



Optimal dividends under a drawdown constraint and a curious square-root rule

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Received: 24 June 2022 / Accepted: 19 January 2023 / Published online: 24 March 2023
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Abstract

In this paper, we address the problem of optimal dividend payout strategies from a surplus process governed by Brownian motion with drift under a drawdown constraint, i.e., the dividend rate can never decrease below a given fraction a of its historical maximum. We solve the resulting two-dimensional optimal control problem and identify the value function as the unique viscosity solution of the corresponding Hamilton–Jacobi–Bellman equation. We then derive sufficient conditions under which a two-curve strategy is optimal, and we show how to determine its concrete form using calculus of variations. We establish a smooth-pasting principle and show how it can be used to prove the optimality of two-curve strategies for sufficiently large initial and maximum dividend rates. We also give a number of numerical illustrations in which the optimality of the two-curve strategy can be established for instances with smaller values of the maximum dividend rate and the concrete form of the curves can be determined. One observes that the resulting drawdown strategies nicely interpolate between the solution for the classical unconstrained dividend problem and that for a ratcheting constraint as recently studied in Albrecher et al. (SIAM J. Financial Math. 13:657–701, 2022). When the maximum allowed dividend rate tends to infinity, we show a surprisingly simple and somewhat intriguing limit result in terms of the parameter a for the surplus level above which, for a sufficiently large current dividend

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rate, a take-the-money-and-run strategy is optimal in the presence of the drawdown constraint.

Keywords Optimal dividends · Viscosity solution · HJB equation · Optimal control

Mathematics Subject Classification (2020) 91G50 · 49L25 · 93E20

JEL Classification G35 · G32 · G33

1 Introduction and model

Assume that the surplus process of a company is given by a Brownian motion with drift,

$$X_t = x + \mu t + \sigma W_t, \quad (1.1)$$

where W is a standard Brownian motion, and $\mu > 0$, $\sigma > 0$ are given constants. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be the complete probability space generated by the Brownian motion $(W_t)_{t \geq 0}$. Assume further that the company uses part of the surplus to pay dividends to the shareholders with rates in a set $[0, \bar{c}]$, where $\bar{c} > 0$ is the maximum dividend rate possible. Let D_t denote the rate at which the company pays dividends at time t ; then the controlled surplus process can be written as

$$X_t^D = X_t - \int_0^t D_s ds.$$

We assume that the process $(D_t)_{t \geq 0}$ is progressively measurable. Thus $\int_0^\cdot D_s ds$ and correspondingly X^D are continuous adapted processes. Note that since $\mathcal{F}_t = \mathcal{F}_{t-}$, any adapted process is predictable.

It is a classical problem in risk theory to find a dividend strategy $D = (D_t)_{t \geq 0}$ that maximises the *reward function* given by the expected sum of discounted dividend payments,

$$J(x; D) = \mathbb{E} \left[\int_0^\tau e^{-qs} D_s ds \right], \quad (1.2)$$

over a set of admissible candidate strategies. Here $q > 0$ is a discount factor and $\tau = \inf\{t \geq 0 : X_t^D < 0\} = \tau_x$ is the ruin time of the controlled process. De Finetti [15] was the first to consider a problem of this kind for a simple random walk, and Gerber [19, 20] considered extensions, including the diffusion setup (1.1) given above; see also Shreve et al. [29]. For a finite maximum dividend rate \bar{c} , this problem was then further investigated by Jeanblanc and Shiryaev [23], Radner and Shepp [27], Asmussen and Taksar [7] and Gerber and Shiu [21]. Since then, a lot of variants of this problem for the process (1.1) and more general underlying risk processes have been considered; see e.g. the surveys Albrecher and Thonhauser [4] and Avanzi [8].

For the diffusion model (1.1), we recently studied in Albrecher et al. [1] this optimal dividend problem under a ratcheting constraint, i.e., under the assumption that the dividend rate can never be decreased over the lifetime of the process, which renders the respective control problem two-dimensional, where the first dimension is the current surplus and the second dimension is the currently employed dividend rate. One motivation to consider that constraint was that it may be psychologically preferable for shareholders to not experience a decrease of dividend payments, and it is interesting to see to what extent such a constraint leads to an overall performance loss.

In this paper, we go one step further and allow reductions of the dividend rate over time, but only up to a certain percentage a of the largest already exercised dividend rate (“drawdown”). More formally, a *dividend drawdown strategy* $D = (D_t)_{t \geq 0}$ with drawdown constraint $a \in [0, 1]$ is one that satisfies $D_t \in [aR_t, \bar{c}]$, where R_t is the running maximum of the dividend rate, that is,

$$R_t := \max\{D_s : 0 \leq s \leq t\} \vee c;$$

here we denote the initial dividend rate by $R_{0-} = c$. A dividend drawdown strategy is called *admissible* if it is progressive with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$.

Define $\Pi_{x,c,a}^{[0,\bar{c}]}$ as the set of all admissible dividend drawdown strategies with initial surplus $x \geq 0$, initial running maximum dividend rate $c \in [0, \bar{c}]$ and drawdown constraint $a \in [0, 1]$. Given $D \in \Pi_{x,c,a}^{[0,\bar{c}]}$, the reward function of this strategy is given by (1.2). Hence, for any triple (x, c, a) , our aim in this paper is to maximise

$$V_a^{\bar{c}}(x, c) = \sup_{D \in \Pi_{x,c,a}^{[0,\bar{c}]}} J(x; D). \tag{1.3}$$

Note that the limit case $a = 1$ corresponds to the ratcheting case (considered previously in Albrecher et al. [1]), and the limit case $a = 0$ corresponds to the optimisation of bounded dividend rates without any drawdown constraint.

Drawdown phenomena have been studied in various contexts in the literature. On the one hand, drawdown times and properties of uncontrolled stochastic processes were investigated in quite some generality (see for instance Landriault et al. [25] for the case of Lévy processes). In the context of control problems, drawdown constraints on the wealth have been considered in portfolio problems in the mathematical finance literature; see for instance Elie and Touzi [18], Chen et al. [14] and Kardaras et al. [24]. For a minimisation of drawdown times of a risk process through dynamic reinsurance, see Brinker [11] and Brinker and Schmidli [12]. Our context, however, is different, as we are interested in implementing a drawdown constraint on the payment structure of the dividend rates, i.e., as a constraint on the admissible dividend policies. In that sense, our approach is more closely related to problems of lifetime consumption in the mathematical finance literature; see Angoshtari et al. [5] who extend Dusenberry’s ratcheting problem of consumption studied by Dybvig [16] to drawdown constraints. However, the concrete model setup and embedding, and also the involved techniques there, are very different from dividend problems of the De Finetti type as studied in this paper.

After deriving some basic analytic properties of the value function $V_a^{\bar{c}}(x, c)$ of our drawdown problem in Sect. 3, we derive a Hamilton–Jacobi–Bellman equation

for $V_a^{\bar{c}}(x, c)$ in Sect. 4 and show that $V_a^{\bar{c}}(x, c)$ is its unique viscosity solution with suitable boundary conditions. We then, in Sect. 5, briefly study in more detail the value function when one already starts at the maximal dividend rate \bar{c} , which serves as a crucial ingredient for the analysis of $V_a^{\bar{c}}(x, c)$ in Sect. 6. Sufficient conditions are given under which the optimal strategy for bounded dividend rates is a two-curve strategy in the space $(0, \infty) \times [0, \bar{c}]$ for (x, c) , which is partitioned by two curves $\gamma^{\bar{c}}(c)$ and $\zeta^{\bar{c}}(c)$ with $\gamma^{\bar{c}}(c) < \zeta^{\bar{c}}(c)$ for all $c \in [0, \bar{c}]$. If for a given c , we have $x < \gamma^{\bar{c}}(c)$, then dividends are paid at rate ac ; if $\gamma^{\bar{c}}(c) \leq x \leq \zeta^{\bar{c}}(c)$, then dividends are paid at rate c ; finally, if $x > \zeta^{\bar{c}}(c)$, then the dividend rate c is increased immediately until $x = \zeta^{\bar{c}}(c_1)$ for some $c_1 \in (c, \bar{c})$ (or $c = \bar{c}$, whichever happens first) is reached. We furthermore establish a smooth-pasting principle for these optimal curves. In Sect. 7, it is shown that the limits of $\gamma^{\bar{c}}(\bar{c})$ and $\zeta^{\bar{c}}(\bar{c})$ as $\bar{c} \rightarrow \infty$ are finite and given by the surprisingly explicit formulas

$$\lim_{\bar{c} \rightarrow \infty} \gamma^{\bar{c}}(\bar{c}) = \frac{\mu}{q} \quad \text{and} \quad \lim_{\bar{c} \rightarrow \infty} \zeta^{\bar{c}}(\bar{c}) = \frac{\mu}{q} \left(1 + \frac{1}{\sqrt{a}} \right). \tag{1.4}$$

This nicely extends the respective limit $2\mu/q$ of the ratcheting curve that was identified for pure ratcheting ($a = 1$) in [1, Lemma 5.21]. In Sect. 8, we then look further into the limiting case and show that for sufficiently large \bar{c} , one has $\gamma^{\bar{c}}(c) \nearrow \gamma^{\bar{c}}(\bar{c})$ and $\zeta^{\bar{c}}(c) \searrow \zeta^{\bar{c}}(\bar{c})$ as $c \rightarrow \bar{c}$. This enables us to establish the general optimality of two-curve strategies whenever the current dividend rate c and the maximal dividend rate \bar{c} are sufficiently large. At the same time, the negative derivative of $\zeta^{\bar{c}}(c)$ close to (sufficiently large) \bar{c} is notably different from the pure ratcheting case ($a = 1$), for which it was shown in [1] that the corresponding derivative is positive for all c close to \bar{c} (and indeed the leading term in the asymptotics of $0 < a < 1$ breaks down for $a = 1$ so that some sort of phase transition happens). The simplicity of the right-hand limit in (1.4) and in particular the appearance of the square-root of the drawdown coefficient a in the right-hand limit are somewhat intriguing. In the absence of an upper limit for the dividend rate, it identifies the minimum surplus level x above which, for sufficiently large current dividend rate, it is preferable to pay out all the surplus x immediately and generate ruin by doing so (a so-called “take-the-money-and-run” strategy, see e.g. Loeffen and Renaud [26]), and that value does not depend on the size of the volatility σ . Consequently, one can get some intuition on this result in the much simpler deterministic model with $\sigma = 0$, which we therefore consider in Sect. 2 before approaching the general case $\sigma > 0$ in the rest of the paper. We give numerical illustrations in Sect. 9, where we establish the optimality of two-curve strategies also for smaller magnitudes of c and \bar{c} by numerically showing that the sufficient conditions from Sect. 6 are satisfied. We obtain the optimal curves by calculus of variation techniques and discuss the properties of the value functions of the drawdown dividend problem and their comparison to classical and ratcheting solutions for various parameter combinations. Finally, Sect. 10 concludes. Appendix A contains the proofs of the results of Sect. 6 together with some auxiliary lemmas, and Appendix B collects some longer formulas appearing in Sects. 7 and 8 in compact form.

2 Some intuition from the deterministic case

Assume **in this section** for simplicity a completely deterministic model

$$X_t = x + \mu t$$

with a positive drift $\mu > 0$ (for the study of such a model in another context in the dividend literature, see e.g. Eisenberg et al. [17]). Then a constant dividend rate $\bar{c} = \mu$ throughout time keeps the surplus at level x for all $t \geq 0$ and correspondingly

$$\mathbb{E} \left[\int_0^\tau e^{-qs} D_s ds \right] = \mathbb{E} \left[\int_0^\infty e^{-qs} \mu ds \right] = \frac{\mu}{q}$$

for any $x > 0$. Consequently, whenever the initial surplus x is larger than μ/q , paying out all the surplus at the beginning (causing immediate ruin) is preferable to any other dividend strategy subject to the constraint $\bar{c} \leq \mu$. At the same time, if a constant dividend rate $\bar{c} > \mu$ is applied, the controlled process leads to ruin at the time $t = x/(\bar{c} - \mu)$ and we obtain instead

$$\begin{aligned} \mathbb{E} \left[\int_0^\tau e^{-qs} D_s ds \right] &= \mathbb{E} \left[\int_0^{x/(\bar{c}-\mu)} e^{-qs} \bar{c} ds \right] \\ &= \frac{\bar{c}}{q} (1 - e^{-q x/(\bar{c}-\mu)}) = x + x \frac{2\mu - qx}{2\bar{c}} + O\left(\frac{1}{\bar{c}^2}\right). \end{aligned}$$

The latter shows that whenever $x > 2\mu/q$, if allowed to do so, paying out all the surplus x immediately (and causing immediate ruin) is preferable to any other constant dividend strategy with large \bar{c} . In other words, the potential gain from later ruin and therefore more dividend income (by exploiting the positive drift, without any risk) is outweighed by the discounting of such later dividend payments. This can also be seen as an intuitive explanation of the limit $2\mu/q$ in Albrecher et al. [1, Lemma 5.21].

Let us now proceed to the case with drawdown. Assume that we start with initial capital $x > b$ for some b to be determined and that we pay dividends at rate $\bar{c} > \mu$ until we reach that lower level b at time $t = (x - b)/(\bar{c} - \mu)$, from which time on we reduce the dividend payments to level $a\bar{c}$ according to our drawdown constraint. In the deterministic model of this section, this then leads to

$$\mathbb{E} \left[\int_0^\tau e^{-qs} D_s ds \right] = \frac{\bar{c}}{q} (1 - e^{-q \frac{x-b}{\bar{c}-\mu}}) + e^{-q \frac{x-b}{\bar{c}-\mu}} \frac{a\bar{c}}{q} (1 - e^{-q \frac{b}{a\bar{c}-\mu}}). \tag{2.1}$$

Taking the derivative with respect to b and setting it to zero gives, after simple calculations, for large \bar{c} the optimal level

$$b^*(\bar{c}) = \frac{a\bar{c} - \mu}{q} \log \frac{a\bar{c}}{a\bar{c} - \mu} = \frac{\mu}{q} - \frac{\mu^2}{2aq\bar{c}} + O\left(\frac{1}{\bar{c}^2}\right).$$

But if one substitutes that value of b into (2.1), an expansion at $\bar{c} = \infty$ gives

$$\mathbb{E} \left[\int_0^\tau e^{-qs} D_s ds \right] = x + \frac{2axq\mu - ax^2q^2 + \mu^2(1 - a)}{2aq\bar{c}} + O\left(\frac{1}{\bar{c}^2}\right). \tag{2.2}$$

The numerator in the second term is negative exactly when

$$x > \frac{\mu}{q} \left(1 + \frac{1}{\sqrt{a}} \right),$$

so that in those cases, it is preferable to immediately pay x as a lump sum dividend and go to ruin immediately (if that is allowed) rather than following the above refracting strategy, as the value x cannot be realised at any later point in time in view of the discounting, despite the continuing deterministic income with drift μ . One may expect that the size of the volatility does not matter when $\bar{c} \rightarrow \infty$, and indeed, as a by-product of the results of this paper, it is shown in Sect. 7 that the same result can be established for the general case $\sigma > 0$; cf. Proposition 7.3. Another way to state this is the following: If one defines $x^*(\bar{c})$ as the surplus value for which, when already currently paying the maximum dividend rate \bar{c} , one is indifferent whether to further increase \bar{c} or not, the above result establishes that $\lim_{\bar{c} \rightarrow \infty} x^*(\bar{c}) = \frac{\mu}{q}(1 + 1/\sqrt{a})$, and it is in terms of that notation that the more general result is proved in Sect. 7.

3 Basic results

Recall the definition (1.3) of our value function $V_a^{\bar{c}}(x, c)$ and denote by $V_a^\infty(x, c)$ the corresponding function when there is no ceiling on dividend rates, i.e., $\bar{c} = \infty$. It is immediate to see that $V_a^{\bar{c}}(0, c) = 0$ for all $c \in [0, \bar{c}]$ and $a \in [0, 1]$.

Remark 3.1 As mentioned in the introduction, the dividend optimisation problem without drawdown constraint has a long history; see e.g. Schmidli [28, Sect. 2.4]. Unlike the drawdown optimisation problem, the problem without the drawdown constraint is one-dimensional. If we denote its value function by $\bar{V}^{\bar{c}}(x)$, then clearly $V_0^{\bar{c}}(x, c) = \bar{V}^{\bar{c}}(x)$ and $V_a^{\bar{c}}(x, c) \leq \bar{V}^{\bar{c}}(x)$ for all $x \geq 0$, $a \in [0, 1]$ and $c \in [0, \bar{c}]$. The function $\bar{V}^{\bar{c}}$ is increasing, concave, twice continuously differentiable with $\bar{V}^{\bar{c}}(0) = 0$ and $\lim_{x \rightarrow \infty} \bar{V}^{\bar{c}}(x) = \bar{c}/q$; so it is Lipschitz with Lipschitz constant $(\bar{V}^{\bar{c}})'(0)$.

Remark 3.2 The dividend optimisation problem without any constraint was addressed by Gerber and Shiu [21] and Schmidli [28, Sect. 2.4]. If $\bar{V}(x)$ denotes its value function, we have $\bar{V}(x) = V_0^\infty(x, c)$ for any $c > 0$. Clearly, $V_a^\infty(x, c) \leq \bar{V}(x)$ for all $a \in [0, 1]$. The function \bar{V} is increasing, concave, twice continuously differentiable with $\bar{V}(0) = 0$ and $x \leq \bar{V}(x) \leq x + \mu/q$; so it is Lipschitz with Lipschitz constant $\bar{V}'(0)$.

Proposition 3.3 *It holds that $V_a^{\bar{c}}(x, c) \nearrow V_a^\infty(x, c)$ as $\bar{c} \rightarrow \infty$.*

Proof It is straightforward that for any $\bar{c}_1 \leq \bar{c}_2$, $V_a^{\bar{c}_1}(x, c) \leq V_a^{\bar{c}_2}(x, c) \leq V_a^\infty(x, c)$ for $0 \leq c \leq \bar{c}_1$. For any $\varepsilon > 0$, take a strategy $D = (D_t)_{t \geq 0} \in \Pi_{x,c,a}^{[0,\infty)}$ with ruin time τ such that $V_a^\infty(x, c) \leq J(x; D) + \varepsilon$. For an increasing sequence $c_n \rightarrow \infty$ with $c_1 > c$,

consider $D^n = (D_t \wedge c_n)_{t \geq 0} \in \Pi_{x,c,a}^{[0,c_n]}$ and let $\tau^n \geq \tau$ be the ruin time of D^n . Then by monotone convergence,

$$\lim_{n \rightarrow \infty} J(x; D_t^n) = \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^{\tau^n} e^{-qs} D_s^n ds \right] \geq \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^{\tau} e^{-qs} D_s^n ds \right] = J(x; D),$$

and so we have the result. □

We now state a straightforward result regarding the boundedness and monotonicity of the value functions.

Proposition 3.4 *In the case $\bar{c} < \infty$, the value function $V_a^{\bar{c}}(x, c)$ is bounded by \bar{c}/q with $\lim_{x \rightarrow \infty} V_a^{\bar{c}}(x) = \bar{c}/q$, nondecreasing in x and nonincreasing in c .*

Proof By Remark 3.1 and Albrecher et al. [1, Theorem 3.3], we have

$$V_1^{\bar{c}}(x, c) \leq V_a^{\bar{c}}(x, c) \leq \bar{V}^{\bar{c}}(x)$$

with $\lim_{x \rightarrow \infty} V_1^{\bar{c}}(x, c) = \lim_{x \rightarrow \infty} \bar{V}^{\bar{c}}(x) = \bar{c}/q$. So $V_a^{\bar{c}}$ is bounded by \bar{c}/q with $\lim_{x \rightarrow \infty} V_a^{\bar{c}}(x, c) = \bar{c}/q$.

On the one hand, $V_a^{\bar{c}}(x, c)$ is nonincreasing in c because given $c_1 < c_2 \leq \bar{c}$, we have $\Pi_{x,c_2,a}^{[0,\bar{c}]} \subseteq \Pi_{x,c_1,a}^{[0,\bar{c}]}$ for any $x \geq 0$. On the other hand, given $0 \leq x_1 < x_2$ and an admissible strategy $D_1 \in \Pi_{x_1,c,a}^{[0,\bar{c}]}$ for any $c \in [0, \bar{c}]$, define $D_2 \in \Pi_{x_2,c,a}^{[0,\bar{c}]}$ as $D_{2,t} = D_{1,t}$ until the ruin time of the controlled process X^{D_1} with $X_0^{D_1} = x_1$, and pay the maximum rate \bar{c} afterwards. Thus $J(x_1; D_1) \leq J(x_2; D_2)$ and we have the result. □

Proposition 3.5 *The function $V_a^\infty(x, c)$ is nondecreasing in x and nonincreasing in c . For the case $a > 0$, we have $\lim_{c \rightarrow \infty} V_a^\infty(x, c) = x$. Moreover, $x \leq V_a^\infty(x, c) \leq x + \mu/q$.*

Proof By Propositions 3.3 and 3.4, $V_a^\infty(x, c)$ is nondecreasing in x and nonincreasing in c . Let us now show that $V_a^\infty(x, c) \geq x$. The function $V_a^\infty(x, c)$ is bounded from below by the reward function resulting from the strategy of paying a constant rate n up to ruin. Defining $\tau_n = \inf\{t : x + (\mu - n)t + \sigma W_t = 0\}$, one gets

$$\begin{aligned} V_a^\infty(x, c) &= \lim_{n \rightarrow \infty} V_a^n(x, c) \geq \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^{\tau_n} e^{-qs} n ds \right] \\ &= \lim_{n \rightarrow \infty} \frac{n}{q} (1 - \mathbb{E}[e^{-q\tau_n}]) = x, \end{aligned}$$

where the last equality follows from Borodin and Salminen [10, Equation 2.0.1].

Finally, let us argue that $\lim_{c \rightarrow \infty} V_a^\infty(x, c) \leq x$. For any $\varepsilon > 0$ and for each c , take $D^c = (D_t^c)_{t \geq 0} \in \Pi_{x,c,a}^{[0,\infty)}$ such that

$$V_a^\infty(x, c) \leq J(x; D^c) + \varepsilon.$$

Then $D^c \geq ac$ and the corresponding ruin time is given by

$$\tau^c = \inf \left\{ t : x + \mu t + \sigma W_t - \int_0^t D_s^c ds = 0 \right\}$$

so that

$$\int_0^{\tau^c} D_s^c ds = x + \mu \tau^c + \sigma W_{\tau^c},$$

hence

$$\tau^c \leq \inf \{ s : x + (\mu - ac)s + \sigma W_s \leq 0 \} = \inf \left\{ s : W_s \leq \frac{-x + (ac - \mu)s}{\sigma} \right\}.$$

This yields for $c > \mu/a$ that $\tau^c < \infty$ a.s. and $\mathbb{E}[\tau^c] \rightarrow 0$ as $c \rightarrow \infty$. Therefore,

$$\begin{aligned} \lim_{c \rightarrow \infty} \mathbb{E} \left[\int_0^{\tau^c} e^{-qs} D_s^c ds \right] &\leq \lim_{c \rightarrow \infty} \mathbb{E} \left[\int_0^{\tau^c} D_s^c ds \right] \\ &= \lim_{c \rightarrow \infty} \mathbb{E}[x + \mu \tau^c + \sigma W_{\tau^c}] \\ &= x + \mu \lim_{c \rightarrow \infty} \mathbb{E}[\tau^c] = x, \end{aligned}$$

and so we have the result. □

The Lipschitz property of the function \bar{V} can now be used to prove a global Lipschitz result on the regularity of the value function.

Proposition 3.6 *In both the restricted case $\bar{c} < \infty$ and the unrestricted case $\bar{c} = \infty$, we have*

$$0 \leq V_a^{\bar{c}}(x_2, c_1) - V_a^{\bar{c}}(x_1, c_2) \leq K((x_2 - x_1) + (c_2 - c_1))$$

for all $0 \leq x_1 \leq x_2$ and $c_1, c_2 \in [0, \bar{c}]$ with $c_1 \leq c_2$, with $K = \max\{\frac{\varepsilon^{-1}}{q}a, 1\} \bar{V}'(0)$.

Proof In the case $\bar{c} < \infty$, by Proposition 3.4, we have

$$0 \leq V_a^{\bar{c}}(x_2, c_1) - V_a^{\bar{c}}(x_1, c_2) \tag{3.1}$$

for all $0 \leq x_1 \leq x_2$ and $c_1, c_2 \in [0, \bar{c}]$ with $c_1 \leq c_2$. Let us now show that there exists $K_1 > 0$ such that

$$V_a^{\bar{c}}(x_2, c) - V_a^{\bar{c}}(x_1, c) \leq K_1(x_2 - x_1) \tag{3.2}$$

for all $0 \leq x_1 \leq x_2$. Take $\varepsilon > 0$ and $D \in \Pi_{x_2, c, a}^{[0, \bar{c}]}$ such that

$$J(x_2; D) \geq V_a^{\bar{c}}(x_2, c) - \varepsilon.$$

The associated control process is given by

$$X_t^D = x_2 + \int_0^t (\mu - D_s) ds + \sigma W_t.$$

Let τ be the ruin time of the process X^D . Define $\tilde{D} \in \Pi_{x_1, c, a}^{[0, \bar{c}]}$ as $\tilde{D}_t = D_t$ and the associated control process

$$X_t^{\tilde{D}} = x_1 + \int_0^t (\mu - D_s) ds + \sigma W_t.$$

Let $\tilde{\tau} \leq \tau$ be the ruin time of the process $X^{\tilde{D}}$; then $X_t^D - X_t^{\tilde{D}} = x_2 - x_1$ for $t \leq \tilde{\tau}$. We can write

$$\begin{aligned} J(x_2; D) - J(x_1; \tilde{D}) &= \mathbb{E} \left[\int_{\tilde{\tau}}^{\tau} e^{-qs} D_s ds \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\int_{\tilde{\tau}}^{\tau} e^{-qs} D_s ds \mid \mathcal{F}_{\tilde{\tau}} \right] \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[e^{-q\tilde{\tau}} \int_0^{\tau - \tilde{\tau}} e^{-qu} D_{\tilde{\tau}+u} du \mid \mathcal{F}_{\tilde{\tau}} \right] \right] \\ &\leq \mathbb{E} \left[\mathbb{E} \left[\int_0^{\tau - \tilde{\tau}} e^{-qu} D_{\tilde{\tau}+u} du \mid \mathcal{F}_{\tilde{\tau}} \right] \right] \\ &\leq V_a^{\bar{c}}(x_2 - x_1, 0). \end{aligned} \tag{3.3}$$

The last inequality of (3.3) involves a shift of stopping times and follows from Claisse et al. [13, Theorem 2]. Indeed, the assumptions of this theorem are satisfied because we can write our controlled process as

$$dX_s = b(s, X, D_s) ds + \sigma(s, X, D_s) dW_s,$$

where $b(s, x, d) = \mu - d$, $\sigma(s, x, d) \equiv \sigma$ and W is a standard Brownian motion. Hence we have

$$\begin{aligned} V_a^{\bar{c}}(x_2, c) - V_a^{\bar{c}}(x_1, c) &\leq J(x_2; D) - J(x_1; \tilde{D}) + \varepsilon \\ &\leq V^{\bar{c}}(x_2 - x_1, 0) + \varepsilon \\ &\leq \bar{V}(x_2 - x_1) + \varepsilon \\ &\leq K_1(x_2 - x_1) + \varepsilon. \end{aligned}$$

So by Remark 3.2, we have (3.2) with $K_1 = \bar{V}'(0)$.

