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Minimax theorems for American options without time-consistency

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Abstract In this paper, we give sufficient conditions guaranteeing the validity of the well-known minimax theorem for the lower Snell envelope. Such minimax results play an important role in the characterisation of arbitrage-free prices of American contingent claims in incomplete markets. Our conditions do not rely on the notions of stability under pasting or time-consistency and reveal some unexpected connection between the minimax result and path properties of the corresponding process of densities. We exemplify our general results in the case of families of measures corresponding to diffusion exponential martingales.

Keywords Minimax · Lower Snell envelope · Time-consistency · Nearly sub-Gaussian random fields · Metric entropies · Simons' lemma

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1 Introduction

Let $0 < T < \infty$ and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathrm{P})$ be a filtered probability space, where $(\mathcal{F}_t)_{0\le t\le T}$ is a right-continuous filtration with $\mathcal{F}_0$ containing only the sets of probability 0 or 1 as well as all the nullsets of $\mathcal{F}_T$. In the sequel, we assume without loss of generality that $\mathcal{F} = \mathcal{F}_T$. Furthermore, let $\mathcal{Q}$ denote a non-empty set of probability measures on $\mathcal{F}$, all absolutely continuous with respect to P. We denote by $\mathcal{L}^1(\mathcal{Q})$ the set of all random variables X on $(\Omega, \mathcal{F}, \mathrm{P})$ which are Q-integrable for every $\mathrm{Q} \in \mathcal{Q}$ and such that $\sup_{\mathrm{Q}\in\mathcal{Q}} \mathbb{E}_{\mathrm{Q}}[|X|] < \infty$. Let $S = (S_t)_{0\le t\le T}$ be a P-semimartingale with respect to $(\mathcal{F}_t)_{0\le t\le T}$ whose trajectories are right-continuous and have finite left limits (càdlàg). Consider also another right-continuous $(\mathcal{F}_t)$-adapted stochastic process $Y = (Y_t)_{0\le t\le T}$ with bounded paths, and let $\mathcal{T}$ stand for the set of all finite stopping times $\tau \le T$ with respect to $(\mathcal{F}_t)_{0\le t\le T}$. We also assume that Y is *quasi-left-upper-semicontinuous* with respect to P, i.e., $\limsup_{n\to\infty} Y_{\tau_n} \le Y_\tau$ P-a.s. holds for any sequence $(\tau_n)_{n\in\mathbb{N}}$ in $\mathcal{T}$ satisfying $\tau_n \nearrow \tau$ for some $\tau \in \mathcal{T}$.

The main objective of our work is to find sufficient conditions for the validity of the minimax result

$$\sup_{\tau\in\mathcal{T}} \inf_{\mathrm{Q}\in\mathcal{Q}} \mathbb{E}_{\mathrm{Q}}[Y_\tau] = \inf_{\mathrm{Q}\in\mathcal{Q}} \sup_{\tau\in\mathcal{T}} \mathbb{E}_{\mathrm{Q}}[Y_\tau]. \tag{1.1}$$

In financial mathematics, this type of result is useful in the characterisation of arbitrage-free prices of American contingent claims in incomplete markets. If $\mathcal{M}$ stands for the family of equivalent local martingale measures with respect to S, i.e.,

$$\mathcal{M} = \{Q \approx P : S \text{ is a local martingale under } Q\},$$

then the set $\Pi(Y)$ of so-called arbitrage-free prices for Y with respect to $\mathcal{M}$ can be defined as the set of all real numbers c fulfilling two properties:

(i) $c \le \mathbb{E}_{\mathrm{Q}}[Y_\tau]$ for some stopping time $\tau \in \mathcal{T}$ and a martingale measure $\mathrm{Q} \in \mathcal{M}$;
(ii) for any stopping time $\tau' \in \mathcal{T}$, there exists some $\mathrm{Q}' \in \mathcal{M}$ such that $c \ge \mathbb{E}_{\mathrm{Q}'}[Y_{\tau'}]$.

