A_1 from (6.2) is unbounded from above.

The case $c \ge 0$ is treated as in [3].

2d) Now consider the case where $\Pi((-\infty, 0)) = 0$, $0 < \Pi(h) < \infty$, and $F((t,\infty)) > 0$ for every $t > 0$ when $c < 0$. In this subcase, we have $V_t = L_t - bt$, where $L_t := \ln(1+x) * \mu_t$ is an increasing process and $b := \Pi(h) - a$. Note that $b > 0$ because otherwise we get a contradiction with the existence of $\beta > 0$ such that $\ln \mathbf{E}[e^{-\beta V_1}] = 0$.

Take arbitrary $N > 0$. Let $s > 0$ be the solution of the equation

$$(|c|/b)(e^{bs} - 1) = N.$$

Take K large enough to ensure that $\mathbf{P}[\xi \le K] > 0$. Choose $t > s$ such that we have $F((s,t)) > 0$. On the non-null set

$$\{L_s = 0, T_1 \in (s,t), e^{L_t - L_s} \ge K e^{bt}, \xi < K\},$$

we have that

$$Q_1 \ge (|c|/b)(e^{bs} - 1) - e^{bt - L_t}\xi \ge N - 1.$$

Thus Q_1 from (6.1) is unbounded from above.

To prove that Q_1/A_1 from (6.2) is unbounded from above, we take $\varepsilon > 0$ and $K > 0$ large enough such that $F((t,\infty)) > 0$ and $F_\xi((0,K)) > 0$. Setting $c_\varepsilon := (|c|/b)(e^{-b\varepsilon} - e^{-2b\varepsilon})$, we have on the non-null set

$$\big\{T_1 > \varepsilon, L_{T_1 - \varepsilon} = 0, L_{T_1} \ge \ln\big((N+K)/c_\varepsilon\big), \xi < K\big\}$$

that

$$Q_1/A_1 \ge (|c|/b)e^{L_{T_1}} e^{-bT_1}(e^{b(T_1 - \varepsilon)} - 1) - \xi \ge c_\varepsilon e^{L_t} - K \ge N.$$

So by Lemma 6.1, Y_∞ is unbounded from above.

If $c \ge 0$, we proceed as in [3], using the assumption that F_ξ charges any neighbourhood of zero.

2e) Finally, we treat the case where $\Pi((0,\infty)) = 0$, $0 < \Pi(|h|) < \infty$, and $F((t,\infty)) > 0$ for every $t > 0$ when $c < 0$. We have again $V_t = L_t - bt$, but now the jump process L is decreasing and the constant $b < 0$.

Fix $N > 0$. Let $s, t > 0$ be such that $F((s,t)) > 0$. On the non-null set

$$\{T_1 \in (s,t), |L_{s/2}| \ge N, L_{s/2} = L_t, \xi < e^{|L_{s/2}|}\},$$

we have

$$Q_1 \geq |c|(T_1/2)e^{|L_{(1/2)T_1}|-|b|T_1} - e^{|L_{(1/2)T_1}|-|b|T_1}\xi \geq |c|(s/2)e^{N-|b|t} - 1.$$

Since N is arbitrary, Q_1 from (6.1) is unbounded from above.

For any $t > 0$ and $K > 0$, on the non-null set $\{T_1 \geq t, L_t = 0, \xi \leq K\}$, we have

$$Q_1/A_1 \leq |c/b|(e^{|b|t} - 1) - K.$$

Thus Q_1/A_1 from (6.2) is unbounded from above and we can use Lemma 6.1.

If $c \geq 0$, we again proceed as in [3].

Remark 6.5 An anonymous referee attracted our attention to the book by Buraczewski et al. [2] on affine distributional equation. Section 2.5 of this book is devoted to the support of the law of Y_∞. In particular, the assumptions of Theorem 2.5.5 (1) and Lemma 2.5.7 (1) provide a sufficient condition for the unboundedness from above of Y_∞. This condition is formulated in terms of the coefficients of the distributional equation that Y_∞ satisfies. We leave here as an open question how to verify it from our (or other) assumptions on the parameters of the model.

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Declarations

Competing Interests The authors declare no competing interests.

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