Let us now show that given $c_1, c_2 \in [0, \bar{c}]$ with $c_1 \leq c_2$, there exists $K_2 > 0$ with

$$V_a^{\bar{c}}(x, c_1) - V_a^{\bar{c}}(x, c_2) \leq K_2(c_2 - c_1). \tag{3.4}$$

Take $\varepsilon > 0$ and $D \in \Pi_{x,c_1,a}^{[0,\bar{c}]}$ such that

$$J(x; D) \geq V_a^{\bar{c}}(x, c_1) - \varepsilon$$

and denote by τ the ruin time of the process X^D . Let us consider $\tilde{D} \in \Pi_{x,c_2,a}^{[0,\bar{c}]}$ as $\tilde{D}_t = \max\{D_t, ac_2\}$, denote by $X^{\tilde{D}}$ the associated controlled surplus process and by $\bar{\tau} \leq \tau$ the corresponding ruin time. We have $\tilde{D}_s - D_s \leq ac_2 - ac_1$ and so $X_{\bar{\tau}}^D = X_{\bar{\tau}}^{\tilde{D}} - X_{\bar{\tau}}^{\tilde{D}} \leq a(c_2 - c_1)\bar{\tau}$. By Remark 3.2, we have

$$\begin{aligned} \mathbb{E}\left[\int_{\bar{\tau}}^{\tau} D_s e^{-qs} ds\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{-q\bar{\tau}} \int_{\bar{\tau}}^{\tau} D_s e^{-q(s-\bar{\tau})} ds \mid \mathcal{F}_{\bar{\tau}}\right]\right] \\ &\leq \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau-\bar{\tau}} D_{u+\bar{\tau}} e^{-qu} du \mid \mathcal{F}_{\bar{\tau}}\right]\right] \\ &\leq \mathbb{E}[V_a^{\bar{c}}(X_{\bar{\tau}}^D, 0)]. \end{aligned}$$

As before, the last inequality involves a shift of stopping times and follows from Claisse et al. [13, Theorem 2]. Then

$$\mathbb{E}\left[\int_{\bar{\tau}}^{\tau} D_s e^{-qs} ds\right] \leq \mathbb{E}[\bar{V}(X_{\bar{\tau}}^D)] \leq \mathbb{E}[\bar{V}((c_2 - c_1)\bar{\tau})] \leq K_1 \mathbb{E}[e^{-q\bar{\tau}} \bar{\tau}(c_2 - c_1)].$$

Hence we can write

$$\begin{aligned} V_a^{\bar{c}}(x, c_1) - V_a^{\bar{c}}(x, c_2) &\leq J(x; D) + \varepsilon - J(x; \tilde{D}) \\ &= \mathbb{E}\left[\int_0^{\bar{\tau}} (D_s - \tilde{D}_s) e^{-qs} ds\right] + \mathbb{E}\left[\int_{\bar{\tau}}^{\tau} D_s e^{-qs} ds\right] + \varepsilon \\ &\leq 0 + \mathbb{E}\left[\int_{\bar{\tau}}^{\tau} D_s e^{-qs} ds\right] + \varepsilon \\ &\leq K_1 E[ae^{-q\bar{\tau}} \bar{\tau}(c_2 - c_1)] + \varepsilon \\ &\leq K_2(c_2 - c_1) + \varepsilon. \end{aligned}$$

So we get (3.4) with $K_2 = K_1 \max_{t \geq 0}(e^{-qt} ta) = K_1 \frac{e^{-1}}{q} a$, $K = K_1 \max\{\frac{e^{-1}}{q} a, 1\}$. We conclude the result from (3.1), (3.2) and (3.4).

In the case $\bar{c} = \infty$, the result follows from Proposition 3.3. □

The following lemma states the dynamic programming principle. Its proof is similar to the one of Azcue and Muler [9, Lemma 1.2].

Lemma 3.7 *Given any stopping time $\tilde{\tau}$, we can write in both the restricted case $\bar{c} < \infty$ and the unrestricted case $\bar{c} = \infty$ that*

$$V_a^{\bar{c}}(x, c) = \sup_{D \in \Pi_{x,c,a}^{[0,\bar{c}]}} \mathbb{E}\left[\int_0^{\tau \wedge \tilde{\tau}} e^{-qs} D_s ds + e^{-q(\tau \wedge \tilde{\tau})} V_a^{\bar{c}}(X_{\tau \wedge \tilde{\tau}}^D, R_{\tau \wedge \tilde{\tau}})\right].$$

We now show a Lipschitz property of $h(a) = V_a^{\bar{c}}(x, c)$ in the drawdown constant $a \in [0, 1]$, for fixed x, c and finite \bar{c} .

Proposition 3.8 *Given $\bar{c} < \infty$ and $a_1, a_2 \in [0, 1]$ with $a_1 < a_2$, there exists $K_3 > 0$ such that*

$$0 \leq V_{a_1}^{\bar{c}}(x, c) - V_{a_2}^{\bar{c}}(x, c) \leq K_3(a_2 - a_1),$$

with $K_3 = \bar{V}'(0) \frac{e^{-1}}{q} \bar{c}$ only depending on \bar{c} . In the case $\bar{c} = \infty$, $V_a^\infty(x, c)$ is continuous in $a \in [0, 1]$.

Proof Consider first the case $\bar{c} < \infty$. Take $\varepsilon > 0$ and $D \in \Pi_{x,c,a_1}^{[0,\bar{c}]}$ such that

$$J(x; D) \geq V_{a_1}^{\bar{c}}(x, c) - \varepsilon.$$

Consider $\tilde{D} \in \Pi_{x,c,a_2}^{[0,\bar{c}]}$ defined as $\tilde{D}_t = \max\{D_t, a_2 R_t\}$. Denote by $X^{\tilde{D}}$ the associated controlled surplus process and by $\bar{\tau} \leq \tau$ the corresponding ruin time. We have $0 \leq \tilde{D}_s - D_s \leq (a_2 - a_1)R_s$ and so

$$X_{\bar{\tau}}^D = X_{\bar{\tau}}^{\tilde{D}} - X_{\bar{\tau}}^{\tilde{D}} \leq \int_0^{\bar{\tau}} (a_2 - a_1)R_s ds = (a_2 - a_1)\bar{\tau}\bar{c}.$$

We can write

$$\begin{aligned} V_{a_1}^{\bar{c}}(x, c) - V_{a_2}^{\bar{c}}(x, c) &= J(x; D) - J(x; \tilde{D}) + \varepsilon \\ &= \mathbb{E} \left[\int_0^{\bar{\tau}} e^{-qs} (D_s - \tilde{D}_s) ds \right] + \mathbb{E} \left[\int_{\bar{\tau}}^\tau e^{-qs} D_s ds \right] + \varepsilon \\ &\leq 0 + \mathbb{E} \left[\mathbb{E} \left[\int_{\bar{\tau}}^\tau e^{-qs} D_s ds \mid \mathcal{F}_{\bar{\tau}} \right] \right] + \varepsilon \\ &= \mathbb{E} \left[\mathbb{E} \left[e^{-q\bar{\tau}} \int_0^{\tau - \bar{\tau}} e^{-qu} D_{\bar{\tau}+u} du \mid \mathcal{F}_{\bar{\tau}} \right] \right] + \varepsilon \\ &\leq \mathbb{E} \left[e^{-q\bar{\tau}} \bar{V}((a_2 - a_1)\bar{\tau}\bar{c}) \right] + \varepsilon \\ &\leq \mathbb{E} \left[e^{-q\bar{\tau}} \bar{V}'(0) ((a_2 - a_1)\bar{\tau}\bar{c}) \right] + \varepsilon \\ &\leq \bar{V}'(0) \frac{e^{-1}}{q} \bar{c} (a_2 - a_1) + \varepsilon, \end{aligned}$$

and we obtain the result with $K_3 = \bar{V}'(0) \frac{e^{-1}}{q} \bar{c}$.

In the case $\bar{c} = \infty$, we want to show that given $\varepsilon > 0$ and $a_1 \geq 0$, there exists $\delta > 0$ such that if $0 < a_2 - a_1 < \delta$, then $V_{a_1}^\infty(x, c) - V_{a_2}^\infty(x, c) < \varepsilon$. Take \bar{c}_0 large enough such that $V_{a_1}^\infty(x, c) - V_{a_1}^{\bar{c}_0}(x, c) < \varepsilon/2$ and $\delta = \varepsilon / (2\bar{V}'(0) \frac{e^{-1}}{q} \bar{c}_0)$. Given any

$a_2 \in (a_1, a_1 + \delta)$, we have

$$\begin{aligned} V_{a_1}^\infty(x, c) - V_{a_2}^\infty(x, c) &= V_{a_1}^\infty(x, c) - V_{a_1}^{\bar{c}_0}(x, c) + V_{a_1}^{\bar{c}_0}(x, c) \\ &\quad - V_{a_2}^{\bar{c}_0}(x, c) + V_{a_2}^{\bar{c}_0}(x, c) - V_{a_2}^\infty(x, c) \\ &\leq \varepsilon/2 + \bar{V}'(0) \frac{e^{-1}}{q} \bar{c}_0(a_2 - a_1) + 0 \leq \varepsilon. \end{aligned} \quad \square$$

Remark 3.9 For $a = 0$, Proposition 3.5 does not hold. Indeed, $V_0^\infty(x, c) = \bar{V}(x)$ so that $\lim_{c \rightarrow \infty} V_0^\infty(x, c) = \bar{V}(x) > x$. Although $\lim_{c \rightarrow \infty} V_a^\infty(x, c) = x$ for $a \in (0, 1]$ and $\lim_{a \rightarrow 0+} V_a^\infty(x, c) = V_0^\infty(x, c)$ by Proposition 3.8, the lack of the Lipschitz property of $V_a^\infty(x, c)$ at $a = 0$ makes it possible that the iterated limits

$$\lim_{c \rightarrow \infty} \left(\lim_{a \rightarrow 0+} V_a^\infty(x, c) \right) = \bar{V}(x) \quad \text{and} \quad \lim_{a \rightarrow 0+} \left(\lim_{c \rightarrow \infty} V_a^\infty(x, c) \right) = x$$

do not coincide.

In the next result, we study the continuity of $V_a^{\bar{c}}(x, c)$ with respect to \bar{c} .

Proposition 3.10 *Given $\bar{c}_1, \bar{c}_2 \in [0, \infty)$ with $\bar{c}_1 < \bar{c}_2$, there exists $K_2 > 0$ such that*

$$0 \leq V_a^{\bar{c}_2}(x, c) - V_a^{\bar{c}_1}(x, c) \leq \frac{1}{q} (\bar{c}_2 - \bar{c}_1)$$

for $c \leq \bar{c}_1$.

Proof Take $\varepsilon > 0$ and $D \in \Pi_{x,c,a}^{[0, \bar{c}_2]}$ such that

$$J(x; D) \geq V_a^{\bar{c}_2}(x, c) - \varepsilon,$$

and denote the ruin time of the process X^D by τ . Let us consider $\tilde{D} \in \Pi_{x,c,a}^{[0, \bar{c}_1]}$ as $\tilde{D}_t = \min\{D_t, \bar{c}_1\} = \bar{c}_1 I_{\{D_t > \bar{c}_1\}} + D_t I_{\{D_t \leq \bar{c}_1\}}$ for $t \leq \tau$ and $\tilde{D}_t = \bar{c}_1$ for $t > \tau$, denote by $X^{\tilde{D}}$ the associated controlled surplus process and by $\bar{\tau} \geq \tau$ the corresponding ruin time. We then have $D_s - \tilde{D}_s \leq \bar{c}_2 - \bar{c}_1$ and can deduce

$$\begin{aligned} V_a^{\bar{c}_2}(x, c) - V_a^{\bar{c}_1}(x, c) &\leq J(x; D) + \varepsilon - J(x; \tilde{D}) \\ &= \mathbb{E} \left[\int_0^\tau (D_s - \tilde{D}_s) e^{-qs} ds \right] - \mathbb{E} \left[\int_\tau^{\bar{\tau}} D_s e^{-qs} ds \right] + \varepsilon \\ &\leq \mathbb{E} \left[\int_0^\tau (D_s - \tilde{D}_s) e^{-qs} ds \right] + \varepsilon \\ &\leq \mathbb{E} \left[\int_0^\tau (\bar{c}_2 - \bar{c}_1) e^{-qs} ds \right] + \varepsilon \\ &= \frac{(\bar{c}_2 - \bar{c}_1)}{q} \mathbb{E}[1 - e^{-q\tau}] + \varepsilon \\ &\leq \frac{(\bar{c}_2 - \bar{c}_1)}{q} + \varepsilon. \end{aligned} \quad \square$$

4 The Hamilton–Jacobi–Bellman equation

In this section, we introduce the Hamilton–Jacobi–Bellman (HJB) equation for the drawdown problem. We show that the value function V defined in (1.3) is the unique viscosity solution of the corresponding HJB equation with suitable boundary conditions.

As stated in the previous section, the limit case $a = 0$ (no drawdown restriction) has been studied for both $\bar{c} < \infty$ and $\bar{c} = \infty$, and the case $a = 1$ (ratcheting) for $\bar{c} < \infty$.

Define

$$\mathcal{L}^k(W)(x, c) := \frac{\sigma^2}{2} \partial_{xx} W(x, c) + (\mu - \kappa) \partial_x W(x, c) - qW(x, c) + \kappa.$$

The HJB equation associated to (1.3) for both $\bar{c} < \infty$ and $\bar{c} = \infty$ is given by

$$\max \left\{ \max_{\kappa \in [ac, c]} \mathcal{L}^k(u)(x, c), \partial_c u(x, c) \right\} = 0 \quad \text{for } x \geq 0 \text{ and } 0 \leq c < \bar{c}. \quad (4.1)$$

The inequality $\max_{\kappa \in [ac, c]} \mathcal{L}^k(u)(x, c) \leq 0$ comes from the dynamic programming principle in Lemma 3.7, considering the strategies of paying dividends at a constant rate $\kappa \in [ac, c]$. For these strategies, the running maximum is c . The inequality $\partial_c u(x, c) \leq 0$ comes from the fact that the value function is nonincreasing in c , and it takes into account the strategies of paying dividends at a constant rate $\kappa \in (c, \bar{c}]$, where the running maximum increases instantaneously from c to κ .

Note that because the operator \mathcal{L}^k is linear in k , an alternative equivalent formulation of (4.1) is

$$\max\{\mathcal{L}^c(u)(x, c), \mathcal{L}^{ac}(u)(x, c), \partial_c u(x, c)\} = 0 \quad \text{for } x \geq 0 \text{ and } 0 \leq c < \bar{c}. \quad (4.2)$$

For the ratcheting case $a = 1$, the HJB equation correspondingly simplifies to

$$\max\{\mathcal{L}^c(u)(x, c), \partial_c u(x, c)\} = 0 \quad \text{for } x \geq 0 \text{ and } 0 \leq c < \bar{c}.$$

Let us introduce the usual notion of viscosity solution for the HJB equation in both cases $0 < \bar{c} < \infty$ or $\bar{c} = \infty$.

Definition 4.1 (a) A locally Lipschitz function $\bar{u} : [0, \infty) \times [0, \bar{c}) \rightarrow \mathbb{R}$ is a *viscosity supersolution* of (4.2) at $(x, c) \in (0, \infty) \times [0, \bar{c})$ if any $(2, 1)$ -differentiable function $\varphi : [0, \infty) \times [0, \bar{c}) \rightarrow \mathbb{R}$ with $\varphi(x, c) = \bar{u}(x, c)$ such that $\bar{u} - \varphi$ reaches the minimum at (x, c) satisfies

$$\max\{\mathcal{L}^c(\varphi)(x, c), \mathcal{L}^{ac}(\varphi)(x, c), \partial_c \varphi(x, y)\} \leq 0.$$

The function φ is called a *test function for a supersolution* at (x, c) .

(b) Similarly, a locally Lipschitz function $\underline{u} : [0, \infty) \times [0, \bar{c}] \rightarrow \mathbb{R}$ is a *viscosity subsolution* of (4.2) at $(x, c) \in (0, \infty) \times [0, \bar{c}]$ if any (2, 1)-differentiable function $\psi : [0, \infty) \times [0, \bar{c}] \rightarrow \mathbb{R}$ with $\psi(x, c) = \underline{u}(x, c)$ such that $\underline{u} - \psi$ reaches the maximum at (x, c) satisfies

$$\max\{\mathcal{L}^c(\psi)(x, c), \mathcal{L}^{ac}(\psi)(x, c), \partial_c \psi(x, c)\} \geq 0.$$

The function ψ is called a *test function for a subsolution* at (x, c) .

(c) A locally Lipschitz function $u : [0, \infty) \times [0, \bar{c}] \rightarrow \mathbb{R}$ is a *viscosity solution* of (4.2) at $(x, c) \in [0, \infty) \times [0, \bar{c}]$ if it is both a supersolution and a subsolution at (x, c) .

4.1 HJB equation with bounded dividend rates

Given $a \in (0, 1]$ and $\bar{c} < \infty$, we define in the sequel, for simplicity of exposition,

$$\Pi_{x,c} := \Pi_{x,c,a}^{[0,\bar{c}]} \quad \text{and} \quad V := V_a^{\bar{c}}. \tag{4.3}$$

Here the state variables are the current surplus and the running maximum dividend rate. The results of this subsection for the case $a = 1$ (ratcheting dividend constraint) were already proved in Albrecher et al. [1].

The next result states that V is a viscosity solution of the HJB equation.

Proposition 4.2 *The function V is a viscosity solution of (4.2) in $(0, \infty) \times [0, \bar{c}]$.*

Proof Let us show first that V is a viscosity supersolution in $(0, \infty) \times [0, \bar{c}]$. By Proposition 3.4, $\partial_c V \leq 0$ in $(0, \infty) \times [0, \bar{c}]$ in the viscosity sense.

Consider now $(x, c) \in (0, \infty) \times [0, \bar{c}]$ and the admissible strategy $D \in \Pi_{x,c}$ which pays dividends at a constant rate $\kappa \in [ac, c]$ up to the ruin time τ . Let X^D be the corresponding controlled surplus process and suppose that there exists a test function φ for a supersolution of (4.2) at (x, c) ; then $\varphi \leq V$ and $\varphi(x, c) = V(x, c)$. We want to prove that $\mathcal{L}^\kappa(\varphi)(x, c) \leq 0$. For that purpose, we consider an auxiliary test function for the supersolution $\tilde{\varphi}$ in such a way that $\tilde{\varphi} \leq \varphi \leq V$ in $[0, \infty) \times [0, \bar{c}]$, $\tilde{\varphi} = \varphi$ in $[0, 2x]$ (so $\mathcal{L}^\kappa(\varphi)(x, c) = \mathcal{L}^\kappa(\tilde{\varphi})(x, c)$) and $\mathcal{L}^\kappa(\tilde{\varphi})(\cdot, c)$ is bounded in $[0, \infty)$. We introduce $\tilde{\varphi}$ because $\mathcal{L}^\kappa(\varphi)(\cdot, c)$ may be unbounded in $[0, \infty)$. We construct $\tilde{\varphi}$ as follows: take $g : [0, \infty) \rightarrow [0, 1]$ twice continuously differentiable with $g = 0$ in $[2x + 1, \infty)$ and $g = 1$ in $[0, 2x]$, and define $\tilde{\varphi}(y, \kappa) = \varphi(y, \kappa)g(y)$. Using Lemma 3.7, we obtain for $h > 0$,

$$\tilde{\varphi}(x, c) = V(x, c) \geq \mathbb{E} \left[\int_0^{\tau \wedge h} \kappa e^{-qs} ds \right] + \mathbb{E}[e^{-q(\tau \wedge h)} \tilde{\varphi}(X_{\tau \wedge h}^D, c)].$$

Hence using Itô’s formula gives

$$\begin{aligned}
 0 &\geq \mathbb{E} \left[\int_0^{\tau \wedge h} e^{-qs} ds \right] + \mathbb{E} [e^{-q(\tau \wedge h)} \tilde{\varphi}(X_{\tau \wedge h}^D, c) - \tilde{\varphi}(x, c)] \\
 &= \mathbb{E} \left[\int_0^{\tau \wedge h} \kappa e^{-qs} ds \right] \\
 &\quad + \mathbb{E} \left[\int_0^{\tau \wedge h} e^{-qs} \left(\frac{\sigma^2}{2} \partial_{xx} \tilde{\varphi}(X_s^D, c) + \partial_x \tilde{\varphi}(X_s^D, c)(\mu - \kappa) - q \tilde{\varphi}(X_s^D, c) \right) ds \right] \\
 &\quad + \mathbb{E} \left[\int_0^{\tau \wedge h} \partial_x \tilde{\varphi}(X_s^D, c) \sigma dW_s \right] \\
 &= \mathbb{E} \left[\int_0^{\tau \wedge h} e^{-qs} \mathcal{L}^\kappa(\tilde{\varphi})(X_s^D, c) ds \right].
 \end{aligned}$$

Since $\tau > 0$ a.s.,

$$\left| \frac{1}{h} \int_0^{\tau \wedge h} e^{-qs} \mathcal{L}^\kappa(\tilde{\varphi})(X_s^D, c) ds \right| \leq \sup_{y \in [0, \infty)} |\mathcal{L}^\kappa(\tilde{\varphi})(y, c)|$$

and

$$\lim_{h \rightarrow 0^+} \frac{1}{h} \int_0^{\tau \wedge h} e^{-qs} \mathcal{L}^\kappa(\tilde{\varphi})(X_s^D, c) ds = \mathcal{L}^\kappa(\tilde{\varphi})(x, c) \quad \text{a.s.}$$

We conclude via dominated convergence that $\mathcal{L}^\kappa(\varphi)(x, c) = \mathcal{L}^\kappa(\tilde{\varphi})(x, c) \leq 0$ for any $\kappa \in [ac, c]$; so V is a viscosity supersolution at (x, c) .

We skip the proof that V is a viscosity subsolution in $(0, \infty) \times [0, \bar{c}]$, because it is similar to that of [1, Proposition 3.1]. □

Let us consider the function

$$v^{\bar{c}} : [0, \infty) \rightarrow [0, \infty), \quad v^{\bar{c}}(x) := V(x, \bar{c}). \tag{4.4}$$

The next proposition gives a comparison result for the viscosity solutions of (4.2) for $\bar{c} > 0$. The proof is similar to that of [1, Lemma 3.2].

Lemma 4.3 *Assume that*

- (i) \underline{u} is a viscosity subsolution and \bar{u} is a viscosity supersolution of the HJB equation (4.2) for all $x > 0$ and all $c \in [0, \bar{c}]$;
- (ii) \underline{u} and \bar{u} are nondecreasing in the variable x and Lipschitz in $[0, \infty) \times [0, \bar{c}]$;
- (iii) $\underline{u}(0, c) = \bar{u}(0, c) = 0, \lim_{x \rightarrow \infty} \underline{u}(x, c) \leq \bar{c}/q \leq \lim_{x \rightarrow \infty} \bar{u}(x, c)$;
- (iv) $\underline{u}(x, \bar{c}) \leq v^{\bar{c}}(x) \leq \bar{u}(x, \bar{c})$ for $x \geq 0$.

Then $\underline{u} \leq \bar{u}$ in $[0, \infty) \times [0, \bar{c}]$.

The following characterisation theorem is a direct consequence of the previous lemma and Propositions 3.4 and 4.2.

Theorem 4.4 *The value function V is the unique function which is nondecreasing in x and a viscosity solution of (4.2) in $(0, \infty) \times [0, \bar{c}]$ with $V(0, c) = 0$, $V(x, \bar{c}) = v^{\bar{c}}(x)$ and $\lim_{x \rightarrow \infty} V(x, c) = \bar{c}/q$ for $c \in [0, \bar{c}]$.*

From Definition 1.3, Lemma 4.3 and Proposition 3.4 together with Proposition 4.2, we also get the following verification theorem.

Theorem 4.5 *Let $\{C_{x,c} \in \Pi_{x,c} : (x, c) \in [0, \infty) \times [0, \bar{c}]\}$ be a family of strategies. If the function $W(x, c) := J(x; C_{x,c})$ is a viscosity supersolution of the HJB equation (4.2) in $(0, \infty) \times [0, \bar{c}]$ with $\lim_{x \rightarrow \infty} W(x, c) = \bar{c}/q$, then W is the value function V . Also, if for each $k \geq 1$, there exists a family $\{C_{x,c}^k \in \Pi_{x,c} : (x, c) \in [0, \infty) \times [0, \bar{c}]\}$ of strategies such that $W(x, c) := \lim_{k \rightarrow \infty} J(x; C_{x,c}^k)$ is a viscosity supersolution of the HJB equation (4.2) in $(0, \infty) \times [0, \bar{c}]$ with $\lim_{x \rightarrow \infty} W(x, c) = \bar{c}/q$, then W is the value function V .*

4.2 HJB equation with unbounded dividend rates

Let us now consider the case $\bar{c} = \infty$ with $a \in (0, 1]$. Since a is fixed, we denote $V^\infty = V_a^\infty$. The proof of the following result is similar to that of the case with bounded dividend rate.

Proposition 4.6 *The function V^∞ is a viscosity solution of (4.2) for any $(x, c) \in (0, \infty) \times [0, \infty)$.*

We now state a comparison result for the unbounded case.

Lemma 4.7 *Assume that*

- (i) \underline{u} is a viscosity subsolution and \bar{u} is a viscosity supersolution of the HJB equation (4.2) for all $x > 0$ and for all $c \in [0, \infty)$;
- (ii) \underline{u} and \bar{u} are nondecreasing in the variable x and Lipschitz in $[0, \infty) \times [0, \infty)$;
- (iii) $\underline{u}(0, c) = \bar{u}(0, c) = 0$;
- (iv) $\underline{u}(x, c) \leq x + \mu/q, x \leq \bar{u}(x, c)$;
- (v) $\lim_{c \rightarrow \infty} \underline{u}(x, c) \leq x \leq \lim_{c \rightarrow \infty} \bar{u}(x, c)$ for $x \geq 0$.

Then $\underline{u} \leq \bar{u}$ in $[0, \infty) \times [0, \infty)$.