The above definition implies that given $c \in \Pi(Y)$, we have

$$\sup_{\tau\in\mathcal{T}} \inf_{\mathrm{Q}\in\mathcal{M}} \mathbb{E}_{\mathrm{Q}}[Y_\tau] \le c \le \sup_{\tau\in\mathcal{T}} \sup_{\mathrm{Q}\in\mathcal{M}} \mathbb{E}_{\mathrm{Q}}[Y_\tau].$$

The following important known result shows that we may characterise the set $\Pi(Y)$ more precisely. It can be found in [23, Theorem 1.20] or [24].

Theorem 1.1 *Suppose that* $\{Y_\tau : \tau \in \mathcal{T}\}$ *is uniformly* Q*-integrable for any* $\mathrm{Q} \in \mathcal{M}$, *and that* $Y = (Y_t)_{0\le t\le T}$ *is upper-semicontinuous in expectation from the left with respect to every* $\mathrm{Q} \in \mathcal{M}$, *that is*, $\limsup_{n\to\infty} \mathbb{E}_{\mathrm{Q}}[Y_{\tau_n}] \le \mathbb{E}_{\mathrm{Q}}[Y_\tau]$ *for any increasing sequence* $(\tau_n)_{n\in\mathbb{N}}$ *converging to some* $\tau \in \mathcal{T}$. *If* $\mathcal{M}$ *denotes the set of equivalent local martingale measures with respect to* S, *then the set* $\Pi(Y)$ *of arbitrage-free prices for* Y *corresponding to* $\mathcal{M}$ *is a real interval with endpoints*

$$\pi_{\inf}(Y) := \inf_{\mathrm{Q}\in\mathcal{M}} \sup_{\tau\in\mathcal{T}} \mathbb{E}_{\mathrm{Q}}[Y_\tau] = \sup_{\tau\in\mathcal{T}} \inf_{\mathrm{Q}\in\mathcal{M}} \mathbb{E}_{\mathrm{Q}}[Y_\tau]$$

and

$$\pi_{\sup}(Y) := \sup_{Q\in\mathcal{M}} \sup_{\tau\in\mathcal{T}} \mathbb{E}_Q[Y_\tau] = \sup_{\tau\in\mathcal{T}} \sup_{Q\in\mathcal{M}} \mathbb{E}_Q[Y_\tau].$$

For a set $\mathcal{Q}$ of P-equivalent probability measures, the so-called *lower Snell envelope of* Y (with respect to $\mathcal{Q}$) is the stochastic process $(U_t^{\downarrow,Y})_{0\le t\le T}$ defined via

$$U_t^{\downarrow,Y} = \operatorname*{ess\,inf}_{Q\in\mathcal{Q}} \operatorname*{ess\,sup}_{\tau\in\mathcal{T},\tau\ge t} \mathbb{E}_Q[Y_\tau|\mathcal{F}_t], \qquad t\in[0,T].$$

Then Theorem 1.1 tells us that the lower Snell envelope with respect to $\mathcal{M}$ at time 0 gives the greatest lower bound for the arbitrage-free price of the corresponding American option.

A natural question is whether a similar characterisation of the lower Snell envelope at time 0 can be proved for sets $\mathcal{Q}$ of equivalent measures which are strictly "smaller" than $\mathcal{M}$. This question can be interesting for at least two reasons. First, the buyer (or the seller) of the option may have some preferences about the set of pricing measures $\mathcal{Q}$ resulting in some additional restrictions on $\mathcal{Q}$ such that $\mathcal{Q}\subseteq\mathcal{M}$. Second, the set of all martingale measures $\mathcal{M}$ may be difficult to describe in a constructive way, as typically only sufficient conditions for the relation $Q\in\mathcal{M}$ are available.