Proof Suppose there is $(x_0, c_0) \in (0, \infty) \times (0, \infty)$ with $\underline{u}(x_0, c_0) - \bar{u}(x_0, c_0) > 0$. Let us define

$$h(c) = 1 + \left(\frac{\underline{u}(x_0, c_0) - \bar{u}(x_0, c_0)}{2\bar{u}(x_0, c_0)} \right) e^{-c} > 1 \quad \text{and} \quad \bar{u}^s(x, c) = sh(c)\bar{u}(x, c)$$

for any $s > 1$. We have

$$\begin{aligned} \underline{u}(x_0, c_0) - \bar{u}^s(x_0, c_0) &= \underline{u}(x_0, c_0) - \left(1 + \frac{\underline{u}(x_0, c_0) - \bar{u}(x_0, c_0)}{2\bar{u}(x_0, c_0)} e^{-c_0} \right) s\bar{u}(x_0, c_0) \\ &= \left(1 - \frac{e^{-c_0} s}{2} \right) (\underline{u}(x_0, c_0) - s\bar{u}(x_0, c_0)) > 0 \end{aligned}$$

for $s \in (1, 2)$.

Let us now show that \bar{u}^s is a strict supersolution. We have that φ is a test function for the supersolution \bar{u} at (x, c) if and only if $\varphi^s := s h(c) \varphi$ is a test function for the supersolution \bar{u}^s at (x, c) . Moreover,

$$\mathcal{L}^\kappa(\varphi^s)(x, c) = s h(c) \mathcal{L}^\kappa(\varphi)(x, c) + \kappa(1 - s h(c)) < 0$$

for $\kappa \in [ac, c]$, and

$$\partial_c \varphi^s(x, c) \leq -s(h(c) - 1)\varphi(x, c) < 0$$

since $\varphi(x, c) = \bar{u}(x, c) \geq x > 0$.

Take $s_0 > 1$ such that $\underline{u}(x_0, c_0) - \bar{u}^{s_0}(x_0, c_0) > 0$. We define

$$M := \sup_{x \geq 0, c \geq 0} (\underline{u}(x, c) - \bar{u}^{s_0}(x, c)).$$

Let us show that

$$\arg \max_{x \geq 0, c \geq 0} (\underline{u}(x, c) - \bar{u}^{s_0}(x, c)) \in (0, b) \times (0, c_1) \tag{4.5}$$

for some positive b and c_1 . Since $\underline{u}(x, c) \leq x + \frac{\mu}{q}$ and $x \leq \bar{u}(x, c)$,

$$\underline{u}(x, c) - \bar{u}^{s_0}(x, c) \leq \left(x + \frac{\mu}{q}\right) - s_0 h(c)x < x(1 - s_0) + \frac{\mu}{q} < 0$$

for x large enough; so there exists $b > x_0$ such that

$$\arg \max_{x \geq 0, c \geq 0} (\underline{u}(x, c) - \bar{u}^{s_0}(x, c)) \in (0, b) \times (0, \infty).$$

Besides, the function

$$g(c) := \max_{x \geq 0} (\underline{u}(x, c) - \bar{u}^{s_0}(x, c)) = \max_{x \in (0, b)} (\underline{u}(x, c) - \bar{u}^{s_0}(x, c))$$

satisfies $\limsup_{c \rightarrow \infty} g(c) \leq 0$ because $\lim_{c \rightarrow \infty} \underline{u}(x, c) \leq x \leq \lim_{c \rightarrow \infty} \bar{u}(x, c)$ for $x \geq 0$. So there exists $c_1 > 0$ with $g(c) \leq \frac{M}{2}$ for $c \geq c_1$ and we deduce (4.5). Hence the maximum is attained in a bounded set, that is,

$$0 < \underline{u}(x_0, c_0) - \bar{u}^{s_0}(x_0, c_0) \leq M = \max_{x \in (0, b) \times (0, c_1)} (\underline{u}(x, c) - \bar{u}^{s_0}(x, c)).$$

This yields a contradiction by following the arguments of the proof of [1, Lemma 3.2]. □

As for bounded dividend rates, the following result is a direct consequence of the previous lemma, Remark 3.5 and Proposition 4.6.

Theorem 4.8 *The value function V^∞ is the unique function which is nondecreasing in x and a viscosity solution of (4.2) in $(0, \infty) \times [0, \infty)$ with $V^\infty(0, c) = 0$, $V^\infty(x, \bar{c}) - x$ bounded and $\lim_{c \rightarrow \infty} V^\infty(x, c) = x$.*

From Definition 1.3, Lemma 4.7 and Remark 3.5 together with Proposition 4.6, we also get the following verification theorem.

Theorem 4.9 Consider a family $\{C_{x,c} \in \Pi_{x,c} : (x, c) \in [0, \infty) \times [0, \infty)\}$ of strategies. If the function $W(x, c) := J(x; C_{x,c})$ is a viscosity supersolution of the HJB equation (4.2) in $(0, \infty) \times [0, \infty)$ with $W(x, c) \geq x$, then W is the value function V^∞ . Also, if for each $k \geq 1$, there exists a family $\{C_{x,c}^k \in \Pi_{x,c} : (x, c) \in [0, \infty) \times [0, \infty)\}$ of strategies such that $W(x, c) := \lim_{k \rightarrow \infty} J(x; C_{x,c}^k)$ is a viscosity supersolution of the HJB equation (4.2) in $(0, \infty) \times [0, \infty)$ with $W(x, c) \geq x$, then W is the value function V^∞ .

5 Refracting dividend strategies and $v^{\bar{c}}$

In the case $0 < \bar{c} < \infty$ and $a \in (0, 1)$, we now want to investigate further the function $v^{\bar{c}}$ (defined in (4.4)) of paying dividends with rates $\kappa \in [a\bar{c}, \bar{c}]$ in an optimal way. The following characterisation is the one-dimensional version of Theorem 4.4.

Proposition 5.1 The function $v^{\bar{c}} : [0, \infty) \rightarrow \mathbb{R}$ is the unique viscosity solution of

$$\max\{\mathcal{L}^{\bar{c}}(W)(x), \mathcal{L}^{a\bar{c}}(W)(x)\} = 0$$

with boundary conditions $W(0) = 0$ and $\lim_{x \rightarrow \infty} W(x) = \bar{c}/q$.

We present in this section a formula for $v^{\bar{c}}$, which turns out to be the reward function of the optimal refracting strategy as derived in Albrecher et al. [3].

The functions W that satisfy $\mathcal{L}^\kappa(W) = 0$ are given by

$$\frac{\kappa}{q} + a_1 e^{\theta_1(\kappa)x} + a_2 e^{\theta_2(\kappa)x} \quad \text{with } a_1, a_2 \in \mathbb{R}, \tag{5.1}$$

where $\theta_1(\kappa) > 0$ and $\theta_2(\kappa) < 0$ are the roots of the characteristic equation

$$\frac{\sigma^2}{2} z^2 + (\mu - \kappa)z - q = 0$$

associated to the operator \mathcal{L}^κ , that is,

$$\theta_{1,2}(\kappa) := \frac{\kappa - \mu \pm \sqrt{(\kappa - \mu)^2 + 2q\sigma^2}}{\sigma^2}. \tag{5.2}$$

Basic properties of $\theta_1(\kappa)$ and $\theta_2(\kappa)$ are that

- 1) $\theta_1(\kappa) = -\theta_2(\kappa)$ if $\kappa = \mu$, and $\theta_1^2(\kappa) \geq \theta_2^2(\kappa)$ if and only if $\kappa - \mu \geq 0$;
- 2) $\theta_1'(\kappa) = \frac{1}{\sigma^2} \left(1 + \frac{\kappa - \mu}{\sqrt{(\kappa - \mu)^2 + 2q\sigma^2}}\right)$ and $\theta_2'(\kappa) = \frac{1}{\sigma^2} \left(1 - \frac{\kappa - \mu}{\sqrt{(\kappa - \mu)^2 + 2q\sigma^2}}\right)$ so that $\theta_1'(\kappa), \theta_2'(\kappa) \in (0, \frac{2}{\sigma^2})$.

The solutions of $\mathcal{L}^\kappa(W) = 0$ with boundary condition $W(0) = 0$ are of the more specific form

$$\frac{\kappa}{q}(1 - e^{\theta_2(\kappa)x}) + A(e^{\theta_1(\kappa)x} - e^{\theta_2(\kappa)x}) \quad \text{with } A \in \mathbb{R}. \tag{5.3}$$

Finally, the unique solution of $\mathcal{L}^\kappa(W) = 0$ with boundary conditions $W(0) = 0$ and $\lim_{x \rightarrow \infty} W(x) = \kappa/q$ corresponds to $A = 0$, so that

$$W(x) = \frac{\kappa}{q}(1 - e^{\theta_2(\kappa)x}).$$

Note that this function is increasing and concave in $[0, \infty)$. In [3, Theorem 3.1], the reward function of a ‘refracting strategy’ that pays $a\bar{c}$ when the current surplus is below a refracting threshold b and pays \bar{c} when the current surplus is above b was shown to be

$$\begin{aligned} v(x, \bar{c}, b) = & \left(B(\bar{c}, b)W_0(x, \bar{c}) + \frac{a\bar{c}}{q}(1 - e^{\theta_2(a\bar{c})x}) \right) I_{\{x < b\}} \\ & + \left(\frac{\bar{c}}{q} + D(\bar{c}, b)e^{\theta_2(\bar{c})x} \right) I_{\{x \geq b\}}, \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} W_0(x, \bar{c}) &= \frac{e^{\theta_1(a\bar{c})x} - e^{\theta_2(a\bar{c})x}}{\sqrt{(\mu - a\bar{c})^2 + 2q\sigma^2}}, \\ B(\bar{c}, b) &= \frac{1}{q} \frac{a\bar{c}e^{\theta_2(a\bar{c})b}(\theta_2(a\bar{c}) - \theta_2(\bar{c})) - (1 - a)\bar{c}\theta_2(\bar{c})}{\partial_x W_0(b, \bar{c}) - \theta_2(\bar{c})W_0(b, \bar{c})}, \\ D(\bar{c}, b) &= B(\bar{c}, b)e^{-\theta_2(\bar{c})b}W_0(b, \bar{c}) - \frac{a\bar{c}}{q}e^{(\theta_2(a\bar{c}) - \theta_2(\bar{c}))b} - \frac{(1 - a)\bar{c}}{q}e^{-\theta_2(\bar{c})b}. \end{aligned} \tag{5.5}$$

The optimal threshold $b^*(\bar{c})$ is given by

$$b^*(\bar{c}) = \arg \max_{b \geq 0} v(x, \bar{c}, b) \tag{5.6}$$

(note that since the underlying process X has no upward jumps, that value does not depend on x). In case (5.6) is positive, then by (5.4) the value of b satisfies

$$\partial_b B(\bar{c}, b) = 0. \tag{5.7}$$

From [3], we know that the threshold can be characterised as the *unique* b such that $v(x, \bar{c}, b)$ is twice continuously differentiable in $x = b$. Since $v(x, \bar{c}, b^*(\bar{c}))$ is twice continuously differentiable with $v(0, \bar{c}, b^*(\bar{c})) = 0$, we therefore obtain $\lim_{x \rightarrow \infty} v(x, \bar{c}, b^*(\bar{c})) = \bar{c}/q$ and $v(x, \bar{c}, b^*(\bar{c}))$ is also a solution of

$$\max\{\mathcal{L}^{\bar{c}}(W)(x), \mathcal{L}^{a\bar{c}}(W)(x)\} = 0.$$

By Proposition 5.1, we have

$$v^{\bar{c}}(x) = v(x, \bar{c}, b^*(\bar{c})),$$

that is, the strategy achieving $v^{\bar{c}}$ has a ‘refracting’ threshold structure with optimal threshold $b^*(\bar{c})$. Note also that since $v^{\bar{c}}$ is twice continuously differentiable at $b^*(\bar{c})$ and $\mathcal{L}(v^{\bar{c}})(b^*(\bar{c})) = \mathcal{L}^{a\bar{c}}(v^{\bar{c}})(b^*(\bar{c})) = 0$, we have $\partial_x v^{\bar{c}}(b^*(\bar{c})) = 1$. Also, since

$$\mathcal{L}^{a\bar{c}}(v^{\bar{c}})(x) - \mathcal{L}^{\bar{c}}(v^{\bar{c}})(x) = c(1 - a)(\partial_x v^{\bar{c}}(x) - 1),$$

we obtain

$$\partial_x v^{\bar{c}}(x) \geq 1 \quad \text{for } x \leq b^*(\bar{c}), \quad \partial_x v^{\bar{c}}(x) \leq 1 \quad \text{for } x \geq b^*(\bar{c}). \quad (5.8)$$

6 Curve strategies and the optimal two-curve strategy for bounded dividend rates

Using the formulas of the previous section, we can find the value function defined in (4.3). The proofs of all the results of this section together with the auxiliary lemmas used in these proofs are deferred to Appendix A. We also include in that appendix some explicit formulas.

Remark 6.1 Before proceeding, note that this optimisation problem is only interesting for $\bar{c} > q\sigma^2/(2\mu)$ as for smaller values of \bar{c} , we know from Asmussen and Taksar [7, Eq. (1.8)] (translated to our notation) that even without a drawdown constraint, it is optimal to pay dividends at the maximal rate \bar{c} until the time of ruin. This is then also the optimal strategy in our situation, as the drawdown constraint does not affect its applicability. Indeed, and as a self-contained derivation of this result in the present context, the reward function of that strategy fulfils

$$\begin{aligned} \mathcal{L}^\kappa \left(\frac{\bar{c}}{q} (1 - e^{\theta_2(\bar{c})x}) \right) (x) &= (\bar{c} - \kappa) \left(-\frac{\bar{c}}{q} \theta_2(\bar{c}) e^{\theta_2(\bar{c})x} - 1 \right) \\ &\leq (\bar{c} - \kappa) \left(-\frac{\bar{c}}{q} \theta_2(\bar{c}) - 1 \right) \leq 0 \end{aligned} \quad (6.1)$$

for both $\kappa = ac$ and $\kappa = c$. So by Proposition 5.1, $v^{\bar{c}}(x) = \frac{\bar{c}}{q}(1 - e^{\theta_2(\bar{c})x}) =: U(x, c)$ and $b^*(\bar{c}) = 0$. By Theorem 4.4, it is then sufficient to prove that

$$\max\{\mathcal{L}^{ac}(U)(x, c), \mathcal{L}^c(U)(x, c), \partial_c U(x, c)\} \leq 0$$

for any $c \in [0, \bar{c}]$; but this follows from (6.1).

In the rest of this paper, we therefore assume that $\bar{c} > \frac{q\sigma^2}{2\mu}$.

Since $V(x, c)$ solves the HJB equation (4.2), it satisfies either $\mathcal{L}^{ac}(V)(x, c) = 0$ or $\mathcal{L}^c(V)(x, c) = 0$ or $\partial_c V(x, c) = 0$. This suggests that the state space $[0, \infty) \times [0, \bar{c}]$ is partitioned into two regions: a *non-change running maximum dividend region* \mathcal{NC}^*

in which the running maximum dividend rate c does not change, and a *change dividend region* \mathcal{CH}^* in which the dividend rate exceeds c (so that the running maximum dividend rate increases). Moreover, the region \mathcal{NC}^* splits into two connected sub-regions: \mathcal{NC}_{ac}^* in which the dividends are paid at the constant rate ac , and \mathcal{NC}_c^* in which the dividends are paid at the constant rate c .

Roughly speaking, the interior of the region \mathcal{NC}_{ac}^* consists of the points (x, c) in the state space where $\mathcal{L}^{ac}(V)(x, c) = 0$, $\mathcal{L}^c(V)(x, c) < 0$ and $\partial_c V(x, c) < 0$; the interior of the region \mathcal{NC}_c^* consists of the points where $\mathcal{L}^c(V)(x, c) = 0$, $\mathcal{L}^{ac}(V)(x, c) < 0$ and $\partial_c V(x, c) < 0$; and the interior of \mathcal{CH}^* consists of the points where $\partial_c V(x, c) = 0$, $\mathcal{L}^c(V)(x, c) < 0$ and $\mathcal{L}^{ac}(V)(x, c) < 0$. We introduce a family of two-curve strategies (or limits of two-curve strategies) where the different dividend payment regions are connected and are defined by two free boundary curves.

Consider a function $\gamma : [0, \bar{c}] \rightarrow (0, \infty)$ which is continuously differentiable, and a function $\zeta : [0, \bar{c}] \rightarrow (0, \infty)$ which is bounded, Riemann-integrable and càdlàg. Define the set

$$\mathcal{B} = \left\{ (\gamma, \zeta) \text{ such that } \gamma \leq \zeta \text{ and } \lim_{c \rightarrow \bar{c}} \zeta(c) = \zeta(\bar{c}) \right\}.$$

In the first part of this section, we define a function $W^{\gamma, \zeta} : [0, \infty) \times [0, \bar{c}] \rightarrow [0, \infty)$ for each $(\gamma, \zeta) \in \mathcal{B}$. We shall see that in some sense, $W^{\gamma, \zeta}(x, c)$ is the reward function of the *two-curve strategy* which pays dividends at the constant rate ac for the points to the left of the curve $\mathcal{R}(\gamma)$, pays dividends at the constant rate c between the curves $\mathcal{R}(\gamma)$ and $\mathcal{R}(\zeta)$, and pays more than c as dividend rate otherwise, where

$$\mathcal{R}(g) = \{(g(c), c) : c \in [0, \bar{c}]\}.$$

Hence the curves $\mathcal{R}(\gamma)$ and $\mathcal{R}(\zeta)$ split the state space $[0, \infty) \times [0, \bar{c}]$ into three connected regions, namely

$$\mathcal{NC}_{ac}(\gamma, \zeta) = \{(x, c) \in [0, \infty) \times [0, \bar{c}] : 0 \leq x < \gamma(c)\}$$

where dividends are paid with the constant rate ac ,

$$\mathcal{NC}_c(\gamma, \zeta) = \{(x, c) \in [0, \infty) \times [0, \bar{c}] : \gamma(c) \leq x < \zeta(c)\}$$

where dividends are paid with the constant rate c , and

$$\mathcal{CH}(\gamma, \zeta) = \{(x, c) \in [0, \infty) \times [0, \bar{c}] : x \geq \zeta(c)\};$$

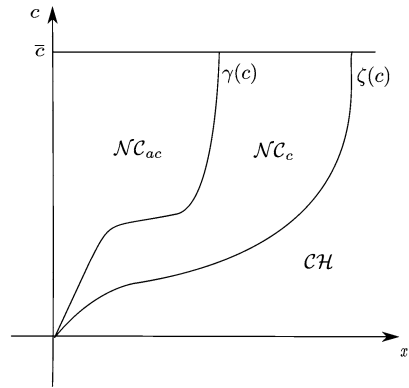
cf. Fig. 1. We set $\mathcal{NC}(\gamma, \zeta) = \mathcal{NC}_{ac}(\gamma, \zeta) \cup \mathcal{NC}_c(\gamma, \zeta)$.

In the second part of this section, we use calculus of variations to look for a pair $(\gamma^0, \zeta^0) \in \mathcal{B}$ which maximises the reward function $W^{\gamma, \zeta}$ among all $(\gamma, \zeta) \in \mathcal{B}$.

In order to define $W^{\gamma, \zeta}$ in the non-change regions $\mathcal{NC}_{ac}(\gamma, \zeta)$ and $\mathcal{NC}_c(\gamma, \zeta)$, we need to introduce some auxiliary functions. Let us consider the set

$$T := \{(y, z) : 0 < y \leq z\},$$

Fig. 1 A two-curve strategy with its regions



and the functions $b_0, b_1 : T \times [0, \infty) \times [0, \bar{c}] \rightarrow \mathbb{R}$ defined by

$$\begin{aligned}
 b_0(y, z, w, c) &:= \frac{b_{00}(y, z, c) + w(e^{(z-y)\theta_1(c)} - e^{(z-y)\theta_2(c)}) b_{01}(y, c)}{q(\theta_1(c) - \theta_2(c))d(y, z, c)}, \\
 b_1(y, z, w, c) &:= \frac{b_{10}(y, z, c) + w(e^{(z-y)\theta_1(c)} - e^{(z-y)\theta_2(c)}) b_{11}(y, c)}{(\theta_1(c) - \theta_2(c))d(y, z, c)}, \tag{6.2}
 \end{aligned}$$

where the functions d and $b_{00}, b_{01}, b_{10}, b_{11}$ are defined in (A.1) and (A.2) below. In Lemma A.1 below, we show that the function d is positive so that b_0 and b_1 are well defined.

In the next result, we define and study the functions $H^{\gamma, \zeta}$ and $A^{\gamma, \zeta}$ that will be used to define $W^{\gamma, \zeta}$.

Proposition 6.2 *For any given $(\gamma, \zeta) \in \mathcal{B}$, there exists a unique continuous function $H^{\gamma, \zeta} : [0, \infty) \times [0, \bar{c}] \rightarrow [0, \infty)$ with $H^{\gamma, \zeta}(\cdot, c)$ continuously differentiable which satisfies for any $c \in [0, \bar{c})$ that*

$$\begin{aligned}
 \mathcal{L}^{ac}(H^{\gamma, \zeta})(x, c) &= 0 \quad \text{for } 0 \leq x < \gamma(c), \\
 \mathcal{L}^c(H^{\gamma, \zeta})(x, c) &= 0 \quad \text{for } \gamma(c) \leq x
 \end{aligned}$$

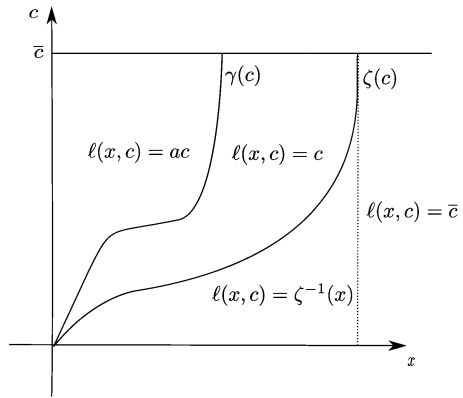
with the boundary conditions $H^{\gamma, \zeta}(0, c) = 0$, $H^{\gamma, \zeta}(x, \bar{c}) = v(x, \bar{c}, \gamma(\bar{c}))$ and $\partial_c H^{\gamma, \zeta}(\zeta(c), c) = 0$ at the points of continuity of ζ . It is given by

$$\begin{aligned}
 H^{\gamma, \zeta}(x, c) &= (f_{10}(x, c) + f_{11}(x, c)A^{\gamma, \zeta}(c))I_{\{x < \gamma(c)\}} \\
 &\quad + (f_{20}(\gamma(c), x, c) + f_{21}(\gamma(c), x, c)A^{\gamma, \zeta}(c))I_{\{x \geq \gamma(c)\}}, \tag{6.3}
 \end{aligned}$$

where $f_{10}, f_{11}, f_{20}, f_{21}$ are defined in (A.3)–(A.6) below,

$$\begin{aligned}
 A^{\gamma, \zeta}(c) &= A^{\gamma, \zeta}(\bar{c})e^{-\int_c^{\bar{c}} b_1(\gamma(s), \zeta(s), \gamma'(s), s)ds} \\
 &\quad - \int_c^{\bar{c}} e^{-\int_c^t b_1(\gamma(s), \zeta(s), \gamma'(s), s)ds} b_0(\gamma(t), \zeta(t), \gamma'(t), t)dt \tag{6.4}
 \end{aligned}$$

Fig. 2 The value of $\ell(x, c)$ in each region



and

$$A^{\gamma, \zeta}(\bar{c}) = \frac{B(\bar{c}, \gamma(\bar{c}))}{\sqrt{(\mu - a\bar{c})^2 + 2q\sigma^2}}, \tag{6.5}$$

where the function B is defined in (5.5) and the functions b_0 and b_1 are defined in (6.2). Moreover, $A^{\gamma, \zeta}$ is differentiable and satisfies

$$(A^{\gamma, \zeta})'(c) = b_0(\gamma(c), \zeta(c), \gamma'(c), c) + b_1(\gamma(c), \zeta(c), \gamma'(c), c)A^{\gamma, \zeta}(c)$$

at the points where ζ is continuous. Moreover, it satisfies the boundary condition (6.5).

Given $(\gamma, \zeta) \in \mathcal{B}$, we define

$$W^{\gamma, \zeta}(x, c) := \begin{cases} H^{\gamma, \zeta}(x, c) & \text{if } (x, c) \in \mathcal{NC}(\zeta), \\ H^{\gamma, \zeta}(x, \ell(x, c)) & \text{if } (x, c) \in \mathcal{CH}(\zeta), \end{cases} \tag{6.6}$$

where $H^{\gamma, \zeta}$ is defined in Proposition 6.2 and

$$\ell(x, c) := \max\{h \in [c, \bar{c}] : \zeta(h) \leq x \text{ for } h \in [c, h]\}$$

for $x \geq \zeta(c)$ and $c \in [0, \bar{c}]$; cf. Fig. 2.

In Appendix A, we show by using a Feynman–Kac argument that $W^{\gamma, \zeta}(x, c)$ is a uniform limit of reward functions of admissible strategies by constructing a sequence of step functions ζ_k converging to ζ . See Definition A.2 and Lemmas A.3 and A.4.

Remark 6.3 Given a $(\gamma, \zeta) \in \mathcal{B}$ where ζ is not a step function, we say that $W^{\gamma, \zeta}$ is the reward function of the *two-curve* strategy $\pi^{(\gamma, \zeta)}$ which starts with an initial surplus x and initial running maximum dividend rate c . Then

- (1) in the case $0 \leq x < \zeta(c)$, it follows the refracting strategy which pays ac when the current surplus is below a refracting threshold $\gamma(c)$, and pays c when the current surplus is above $\gamma(c)$, until either reaching the curve $\mathcal{R}(\zeta)$ or ruin (whichever comes first);

(2) in the case $x > \zeta(c)$, it increases immediately the dividend rate from c to $\ell(x, c)$;

(3) in the case $x = \zeta(c)$, it can be seen as the limit of admissible strategies $\pi_{x,c}^{(\gamma,\zeta_k)} \in \Pi_{x,c}$ arising from Lemma A.4 in Appendix A.

We now look for the maximum of $W^{\gamma,\zeta}$ among $(\gamma, \zeta) \in \mathcal{B}$. Let us see that if there exists a pair $(\gamma_0, \zeta_0) \in \mathcal{B}$ such that

$$A^{\gamma_0,\zeta_0}(0) = \max\{A^{\gamma,\zeta}(0) : (\gamma, \zeta) \in \mathcal{B}\}, \tag{6.7}$$

then $W^{\gamma_0,\zeta_0}(x, c) \geq W^{\gamma,\zeta}(x, c)$ for all $(x, c) \in [0, \infty) \times [0, \bar{c}]$ and $(\gamma, \zeta) \in \mathcal{B}$.

From Lemma A.1 and $\theta_2 > 0 > \theta_1$, we obtain that f_{11} and f_{21} defined in (A.4) and (A.6) are positive, and so by (6.3) and (6.6),

$$\arg \max_{(\gamma,\zeta) \in \mathcal{B}} W^{\gamma,\zeta}(x, c) = \arg \max_{(\gamma,\zeta) \in \mathcal{B}} A^{\gamma,\zeta}(c).$$

This implies that $\arg \max_{(\gamma,\zeta) \in \mathcal{B}} W^{\gamma,\zeta}(x, c)$ does not depend on the initial capital x . In addition, it also does not depend on the initial value c ; this follows from Lemma A.5, where we prove that the pair of functions (γ_0, ζ_0) which maximises (6.7) also maximises $A^{\gamma,\zeta}(c)$ for any $c \in [0, \bar{c}]$. Let us now find the implicit equation for the function A^{γ_0,ζ_0} for (γ_0, ζ_0) satisfying (6.7).

Proposition 6.4 *If a pair (γ_0, ζ_0) as in (6.7) exists, then $A^{\gamma_0,\zeta_0}(c)$ satisfies*

$$b_{1z}(c)A^{\gamma_0,\zeta_0}(c) + b_{0z}(c) = 0 \quad \text{for all } c \in [0, \bar{c}], \tag{6.8}$$

$$b_{1w}(c)A^{\gamma_0,\zeta_0}(c) + b_{0w}(c) = 0 \quad \text{for all } c \in [0, \bar{c}] \tag{6.9}$$

$$b_{1y}(c)A^{\gamma_0,\zeta_0}(c) + b_{0y}(c) = 0 \quad \text{for all } c \in [0, \bar{c}], \tag{6.10}$$

where for $i = 0, 1$,

$$\begin{aligned} b_i(s) &:= b_i(\gamma_0(s), \zeta_0(s), \gamma'_0(s), s), \\ b_{iu}(s) &:= \partial_u b_i(\gamma_0(s), \zeta_0(s), \gamma'_0(s), s) \quad \text{for } u \in \{z, y, w\}. \end{aligned}$$

Moreover, $\gamma_0(\bar{c}) = b^*(\bar{c})$ is the optimal threshold defined in (5.6) and

$$A^{\gamma_0,\zeta_0}(\bar{c}) = \frac{B(\bar{c}, b^*(\bar{c}))}{\sqrt{(\mu - ac)^2 + 2q\sigma^2}}.$$

Proposition 6.5 *Consider the functions C_0 and C_{ij} for $i = 1, 2$ and $j = 0, 1, 2$ defined in (A.8)–(A.11). If $(\gamma_0, \zeta_0) \in \mathcal{B}$ defined in (6.7) satisfies that ζ_0 is continuous and*

$$C_{11}(\gamma_0(c), \zeta_0(c), c)C_{22}(\gamma_0(c), \zeta_0(c), c) \neq 0 \tag{6.11}$$

for $c \in [0, \bar{c}]$, then γ_0 and ζ_0 are infinitely differentiable and (γ_0, ζ_0) is a solution of the system of ODEs

$$\begin{cases} \gamma'(c) = \frac{C_{10}(\gamma(c), \zeta(c), c)}{C_{11}(\gamma(c), \zeta(c), c)}, \\ \zeta'(c) = \frac{C_{20}(\gamma(c), \zeta(c), c)C_{11}(\gamma(c), \zeta(c), c)}{C_{11}(\gamma(c), \zeta(c), c)C_{22}(\gamma(c), \zeta(c), c)} \\ \quad - \frac{C_{21}(\gamma(c), \zeta(c), c)C_{10}(\gamma(c), \zeta(c), c)}{C_{11}(\gamma(c), \zeta(c), c)C_{22}(\gamma(c), \zeta(c), c)}, \end{cases} \tag{6.12}$$

with boundary conditions

$$\gamma_0(\bar{c}) = b^*(\bar{c}) \quad \text{and} \quad C_0(b^*(\bar{c}), \zeta_0(\bar{c}), \bar{c}) = 0, \tag{6.13}$$

where $b^*(\bar{c})$ is the optimal threshold defined in (5.6).

Let us study the uniqueness of the solution of (6.12) with boundary condition (6.13). We know that if $(\gamma, \zeta) \in \mathcal{B}$ is a solution, then $\gamma(\bar{c}) = b^*(\bar{c})$, the optimal threshold defined in (5.6). In order to obtain $\zeta(\bar{c})$, we have to find a zero of $C_0(b^*(\bar{c}), \cdot, \bar{c})$ in $(b^*(\bar{c}), \infty)$. Let us assume that there exists a unique zero $z^*(\bar{c})$ of $C_0(b^*(\bar{c}), \cdot, \bar{c})$ in $(b^*(\bar{c}), \infty)$. In the next result, we show that under this assumption, the existence of a solution (γ, ζ) of (6.12) implies its uniqueness.

In Sect. 7, we show that there is a unique zero $z^*(\bar{c})$ of $C_0(b^*(\bar{c}), \cdot, \bar{c})$ in $(b^*(\bar{c}), \infty)$ for \bar{c} large enough. Also, we check this assumption in the numerical examples for each set of parameters.

Proposition 6.6 *Assume that there exists a unique zero $z^*(\bar{c})$ of $C_0(b^*(\bar{c}), \cdot, \bar{c})$ in $(b^*(\bar{c}), \infty)$. If $(\gamma_1, \zeta_1) \in \mathcal{B}$ and $(\gamma_2, \zeta_2) \in \mathcal{B}$ are two solutions of the system (6.12) of differential equations with boundary conditions (6.13), then $(\gamma_1, \zeta_1) = (\gamma_2, \zeta_2)$.*

Let us now introduce a lower bound \underline{c} for the dividend rate (to be specified later), and denote by $(\bar{\gamma}, \bar{\zeta})$ a solution of (6.12) in $[\underline{c}, \bar{c}]$ with boundary conditions (6.13).

Remark 6.7 Since the functions C_{ij} defined in (A.10) and (A.11) are infinitely differentiable, a recursive argument establishes that $\bar{\gamma}$ and $\bar{\zeta}$ are also infinitely differentiable.

The next result says that the reward function $W^{\bar{\gamma}, \bar{\zeta}}$ satisfies a smooth-pasting property on the two free-boundary curves. Note that this extends Albrecher et al. [1, Prop. 5.13] from the ratcheting case with one free boundary to our present drawdown case. For a general account on conditions for smooth-pasting when the value function is not necessarily smooth, see e.g. Guo and Tomecek [22].

Proposition 6.8 *If a pair of infinitely differentiable functions $(\gamma, \zeta) \in \mathcal{B}$ satisfies*

$$\begin{aligned} \partial_{xx} W^{\gamma, \zeta}(\gamma(c)^+, c) &= \partial_{xx} W^{\gamma, \zeta}(\gamma(c)^-, c) & \text{for } c \in [\underline{c}, \bar{c}], \\ \partial_{cx} W^{\gamma, \zeta}(\zeta(c), c) &= \partial_{cc} W^{\gamma, \zeta}(\zeta(c), c) = 0 & \text{for } c \in [\underline{c}, \bar{c}], \end{aligned}$$

then (γ, ζ) is a solution of both (6.8) and (6.9) in $[\underline{c}, \bar{c}]$ with boundary conditions (6.13). Moreover, $\partial_x(W^{\gamma, \zeta})(\gamma(c), c) = 1$ for $c \in [\underline{c}, \bar{c}]$. Conversely, let $(\bar{\gamma}, \bar{\zeta})$ be a solution of (6.12) in $[\underline{c}, \bar{c}]$ with boundary conditions (6.13). Then $W^{\bar{\gamma}, \bar{\zeta}}$ satisfies the smooth-pasting properties

$$\begin{aligned} \partial_{xx} W^{\bar{\gamma}, \bar{\zeta}}(\bar{\gamma}(c)^+, c) &= \partial_{xx} W^{\bar{\gamma}, \bar{\zeta}}(\bar{\gamma}(c)^-, c) && \text{for } c \in [\underline{c}, \bar{c}], \\ \partial_{cx} W^{\bar{\gamma}, \bar{\zeta}}(\bar{\zeta}(c), c) &= \partial_{cc} W^{\bar{\gamma}, \bar{\zeta}}(\bar{\zeta}(c), c) = 0 && \text{for } c \in [\underline{c}, \bar{c}]. \end{aligned}$$

The next result shows more regularity for $W^{\bar{\gamma}, \bar{\zeta}}$ if $\bar{\zeta}$ is strictly monotone.

Proposition 6.9 *If $(\bar{\gamma}, \bar{\zeta})$ is a solution of (6.12) in $[\underline{c}, \bar{c}]$ with boundary conditions (6.13) and $\bar{\zeta}'(c) \neq 0$ in $[\underline{c}, \bar{c}]$, then $W^{\bar{\gamma}, \bar{\zeta}}$ is (2, 1)-differentiable.*

The next theorem is the main result of this section.

Theorem 6.10 *Let $(\bar{\gamma}, \bar{\zeta})$ be a solution of (6.12) in $[\underline{c}, \bar{c}]$ with boundary conditions (6.13) such that the function $W^{\bar{\gamma}, \bar{\zeta}}$ is (2, 1)-differentiable and satisfies*

$$\begin{aligned} \partial_c W^{\bar{\gamma}, \bar{\zeta}}(x, c) &\leq 0 && \text{for } x \in [0, \bar{\zeta}(c)], \\ \partial_x W^{\bar{\gamma}, \bar{\zeta}}(x, c) &\geq 1 && \text{for } x \in [0, \bar{\gamma}(c)], \\ \partial_x W^{\bar{\gamma}, \bar{\zeta}}(x, c) &\leq 1 && \text{for } x \in [\bar{\gamma}(c), \bar{\zeta}(c)] \end{aligned}$$

for $c \in [\underline{c}, \bar{c}]$. Then $W^{\bar{\gamma}, \bar{\zeta}} = V$.

Remark 6.11 We conjecture that there is always a unique zero $z^*(\bar{c})$ of $C_0(b^*(\bar{c}), \cdot, \bar{c})$ in $(b^*(\bar{c}), \infty)$ for $\bar{c} > q\sigma^2/(2\mu)$, that there exists a solution $(\bar{\gamma}, \bar{\zeta}) \in \mathcal{B}$ of the system of differential equations (6.12) satisfying the boundary conditions (6.13), and that the reward function $W^{\bar{\gamma}, \bar{\zeta}}$ is a viscosity supersolution of the HJB equation (4.2). If that holds, $(\bar{\gamma}, \bar{\zeta}) = (\gamma_0, \zeta_0)$ and $W^{\bar{\gamma}, \bar{\zeta}}$ is the value function V . Moreover, the optimal strategy is then a two-curve strategy. In Sect. 8, we show that that this conjecture always holds in $[\underline{c}, \bar{c}]$ for \bar{c} large enough and some suitable $\underline{c} < \bar{c}$, and in Sect. 9, we also show numerically that it is true for further instances.

7 Asymptotic values as $\bar{c} \rightarrow \infty$

The symbolic computations of this section are highly involved; so we use the Wolfram Mathematica software to obtain Taylor expansions. Note that all results of this section are derived for $0 < a < 1$, and the resulting expressions need not necessarily be applicable for the limit to $a = 1$, as dominant terms in the asymptotics may change.

Recall the boundary condition $C_0(b^*(\bar{c}), \cdot, \bar{c}) = 0$ of the differential equation (6.12); cf. (6.13). Note that for $\bar{c} > \frac{q\sigma^2}{2\mu}$, we have from Remark 6.1 that $b^*(\bar{c})$ is

the unique positive b satisfying (5.7). In this section, we show that there is a unique zero $z^*(\bar{c})$ of $C_0(b^*(\bar{c}), \cdot, \bar{c})$ in $(b^*(\bar{c}), \infty)$ for \bar{c} large enough and that

$$\lim_{\bar{c} \rightarrow \infty} (b^*(\bar{c}), z^*(\bar{c})) = \left(\frac{\mu}{q}, \frac{\mu}{q} \left(1 + \frac{1}{\sqrt{a}} \right) \right).$$

We also show $\searrow\text{-}\lim_{\bar{c} \rightarrow \infty} V_a^{\bar{c}}(x, \bar{c}) = x$ for $0 < x < \lim_{\bar{c} \rightarrow \infty} z^*(\bar{c}) = \frac{\mu}{q} \left(1 + \frac{1}{\sqrt{a}} \right)$ and $\nearrow\text{-}\lim_{\bar{c} \rightarrow \infty} V_a^{\bar{c}}(x, \bar{c}) = x$ for $x > \lim_{\bar{c} \rightarrow \infty} z^*(\bar{c}) = \frac{\mu}{q} \left(1 + \frac{1}{\sqrt{a}} \right)$.

In the rest of the section, we denote $V_a^{\bar{c}}$ by $V^{\bar{c}}$.

Proposition 7.1 *It holds that $\lim_{\bar{c} \rightarrow \infty} b^*(\bar{c}) = \mu/q$. More precisely, the Taylor expansion of $b^*(\bar{c})$ at $\bar{c} = \infty$ is given by*

$$b^*(\bar{c}) = \frac{\mu}{q} - \frac{\mu^2 + aq\sigma^2}{2aq} \frac{1}{\bar{c}} + O\left(\frac{1}{\bar{c}^2}\right). \tag{7.1}$$

Proof We have from (5.6) that

$$\partial_b B(\bar{c}, b^*(\bar{c})) = 0. \tag{7.2}$$

But

$$\partial_b B(\bar{c}, b) = \frac{c\sqrt{(\mu - a\bar{c})^2 + 2q\sigma^2}}{q(e^{\theta_1(a\bar{c})b}(\theta_1(a\bar{c}) - \theta_2(\bar{c})) + e^{\theta_2(a\bar{c})b}(\theta_2(\bar{c}) - \theta_2(a\bar{c})))^2} E(\bar{c}, b),$$

where

$$\begin{aligned} E(\bar{c}, b) &= e^{\theta_1(a\bar{c})b} (a - 1)\theta_2(\bar{c})(\theta_2(\bar{c}) - \theta_1(a\bar{c}))\theta_1(a\bar{c}) \\ &\quad + e^{\theta_2(a\bar{c})b} (1 - a)\theta_2(\bar{c})(\theta_2(\bar{c}) - \theta_2(a\bar{c}))\theta_2(a\bar{c}) \\ &\quad + e^{(\theta_1(a\bar{c}) + \theta_2(a\bar{c}))b} a(\theta_2(\bar{c}) - \theta_2(a\bar{c}))(\theta_2(\bar{c}) - \theta_1(a\bar{c}))(\theta_2(a\bar{c}) - \theta_1(a\bar{c})). \end{aligned} \tag{7.3}$$

Let us define $F_0(\bar{c}, b) := E(\bar{c}, b)/e^{\theta_1(a\bar{c})b}$. The Taylor expansions of $\theta_1(c)$ and $\theta_2(c)$ in (5.2) at $c = \infty$ are given by

$$\begin{aligned} \theta_1(c) &= \frac{2}{\sigma^2}c - \frac{2\mu}{\sigma^2} + q\frac{1}{c} + O\left(\frac{1}{c^2}\right), \\ \theta_2(c) &= -q\frac{1}{c} - q\mu\frac{1}{c^2} + O\left(\frac{1}{c^3}\right). \end{aligned} \tag{7.4}$$

Let us prove first that there is no sequence $b^*(c_n) \rightarrow \infty$ with $c_n \rightarrow \infty$. Using (7.4), we obtain

$$\lim_{n \rightarrow \infty} F_0(c_n, b^*(c_n)) = \lim_{n \rightarrow \infty} \frac{4(a - 1)a^2q}{\sigma^4} c_n (1 - e^{-\frac{qb^*(c_n)}{ac_n}}).$$

First, let us assume that $b^*(c_n) = c_n\alpha_n$ with $\alpha_n \rightarrow \bar{\alpha} \in (0, \infty)$. Then since $1 - e^{-\frac{q\bar{\alpha}}{a}} > 0$ and $a < 1$,

$$0 = \lim_{n \rightarrow \infty} F_0(c_n, b^*(c_n)) = -\infty,$$

which is a contradiction. Secondly, let us assume that $b^*(c_n) = c_n\alpha_n$ with $\alpha_n \rightarrow \infty$. Then since $e^{-\frac{qb^*(c_n)}{ac_n}} \rightarrow 0$, we have $0 = \lim_{n \rightarrow \infty} F_0(c_n, b^*(c_n)) = -\infty$ which is also a contradiction. Finally, let us assume that $b^*(c_n) = c_n\alpha_n$ with $\alpha_n \rightarrow 0+$. Then

$$0 = \lim_{n \rightarrow \infty} F_0(c_n, b^*(c_n)) = \lim_{n \rightarrow \infty} \frac{4(a-1)a^2q}{\sigma^4} \frac{1 - e^{-\frac{q\alpha_n}{a}}}{\alpha_n} b^*(c_n) = -\infty$$

gives again a contradiction. Hence $\limsup_{\bar{c} \rightarrow \infty} b^*(\bar{c}) < \infty$.

Let us define the function $H_0 : [0, \infty) \times (0, \infty) \rightarrow \mathbb{R}$ as

$$H_0(u, b) := \begin{cases} \frac{4(a-1)aq(qb-\mu)}{\sigma^4} & \text{for } u = 0, \\ F_0(\frac{1}{u}, b) & \text{for } u > 0. \end{cases}$$

Then $H_0(u, b)$ is infinitely continuously differentiable because it is infinitely continuously differentiable for $u > 0$ and $\lim_{u \rightarrow 0+} F_0(\frac{1}{u}, b) = 4(a-1)aq(qb-\mu)/\sigma^4 < \infty$. Moreover, its first-order Taylor expansion at $u = 0$ is given by

$$H_0(u, b) = \frac{4(a-1)aq(qb-\mu)}{\sigma^4} + \frac{2(a-1)q - q^2b^2 + 2(1-a)\mu^2 + aq(2b\mu + \sigma^2)}{\sigma^4}u + O(u^2).$$

From (7.2), we obtain $H_0(u, b^*(1/u)) = 0$ for $u > 0$. Let us show that we have $\lim_{u \rightarrow 0+} b^*(1/u) = \mu/q$. We have already seen that $b^*(1/u)$ is bounded for $u \in [0, \varepsilon)$ for some $\varepsilon > 0$. Take any sequence $u_n \rightarrow 0+$ with $\lim_{n \rightarrow \infty} b^*(1/u_n) = b_0 < \infty$; then

$$\lim_{n \rightarrow \infty} H_0(u_n, b^*(1/u_n)) = H_0(0, b_0) = \frac{4(a-1)aq(qb_0-\mu)}{\sigma^4} = 0$$

and so $b_0 = \mu/q$. Using that $\partial_b H_0(0, b) = \frac{4(a-1)aq^2}{\sigma^4} \neq 0$, we conclude by the implicit function theorem that the function $h : [0, \infty) \rightarrow \mathbb{R}$ defined as $h(0) = \frac{\mu}{q}$ and $h(u) = b^*(\frac{1}{u})$ for $u > 0$ is infinitely continuously differentiable, and the result follows. □

Proposition 7.2 *There exists a unique zero $z^*(\bar{c})$ of $C_0(b^*(\bar{c}), \cdot, \bar{c})$ in $(b^*(\bar{c}), \infty)$ for \bar{c} large enough with $\lim_{\bar{c} \rightarrow \infty} z^*(\bar{c}) = \frac{\mu}{q}(1 + \frac{1}{\sqrt{a}})$. More precisely, $z^*(\bar{c})$ is infinitely continuously differentiable for \bar{c} large enough, and its first-order Taylor expansion at*

$\bar{c} = \infty$ is given by

$$z^*(\bar{c}) = \frac{\mu}{q} \left(1 + \frac{1}{\sqrt{a}} \right) + \frac{(1 - 2\sqrt{a} - 3a)\mu^2 - 3(1 + \sqrt{a^3}/2)q\sigma^2}{3q\sqrt{a^3}} \frac{1}{\bar{c}} + O\left(\frac{1}{\bar{c}^2}\right). \tag{7.5}$$

Proof Considering the functions $C_0(y, z, c)$ defined in (A.8) and $E(c, y)$ defined in (7.3), we define

$$\begin{aligned} \tilde{C}_0(y, z, c) &= \left(ce^{2(z-y)\theta_1(c)+y\theta_1(ac)} (\theta_2(c) - \theta_1(c)) (\theta_1(ac) - \theta_2(c)) \theta'_1(c) \right) E(c, y) \\ &\quad - \left(ce^{2(z-y)\theta_1(c)+y\theta_2(ac)} (\theta_2(ac) - \theta_2(c)) (\theta_2(c) - \theta_1(c)) \theta'_1(c) \right) E(c, y) \\ &\quad + \frac{C_0(y, z, c)}{(\theta_1(c) - \theta_2(c))}. \end{aligned}$$

Since $E(\bar{c}, b^*(\bar{c})) = 0$, $\theta_2(c) - \theta_1(c) < 0$ and $d(y, z, c) > 0$, we see that the property $C_0(b^*(\bar{c}), z^*(\bar{c}), \bar{c}) = 0$ is equivalent to $\tilde{C}_0(b^*(\bar{c}), z^*(\bar{c}), \bar{c}) = 0$. We can write

$$\tilde{C}_0(y, z, c) = \sum_{i=1}^{16} m_i(y, z, c) e^{g_i(y, z, c)}, \tag{7.6}$$

where the $m_i(y, z, c)$ are of the form

$$m_i(y, z, c) = m_{i0}(y, c) + m_{i1}(y, c)z$$

and $m_{i0}(y, c)$, $m_{i1}(y, c)$ are polynomials in $\theta_1(c)$, $\theta_2(c)$, $\theta_1(ac)$, $\theta_2(ac)$, $\theta'_1(c)$, $\theta'_2(c)$, $\theta'_1(ac)$, $\theta'_2(ac)$, y , c , a . The functions $g_i(y, z, c)$ in (7.6) are positive linear combinations of $(z - y)\theta_1(c)$, $(z - y)\theta_2(c)$, $y\theta_1(ac)$ and $y\theta_2(ac)$, with the concrete form given in Appendix B. Define

$$F_1(y, z, c) := \frac{\tilde{C}_0(y, z, c)}{e^{g_{12}(y, z, c)}}.$$

Let us show first that there is no sequence (z_n, c_n) with $z_n > b^*(c_n)$ such that $C_0(b^*(c_n), z_n, c_n) = 0$, $c_n \rightarrow \infty$ and $z_n \rightarrow \infty$. From the definitions of the exponents g_i given in Appendix B and the expressions (7.4), we have that

$$\begin{aligned} &\lim_{n \rightarrow \infty} F_1(b^*(c_n), z_n, c_n) \\ &= \lim_{n \rightarrow \infty} \sum_{i=12}^{14} m_i(b^*(c_n), z_n, c_n) e^{g_i(b^*(c_n), z_n, c_n) - g_{12}(b^*(c_n), z_n, c_n)}, \end{aligned}$$

because the other terms are negligible. We can write

$$\begin{aligned}
 m_{12,0}(y, c) &= \frac{64(a-1)a^2}{\sigma^{12}}c^6 + O(c^5), \quad m_{12,1}(y, c) = O(c^4), \\
 m_{13,0}(y, c) &= \frac{64(a-1)^2a^2}{\sigma^{12}}c^6 + O(c^5), \quad m_{13,1}(y, c) = \frac{128(a-1)^2a^2q}{\sigma^{12}}c^5 \\
 &\quad + O(c^4) \\
 m_{14,0}(y, c) &= -\frac{64(a-1)a^3}{\sigma^{12}}c^6 + O(c^5), \quad m_{14,1}(y, c) = -\frac{64a^2(1-3a+2a^2)q}{\sigma^{12}}c^5 \\
 &\quad + O(c^4)
 \end{aligned}$$

and

$$\begin{aligned}
 g_{13}(y, z, c) - g_{12}(y, z, c) &= -q(z-y)\frac{1}{c} + (1+z)O\left(\frac{1}{c^2}\right), \\
 g_{14}(y, z, c) - g_{12}(y, z, c) &= -\left(\frac{qy}{a} + q(z-y)\right)\frac{1}{c} + (1+z)O\left(\frac{1}{c^2}\right).
 \end{aligned}$$

If $z_n \rightarrow \infty, c_n \rightarrow \infty$ with $b^*(c_n) \rightarrow \frac{\mu}{q}$, we deduce that

$$0 = \lim_{n \rightarrow \infty} F_1(b^*(c_n), z_n, c_n) = \lim_{n \rightarrow \infty} \frac{64(a-1)a^2c_n^6}{\sigma^{12}}e^{-q\frac{z_n}{c_n}}\left(e^{q\frac{z_n}{c_n}} - 1 - q\frac{z_n}{c_n}\right).$$

First, assume that $z_n = c_n\alpha_n$ with $\alpha_n \rightarrow \bar{\alpha} \in (0, \infty)$. Then $e^{-q\bar{\alpha}}(e^{q\bar{\alpha}} - 1 - q\bar{\alpha}) > 0$ and $a < 1$ yield

$$0 = \lim_{n \rightarrow \infty} F_1(b^*(c_n), z_n, c_n) = -\infty,$$

which is a contradiction. Secondly, assume that $z_n = c_n\alpha_n$ with $\alpha_n \rightarrow \infty$. Then since

$$e^{-q\alpha_n}(e^{q\alpha_n} - 1 - q\alpha_n) = 1 - (1 + q\alpha_n)e^{-q\alpha_n} \rightarrow 1,$$

we have

$$0 = \lim_{n \rightarrow \infty} F_1(b^*(c_n), z_n, c_n) = -\infty$$

which is also a contradiction. Finally, assume that $z_n = c_n\alpha_n$ with $\alpha_n \rightarrow 0+$. Then

$$\begin{aligned}
 0 &= \lim_{n \rightarrow \infty} F_1(b^*(c_n), z_n, c_n) \\
 &= \lim_{n \rightarrow \infty} \frac{64(a-1)a^2}{\sigma^{12}}q^2e^{-q\alpha_n}\frac{e^{q\alpha_n} - 1 - q\alpha_n}{q^2\alpha_n^2}z_n^2c_n^4 = -\infty,
 \end{aligned}$$

which is also a contradiction. Hence there is no such sequence (z_n, c_n) .