A careful inspection of the proof of Theorem 1.1 reveals that it essentially relies on the minimax identity (1.1) with $\mathcal{Q}=\mathcal{M}$ which is routinely proved in the literature using a special property of $\mathcal{M}$ which is known as stability under pasting. To recall, a set $\mathcal{Q}$ of probability measures on $\mathcal{F}$ is called *stable under pasting with respect to* $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\le t\le T},\mathrm{P})$ if all measures in $\mathcal{Q}$ are equivalent to P and for every $Q_1, Q_2\in\mathcal{Q}$ as well as $\tau\in\mathcal{T}$, the *pasting of* Q_1 *and* Q_2 *in* τ, that is, the probability measure Q_3 defined by the pasting procedure

$$Q_3[A] := \mathbb{E}_{Q_1}\big[Q_2[A|\mathcal{F}_\tau]\big], \qquad A\in\mathcal{F},$$

belongs to $\mathcal{Q}$. Stability under pasting implies that the set $\mathcal{Q}$ is rather "big" if we exclude the trivial case where it consists of only one element. Let us mention that the property of stability under pasting is closely related to the concept of time-consistency. As in [10], we call a set $\mathcal{Q}$ of probability measures on $\mathcal{F}$ which are all equivalent to P *time-consistent with respect to* $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\le t\le T},\mathrm{P})$ if for any $\tau,\sigma\in\mathcal{T}$ with $\tau\le\sigma$ and any P-essentially bounded random variables X, Z, we have the implication

$$\operatorname*{ess\,inf}_{Q\in\mathcal{Q}} \mathbb{E}_Q[X|\mathcal{F}_\sigma] \le \operatorname*{ess\,inf}_{Q\in\mathcal{Q}} \mathbb{E}_Q[Z|\mathcal{F}_\sigma] \Longrightarrow \operatorname*{ess\,inf}_{Q\in\mathcal{Q}} \mathbb{E}_Q[X|\mathcal{F}_\tau] \le \operatorname*{ess\,inf}_{Q\in\mathcal{Q}} \mathbb{E}_Q[Z|\mathcal{F}_\tau].$$

Other contributions to the minimax-relationship (1.1) use the property of *recursiveness* (see [3, 4, 5, 9])

$$\operatorname*{ess\,inf}_{Q\in\mathcal{Q}} \mathbb{E}_Q\left[\operatorname*{ess\,inf}_{Q\in\mathcal{Q}} \mathbb{E}_Q[X|\mathcal{F}_\sigma]\,\middle|\,\mathcal{F}_\tau\right] = \operatorname*{ess\,inf}_{Q\in\mathcal{Q}} \mathbb{E}_Q[X|\mathcal{F}_\tau],$$

assumed for stopping times $\sigma,\tau\in\mathcal{T}$ with $\tau\le\sigma$ and any P-essentially bounded random variable X. It may be easily verified that recursiveness and time-consistency are equivalent (see e.g. [10, proof of Theorem 12]). Moreover, stability under pasting generally implies time-consistency (see Proposition 4.1 below and also [11, Theorem 6.51] for the time-discrete case). To the best of our knowledge, all studies of the minimax-relationship (1.1) so far considered only time-consistent sets $\mathcal{Q}$ (see Sect. 4 for a further discussion on this issue).

In this paper, we formulate conditions of a different kind on the family $\mathcal{Q}$ which do not rely on the notions of consistency or stability, but still ensure the minimax relation (1.1). The key is to impose a certain condition on the range of the mapping

$$\mu_{\mathcal{Q}} : \mathcal{F} \to \ell^{\infty}(\mathcal{Q}),\ \ \mu_{\mathcal{Q}}(A)(\mathrm{Q}) := \mathrm{Q}[A],$$

where $\ell^{\infty}(\mathcal{Q})$ denotes the space of all bounded real-valued mappings on $\mathcal{Q}$. This $\mu_{\mathcal{Q}}$ is a so-called vector measure satisfying $\mu_{\mathcal{Q}}(A_1 \cup A_2) = \mu_{\mathcal{Q}}(A_1) + \mu_{\mathcal{Q}}(A_2)$ for disjoint sets $A_1, A_2 \in \mathcal{F}$. We refer to $\mu_{\mathcal{Q}}$ as the *vector measure associated with* $\mathcal{Q}$.