Using the Taylor expansions of $\theta_1(\bar{c})$, $\theta_2(\bar{c})$ and $b^*(\bar{c})$ at $\bar{c} = \infty$ given in (7.4) and Proposition 7.1, we find that the function

$$H_1(z, u) = \begin{cases} \frac{32(a-1)a(a(\mu-qz)^2-\mu^2)}{\sigma^{12}} & \text{for } u = 0, \\ u^4 F_1(b^*(\frac{1}{u}), z, \frac{1}{u}) & \text{for } u > 0 \end{cases}$$

is infinitely continuously differentiable, because it is infinitely continuously differentiable for $u > 0$ and

$$\lim_{u \rightarrow 0^+} u^4 F_1\left(b^*\left(\frac{1}{u}\right), z, \frac{1}{u}\right) = \frac{32(a-1)a(a(\mu-qz)^2-\mu^2)}{\sigma^{12}} < \infty.$$

Moreover, its first-order power series expansion is given by

$$\begin{aligned} H_1(z, u) &= \frac{32(a-1)a(a(\mu-qz)^2-\mu^2)}{\sigma^{12}} \\ &\quad - u \frac{32(a-1)}{3\sigma^{12}} \left(-4\mu^3 + 3a\mu(2q^2z^2 - 6qz\mu + \mu^2) \right. \\ &\quad \left. + a^2(qz - \mu)(2q^2z^2 - \mu^2 - q(z\mu + 3\sigma^2)) \right) \\ &\quad + O(u^2). \end{aligned}$$

Since the only zero of $H_1(z, 0)$ in $[\frac{\mu}{q}, \infty)$ is $\frac{\mu}{q}(1 + \frac{1}{\sqrt{a}})$ and

$$\partial_z H_1(z, 0) = \partial_z \frac{32(a-1)a(a(\mu-qz)^2-\mu^2)}{\sigma^{12}} = \frac{64(1-a)a^2q(\mu-qz)}{\sigma^{12}} \neq 0$$

for $z \geq \frac{\mu}{q}$, we conclude by the implicit function theorem that there exist $\varepsilon > 0$ and a unique infinitely continuously differentiable function $g : [0, \varepsilon) \rightarrow \mathbb{R}$ with $g(0) = \frac{\mu}{q}(1 + \frac{1}{\sqrt{a}})$ and $H_1(g(u), u) = 0$ for $u \in [0, \varepsilon)$. In addition, $g(u)$ is the unique zero of $H_1(\cdot, u)$ in a neighbourhood U of $(\frac{\mu}{q}(1 + \frac{1}{\sqrt{a}}), 0)$. Moreover, the first-order Taylor expansion of g at $u = 0$ is given by

$$g(u) = \frac{\mu}{q} \left(1 + \frac{1}{\sqrt{a}} \right) + \frac{(2 - 4\sqrt{a}\mu^2 - 6a)\mu^2 - 3(2 + \sqrt{a^3})q\sigma^2}{6q\sqrt{a^3}} u + O(u^2). \tag{7.7}$$

Let us now show that $g(u)$ is the only zero of $H_1(\cdot, u)$ in $(b^*(1/u), \infty)$ for u small enough. If this were not the case, there would be a sequence $(z_n, u_n)_{n \geq 1}$ with $z_n > b^*(1/u_n)$, $z_n \neq g(u_n)$ such that $u_n \searrow 0$ and $H_1(z_n, u_n) = 0$. If there exists a convergent subsequence (z_{n_k}) with $z_{n_k} \rightarrow z_0 \in [\frac{\mu}{q}, \infty)$, then $H_1(z_0, 0) = 0$ by continuity and so $z_0 = g(0) = \frac{\mu}{q}(1 + \frac{1}{\sqrt{a}})$ which is a contradiction because $(z_{n_k}, u_{n_k}) \notin U$ for k large enough. So $z_n \rightarrow \infty$ and this is also a contradiction. So from (7.7), we get the result. \square

Proposition 7.3 *There exists a unique zero $x^*(\bar{c})$ of $\partial_{\bar{c}}V^{\bar{c}}(x, \bar{c})$ in $(0, \infty)$ for \bar{c} large enough with $x^*(\bar{c}) > b^*(\bar{c})$ and $\lim_{\bar{c} \rightarrow \infty} x^*(\bar{c}) = \frac{\mu}{q} \left(1 + \frac{1}{\sqrt{a}}\right)$. More precisely, $x^*(\bar{c})$ is infinitely continuously differentiable for \bar{c} large enough and its first-order Taylor expansion at $\bar{c} = \infty$ is given by*

$$x^*(\bar{c}) = \frac{\mu}{q} \left(1 + \frac{1}{\sqrt{a}}\right) + \frac{(1 - 2\sqrt{a} - 3a)\mu^2 - 3(1 + \sqrt{a^3})q\sigma^2}{3\sqrt{a^3}q} \frac{1}{\bar{c}} + O\left(\frac{1}{\bar{c}^2}\right). \tag{7.8}$$

Moreover, we have

$$\begin{aligned} \searrow \lim_{\bar{c} \rightarrow \infty} V^{\bar{c}}(x, \bar{c}) &= x && \text{for } 0 < x < \frac{\mu}{q} \left(1 + \frac{1}{\sqrt{a}}\right), \\ \nearrow \lim_{\bar{c} \rightarrow \infty} V^{\bar{c}}(x, \bar{c}) &= x && \text{for } x > \frac{\mu}{q} \left(1 + \frac{1}{\sqrt{a}}\right). \end{aligned}$$

Proof From (5.4), we have that

$$\begin{aligned} V^{\bar{c}}(x, \bar{c}) = v(x, \bar{c}, b^*(\bar{c})) &= \left(B(\bar{c}, b^*(\bar{c}))W_0(x, \bar{c}) + \frac{a\bar{c}}{q}(1 - e^{\theta_2(a\bar{c})x}) \right) I_{\{x < b^*(\bar{c})\}} \\ &+ \left(\frac{\bar{c}}{q} + D(\bar{c}, b^*(\bar{c}))e^{\theta_2(\bar{c})x} \right) I_{\{x \geq b^*(\bar{c})\}}, \end{aligned}$$

and from Proposition 7.1, we know that $\nearrow \lim_{\bar{c} \rightarrow \infty} b^*(\bar{c}) = \mu/q$.

Take $x < \mu/q$. Then $x < b^*(\bar{c})$ for \bar{c} large enough; so we have

$$V^{\bar{c}}(x, \bar{c}) = B(\bar{c}, b^*(\bar{c}))W_0(x, \bar{c}) + \frac{a\bar{c}}{q}(1 - e^{\theta_2(a\bar{c})x})$$

and so

$$\partial_{\bar{c}}V^{\bar{c}}(x, \bar{c}) = \frac{F_2(x, \bar{c})}{\ell_0(b^*(\bar{c}), \bar{c})},$$

where

$$\begin{aligned} \ell_0(b, c) &= q((\mu - ac)^2 + 2q\sigma^2) \\ &\times \left(e^{b\theta_1(ac)}(\theta_1(ac) - \theta_2(c)) + e^{b\theta_2(ac)}(\theta_2(c) - \theta_2(ac)) \right)^2 > 0 \end{aligned} \tag{7.9}$$

and

$$F_2(x, \bar{c}) := \sum_{i=1}^{11} \ell_i(x, b^*(\bar{c}), \bar{c})e^{h_i(x, b^*(\bar{c}), \bar{c})}.$$

Here $\ell_i(x, b, c)$ are polynomials in $\theta_1(c), \theta_2(c), \theta_1(ac), \theta_2(ac), \theta'_1(c), \theta'_2(c), \theta'_1(ac), \theta'_2(ac), x, b, c, a$, and $h_i(x, b, c), i = 1, \dots, 11$, are positive linear combinations

of $b\theta_1(ac)$, $x\theta_1(ac)$, $b\theta_2(ac)$ and $x\theta_2(ac)$ stated in detail in Appendix B. Since $\lim_{\bar{c} \rightarrow \infty} b^*(\bar{c}) = \mu/q$, the Taylor expansion of $F_2(x, \bar{c})/\bar{c}^2$ at $\bar{c} = \infty$ is given by

$$\frac{F_2(x, \bar{c})}{\bar{c}^2} = \frac{2a^3 xq(xq - 2\mu)}{\sigma^4} + O\left(\frac{1}{\bar{c}}\right).$$

Since $xq - 2\mu < 0$, we have $\partial_{\bar{c}} V^{\bar{c}}(x, \bar{c}) < 0$ for \bar{c} large enough, and so we obtain $\lim_{\bar{c} \rightarrow \infty} V^{\bar{c}}(x, \bar{c}) = x^+$ for $x < \frac{\mu}{q}$.

Take now $x \geq \mu/q > b^*(\bar{c})$. Then

$$V^{\bar{c}}(x, \bar{c}) = \left(\frac{\bar{c}}{q} + D(\bar{c}, b^*(\bar{c}))\right) e^{\theta_2(\bar{c})x}$$

and so

$$\partial_{\bar{c}} V^{\bar{c}}(x, \bar{c}) = \frac{F_3(x, \bar{c})}{\ell_0(b^*(\bar{c}), \bar{c})},$$

where

$$F_3(x, \bar{c}) = \sum_{i=1}^8 \bar{\ell}_i(x, b^*(\bar{c}), \bar{c}) e^{k_i(x, b^*(\bar{c}), \bar{c})},$$

$\ell_0(b, \bar{c})$ is defined in (7.9), $\bar{\ell}_i(x, b, c)$ are polynomials in $\theta_1(c)$, $\theta_2(c)$, $\theta_1(ac)$, $\theta_2(ac)$, $\theta'_1(c)$, $\theta'_2(c)$, $\theta'_1(ac)$, $\theta'_2(ac)$, x, b, c, a , and $k_i(x, b, c)$, $i = 1, \dots, 8$, are positive linear combinations of $b\theta_1(ac)$, $b\theta_2(ac)$ and $(x - b)\theta_2(c)$ as detailed in Appendix B. Since $\lim_{\bar{c} \rightarrow \infty} b^*(\bar{c}) = \mu/q$, the Taylor expansion of $F_3(x, \bar{c})/\bar{c}^2$ at $\bar{c} = \infty$ is given by

$$\begin{aligned} \frac{F_3(x, \bar{c})}{\bar{c}^2} &= \frac{2a^3}{\sigma^4} (aq^2x^2 - 2aq\mu x + \mu^2(a - 1)) \\ &+ \frac{4a^2}{3\sigma^4} (-3a(2qx - 3\mu)(qx - \mu)\mu + 5\mu^3 + 3q\mu\sigma^2 \\ &\quad - a^2(qx - \mu)(q^2x^2 - 5qx\mu + 4\mu^2 - 3q\sigma^2)) \frac{1}{\bar{c}} \\ &+ O\left(\frac{1}{\bar{c}^2}\right). \end{aligned}$$

So $F_3(x^*(\bar{c}), \bar{c}) = 0$ yields that the Taylor expansion of $x^*(\bar{c})$ is given by (7.8).

Finally, for \bar{c} large enough, we get $\partial_{\bar{c}} V^{\bar{c}}(x, \bar{c}) < 0$ for $x \in [\frac{\mu}{q}, \frac{\mu}{q}(1 + \frac{1}{\sqrt{a}})]$ and $\partial_{\bar{c}} V^{\bar{c}}(x, \bar{c}) > 0$ for $x > \frac{\mu}{q}(1 + \frac{1}{\sqrt{a}})$. So the result follows. \square

Remark 7.4 Note that

$$z^*(\bar{c}) - x^*(\bar{c}) = \frac{\sigma^2}{2} \frac{1}{\bar{c}} + O\left(\frac{1}{\bar{c}^2}\right). \tag{7.10}$$

So $z^*(\bar{c}) > x^*(\bar{c})$ for \bar{c} large enough, and we have asymptotic equivalence for these two quantities when $\bar{c} \rightarrow \infty$. At the same time, the inequality $z^*(\bar{c}) \geq x^*(\bar{c})$ can easily be seen to hold for any \bar{c} from the following argument: We have

$$\begin{aligned} V^{\bar{c}}(x, \bar{c}) - V^{\bar{c}-h}(x, \bar{c} - h) &= V^{\bar{c}}(x, \bar{c}) - V^{\bar{c}}(x, \bar{c} - h) \\ &\quad + V^{\bar{c}}(x, \bar{c} - h) - V^{\bar{c}-h}(x, \bar{c} - h) \\ &\geq V^{\bar{c}}(x, \bar{c}) - V^{\bar{c}}(x, \bar{c} - h), \end{aligned}$$

since $V^{\bar{c}}(x, c)$ is nondecreasing in \bar{c} by Proposition 3.10. Dividing by h and taking the limit as h goes to zero, we get

$$\partial_{\bar{c}} V^{\bar{c}}(x, \bar{c}) \geq V_c^{\bar{c}}(x, c)|_{c=\bar{c}}.$$

Hence $\partial_{\bar{c}} V^{\bar{c}}(z^*(\bar{c}), \bar{c}) \geq V_c^{\bar{c}}(z^*(\bar{c}), c)|_{c=\bar{c}} = 0$, and then the value $x^*(\bar{c})$ where $\partial_{\bar{c}} V^{\bar{c}}(\cdot, \bar{c})$ changes from negative to positive satisfies $x^*(\bar{c}) \leq z^*(\bar{c})$.

Remark 7.5 One observes from (7.5) that for very small values of a , the coefficient of $1/\bar{c}$ in the asymptotic expansion is positive so that the limit $\mu(1 + 1/\sqrt{a})/q$ is approached from the right, whereas for larger values of a , that coefficient is negative and the limit is approached from the left as \bar{c} becomes large; see also the numerical illustrations in Sect. 9. It may also be instructive to derive the higher-order limiting behaviour of $x^*(\bar{c})$ established in Proposition 7.3 in a direct way for the deterministic case discussed in Sect. 2. Concretely, including one more term in the expansion (2.2) gives

$$\begin{aligned} x + \frac{2axq\mu - ax^2q^2 + \mu^2(1 - a)}{2aq\bar{c}} \\ + \frac{\mu^3 + 3a\mu^2(\mu - xq) + a^2(xq - 4\mu)(\mu - xq)^2}{6a^2q\bar{c}^2} + \left(\frac{1}{\bar{c}^3}\right), \end{aligned}$$

and substituting $x = \frac{\mu}{q}(1 + \frac{1}{\sqrt{a}}) + \frac{a_0}{\bar{c}}$ (for an $a_0 \in \mathbb{R}$ to be identified) into this expression gives

$$\frac{3\sqrt{a^3}q\mu a_0 + \mu^3(2\sqrt{a} + 3a - 1)}{3a^2q\bar{c}^3} + O\left(\frac{1}{\bar{c}^4}\right).$$

This fraction equals zero for $a_0 = \frac{(1-2\sqrt{a}-3a)\mu^2}{3\sqrt{a^3}q}$ so that we obtain

$$x^*(\bar{c}) = \frac{\mu}{q}\left(1 + \frac{1}{\sqrt{a}}\right) + \frac{(1 - 2\sqrt{a} - 3a)\mu^2}{3\sqrt{a^3}q\bar{c}} + O\left(\frac{1}{\bar{c}^2}\right),$$

which exactly corresponds to (7.8) for $\sigma = 0$. This formula shows that in the deterministic case, indeed the limit $\mu(1 + 1/\sqrt{a})/q$ is approached from the right for $a < 1/9$ and from the left for $a > 1/9$ as $\bar{c} \rightarrow \infty$.

8 Optimal strategies for \bar{c} large

In the next result, we show that for \bar{c} large enough, there exists a unique solution of (6.12) with boundary conditions (6.13) and that $\bar{\zeta}' < 0$ and $\bar{\gamma}' > 0$ in a neighbourhood of \bar{c} . We emphasise again that **for all results in this section**, we assume $a < 1$.

Proposition 8.1 *For \bar{c} large enough, we can find $\underline{c} \in [0, \bar{c})$ such that there exists a unique solution $(\bar{\gamma}(c), \bar{\zeta}(c))$ of (6.12) with boundary conditions (6.13) in $[\underline{c}, \bar{c}]$, and $\bar{\gamma}$ is strictly increasing and $\bar{\zeta}$ is strictly decreasing in $[\underline{c}, \bar{c}]$, respectively.*

Proof In order to prove that there exists a unique solution $(\bar{\gamma}(c), \bar{\zeta}(c))$ of (6.12) in $[\underline{c}, \bar{c}]$ for some $\underline{c} < \bar{c}$, it suffices to show that

$$C_{11}(b^*(\bar{c}), z^*(\bar{c}), \bar{c}) \neq 0 \quad \text{and} \quad C_{22}(b^*(\bar{c}), z^*(\bar{c}), \bar{c}) \neq 0$$

for \bar{c} large enough. Combining (7.1) and (7.5) with the formulas of $C_{11}(y, z, c)$ and $C_{22}(y, z, c)$ given in (A.9) and (A.11), we obtain that

$$C_{11}(b^*(\bar{c}), z^*(\bar{c}), \bar{c}) = -\frac{32(1-a)^2 a q}{\sigma^{10}} \bar{c}^5 + O(\bar{c}^4),$$

$$C_{22}(b^*(\bar{c}), z^*(\bar{c}), \bar{c}) e^{(z^*(\bar{c}) - b^*(\bar{c}))\theta_1(\bar{c})} = \frac{32(1-a)q\mu}{\sqrt{a}\sigma^{10}} \bar{c}^3 + O(\bar{c}^2),$$

and so

$$C_{11}(b^*(\bar{c}), z^*(\bar{c}), \bar{c}) < 0 \quad \text{and} \quad C_{22}(b^*(\bar{c}), z^*(\bar{c}), \bar{c}) > 0$$

for \bar{c} large enough.

In order to prove that $\bar{\gamma}(c)$ is increasing and $\bar{\zeta}(c)$ is decreasing in $[\underline{c}, \bar{c}]$ for \bar{c} large enough and some $\underline{c} < \bar{c}$, we use the differential equations (6.12) at $c = \bar{c}$ and the power series expansion of C_{ij} to show that

$$\gamma'(\bar{c}) = \frac{aq\sigma^2 + \mu^2}{2aq} \frac{1}{\bar{c}^2} + O\left(\frac{1}{\bar{c}^3}\right),$$

$$\zeta'(\bar{c}) = -\frac{3q\sigma^4}{4} \frac{1}{\bar{c}^4} + O\left(\frac{1}{\bar{c}^5}\right)$$

for \bar{c} large enough; so we have the result. □

In the following result, we show that the reward function $W^{\bar{\gamma}, \bar{\zeta}}$ of the two-curve strategy given by the solutions of $(\bar{\gamma}, \bar{\zeta})$ of (6.12) with boundary conditions (6.13) is the value function in $[0, \infty) \times [\underline{c}, \bar{c}]$ for \bar{c} large enough and some $\underline{c} < \bar{c}$. So the optimal strategy is a two-curve strategy.

Theorem 8.2 *There exist a large $\bar{c} > q\sigma^2/(2\mu)$ and some $\underline{c} < \bar{c}$ such that $W^{\bar{\gamma}, \bar{\zeta}} = V$ in $[0, \infty) \times [\underline{c}, \bar{c}]$.*

Proof By Proposition 8.1, there exist \bar{c} large enough and some $\underline{c} < \bar{c}$ such that $\bar{\zeta}'(c) \neq 0$, and so by Proposition 6.9, $W^{\bar{\gamma}, \bar{\zeta}}$ is $(2, 1)$ -differentiable in $[0, \infty) \times [\underline{c}, \bar{c}]$. Using Theorem 6.10, in order to prove the result, it is sufficient to show that

$$\begin{aligned} \partial_x W^{\bar{\gamma}, \bar{\zeta}}(x, c) &\geq 1 && \text{for } x \in [0, \bar{\gamma}(c)), \\ \partial_x W^{\bar{\gamma}, \bar{\zeta}}(x, c) &\leq 1 && \text{for } x \in [\bar{\gamma}(c), \bar{\zeta}(c)], \\ \partial_c W^{\bar{\gamma}, \bar{\zeta}}(x, c) &\leq 0 && \text{for } x \in [0, \bar{\zeta}(c)] \end{aligned} \tag{8.1}$$

for $c \in [\underline{c}, \bar{c}]$. We have from Proposition 6.8 that $\partial_x(W^{\bar{\gamma}, \bar{\zeta}})(\bar{\gamma}(c), c) = 1$ for $c \in [\underline{c}, \bar{c}]$, and the Taylor expansion of $\partial_{xx}v^{\bar{c}}(x)$ at $\bar{c} = \infty$ is given by

$$\begin{cases} -\frac{q}{ac} + \frac{q(qx-2\mu)}{a^2c^2} + O(\frac{1}{c^3}) & \text{if } x < b^*(\bar{c}), \\ -\frac{q}{c} + \frac{q(qx-2\mu)}{c^2} + O(\frac{1}{c^3}) & \text{if } x \geq b^*(\bar{c}), \end{cases}$$

which is negative so that $\partial_{xx}v^{\bar{c}}(x) < 0$ for \bar{c} large enough. Since $\partial_{xx}(W^{\bar{\gamma}, \bar{\zeta}})(x, c)$ is continuous, there exists $\underline{c} < \bar{c}$ such that $\partial_{xx}(W^{\bar{\gamma}, \bar{\zeta}})(x, c) < 0$ in (x, c) for $c \in [\underline{c}, \bar{c}]$ and $x \in [0, \bar{\zeta}(c)]$. We conclude that (8.1) holds for $c \in [\underline{c}, \bar{c}]$ and \bar{c} large enough.

Let us show that for \bar{c} large enough and some $\underline{c} < \bar{c}$, we have $\partial_c W^{\bar{\gamma}, \bar{\zeta}}(x, c) \leq 0$ for $c \in [\underline{c}, \bar{c}]$ and $0 \leq x \leq \bar{\zeta}(c)$. We prove first that $\partial_c W^{\bar{\gamma}, \bar{\zeta}}(x, c) \leq 0$ for $x \in [\bar{\gamma}(c), \bar{\zeta}(c)]$. We have

$$\begin{aligned} \partial_c W^{\bar{\gamma}, \bar{\zeta}}(x, c) &= \partial_c H^{\bar{\gamma}, \bar{\zeta}}(x, c) \\ &= \frac{d}{dc} \left(f_{20}(\bar{\gamma}(c), x, c) \right) + \frac{d}{dc} \left(f_{21}(\bar{\gamma}(c), x, c) \right) A^{\bar{\gamma}, \bar{\zeta}}(c) \\ &\quad + f_{21}(\bar{\gamma}(c), x, c) (A^{\bar{\gamma}, \bar{\zeta}})'(c) \\ &= f_{21}(\bar{\gamma}(c), x, c) \left(-b_0(\bar{\gamma}(c), x, \bar{\gamma}'(c), c) \right. \\ &\quad \left. - b_1(\bar{\gamma}(c), x, \bar{\gamma}'(c), c) A^{\bar{\gamma}, \bar{\zeta}}(c) + (A^{\bar{\gamma}, \bar{\zeta}})'(c) \right), \end{aligned}$$

and by Lemma A.1, $f_{21}(y, x, c) > 0$ for $x > y$. So we should prove that

$$G(x, \bar{c}) := -b_0(\bar{\gamma}(\bar{c}), x, \bar{\gamma}'(\bar{c}), \bar{c}) - b_1(\bar{\gamma}(\bar{c}), x, \bar{\gamma}'(\bar{c}), \bar{c}) A^{\bar{\gamma}, \bar{\zeta}}(\bar{c}) + (A^{\bar{\gamma}, \bar{\zeta}})'(\bar{c}) < 0$$

for $x \in [\bar{\gamma}(c), \bar{\zeta}(c)]$. By Proposition 6.8, $0 = \partial_c H^{\bar{\gamma}, \bar{\zeta}}(x, \bar{c}) = \partial_{cx} H^{\bar{\gamma}, \bar{\zeta}}(x, \bar{c})$, and so we have $G(\bar{\zeta}(\bar{c}), \bar{c}) = \partial_x G(\bar{\zeta}(\bar{c}), \bar{c}) = 0$. Then it is sufficient to prove that $\partial_{xx}G(x, c) < 0$ for $x \in [\bar{\gamma}(c), \bar{\zeta}(c)]$. We first show that $\partial_{xx}G(x, \bar{c}) < 0$ for $x \in [\bar{\gamma}(\bar{c}), \bar{\zeta}(\bar{c})]$ for \bar{c} large enough, and then the result follows for $c \in [\underline{c}, \bar{c}]$ for some $\underline{c} < \bar{c}$ by continuity arguments in a compact set. Using $\bar{\gamma}(\bar{c}) = b^*(\bar{c})$, $\bar{\zeta}(\bar{c}) = z^*(\bar{c})$, (7.1) and (7.5), we obtain that the Taylor expansion at $\bar{c} = \infty$ of

$$h(x, \bar{c}) := \frac{\partial_{xx}G(x, \bar{c})}{e^{-(x-\bar{\gamma}(\bar{c}))\theta_1(\bar{c})-\bar{\gamma}(\bar{c})\theta_1(a\bar{c})-\bar{\gamma}(\bar{c})\theta_2(\bar{c})}}$$

is given by

$$h(x, \bar{c}) = \frac{2(a(qx - \mu)^2 - \mu^2)}{a^2q\sigma^4} + \frac{12q\mu\sigma^2 - 10\mu^3 - 4a^2(qx - \mu)^2(qx + 2\mu) + 6a(q^2x^2\mu + 3\mu^2)}{3a^3q\sigma^4} \frac{1}{\bar{c}} + O\left(\frac{1}{\bar{c}^2}\right),$$

and the Taylor expansion of $h(z^*(\bar{c}), \bar{c})$ at $\bar{c} = \infty$ is given by

$$h(z^*(\bar{c}), \bar{c}) = -\frac{2\mu}{a^{\frac{3}{2}}\sigma^2} \frac{1}{\bar{c}} + O\left(\frac{1}{\bar{c}^2}\right).$$

Since

$$\partial_x h(x, \bar{c}) = \frac{4(a(\bar{c} - qx - \mu)(qx - \mu) + qx\mu)}{a^2\sigma^4} \frac{1}{\bar{c}} + O\left(\frac{1}{\bar{c}^2}\right)$$

is positive and $h(z^*(\bar{c}), \bar{c}) < 0$ for \bar{c} large enough, we conclude that $\partial_{xx}G(x, \bar{c}) < 0$ for $x \in [\bar{\gamma}(\bar{c}), \bar{\zeta}(\bar{c})]$. Let us show that for \bar{c} large enough and some $\underline{c} < \bar{c}$, it holds that $\partial_c W^{\bar{\gamma}, \bar{\zeta}}(x, c) \leq 0$ for $x \in [0, \bar{\gamma}(c)]$ and $c \in [\underline{c}, \bar{c}]$. We can write

$$\begin{aligned} \partial_c W^{\bar{\gamma}, \bar{\zeta}}(x, c) &= \partial_c (f_{10}(x, c) + f_{11}(x, c)A^{\bar{\gamma}, \bar{\zeta}}(c)) \\ &= f_{11}(x, c) \left(\frac{\partial_c f_{10}(x, c)}{f_{11}(x, c)} + \frac{\partial_c f_{11}(x, c)}{f_{11}(x, c)} A^{\bar{\gamma}, \bar{\zeta}}(c) + (A^{\bar{\gamma}, \bar{\zeta}})'(c) \right), \end{aligned}$$

where $f_{11}(x, c) > 0$; so we should prove that

$$G_1(x, \bar{c}) := \frac{\partial_c f_{10}(x, c)}{f_{11}(x, c)} + \frac{\partial_c f_{11}(x, c)}{f_{11}(x, c)} A^{\bar{\gamma}, \bar{\zeta}}(c) + (A^{\bar{\gamma}, \bar{\zeta}})'(c) < 0$$

for $x \in [0, \bar{\gamma}(c)]$. We have shown that $\partial_c W^{\bar{\gamma}, \bar{\zeta}}(\bar{\gamma}(c), c) < 0$ so that $G_1(\bar{\gamma}(c), \bar{c}) < 0$; then it suffices to prove that $\partial_x G_1(x, c) > 0$ for $x \in [0, \bar{\gamma}(c)]$. We show first that $\partial_x G_1(x, \bar{c}) > 0$ for $x \in [0, \bar{\gamma}(\bar{c})]$ for \bar{c} large enough; then the result follows for $c \in [\underline{c}, \bar{c}]$ with some $\underline{c} < \bar{c}$ by continuity arguments in a compact set. Using $\bar{\gamma}(\bar{c}) = b^*(\bar{c})$, (7.1) and (7.5), we obtain that the Taylor expansion at $\bar{c} = \infty$ of

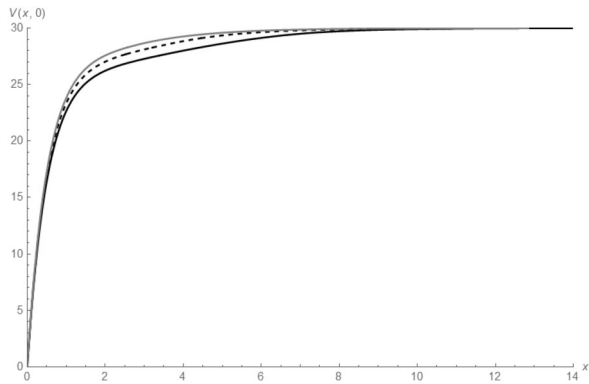
$$h_1(x, \bar{c}) := e^{x\theta_1(a\bar{c})} \partial_x G_1(x, \bar{c})$$

is given by

$$h_1(x, \bar{c}) = \frac{x(2\mu - qx)}{\sigma^2} \frac{1}{\bar{c}} + O\left(\frac{1}{\bar{c}^2}\right),$$

which is positive for \bar{c} large enough and $x \leq b^*(\bar{c}) < \frac{\mu}{q} < \frac{2\mu}{q}$. □

Fig. 3 $V_0^3(x, 0)$ (gray solid), $V_{0.5}^3(x, 0)$ (dashed) and $V_1^3(x, 0)$ (black solid) as a function of x



9 Numerical examples

In this section, we consider some numerical illustrations for the case $q = 0.1$, $\mu = 4$ and $\sigma = 2$.

9.1 Bounded case

Let us first consider the case with an upper bound $\bar{c} = 3$ for the dividend rate. In this case, we are able to derive the value function and the optimal strategies for the problem with drawdown constraints $a = 0.2$, $a = 0.5$ and $a = 0.8$. Indeed, they are of two-curve type as conjectured in Remark 6.11. The obtained reward function and optimal dividend strategies then also allow us to compare them with those for the (already previously known) extreme cases $a = 0$ (classical dividend problem without any constraint) and $a = 1$ (dividend problem with ratcheting constraint). To obtain the value functions $V_a^{\bar{c}}$ for each set of parameters, we proceed as follows:

- 1) We check that there exists a unique zero $z^*(\bar{c})$ of $C_0(b^*(\bar{c}), \cdot, \bar{c})$ in $(b^*(\bar{c}), \infty)$.
- 2) We obtain the curves $\bar{\gamma}$ and $\bar{\zeta}$ by solving numerically, by the Euler method, the system (6.12) of ordinary differential equations with boundary conditions (6.13).
- 3) We check numerically that the pair $(\bar{\gamma}, \bar{\zeta})$ satisfies (6.11) for $c \in [0, \bar{c}]$. So by Proposition 6.6, we are approximating the unique solution $(\bar{\gamma}, \bar{\zeta})$. We also verify that $\bar{\zeta}$ is nondecreasing.
- 4) We check that the function $W^{\bar{\gamma}, \bar{\zeta}}$ defined in (6.6) satisfies the conditions of Theorem 4.4. Hence $W^{\bar{\gamma}, \bar{\zeta}}$ is the value function $V_a^{\bar{c}}$, and the optimal strategy is indeed a two-curve strategy given by $(\bar{\gamma}, \bar{\zeta}) \in \mathcal{B}$.

Figure 3 depicts the graphs of $V_a^{\bar{c}}(x, 0)$ with $\bar{c} = 3$ for $a = 0$ (no restrictions, gray solid), $a = 0.5$ (dashed) and $a = 1$ (ratcheting, black solid) as a function of x . One can nicely see how the drawdown case is – in terms of performance – a compromise between the unconstrained case and the stronger constraint of ratcheting.

In order to see the impact of the drawdown restriction more clearly, we plot in Fig. 4 the difference between $V_0^{\bar{c}}(x)$ (the unconstrained value function) and $V_a^{\bar{c}}(x, 0)$

Fig. 4 $V_0^3(x) - V_a^3(x, 0)$ for $a = 0.2$ (dotted), $a = 0.5$ (dashed), $a = 0.8$ (dotted–dashed) and $a = 1$ (solid)

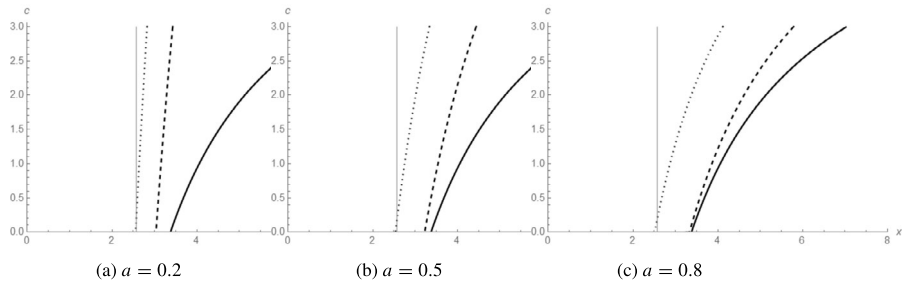
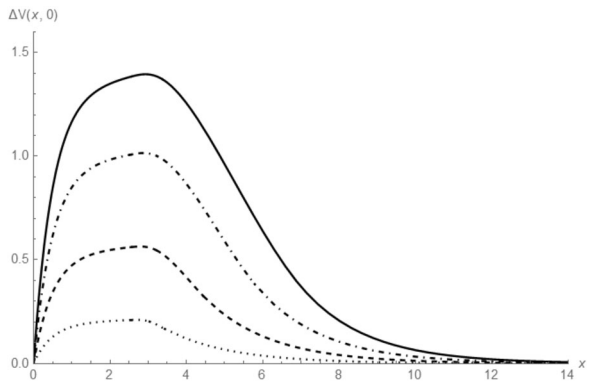


Fig. 5 Optimal drawdown curves $(\bar{\gamma}(c), c)$ (dotted) and $(\bar{\zeta}(c), c)$ (dashed) for $a = 0.2, 0.5, 0.8$, together with the optimal threshold of the unconstrained problem ($a = 0$, solid gray) and the optimal ratcheting curve $(\bar{\xi}(c), c)$ ($a = 1$, solid black)

as a function of x for increasingly restrictive drawdown levels $a = 0.2$ (dotted), $a = 0.5$ (dashed), $a = 0.8$ (dotted–dashed) and finally $a = 1$ (ratcheting, solid). One observes that in particular for smaller values of x , the relaxation of ratcheting towards the drawdown constraint improves the performance of the resulting strategy quite a bit, although the relative gap between the performance of the ratcheting and the unconstrained cases is anyway not so big (cf. Fig. 3). The latter speaks in favour of the consideration of such strategies, as ratcheting and drawdown may be important for shareholders from a psychological point of view, and the efficiency loss when introducing these constraints is quite minor. In particular, if for a given initial surplus level x , one has a target efficiency loss one is willing to accept, results like Fig. 4 can help to identify the corresponding drawdown coefficient a that can still guarantee such a performance.

In terms of the nature of the optimal strategy (which indeed turns out to be of two-curve type), Fig. 5 shows the optimal drawdown curves $(\bar{\gamma}(c), c)$ (dotted) and $(\bar{\zeta}(c), c)$ (dashed) for $a = 0.2, a = 0.5$ and $a = 0.8$, respectively. In all plots, we also depict the optimal threshold of the unconstrained dividend problem $a = 0$ (solid gray) and the optimal ratcheting curve $(\bar{\xi}(c), c)$ for $a = 1$ (solid black). To that end, recall from Asmussen and Taksar [7] that the optimal threshold for $a = 0$ is

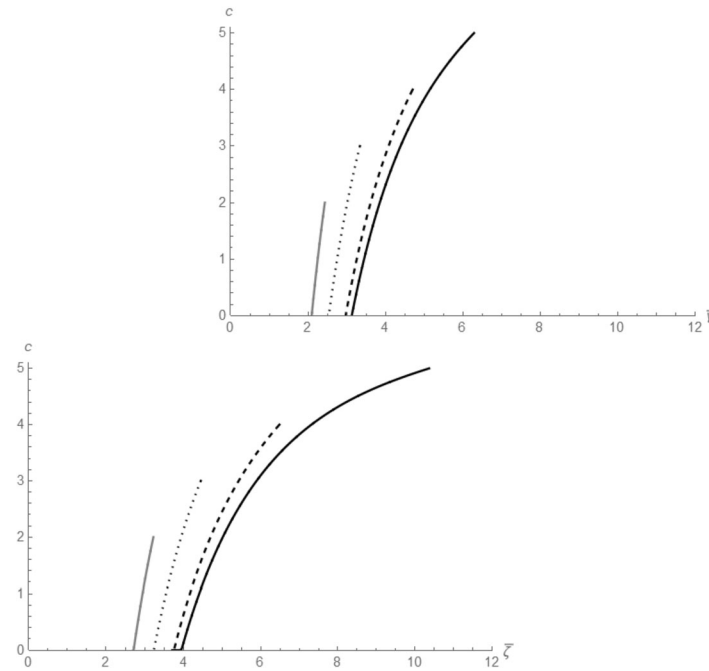


Fig. 6 The curves $\bar{\gamma}^{\bar{c}}(c)$ (left) and $\bar{\zeta}^{\bar{c}}(c)$ (right) for $a = 0.5$: $\bar{c} = 2$ (solid gray), $\bar{c} = 3$ (dotted), $\bar{c} = 4 = \mu$ (dashed) and $\bar{c} = 5$ (solid black)

given by

$$\frac{1}{\theta_1(0) - \theta_2(0)} \log \frac{\theta_2(0) (\theta_2(0) - \theta_2(\bar{c}))}{\theta_1(0) (\theta_1(0) - \theta_2(\bar{c}))},$$

whereas the optimal strategy in the ratcheting case is given by a one-curve strategy which is obtained numerically according to the results in Albrecher et al. [1]. One can nicely see how the two curves $(\bar{\gamma}(c), c)$ and $(\bar{\zeta}(c), c)$ move towards the right as a increases, interpolating between the unconstrained and the ratcheting cases. Note that the resulting two-curve shapes are somewhat reminiscent of some figures obtained in Guo and Tomecek [22] for other types of singular control problems, where also a smooth-fit principle was established.

Notice also that the location of these curves can vary considerably as the maximally allowed dividend rate \bar{c} changes. Figure 6 depicts $\bar{\gamma}^{\bar{c}}(c)$ and $\bar{\zeta}^{\bar{c}}(c)$ for $a = 0.5$ for \bar{c} growing from 2 to 5. In particular, when \bar{c} is larger, the necessary surplus level x to switch to higher dividend rates is larger as well. Figure 7 shows the corresponding value functions for these increasing values of \bar{c} ($a = 0.5$). Recall that while the draw-down constraint is not a major efficiency loss when compared to the unconstrained case for the same \bar{c} (cf. Fig. 3 for the case $\bar{c} = 3$), the size of \bar{c} itself naturally has a considerable impact on the size of the value function.

Fig. 7 $V_{0.5}^2(x, 0)$ (solid gray), $V_{0.5}^3(x, 0)$ (dotted), $V_{0.5}^4(x, 0)$ (dashed) and $V_{0.5}^5(x, 0)$ (black solid) as a function of x

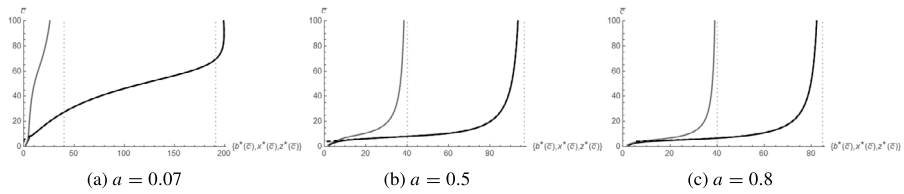
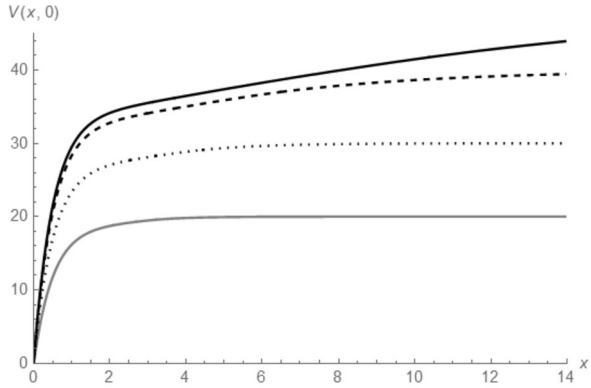


Fig. 8 The boundary condition values $b^*(\bar{c})$ (grey), $z^*(\bar{c})$ (solid) and $x^*(\bar{c})$ (dashed) as a function of \bar{c} for different values of a

9.2 Boundary conditions

Let us now investigate the situation when the maximally allowed dividend rate \bar{c} becomes large. In addition to $a = 0.5$ and $a = 0.8$, we now also consider a smaller drawdown level $a = 0.07$ (in order to illustrate the different monotonicity for small values of a , cf. Remark 7.5). One finds numerically that there exists a unique zero $z^*(\bar{c})$ of $C_0(b^*(\bar{c}), \cdot, \bar{c})$ in $(b^*(\bar{c}), \infty)$ for any $\bar{c} \geq 0$. We have also found that there exists a unique zero $x^*(\bar{c})$ in $(0, \infty)$ of $\partial_{\bar{c}} V^{\bar{c}}(\cdot, \bar{c})$ for $\bar{c} \geq 5.17$ for $a = 0.07$, for $\bar{c} \geq 3.45$ for $a = 0.5$, and for $\bar{c} \geq 2.52$ for $a = 0.8$. Recall that we have proved in Propositions 7.1–7.3 that $\lim_{\bar{c} \rightarrow \infty} z^*(\bar{c}) = \lim_{\bar{c} \rightarrow \infty} x^*(\bar{c}) = \mu(1 + 1/\sqrt{a})/q$ and $\lim_{\bar{c} \rightarrow \infty} b^*(\bar{c}) = \mu/q$.

Figure 8 shows the curves of the boundary conditions $(b^*(\bar{c}), \bar{c})$, $(z^*(\bar{c}), \bar{c})$ and $(x^*(\bar{c}), \bar{c})$ for $a = 0.07$, $a = 0.5$ and $a = 0.8$, respectively. In the case $a = 0.07$, one sees how the limit $\mu(1 + 1/\sqrt{a})/q = 191.2$ (vertical dotted line) is indeed approached from the right as $\bar{c} \rightarrow \infty$, whereas for $a = 0.5$ and $a = 0.8$, the respective limits 96.57 and 84.72 (vertical dotted line) are approached from the left; cf. Remark 7.5. It is important to keep in mind that these plots only depict the boundary value for each choice of \bar{c} , and are not to be confused with the optimal drawdown curves in Fig. 5. Note that $x^*(\bar{c})$ and $z^*(\bar{c})$ are – already for moderate values of \bar{c} – almost identical, with $z^*(\bar{c}) > x^*(\bar{c})$; see Fig. 9 for a graph of the difference $z^*(\bar{c}) - x^*(\bar{c})$ for $a = 0.07$, $a = 0.5$ and $a = 0.8$, respectively. From the latter, one nicely sees $z^*(\bar{c}) > x^*(\bar{c})$ (cf. Remark 7.4) as well as the asymptotic equivalence (7.10) of the two quantities.

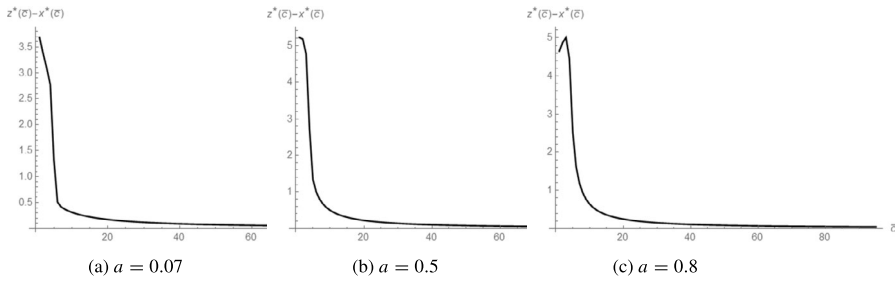


Fig. 9 The difference $z^*(\bar{c}) - x^*(\bar{c})$ as a function of \bar{c} for different values of a

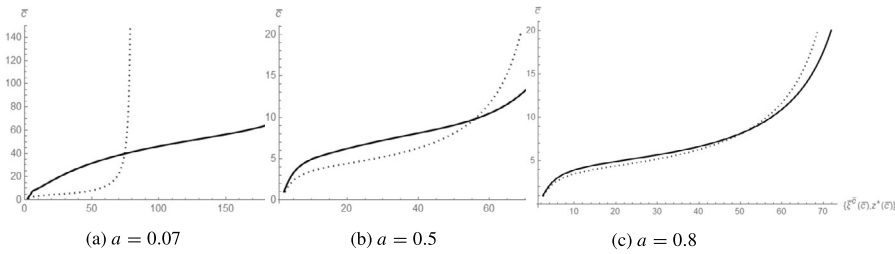


Fig. 10 The boundary values $z^*(\bar{c})$ (solid) and the optimal ratcheting boundary value $\bar{\xi}^{\bar{c}}(\bar{c})$ (dotted) as \bar{c} grows, for different values of a

In Fig. 5, we saw that the curve $\bar{\zeta}^{\bar{c}}(c)$ is to the left of the ratcheting curve $\bar{\xi}^{\bar{c}}(c)$. At the same time, for large values of \bar{c} , we know that $\bar{\zeta}^{\bar{c}}(c)$ must be to the right of $\bar{\xi}^{\bar{c}}(c)$, as

$$\lim_{\bar{c} \rightarrow \infty} \bar{\zeta}^{\bar{c}}(\bar{c}) = \frac{2\mu}{q} < \frac{\mu}{q} \left(1 + \frac{1}{\sqrt{a}} \right) = \lim_{\bar{c} \rightarrow \infty} z^*(\bar{c}).$$

It is therefore of interest to see when this crossing for the limiting value takes place. Figure 10 depicts $z^*(\bar{c})$ (solid) and $\bar{\xi}^{\bar{c}}(\bar{c})$ (dotted) for $a = 0.07$, $a = 0.5$ and $a = 0.8$, respectively. We see that indeed $z^*(\bar{c}) < \bar{\xi}^{\bar{c}}(\bar{c})$ for \bar{c} small, and $z^*(\bar{c}) > \bar{\xi}^{\bar{c}}(\bar{c})$ for \bar{c} large. Moreover, we obtain numerically that the intersection point of the curves $z^*(\bar{c})$ and $\bar{\xi}^{\bar{c}}(\bar{c})$ occurs at $\bar{c} = 39.70$ for $a = 0.07$, at $\bar{c} = 9.74$ for $a = 0.5$, and at $\bar{c} = 8.37$ for $a = 0.8$ for the given set of parameters. That is, if \bar{c} exceeds that threshold, the possibility of the drawdown increases the level of the surplus above which one starts to pay the maximal dividend rate, when compared to pure ratcheting, and it is intuitive that the difference is less pronounced as a increases.

10 Conclusions

In this paper, we have addressed the problem of optimal dividends under a drawdown constraint. We have shown that the value function can be expressed as the

unique viscosity solution of a two-dimensional Hamilton–Jacobi–Bellman equation and have derived conditions under which the optimal strategy is of a two-curve form. We conjecture that these conditions are in fact always fulfilled and – using a smooth-fit principle – have proved this for large values of current and maximal dividend rate c and \bar{c} , respectively. For concrete numerical examples, we have also proved the optimality of two-curve strategies numerically for small values of c and \bar{c} , and have shown how to identify the resulting optimal curves. This turns out to be a very challenging and technical task, involving the numerical solution of a highly involved system of ordinary differential equations and its boundary conditions. We have illustrated how this can be concretely implemented for a moderate size of \bar{c} ; for high values of \bar{c} , this is numerically difficult because the formulas involve algebraic sums with terms with exponentials with very large exponents and the computations require very high numerical precision. We furthermore have shown that when \bar{c} tends to infinity, the curves converge to a finite limit, the size of which follows a surprisingly simple and intriguing formula in terms of the square-root of the drawdown percentage a for any volatility parameter σ . The latter fact also allows getting some intuition on the nature of this limit from the deterministic limit case $\sigma = 0$. Altogether, this paper is the first to explicitly address a drawdown constraint for a control problem in this context, and it has turned out that the resulting strategies smoothly interpolate between the unconstrained problem and the situation with ratcheting constraints, allowing to get some quantitative insight in the efficiency gain when relaxing the ratcheting. It will be interesting to see whether other dividend – and more generally control – problems can be extended in a similar way. In particular, extending the results from the Brownian risk model to a compound Poisson surplus process may be an interesting endeavour, which would lead to a relaxation of the ratcheting problem studied in Albrecher et al. [2]. Another future direction of research may be to extend the approach of the present paper to incorporate constraints on the dividend rate in terms of an average of its previous values, for instance along the lines of Angoshtari et al. [6].

Appendix A: Results and proofs for Sect. 6

We first state some definitions and formulas used to define the functions b_0 and b_1 introduced in (6.2). We set

$$\begin{aligned} d(y, z, c) &= e^{(z-y)\theta_1(c)+y\theta_2(ac)}(\theta_2(c) - \theta_2(ac)) - e^{(z-y)\theta_1(c)+y\theta_1(ac)}(\theta_2(c) - \theta_1(ac)) \\ &\quad + e^{(z-y)\theta_2(c)+y\theta_2(ac)}(\theta_2(ac) - \theta_1(c)) + e^{(z-y)\theta_2(c)+y\theta_1(ac)}(\theta_1(c) - \theta_1(ac)), \quad (\text{A.1}) \end{aligned}$$

$$\begin{aligned}
 b_{11}(y, z, c) &= -(\theta_1(c) - \theta_2(c)) \\
 &\quad \times \left(e^{y\theta_1(ac)} (\theta_1(ac) - \theta_1(c)) (\theta_1(ac) - \theta_2(c)) \right. \\
 &\quad \left. + e^{y\theta_2(ac)} (\theta_2(c) - \theta_2(ac)) (\theta_2(ac) - \theta_1(c)) \right). \quad (\text{A.2})
 \end{aligned}$$

$$\begin{aligned}
 b_{01}(y, z, c) &= c(\theta_1(c) - \theta_2(c)) \\
 &\quad \times \left(a e^{y\theta_2(ac)} (\theta_2(c) - \theta_2(ac)) (\theta_1(c) - \theta_2(ac)) + (1 - a)\theta_1(c)\theta_2(c) \right),
 \end{aligned}$$

$$\begin{aligned}
 &b_{10}(y, z, c) \\
 &= e^{(z-y)\theta_1(c)+y\theta_1(ac)} \left((\theta_1(c) - \theta_2(c)) \left(\theta_2'(c) + (z-y)(\theta_2(c) - \theta_1(ac))\theta_1'(c) \right. \right. \\
 &\quad \left. \left. - a\theta_1'(ac) \left(1 + y(\theta_1(ac) - \theta_2(c)) \right) \right) \right. \\
 &\quad \left. + (\theta_1(ac) - \theta_2(c)) (\theta_1'(c) - \theta_2'(c)) \right) \\
 &\quad - e^{(z-y)\theta_1(c)+y\theta_2(ac)} \left((\theta_1(c) - \theta_2(c)) \left(\theta_2'(c) + (z-y)(\theta_2(c) - \theta_2(ac))\theta_1'(c) \right. \right. \\
 &\quad \left. \left. - a\theta_2'(ac) \left(1 + y(\theta_2(ac) - \theta_2(c)) \right) \right) \right. \\
 &\quad \left. + (\theta_2(c) - \theta_2(ac)) (\theta_1'(c) - \theta_2'(c)) \right) \\
 &\quad - e^{(z-y)\theta_2(c)+y\theta_1(ac)} \left((\theta_1(c) - \theta_2(c)) \left(\theta_1'(c) + (z-y)(\theta_1(c) - \theta_1(ac))\theta_2'(c) \right. \right. \\
 &\quad \left. \left. - a\theta_1'(ac) \left(1 + y(\theta_1(ac) - \theta_1(c)) \right) \right) \right. \\
 &\quad \left. + (\theta_1(c) - \theta_1(ac)) (\theta_1'(c) - \theta_2'(c)) \right) \\
 &\quad + e^{(z-y)\theta_2(c)+y\theta_2(ac)} \left((\theta_1(c) - \theta_2(c)) \left(\theta_1'(c) + (z-y)(\theta_1(c) - \theta_2(ac))\theta_2'(c) \right. \right. \\
 &\quad \left. \left. - a\theta_2'(ac) \left(1 + y(\theta_2(ac) - \theta_1(c)) \right) \right) \right. \\
 &\quad \left. + (\theta_2(ac) - \theta_1(c)) (\theta_1'(c) - \theta_2'(c)) \right),
 \end{aligned}$$

$$\begin{aligned}
 & b_{00}(y, z, c) \\
 &= e^{(z-y)\theta_2(c)}(1-a)\left((\theta_1(c)-\theta_2(c))\theta_1(c)(1+c(z-y)\theta'_2(c))\right. \\
 &\quad \left.+c(\theta_1(c)\theta'_2(c)-\theta'_1(c)\theta_2(c))\right) \\
 &\quad - e^{(z-y)\theta_1(c)}(1-a)\left((\theta_1(c)-\theta_2(c))\theta_2(c)(1+c(z-y)\theta'_1(c))\right. \\
 &\quad \left.+c(\theta_1(c)\theta'_2(c)-\theta'_1(c)\theta_2(c))\right) \\
 &\quad - ae^{(z-y)\theta_1(c)+y\theta_2(ac)}\left((\theta_1(c)-\theta_2(c))(\theta_2(c)-\theta_2(ac))\right. \\
 &\quad \quad \times (1+c(z-y)\theta'_1(c)+acy\theta'_2(ac)) \\
 &\quad \quad +c(\theta_1(c)\theta'_2(c)-\theta'_1(c)\theta_2(c)) \\
 &\quad \quad \left.+c\theta_2(ac)(\theta'_1(c)-\theta'_2(c))-ac\theta'_2(ac)(\theta_1(c)-\theta_2(c))\right) \\
 &\quad + ae^{(z-y)\theta_2(c)+y\theta_2(ac)}\left((\theta_1(c)-\theta_2(c))(\theta_1(c)-\theta_2(ac))\right. \\
 &\quad \quad \times (1+c(z-y)\theta'_2(c)+acy\theta'_2(ac)) \\
 &\quad \quad +c(\theta_1(c)\theta'_2(c)-\theta'_1(c)\theta_2(c)) \\
 &\quad \quad \left.+c\theta_2(ac)(\theta'_1(c)-\theta'_2(c))-ac\theta'_2(ac)(\theta_1(c)-\theta_2(c))\right) \\
 &\quad - (\theta_1(c)-\theta_2(c))^2.
 \end{aligned}$$