The paper is organised as follows. In Sect. 2, we present our main result concerning the sets $\mathcal{Q}$ whose associated vector measures have relatively compact range. Next we deduce another criterion in terms of path properties of the corresponding process of densities $(d\mathrm{Q}/d\mathrm{P})_{\mathrm{Q}\in\mathcal{Q}}$. The latter characterisation is especially useful for the case of suitably parametrised families of local martingale measures. Specifically in Sect. 3, we formulate an easy-to-check criterion for the case of processes of densities corresponding to nearly sub-Gaussian families of local martingales. In Sect. 4, we discuss related results from the literature. Section 5 contains a general minimax result for lower Snell envelopes. The proofs of all relevant results are gathered in Sect. 6, whereas the Appendix presents some auxiliary results on path properties of nearly sub-Gaussian random fields.

2 Main results

Let us emphasise that $\mathcal{Q}$ consists of probability measures on $\mathcal{F}$ which are absolutely continuous to P, but need not be equivalent. Throughout this paper, we assume that

$$(\Omega, \mathcal{F}_t, \mathrm{P}|_{\mathcal{F}_t}) \text{ is atomless for every } t > 0. \tag{2.1}$$

Concerning the process Y, we assume that

$$Y^* := \sup_{t\in[0,T]} |Y_t| \in \mathcal{L}^1(\mathcal{Q}), \tag{2.2}$$

and often also

$$\lim_{a\to\infty} \sup_{\mathrm{Q}\in\mathcal{Q}} \mathbb{E}_{\mathrm{Q}}[Y^* \mathbb{1}_{\{Y^*>a\}}] = 0. \tag{2.3}$$

Moreover, the space $\ell^{\infty}(\mathcal{Q})$ is endowed with the sup-norm $\|\cdot\|_{\infty}$. Using the notation $\mathrm{co}(\mathcal{Q})$ for the convex hull of $\mathcal{Q}$, our main minimax result reads as follows.

Theorem 2.1 *Let the range of $\mu_{\mathcal{Q}}$ be relatively $\|\cdot\|_\infty$-compact. If $Y = (Y_t)_{0\leq t\leq T}$ fulfils* (2.3) *and if* (2.1) *holds*, *then*

$$\sup_{\tau\in\mathcal{T}}\inf_{\mathrm{Q}\in\mathcal{Q}}\mathbb{E}_{\mathrm{Q}}[Y_\tau] = \sup_{\tau\in\mathcal{T}}\inf_{\mathrm{Q}\in\mathrm{co}(\mathcal{Q})}\mathbb{E}_{\mathrm{Q}}[Y_\tau] = \inf_{\mathrm{Q}\in\mathrm{co}(\mathcal{Q})}\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathrm{Q}}[Y_\tau]. \tag{2.4}$$

The proof of Theorem 2.1 may be found in Sect. 6.4.

Remark 2.2 Obviously, condition (2.3) implies (2.2). On our way to verifying Theorem 2.1, we establish some auxiliary results which are interesting in their own right. They rely on the weaker condition (2.2) only.

Remark 2.3 Let $\mathcal{Q}$ be relatively compact with respect to the topology of total variation, that is, the topology with metric d_{tv} defined by

$$d_{\mathrm{tv}}(\mathrm{Q}_1,\mathrm{Q}_2) := \sup_{A\in\mathcal{F}}|\mathrm{Q}_1(A)-\mathrm{Q}_2(A)|.$$

Then it is already known that $\{\mu_{\mathcal{Q}}(A) : A\in\mathcal{F}\}$ is relatively $\|\cdot\|_\infty$-compact (cf. [2]). Moreover, if each member of $\mathcal{Q}$ is equivalent to P, then the set $\mathcal{Q}$ is *not* time-consistent with respect to $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\leq t\leq T},\mathrm{P})$ whenever it has more than one element, $(\Omega,\mathcal{F}_t,\mathrm{P}|_{\mathcal{F}_t})$ is atomless and $L^1(\Omega,\mathcal{F}_t,\mathrm{P}|_{\mathcal{F}_t})$ is weakly separable for every $t>0$. This is shown in Sect. 6.9. The above conditions on the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ are always satisfied if it is assumed to be the standard augmentation of the natural filtration induced by some d-dimensional right-continuous stochastic process $Z=(Z_t)_{0\leq t\leq T}$ on the probability space $(\Omega,\mathcal{F},\mathrm{P})$ such that the marginals Z_t have absolutely continuous distributions for any $t>0$, Z_0 is constant P-a.s. and $\mathcal{F}_0$ is trivial (see [7, Remark 2.3] or [8, Remark 3]).