Lemma A.1 *The function $d(y, z, c)$ defined in (A.1) is positive in $T \times [0, \bar{c}]$, and so b_0 and b_1 introduced in (6.2) are well defined.*

Proof Using $\theta_1 > 0 > \theta_2$ and $\theta'_1, \theta'_2 > 0$, define $g(y, h, c) = d(y, y + h, c)/e^{h\theta_2(c)}$. Then

$$g(y, 0, c) = (e^{y\theta_1(ac)} - e^{y\theta_2(ac)})(\theta_1(c) - \theta_2(c)) > 0$$

for $y > 0$ and

$$\begin{aligned}
 \partial_h g(y, h, c) &= (\theta_2(c) - \theta_1(c))e^{h(\theta_1(c)-\theta_2(c))} \\
 &\quad \times \left(e^{y\theta_1(ac)}(\theta_2(c) - \theta_1(ac)) + e^{y\theta_2(ac)}(\theta_2(ac) - \theta_2(c))\right) > 0
 \end{aligned}$$

for $y \geq 0$. So the result holds. □

To prove Proposition 6.2, we introduce some auxiliary formulas. Set

$$f_{10}(x, c) := \frac{ca}{q}(1 - e^{\theta_2(ac)x}), \tag{A.3}$$

$$f_{11}(x, c) := e^{\theta_1(ac)x} - e^{\theta_2(ac)x}, \tag{A.4}$$

$$\begin{aligned}
 f_{20}(y, x, c) &:= \frac{c}{q(\theta_2(c) - \theta_1(c))} \\
 &\times \left(\theta_2(c) + (a - 1)e^{\theta_1(c)(x-y)}\theta_2(c) \right. \\
 &\quad \left. + ae^{y\theta_2(ac)} \left(-e^{\theta_2(c)(x-y)}\theta_2(ac) + e^{\theta_1(c)(x-y)}(\theta_2(ac) - \theta_2(c)) \right) \right. \\
 &\quad \left. + \theta_1(c) \left(-1 + e^{\theta_2(c)(x-y)}(1 + a(e^{y\theta_2(ac)} - 1)) \right) \right), \tag{A.5}
 \end{aligned}$$

$$f_{21}(y, x, c) := \frac{d(y, x, c)}{\theta_1(c) - \theta_2(c)}. \tag{A.6}$$

Proof of Proposition 6.2 As the function $H^{\gamma, \zeta}(\cdot, c)$ is continuously differentiable at $x = \gamma(c)$ and satisfies $H^{\gamma, \zeta}(0, c) = 0$, $\mathcal{L}^{ac}(H^{\gamma, \zeta})(x, c) = 0$ for $0 \leq x < \gamma(c)$ and $\mathcal{L}^c(H^{\gamma, \zeta})(x, c) = 0$ for $\gamma(c) \leq x$, there exists by (5.1) and (5.3) a function $A(c)$ with

$$\begin{aligned}
 H^{\gamma, \zeta}(x, c) &= (f_{10}(x, c) + f_{11}(x, c)A(c))I_{\{x < \gamma(c)\}} \\
 &\quad + f_{20}(\gamma(c), x, c) + f_{21}(\gamma(c), x, c)A(c))I_{\{x \geq \gamma(c)\}}.
 \end{aligned}$$

Let us find a formula for the function $A(c)$. Since

$$H^{\gamma, \zeta}(x, \bar{c}) = v(x, \bar{c}, \gamma(\bar{c})),$$

we obtain by (5.4) that

$$A(\bar{c}) = \frac{B(\bar{c}, \gamma(\bar{c}))}{\sqrt{(\mu - a\bar{c})^2 + 2q\sigma^2}}.$$

Using that $\partial_c H^{\gamma, \zeta}(x, c)|_{x=\zeta(c)} = 0$ for all $c \in [0, \bar{c}]$ and that $\zeta(c) > \gamma(c)$, we get

$$\begin{aligned}
 0 &= \partial_c H^{\gamma, \zeta}(x, c)|_{x=\zeta(c)} = \frac{d}{dc} \left(f_{20}(\gamma(c), x, c) + f_{21}(\gamma(c), x, c)A(c) \right) \Big|_{x=\zeta(c)} \\
 &= \left(\frac{d}{dc} \left(f_{20}(\gamma(c), x, c) \right) + \frac{d}{dc} \left(f_{21}(\gamma(c), x, c) \right) A(c) \right. \\
 &\quad \left. + f_{21}(\gamma(c), x, c) A'(c) \right) \Big|_{x=\zeta(c)},
 \end{aligned}$$

and $f_{21}(y, x, c) = \frac{d(y, x, c)}{\theta_1(c) - \theta_2(c)} > 0$ for $x > y$ by Lemma A.1. So we obtain

$$\begin{aligned}
 A'(c) &= \frac{-\frac{d}{dc}(f_{20}(\gamma(c), x, c))}{f_{21}(\gamma(c), x, c)} \Big|_{x=\zeta(c)} + \frac{-\frac{d}{dc}(f_{21}(\gamma(c), x, c))}{f_{21}(\gamma(c), x, c)} \Big|_{x=\zeta(c)} A(c) \\
 &= b_0(\gamma(c), \zeta(c), \gamma'(c), c) + b_1(\gamma(c), \zeta(c), \gamma'(c), c)A(c)
 \end{aligned}$$

at the points where ζ is continuous, with b_0 and b_1 defined in (6.2). Since ζ is Riemann-integrable, it is differentiable almost everywhere. Note that the function

A is the unique solution of this ODE. Since the function A depends on γ and ζ , we call it $A^{\gamma,\zeta}$. Hence we have the result. \square

Definition A.2 Let $\zeta : [0, \bar{c}] \rightarrow [0, \infty)$ be a step function, that is,

$$\zeta(c) := \sum_{i=1}^{n-1} z_i I_{[c_i, c_{i+1})}(c)$$

with $0 = c_1 < c_2 < \dots < c_n = \bar{c}$ and $z_i > 0$. Then the two-curve strategy defined at the beginning of Sect. 6 and described in Fig. 1, starting with an initial surplus x and initial running maximum dividend rate c , is given as follows:

(1) If $0 \leq x < \zeta(c)$, that is, $(x, c) \in \mathcal{NC}(\gamma, \zeta)$, follow the refracting strategy which pays at the rate ac when the current surplus is below a refracting threshold $\gamma(c)$ and at c when the current surplus is above $\gamma(c)$, until either reaching the curve $\mathcal{R}(\zeta)$ or ruin (whichever comes first).

(2) If $x \geq \zeta(c)$, that is, $(x, c) \in \mathcal{CH}(\gamma, \zeta)$, increase immediately the dividend rate to $\ell(x, c) \in \{c_2, \dots, c_n\}$; note that

$$\ell(x, c) = \max\{c_i \geq c : z_k \leq x \text{ for } h(c) \leq k \leq i - 1\},$$

where $h(c) := \min\{h : c_h \geq c\}$. We denote this two-curve strategy by $\pi^{(\gamma,\zeta)}$.

In the next result, we show by using a Feynman–Kac argument that if ζ is a step function, then $W^{\gamma,\zeta}$ is the reward function of a two-curve strategy. In the lemma that follows, we show that in the general case $(\gamma, \zeta) \in \mathcal{B}$, $W^{\gamma,\zeta}$ is the limit of reward functions of two-curve strategies.

Lemma A.3 Consider $(\gamma, \zeta) \in \mathcal{B}$, with ζ being a step function. Let $D^{x,c} \in \Pi_{x,c}^{[0,\bar{c}]}$ be the admissible strategy corresponding to the two-curve strategy $\pi^{(\gamma,\zeta)}$ starting in (x, c) . Calling $j(x, c) := J(x; D^{x,c})$, we obtain that j is continuous in $[0, \infty) \times [0, \bar{c}]$ and $j(x, c) = W^{\gamma,\zeta}(x, c)$.

Proof We prove inductively that $j(x, c_i)$ is continuous in x for $i = 1, \dots, n$. First, $j(\cdot, c_i)$ is differentiable in $[0, z_i)$ because it corresponds to the reward function of a refracting dividend strategy at $x = \gamma(c_i)$ with a given boundary condition at $x = z_i$ (see for instance Albrecher et al. [3, Theorem 3.1]). In the case $i = n$, $j(x, c_n) = v(x, c, \gamma(\bar{c}))$ which is continuous in x ; in the case $i < n$, $j(x, c_i)$ is continuous in x for $x \leq z_i$ because

$$j(x, c_i) = (f_{10}(x, c_i) + f_{11}(x, c_i)A_i)I_{\{x < \gamma(c_i)\}} + (f_{20}(\gamma(c_i), x, c_i) + f_{21}(\gamma(c_i), x, c)A_i)I_{\{x \geq \gamma(c_i)\}}$$

for some constant A_i , and $j(x, c_i) = j(x, c_{i+1})$ for $x \geq z_i$. As $j(x, c) = j(x, c_{i+1})$ for $c \in (c_i, c_{i+1})$, we conclude that j is continuous in $[0, \infty) \times [0, \bar{c}]$.

Let us now show that $j(x, c)$ satisfies the assumptions of Proposition 6.2 and so $j(x, c) = H^{\gamma,\zeta}(x, c) = W^{\gamma,\zeta}(x, c)$ for $0 \leq x \leq \zeta(c)$. Indeed, it is straightforward that

$j(\cdot, \bar{c}) = H^{\gamma, \zeta}(\cdot, \bar{c}) = v(\cdot, c, \gamma(\bar{c}))$, $j(\cdot, c)$ is continuously differentiable for any $c \in [0, \bar{c})$, $\mathcal{L}^{ac}(j)(x, c) = 0$ for $0 \leq x < \gamma(c)$, $\mathcal{L}^c(j)(x, c) = 0$ for $\gamma(c) \leq x \leq \zeta(c)$ and $j(0, c) = 0$. Also $\partial_c j(\zeta(c), c) = 0$ at the points of continuity of ζ because $j(x, c) = j(x, c_{i+1})$ for $x \geq \zeta(c) = z_i$ in the case $c \in (c_i, c_{i+1})$.

From the definition of $\pi^{(\gamma, \zeta)}$, it is straightforward that $j(x, c) = H^{\gamma, \zeta}(x, \ell(x, c))$ if $x \geq \zeta(c)$; so we get the result. □

In the next result, we show that for any $(\gamma, \zeta) \in \mathcal{B}$, the function $W^{\gamma, \zeta}$ is the uniform limit of reward functions of curve strategies, where the ζ_k are step functions with $\zeta_k \rightarrow \zeta$.

Lemma A.4 *Given $(\gamma, \zeta) \in \mathcal{B}$, there exists a sequence of right-continuous step functions $\zeta_k : [0, \bar{c}] \rightarrow [0, \infty)$ such that $W^{\gamma, \zeta_k}(x, c)$ converges uniformly to $W^{\gamma, \zeta}(x, c)$.*

Proof Since ζ is a Riemann-integrable càdlàg function, we can approximate it uniformly by right-continuous step functions. Namely, take a sequence of finite sets $\mathcal{S}^k = \{c_1^k, c_2^k, \dots, c_{n_k}^k\}$ with $0 = c_1^k < c_2^k < \dots < c_{n_k}^k = \bar{c}$ and consider the right-continuous step functions

$$\zeta_k(c) = \sum_{i=1}^{n_k-1} \zeta(c_i^k) I_{[c_i^k, c_{i+1}^k)}(c)$$

with $\delta(\mathcal{S}^k) = \max_{i=1, \dots, n_k-1} (c_{i+1}^k - c_i^k) \rightarrow 0$. We have that $\zeta_k \rightarrow \zeta$ uniformly, and so both $A^{\gamma, \zeta_k} \rightarrow A^{\gamma, \zeta}$ and $W^{\gamma, \zeta_k} \rightarrow W^{\gamma, \zeta}$ uniformly. □

Lemma A.5 *For a given $c \in [0, \bar{c})$, consider $\gamma : [c, \bar{c}] \rightarrow (0, \infty)$ which is continuously differentiable and $\zeta : [c, \bar{c}] \rightarrow (0, \infty)$ which is bounded, Riemann-integrable and càdlàg. Define the set*

$$\mathcal{B}_c = \left\{ (\gamma, \zeta) \text{ such that } \gamma \leq \zeta \text{ in } [c, \bar{c}] \text{ and } \lim_{c \rightarrow \bar{c}-} \zeta(c) = \zeta(\bar{c}-) \right\}.$$

If $(\gamma_0, \zeta_0) \in \mathcal{B}$ satisfies (6.7), then for any $c \in [0, \bar{c})$,

$$A^{\gamma_0, \zeta_0}(c) = \max\{A^{\gamma, \zeta}(c) : \zeta \in \mathcal{B}_c\}.$$

Proof Given $(\gamma, \zeta) \in \mathcal{B}$, we can write

$$\begin{aligned} A^{\gamma, \zeta}(c) &= A^{\gamma, \zeta}(\bar{c}) e^{-\int_c^{\bar{c}} b_1(\gamma(s), \zeta(s), \gamma'(s), s) ds} \\ &\quad - \int_c^{\bar{c}} e^{-\int_c^t b_1(\gamma(s), \zeta(s), \gamma'(s), s) ds} b_0(\gamma(t), \zeta(t), \gamma'(t), t) dt, \\ A^{\gamma, \zeta}(0) &= - \int_0^c e^{-\int_0^t b_1(\gamma(s), \zeta(s), \gamma'(s), s) ds} b_0(\gamma(t), \zeta(t), \gamma'(t), t) dt \\ &\quad + (e^{-\int_0^c b_1(\gamma(s), \zeta(s), \gamma'(s), s) ds}) A^{\gamma, \zeta}(c). \end{aligned}$$

Note that the pair $(\gamma_0|_{[c,\bar{c}]}, \zeta_0|_{[c,\bar{c}]})$ is in \mathcal{B}_c . Indeed, given any $(\gamma, \zeta) \in \mathcal{B}_c$ and any function $\chi : [0, \bar{c}] \rightarrow [0, 1]$ which is continuously differentiable with $\chi = 0$ in $[0, c]$, we define the two functions $\gamma_1(s) = \gamma_0(s)I_{\{0 \leq s < c\}} + \gamma(s)\chi(s)I_{\{c \leq s \leq \bar{c}\}}$ and $\zeta_1(s) = \zeta_0(s)I_{\{0 \leq s < c\}} + \zeta(s)\chi(s)I_{\{c \leq s \leq \bar{c}\}}$. Then $(\gamma_1, \zeta_1) \in \mathcal{B}$. Moreover,

$$A^{\gamma_0, \zeta_0}(0) \geq A^{\gamma_1, \zeta_1}(0) = - \int_0^c e^{-\int_0^t b_1(\gamma_0(s), \zeta_0(s), s) ds} b_0(\gamma_0(t), \zeta_0(t), t) dt + (e^{-\int_0^c b_1(\gamma_0(s), \zeta_0(s), s) ds}) A^{\gamma\chi, \zeta\chi}(c).$$

Hence

$$\begin{aligned} A^{\gamma_0, \zeta_0}(0) &\geq - \int_0^c e^{-\int_0^t b_1(\gamma_0(s), \zeta_0(s), s) ds} b_0(\gamma_0(t), \zeta_0(t), t) dt + e^{-\int_0^c b_1(\gamma_0(s), \zeta_0(s), s) ds} \sup_{(\gamma, \zeta) \in \mathcal{B}_c, \chi} A^{\gamma\chi, \zeta\chi}(c) \\ &= - \int_0^c e^{-\int_0^t b_1(\gamma_0(s), \zeta_0(s), s) ds} b_0(\gamma_0(t), \zeta_0(t), t) dt + e^{-\int_0^c b_1(\gamma_0(s), \zeta_0(s), s) ds} \sup_{(\gamma, \zeta) \in \mathcal{B}_c} A^{\gamma, \zeta}(c) \\ &\geq - \int_0^c e^{-\int_0^t b_1(\gamma_0(s), \zeta_0(s), s) ds} b_0(\gamma_0(t), \zeta_0(t), t) dt + e^{-\int_0^c b_1(\gamma_0(s), \zeta_0(s), s) ds} A^{\gamma_0, \zeta_0}(c) \\ &= A^{\gamma_0, \zeta_0}(0), \end{aligned}$$

and so we have $\sup_{(\gamma, \zeta) \in \mathcal{B}_c} A^{\gamma, \zeta}(c) = A^{\gamma_0, \zeta_0}(c)$. □

Proof of Proposition 6.4 Consider any function $(\gamma_1, \zeta_1) \in \mathcal{B}$ with $\gamma_1(\bar{c}) = \zeta_1(\bar{c}) = 0$. Then

$$\begin{aligned} &A^{\gamma_0 + \eta\gamma_1, \zeta_0 + \varepsilon\zeta_1}(0) \\ &= A^{\gamma_0, \zeta_0}(\bar{c}) e^{-\int_0^{\bar{c}} b_1(\gamma_0(t) + \eta\gamma_1(t), \zeta_0(t) + \varepsilon\zeta_1(t), \gamma_0'(t) + \eta\gamma_1'(t), t) dt} \\ &\quad - \int_0^{\bar{c}} e^{-\int_0^s b_1(\gamma_0(u) + \eta\gamma_1(u), \zeta_0(u) + \varepsilon\zeta_1(u), \gamma_0'(u) + \eta\gamma_1'(u), u) du} \\ &\quad \times b_0(\gamma_0(s) + \eta\gamma_1(s), \zeta_0(s) + \varepsilon\zeta_1(s), \gamma_0'(s) + \eta\gamma_1'(s), s) ds. \end{aligned}$$

Taking the derivative with respect to ε at $\eta = \varepsilon = 0$, we get

$$\begin{aligned}
0 &= \partial_\varepsilon A^{\gamma_0 + \eta\gamma_1, \zeta_0 + \varepsilon\zeta_1}(0) \Big|_{\eta=0, \varepsilon=0} \\
&= \int_0^{\bar{c}} \left(e^{-\int_0^s b_1(u)du} b_0(s) \left(\int_0^s \zeta_1(u) b_{1z}(u) du \right) \right) ds \\
&\quad - \int_0^{\bar{c}} \left(e^{-\int_0^s b_1(u)du} \zeta_1(s) b_{0z}(s) \right) ds \\
&\quad - A^{\gamma_0, \zeta_0}(\bar{c}) e^{-\int_0^{\bar{c}} b_1(t)dt} \int_0^{\bar{c}} \zeta_1(s) b_{0z}(s) ds.
\end{aligned}$$

Using integration by parts, we obtain

$$\begin{aligned}
&\int_0^{\bar{c}} \left(e^{-\int_0^s b_1(u)du} b_0(s) \left(\int_0^s \zeta_1(u) b_{1z}(u) du \right) \right) ds \\
&= \int_0^{\bar{c}} \left(\zeta_1(s) b_{1z}(s) \left(\int_s^{\bar{c}} e^{-\int_0^t b_1(u)du} b_0(t) dt \right) \right) ds,
\end{aligned}$$

and so

$$\begin{aligned}
0 &= \int_0^{\bar{c}} \zeta_1(s) \left(b_{1z}(s) \int_s^{\bar{c}} e^{-\int_0^t b_1(u)du} b_0(t) dt - e^{-\int_0^s b_1(u)du} b_{0z}(s) \right) \\
&\quad - A^{\gamma_0, \zeta_0}(\bar{c}) e^{-\int_0^{\bar{c}} b_1(t)dt} b_{0z}(s) \Big) ds.
\end{aligned}$$

Since this holds for any ζ_1 with $\zeta_1(\bar{c}) = 0$, we get using (6.4) that for any $c \in [0, \bar{c})$,

$$\begin{aligned}
0 &= b_{1z}(c) \int_c^{\bar{c}} e^{-\int_0^t b_1(u)du} b_0(t) dt - e^{-\int_0^c b_1(u)du} b_{0z}(c) \\
&\quad - A^{\gamma_0, \zeta_0}(\bar{c}) e^{-\int_0^{\bar{c}} b_1(t)dt} b_{0z}(c) \\
&= e^{-\int_0^c b_1(u)du} \left(-b_{1z}(c) A^{\gamma_0, \zeta_0}(c) - b_{0z}(c) \right),
\end{aligned}$$

and so we deduce (6.8).

Taking the derivative with respect to η at $\eta = \varepsilon = 0$, we get

$$\begin{aligned}
0 &= \partial_\eta A^{\gamma_0 + \eta\gamma_1, \zeta_0 + \varepsilon\zeta_1}(0) \Big|_{\eta=0, \varepsilon=0} \\
&= \int_0^{\bar{c}} \left(e^{-\int_0^s b_1(u)du} b_0(s) \left(\int_0^s \gamma_1'(u) b_{1w}(u) du \right) \right) ds \\
&\quad + \int_0^{\bar{c}} \left(e^{-\int_0^s b_1(u)du} b_0(s) \left(\int_0^s \gamma_1(u) b_{1y}(u) du \right) \right) ds \\
&\quad - \int_0^{\bar{c}} \left(e^{-\int_0^s b_1(u)du} \gamma_1'(s) b_{0w}(s) \right) ds - \int_0^{\bar{c}} \left(e^{-\int_0^s b_1(u)du} \gamma_1(s) b_{0y}(s) \right) ds \\
&\quad - A^{\gamma_0, \zeta_0}(\bar{c}) e^{-\int_0^{\bar{c}} b_1(t)dt} \int_0^{\bar{c}} \gamma_1'(t) b_{1z}(t) dt \\
&\quad - A^{\gamma_0, \zeta_0}(\bar{c}) e^{-\int_0^{\bar{c}} b_1(t)dt} \int_0^{\bar{c}} \gamma_1(t) b_{1y}(t) dt.
\end{aligned}$$

Using integration by parts, we obtain

$$\begin{aligned}
 0 = & \gamma_1(0) \left(-b_{1w}(0) \int_0^{\bar{c}} e^{-\int_0^t b_1(u)du} b_0(t)dt + b_{0w}(0) \right. \\
 & \left. + e^{-\int_0^{\bar{c}} b_1(t)dt} A^{\gamma_0, \zeta_0}(\bar{c}) b_{1w}(0) \right) \\
 & + \gamma_1(\bar{c}) \left(-e^{-\int_0^{\bar{c}} b_1(t)dt} b_{0w}(\bar{c}) - e^{-\int_0^{\bar{c}} b_1(t)dt} A^{\gamma_0, \zeta_0}(\bar{c}) b_{1w}(\bar{c}) \right) \\
 & + \int_0^{\bar{c}} \gamma_1(s) \left(\frac{d}{ds} b_{1w}(s) \right) \left(e^{-\int_s^{\bar{c}} b_1(t)dt} A^{\gamma_0, \zeta_0}(\bar{c}) - \int_s^{\bar{c}} e^{-\int_s^t b_1(u)du} b_0(t)dt \right) ds \\
 & + \int_0^{\bar{c}} \gamma_1(s) \left(b_{1y}(s) \left(\int_s^{\bar{c}} e^{-\int_s^t b_1(u)du} b_0(t)dt - e^{-\int_s^{\bar{c}} b_1(t)dt} A^{\gamma_0, \zeta_0}(\bar{c}) \right) \right) ds \\
 & + \int_0^{\bar{c}} \gamma_1(s) \left(b_{1w}(s) b_0(s) + \frac{d}{ds} b_{1w}(s) - b_{0w}(s) b_1(s) - b_{0y}(s) \right) ds.
 \end{aligned}$$

Using that $\gamma_1(\bar{c}) = 0$ and (6.4), we get

$$\begin{aligned}
 0 = & \gamma_1(0) (b_{1w}(0) A^{\gamma_0, \zeta_0}(0) + b_{0w}(0)) \\
 & + \int_0^{\bar{c}} \gamma_1(s) e^{-\int_0^s b_1(u)du} \left(\left(\frac{d}{ds} b_{1w}(s) - b_{1y}(s) \right) A^{\gamma_0, \zeta_0}(s) + \frac{d}{ds} b_{0w}(s) \right. \\
 & \left. - b_{0y}(s) + b_{1w}(s) b_0(s) - b_{0w}(s) b_1(s) \right) ds.
 \end{aligned}$$

Since this holds for any γ_1 with $\gamma_1(\bar{c}) = 0$, we obtain

$$\left(\frac{d}{ds} b_{1w}(s) - b_{1y}(s) \right) A^{\gamma_0, \zeta_0}(s) + \frac{d}{ds} b_{0w}(s) - b_{0y}(s) + b_{1w}(s) b_0(s) - b_{0w}(s) b_1(s) = 0 \tag{A.7}$$

for all $c \in [0, \bar{c}]$ and

$$b_{1w}(0) A^{\gamma_0, \zeta_0}(0) + b_{0w}(0) = 0.$$

By Lemma A.5, taking the derivative $0 = \partial_\eta A^{\gamma_0 + \eta\gamma_1, \zeta_0 + \varepsilon\zeta_1}(c) \Big|_{\eta=0, \varepsilon=0}$, we also obtain that (6.9) holds. Note that with (6.9), we have

$$\begin{aligned}
 0 = & \frac{d}{ds} (b_{1w}(s) A^{\gamma_0, \zeta_0}(s) + b_{0w}(s)) \\
 = & \left(\frac{d}{ds} b_{1w}(s) \right) A^{\gamma_0, \zeta_0}(s) + \frac{d}{ds} b_{0w}(s) + b_0(s) b_{1w}(s) - b_1(s) b_{0w}(s)
 \end{aligned}$$

and so, from (A.7)

$$\begin{aligned}
0 &= \left(\frac{d}{ds} b_{1w}(s) \right) A^{\gamma_0, \zeta_0}(s) - b_{1y}(s) A^{\gamma_0, \zeta_0}(s) + \frac{d}{ds} b_{0w}(s) - b_{0y}(s) + b_{1w}(s) b_0(s) \\
&\quad - b_{0w}(s) b_1(s) \\
&= \left(\frac{d}{ds} b_{1w}(s) \right) A^{\gamma_0, \zeta_0}(s) + \frac{d}{ds} b_{0w}(s) + b_{1w}(s) b_0(s) - b_{0w}(s) b_1(s) \\
&\quad - b_{1y}(s) A^{\gamma_0, \zeta_0}(s) - b_{0y}(s) \\
&= \frac{d}{ds} (b_{1w}(s) A^{\gamma_0, \zeta_0}(s) + b_{0w}(s)) - (b_{1y}(s) A^{\gamma_0, \zeta_0}(s) + b_{0y}(s)) \\
&= -(b_{1y}(s) A^{\gamma_0, \zeta_0}(s) + b_{0y}(s)),
\end{aligned}$$

from which we deduce (6.10). \square

To prove Proposition 6.5, we next introduce some auxiliary formulas. Set

$$C_0(y, z, c) := b_{11}(y, c) \partial_z \left(\frac{b_{00}(y, z, c)}{d(y, z, c)} \right) - b_{01}(y, c) \partial_z \left(\frac{b_{10}(y, z, c)}{d(y, z, c)} \right), \quad (\text{A.8})$$

$$\begin{aligned}
C_{11}(y, z, c) &:= b_{11}(y, c) \partial_y \left(\frac{(e^{(z-y)\theta_1(c)} - e^{(z-y)\theta_2(c)}) b_{01}(y, c)}{d(y, z, c)} \right) \\
&\quad - b_{01}(y, c) \partial_y \left(\frac{(e^{(z-y)\theta_1(c)} - e^{(z-y)\theta_2(c)}) b_{11}(y, c)}{d(y, z, c)} \right), \quad (\text{A.9})
\end{aligned}$$

$$C_{10}(y, z, c) := b_{01}(y, c) \partial_y \left(\frac{b_{10}(y, z, c)}{d(y, z, c)} \right) - b_{11}(y, c) \partial_y \left(\frac{b_{00}(y, z, c)}{d(y, z, c)} \right), \quad (\text{A.10})$$

$$\begin{aligned}
C_{21}(y, z, c) &:= \partial_y C_0(y, z, c) \\
&= \partial_y \left(b_{11}(y, c) \partial_z \left(\frac{b_{00}(y, z, c)}{d(y, z, c)} \right) - b_{01}(y, c) \partial_z \left(\frac{b_{10}(y, z, c)}{d(y, z, c)} \right) \right), \\
C_{22}(y, z, c) &:= \partial_z C_0(y, z, c) \\
&= \partial_z \left(b_{11}(y, c) \partial_z \left(\frac{b_{00}(y, z, c)}{d(y, z, c)} \right) - b_{01}(y, c) \partial_z \left(\frac{b_{10}(y, z, c)}{d(y, z, c)} \right) \right), \quad (\text{A.11})
\end{aligned}$$

$$\begin{aligned}
C_{20}(y, z, c) &:= -\partial_c C_0(y, z, c) \\
&= -\partial_c \left(b_{11}(y, c) \partial_z \left(\frac{b_{00}(y, z, c)}{d(y, z, c)} \right) - b_{01}(y, c) \partial_z \left(\frac{b_{10}(y, z, c)}{d(y, z, c)} \right) \right).
\end{aligned}$$

Proof of Proposition 6.5 From (6.8) and (6.9), we have

$$(b_{1w} b_{0z} - b_{0w} b_{1z})(\gamma_0(c), \zeta_0(c), \gamma'_0(c), c) = 0 \quad \text{for } c \in [0, \bar{c}].$$

From (6.2), we can write

$$\begin{aligned} & (b_{1w}b_{0z} - b_{0w}b_{1z})(y, z, w, c) \\ &= \frac{e^{(z-y)\theta_1(c)} - e^{(z-y)\theta_2(c)}}{q(\theta_1(c) - \theta_2(c))^2 d(y, z, c)} \\ & \quad \times \left(b_{11}(y, c)\partial_z\left(\frac{b_{00}(y, z, c)}{d(y, z, c)}\right) - b_{01}(y, c)\partial_z\left(\frac{b_{10}(y, z, c)}{d(y, z, c)}\right) \right) \\ &= \frac{e^{(z-y)\theta_1(c)} - e^{(z-y)\theta_2(c)}}{q(\theta_1(c) - \theta_2(c))^2 d(y, z, c)} C_0(y, z, c), \end{aligned}$$

which does not depend on w . So we conclude that

$$C_0(\gamma_0(c), \zeta_0(c), c) = 0 \quad \text{for } c \in [0, \bar{c}]. \tag{A.12}$$

Moreover, from Proposition 6.4, we get (6.13). From (6.9) and (6.10), we have

$$(b_{1w}b_{0y} - b_{0w}b_{1y})(\gamma_0(c), \zeta_0(c), \gamma'_0(c), c) = 0 \quad \text{for } c \in [0, \bar{c}],$$

and we can write from (6.2) that

$$\begin{aligned} (b_{1w}b_{0y} - b_{0w}b_{1y})(y, z, w, c) &= \frac{e^{(z-y)\theta_1(c)} - e^{(z-y)\theta_2(c)}}{q(\theta_1(c) - \theta_2(c))^2 d(y, z, c)} \\ & \quad \times (wC_{11}(y, z, c) - C_{10}(y, z, c)). \end{aligned}$$

So since $C_{11}(\gamma_0(c), \zeta_0(c), c) \neq 0$, we get the first equation of (6.12). Taking the derivative of (A.12) with respect to c and using that ζ_0 is continuous, γ_0 is continuously differentiable and $C_{22}(\gamma_0(c), \zeta_0(c), c) \neq 0$, we obtain that ζ_0 is continuously differentiable and

$$\begin{aligned} 0 &= \partial_y C_0(\gamma_0(c), \zeta_0(c), c)\gamma'_0(c) + \partial_z C_0(\gamma_0(c), \zeta_0(c), c)\zeta'_0(c) \\ & \quad + \partial_c C_0(\gamma_0(c), \zeta_0(c), c) \\ &= C_{21}(\gamma_0(c), \zeta_0(c), c)\gamma'_0(c) + C_{22}(\gamma_0(c), \zeta_0(c), c)\zeta'_0(c) - C_{20}(\gamma_0(c), \zeta_0(c), c). \end{aligned}$$

Using the first equation of (6.12), we get the second equation of (6.12). By a recursive argument, we finally obtain that γ_0 and ζ_0 are infinitely differentiable. \square

Proof of Proposition 6.6 Consider

$$c_m = \min \{c \in [0, \bar{c}] : (\gamma_1(d), \zeta_1(c)) = (\gamma_2(d), \zeta_2(d)) \text{ for } d \in [c, \bar{c}]\}.$$

Let us call

$$\begin{aligned} F_1(y, z, c) &= (C_{10}(y, z, c)C_{22}(y, z, c), C_{20}(y, z, c)C_{11}(y, z, c) \\ & \quad - C_{21}(y, z, c)C_{10}(y, z, c)), \\ F_2(y, z, c) &= C_{11}(y, z, c)C_{22}(y, z, c) \end{aligned}$$

and $F(y, z, c) = F_1(y.z, c)/F_2(y.z, c)$. Note that F_1, F_2 and F are infinitely differentiable,

$$(\gamma'_i(c), \zeta'_i(c)) = F(\gamma_i(c), \zeta_i(c), c) \quad \text{for } c \in [0, \bar{c}]$$

and $(\gamma_i(c_m), \zeta_i(c_m)) = (\gamma_i(c_m), \zeta_i(c_m))$ for $i = 1, 2$ and

$$F_2(\gamma_1(c_m), \zeta_1(c_m), c_m) = F_2(\gamma_2(c_m), \zeta_2(c_m), c_m) \neq 0.$$

If $c_m = 0$, we have the result. On the other hand, for $c_m > 0$, using the Picard–Lindelöf theorem, there exists a unique solution of (6.12) with boundary condition $\zeta(c_m) = \zeta_1(c_m)$ in $[\max\{c_m - \delta, 0\}, c_m]$ for some $\delta > 0$, which is a contradiction. \square

Proof of Proposition 6.8 Take a pair of infinitely differentiable functions $(\gamma, \zeta) \in \mathcal{B}$ and consider the function $H^{\gamma, \zeta}(x, c)$ introduced in Proposition 6.2. First note that $H^{\gamma, \zeta}$ satisfies $\mathcal{L}^{ac}(H^{\gamma, \zeta})(x, c) = 0$ for $0 \leq x \leq \gamma(c)$, $\mathcal{L}^c(H^{\gamma, \zeta})(x, c) = 0$ for $x \geq \gamma(c)$, $H^{\gamma, \zeta}(0, c) = 0$, $\partial_c H^{\gamma, \zeta}(x, c)|_{x=\zeta(c)} = 0$ and $H^{\gamma, \zeta}(x, \bar{c}) = v(x, \bar{c}, \gamma(\bar{c}))$. So we have for $x > \gamma(c)$ that

$$\begin{aligned} & \partial_c H^{\gamma, \zeta}(x, c) \\ &= f_{21}(\gamma(c), x, c) \left(-b_0(\gamma(c), x, \gamma'(c), c) - A^{\gamma, \zeta}(c)b_1(\gamma(c), x, \gamma'(c), c) \right), \\ & \partial_{cx} H^{\gamma, \zeta}(x, c)|_{x=\zeta(c)} \\ &= f_{21}(\gamma(c), \zeta(c), c) \left(\partial_x b_0(\gamma(c), x, \gamma'(c), c)|_{x=\zeta(c)} \right. \\ & \qquad \qquad \qquad \left. - A^{\gamma, \zeta}(c)\partial_x b_1(\gamma(c), x, \gamma'(c), c)|_{x=\zeta(c)} \right). \end{aligned}$$

Since $f_{21}(y, x, c) = d(y, x, c)/(\theta_1(c) - \theta_2(c)) > 0$ for $x > y$ by Lemma A.1, we obtain that $\partial_{cx} H^{\gamma, \zeta}(x, c)|_{x=\zeta(c)} = 0$ if and only if (6.8) holds for (γ, ζ) in $[\underline{c}, \bar{c}]$. As $W^{\gamma, \zeta}(x, c) = H^{\gamma, \zeta}(x, c)$ for $x < \zeta(c)$ and $W^{\gamma, \zeta}(x, c) = H^{\gamma, \zeta}(x, C(x, c))$ for $x \geq \zeta(c)$, we get $\partial_c W^{\gamma, \zeta}(x, c) = 0$ for $x \geq \zeta(c)$ and hence $\partial_{cx} W^{\gamma, \zeta}(\zeta(c), c) = 0$. Moreover, $\partial_c H^{\gamma, \zeta}(\zeta(c), c) = 0$ for $c \in [\underline{c}, \bar{c}]$, and so

$$\begin{aligned} 0 &= \frac{d}{dc} \left(\partial_c H^{\gamma, \zeta}(\zeta(c), c) \right) \\ &= \partial_{cc} H^{\gamma, \zeta}(\zeta(c), c) + \partial_{cx} H^{\gamma, \zeta}(\zeta(c), c)\zeta'(c) \\ &= \partial_{cc} H^{\gamma, \zeta}(\zeta(c), c). \end{aligned}$$

In all, since $W^{\gamma, \zeta}(x, c) = H^{\gamma, \zeta}(x, C(x, c))$ if $x \geq \zeta(c)$, we get $\partial_{cc} W^{\gamma, \zeta}(x, c) = 0$ if $x \geq \zeta(c)$ and so $\partial_{cc} W^{\gamma, \zeta}(\zeta(c), c) = 0$.

Secondly, we have that $\partial_y f_{11}(y, c) \neq 0$, $b_{11}(y, z, c) > 0$ and

$$-\frac{b_{01}(y, c)}{qb_{11}(y, c)} = \frac{1 - \partial_y f_{10}(y, c)}{\partial_y f_{11}(y, c)}.$$

So

$$A^{\gamma, \zeta}(c) = -\frac{\partial_w b_0(\gamma(c), \zeta(c), \gamma'(c), c)}{\partial_w b_1(\gamma(c), \zeta(c), \gamma'(c), c)} = -\frac{b_{01}(\gamma(c), c)}{qb_{11}(\gamma(c), c)} = \frac{1 - \partial_y f_{10}(\gamma(c), c)}{\partial_y f_{11}(\gamma(c), c)},$$

and then

$$\begin{aligned} \partial_x(W^{\gamma, \zeta})(\gamma(c), c) &= \partial_x(W^{\gamma, \zeta})(\gamma(c)^-, c) \\ &= \partial_y f_{10}(\gamma(c), c) + A^{\gamma, \zeta}(c) \partial_y f_{11}(\gamma(c), c) = 1 \end{aligned}$$

if and only if (6.9) holds for (γ, ζ) in $[\underline{c}, \bar{c}]$. Note that since

$$\mathcal{L}^c W^{\gamma, \zeta}(\gamma(c)^+, c) = \mathcal{L}^{ac} W^{\gamma, \zeta}(\gamma(c)^-, c) = 0$$

and $\partial_x(W^{\gamma, \zeta})(\gamma(c), c) = 1$, we obtain

$$\begin{aligned} 0 &= \mathcal{L}^c W^{\gamma, \zeta}(\gamma(c)^+, c) - \mathcal{L}^{ac} W^{\gamma, \zeta}(\gamma(c)^-, c) \\ &= \frac{\sigma^2}{2} \left(\partial_{xx} W^{\gamma, \zeta}(\gamma(c)^+, c) - \partial_{xx} W^{\gamma, \zeta}(\gamma(c)^-, c) \right). \end{aligned}$$

Therefore, when $(\bar{\gamma}, \bar{\zeta})$ is a solution of (6.12), it satisfies (6.8) and (6.9), and so

$$\begin{aligned} \partial_{xx} W^{\bar{\gamma}, \bar{\zeta}}(\bar{\gamma}(c)^+, c) &= \partial_{xx} W^{\bar{\gamma}, \bar{\zeta}}(\bar{\gamma}(c)^-, c) \quad \text{for } c \in [\underline{c}, \bar{c}], \\ \partial_{cx} W^{\bar{\gamma}, \bar{\zeta}}(\bar{\zeta}(c), c) &= \partial_{cc} W^{\bar{\gamma}, \bar{\zeta}}(\bar{\zeta}(c), c) = 0 \quad \text{for } c \in [\underline{c}, \bar{c}]. \end{aligned} \quad \square$$

Proof of Proposition 6.9 It holds that

$$\begin{aligned} f_{10}(x, c)|_{x=y} &= f_{20}(y, x, c)|_{x=y}, \\ f_{11}(x, c)|_{x=y} &= f_{21}(y, x, c)|_{x=y}, \\ \partial_x f_{10}(x, c)|_{x=y} &= \partial_x f_{20}(y, x, c)|_{x=y}, \\ \partial_x f_{11}(x, c)|_{x=y} &= \partial_x f_{21}(y, x, c)|_{x=y}, \\ \partial_c f_{10}(x, c)|_{x=y} &= \partial_c f_{20}(y, x, c)|_{x=y}, \\ \partial_c f_{11}(x, c)|_{x=y} &= \partial_c f_{21}(y, x, c)|_{x=y}, \\ \partial_{cx} f_{10}(x, c)|_{x=y} &= \partial_{cx} f_{20}(y, x, c)|_{x=y}, \\ \partial_{cx} f_{11}(x, c)|_{x=y} &= \partial_{cx} f_{21}(y, x, c)|_{x=y}, \\ \partial_y f_{20}(y, x, c)|_{x=y} &= \partial_y f_{21}(y, x, c)|_{x=y} = 0, \\ \partial_{cy} f_{20}(y, x, c)|_{x=y} &= \partial_{cy} f_{21}(y, x, c)|_{x=y} = 0. \end{aligned}$$

By Proposition 6.8, $W_{xx}^{\bar{\gamma}, \bar{\zeta}}(x, c)$ is continuous at $x = \bar{\gamma}(c)$, and so $W^{\bar{\gamma}, \bar{\zeta}}$ is $(2, 1)$ -differentiable for $x < \bar{\zeta}(c)$.

If $\bar{\zeta}'(c) > 0$ in $[\underline{c}, \bar{c}]$, the inverse $\bar{\zeta}^{-1}$ exists and $\ell(x, c)$ can be written as

$$\ell(x, c) = \begin{cases} \bar{c} & \text{if } \bar{\zeta}(\bar{c}) \leq x, \\ \bar{\zeta}^{-1}(x) & \text{if } \bar{\zeta}(c) \leq x < \bar{\zeta}(\bar{c}). \end{cases}$$

In order to show that $W^{\bar{\nu}, \bar{\zeta}}$ is $(2, 1)$ -differentiable, it is enough to prove that

$$\partial_{xx} W^{\bar{\zeta}}(x^+, c) = \partial_{xx} W^{\bar{\zeta}}(x^-, c)$$

for $\bar{\zeta}(c) \leq x < \bar{\zeta}(\bar{c})$. We have by Proposition 6.8 that

$$\begin{aligned} \partial_x W^{\bar{\nu}, \bar{\zeta}}(x^+, c) &= \partial_x H^{\bar{\nu}, \bar{\zeta}}(x, \bar{\zeta}^{-1}(x)) + \partial_c H^{\bar{\nu}, \bar{\zeta}}(x, \bar{\zeta}(x))(\bar{\zeta}^{-1})'(x) \\ &= \partial_x H^{\bar{\nu}, \bar{\zeta}}(x, \bar{\zeta}^{-1}(x)) \\ &= \partial_x W^{\bar{\nu}, \bar{\zeta}}(x^-, c). \end{aligned}$$

Consequently,

$$\begin{aligned} \partial_{xx} W^{\bar{\nu}, \bar{\zeta}}(x^+, c) &= \partial_{xx} H^{\bar{\nu}, \bar{\zeta}}(x, \bar{\zeta}^{-1}(x)) + \partial_{cx} H^{\bar{\nu}, \bar{\zeta}}(x, \bar{\zeta}^{-1}(x))(\bar{\zeta}^{-1})'(x) \\ &= \partial_{xx} H^{\bar{\nu}, \bar{\zeta}}(x, \bar{\zeta}^{-1}(x)) \\ &= \partial_{xx} W^{\bar{\nu}, \bar{\zeta}}(x^-, c). \end{aligned}$$

If $\bar{\zeta}'(c) < 0$ in $[\underline{c}, \bar{c}]$, we have $\ell(x, c) = \bar{c}$ for $x \geq \bar{\zeta}(c)$. To show that $W^{\bar{\nu}, \bar{\zeta}}$ is $(2, 1)$ -differentiable, it is sufficient to prove that $\partial_{xx} W^{\bar{\zeta}}(x^+, c) = \partial_{xx} W^{\bar{\zeta}}(x^-, c)$ for $x = \bar{\zeta}(c)$. Since $\ell(x, c) = \bar{c}$,

$$H^{\bar{\nu}, \bar{\zeta}}(\bar{\zeta}(c), c) = W^{\bar{\nu}, \bar{\zeta}}(\bar{\zeta}(c), c) = v^{\bar{c}}(\bar{\zeta}(c))$$

so that

$$H_x^{\bar{\nu}, \bar{\zeta}}(\bar{\zeta}(c), c)\bar{\zeta}'(c) + H_c^{\bar{\nu}, \bar{\zeta}}(\bar{\zeta}(c), c) = (v^{\bar{c}})'(\bar{\zeta}(c))\bar{\zeta}'(c).$$

Since $H_c^{\bar{\nu}, \bar{\zeta}}(\bar{\zeta}(c), c) = W_c^{\bar{\nu}, \bar{\zeta}}(\bar{\zeta}(c), c) = 0$ and $\bar{\zeta}'(c) < 0$, we get

$$W_x^{\bar{\nu}, \bar{\zeta}}\left((\bar{\zeta}(c))^{-}, c\right) = H_x^{\bar{\nu}, \bar{\zeta}}(\bar{\zeta}(c), c) = (v^{\bar{c}})'(\bar{\zeta}(c)) = W_x^{\bar{\nu}, \bar{\zeta}}\left((\bar{\zeta}(c))^{+}, c\right)$$

so that $H_x^{\bar{\nu}, \bar{\zeta}}(\bar{\zeta}(c), c) = (v^{\bar{c}})'(\bar{\zeta}(c))$. Hence, taking the derivative one more time with respect to c , we obtain

$$H_{xx}^{\bar{\nu}, \bar{\zeta}}(\bar{\zeta}(c), c)\bar{\zeta}'(c) + H_{xc}^{\bar{\nu}, \bar{\zeta}}(\bar{\zeta}(c), c) = (v^{\bar{c}})''(\bar{\zeta}(c))\bar{\zeta}'(c).$$

By Proposition 6.8, $H_{xc}^{\bar{\nu}, \bar{\zeta}}(\bar{\zeta}(c), c) = H_{cx}^{\bar{\nu}, \bar{\zeta}}(\bar{\zeta}(c), c) = 0$ and $\bar{\zeta}'(c) < 0$, and we get

$$W_{xx}^{\bar{\nu}, \bar{\zeta}}\left((\bar{\zeta}(c))^{-}, c\right) = H_{xx}^{\bar{\nu}, \bar{\zeta}}(\bar{\zeta}(c), c) = (v^{\bar{c}})''(\bar{\zeta}(c)) = W_{xx}^{\bar{\nu}, \bar{\zeta}}\left((\bar{\zeta}(c))^{+}, c\right). \quad \square$$

Proof of Theorem 6.10 As $\bar{\zeta}$ is continuous in $[\underline{c}, \bar{c}]$, there exists $M = \max_{c \in [\underline{c}, \bar{c}]} \bar{\zeta}(c)$. By definition, if $x \geq M$, then $\ell(x, c) = \bar{c}$ and $W^{\bar{\gamma}, \bar{\zeta}}(x, c) = v^{\bar{c}}(x)$, and therefore $\lim_{x \rightarrow \infty} W^{\bar{\gamma}, \bar{\zeta}}(x, c) = \lim_{x \rightarrow \infty} v^{\bar{c}}(x) = \bar{c}/q$.

By (5.8), we have that

$$\partial_x v^{\bar{c}}(x) \leq 1 \quad \text{for } x \geq \bar{\zeta}(\bar{c}) \quad \text{and} \quad \partial_x v^{\bar{c}}(b^*(\bar{c})) = 1. \tag{A.13}$$

Since

$$\begin{aligned} \mathcal{L}^{ac}(W^{\bar{\gamma}, \bar{\zeta}})(x, c) - \mathcal{L}^c(W^{\bar{\gamma}, \bar{\zeta}})(x, c) &= (c - ac)\partial_x W^{\bar{\gamma}, \bar{\zeta}}(x, c) + (ac - c) \\ &= c(1 - a)(\partial_x W^{\bar{\gamma}, \bar{\zeta}}(x, c) - 1), \end{aligned}$$

we get $\mathcal{L}^{ac}(W^{\bar{\gamma}, \bar{\zeta}})(x, c) \leq \mathcal{L}^c(W^{\bar{\gamma}, \bar{\zeta}})(x, c) = 0$ for $x \in [\bar{\gamma}(c), \bar{\zeta}(c)]$ and

$$\mathcal{L}^c(W^{\bar{\gamma}, \bar{\zeta}})(x, c) \leq \mathcal{L}^{ac}(W^{\bar{\gamma}, \bar{\zeta}})(x, c) = 0$$

for $x \in [0, \bar{\gamma}(c)]$.

By Theorem 4.4, it remains to prove that we have $\mathcal{L}^{ac}W^{\bar{\gamma}, \bar{\zeta}}(x, c) \leq 0$ and $\mathcal{L}^cW^{\bar{\gamma}, \bar{\zeta}}(x, c) \leq 0$ for $x \geq \bar{\zeta}(c)$, $c \in [\underline{c}, \bar{c}]$. But

$$\ell(x, c) = \max\{h \in [c, \bar{c}] : \bar{\zeta}(h) \leq x \text{ for } h \in [c, h]\}$$

satisfies $\ell(x, c) \geq c$, and also either $\ell(x, c) = \bar{c}$ or $\bar{\zeta}(\ell(x, c)) = x$. So we obtain $\mathcal{L}^\alpha W^{\bar{\gamma}, \bar{\zeta}}(x, \alpha)|_{\alpha=\ell(x, c)} = 0$ and then

$$\begin{aligned} \mathcal{L}^c W^{\bar{\gamma}, \bar{\zeta}}(x, c) &= \mathcal{L}^{\ell(x, c)} W^{\bar{\gamma}, \bar{\zeta}}(x, c) + (\ell(x, c) - c) \left(\partial_x W^{\bar{\gamma}, \bar{\zeta}}(x, \ell(x, c)) - 1 \right) \\ &= (\ell(x, c) - c) \left(\partial_x W^{\bar{\gamma}, \bar{\zeta}}(x, \ell(x, c)) - 1 \right) \leq 0, \end{aligned}$$

because (A.13) and $\partial_x W^{\bar{\gamma}, \bar{\zeta}}(\bar{\zeta}(c), c) \leq 1$ for $c \in [\underline{c}, \bar{c}]$ imply $\partial_x W^{\bar{\gamma}, \bar{\zeta}}(x, \ell(x, c)) \leq 1$. Also,

$$\begin{aligned} \mathcal{L}^{ac}(W^{\bar{\gamma}, \bar{\zeta}})(x, c) - \mathcal{L}^c(W^{\bar{\gamma}, \bar{\zeta}})(x, c) &= c(1 - a)(\partial_x W^{\bar{\gamma}, \bar{\zeta}}(x, c) - 1) \\ &= c(1 - a) \left(\partial_x W^{\bar{\gamma}, \bar{\zeta}}(x, \ell(x, c)) - 1 \right) \leq 0 \end{aligned}$$

for $x \geq \bar{\zeta}(c)$, $c \in [\underline{c}, \bar{c}]$. □

Appendix B: Some formulas of Sects. 7 and 8

In the following, we state in a compact way some formulas referred to in Sects. 7 and 8.

$$\begin{pmatrix} g_1(y, z, c) \\ g_2(y, z, c) \\ g_3(y, z, c) \\ g_4(y, z, c) \\ g_5(y, z, c) \\ g_6(y, z, c) \\ g_7(y, z, c) \\ g_8(y, z, c) \\ g_9(y, z, c) \\ g_{10}(y, z, c) \\ g_{11}(y, z, c) \\ g_{12}(y, z, c) \\ g_{13}(y, z, c) \\ g_{14}(y, z, c) \\ g_{15}(y, z, c) \\ g_{16}(y, z, c) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} (z-y)\theta_1(c) \\ (z-y)\theta_2(c) \\ y\theta_1(ac) \\ y\theta_2(ac) \end{pmatrix},$$

$$\begin{pmatrix} h_1(x, b, c) \\ h_2(x, b, c) \\ h_3(x, b, c) \\ h_4(x, b, c) \\ h_5(x, b, c) \\ h_6(x, b, c) \\ h_7(x, b, c) \\ h_8(x, b, c) \\ h_9(x, b, c) \\ h_{10}(x, b, c) \\ h_{11}(x, b, c) \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} b\theta_1(ac) \\ x\theta_1(ac) \\ b\theta_2(ac) \\ x\theta_2(ac) \end{pmatrix},$$

$$\begin{pmatrix} k_1(x, b, c) \\ k_2(x, b, c) \\ k_3(x, b, c) \\ k_4(x, b, c) \\ k_5(x, b, c) \\ k_6(x, b, c) \\ k_7(x, b, c) \\ k_8(x, b, c) \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} b\theta_1(ac) \\ b\theta_2(ac) \\ (x-b)\theta_2(c) \end{pmatrix}.$$

Funding Note Open access funding provided by University of Lausanne. H. Albrecher acknowledges financial support from the Swiss National Science Foundation Project 200021_191984.

Declarations

Competing Interests The authors declare no competing interests.

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