Let us now present a simple sufficient criterion guaranteeing the validity of the minimax relation (2.4). It turns out that under these conditions, $\mathcal{Q}$ fails to be time-consistent.

Theorem 2.4 *Let* (2.1) *and* (2.3) *be fulfilled. Furthermore*, *let d denote a totally bounded semimetric on $\mathcal{Q}$ and let $(d\mathrm{Q}/d\mathrm{P})_{\mathrm{Q}\in\mathcal{Q}}$ have* P*-a.s. d-uniformly continuous paths*. *If $\sup_{\mathrm{Q}\in\mathcal{Q}}(d\mathrm{Q}/d\mathrm{P})\leq U$* P*-a.s. for some* P*-integrable random variable U*, *then*

$$\sup_{\tau\in\mathcal{T}}\inf_{\mathrm{Q}\in\mathcal{Q}}\mathbb{E}_{\mathrm{Q}}[Y_\tau] = \sup_{\tau\in\mathcal{T}}\inf_{\mathrm{Q}\in\mathrm{co}(\mathcal{Q})}\mathbb{E}_{\mathrm{Q}}[Y_\tau] = \inf_{\mathrm{Q}\in\mathrm{co}(\mathcal{Q})}\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathrm{Q}}[Y_\tau].$$

If in addition $L^1(\Omega,\mathcal{F}_t,\mathrm{P}|_{\mathcal{F}_t})$ is weakly separable for every $t>0$, *then $\mathcal{Q}$ is not time-consistent with respect to $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\leq t\leq T},\mathrm{P})$ whenever it consists of more than one element and all elements of $\mathcal{Q}$ are equivalent to* P.

The proof of Theorem 2.4 is relegated to Sect. 6.5.

Remark 2.5 The most restrictive condition of Theorem 2.4 is the uniform continuity of paths of the process $(d\mathrm{Q}/d\mathrm{P})_{\mathrm{Q}\in\mathcal{Q}}$. In order to verify this condition, one obviously needs to put some constraints on the complexity of the set $\mathcal{Q}$. In the next section, we therefore turn to suitably parametrised (possibly by a functional parameter) families of measures. Such parametrisations naturally arise in mathematical finance when considering families of martingale pricing models.

3 Applications to parametrised families

Fix a semimetric space (Θ, d_Θ) with finite diameter Δ. Moreover, let us assume

$$\mathcal{Q} = \{\mathrm{Q}_\theta : \theta \in \Theta\} \qquad \text{and} \qquad \mathrm{Q}_\theta \neq \mathrm{Q}_\vartheta \text{ for } \theta \neq \vartheta. \tag{3.1}$$

Then d_Θ induces in a natural way a semimetric d on $\mathcal{Q}$ which is totally bounded if and only if d_Θ fulfils this property. We want to find conditions such that $\mathcal{Q}$ meets the requirements of Theorem 2.4. To this end, we consider a situation where the processes of densities corresponding to the probability measures from $\mathcal{Q}$ are related to a *nearly sub-Gaussian family of (local) martingales* $X^\theta := (X_t^\theta)_{0\le t\le T}$, $\theta \in \Theta$, that is, each X^θ is a centered (local) martingale for which we assume that there is some $C \ge 1$ such that

$$\sup_{t\in[0,T]} \mathbb{E}\left[\exp\left(\lambda(X_t^\theta - X_t^\vartheta)\right)\right] \le C \exp\left(\lambda^2 d_\Theta^2(\theta, \vartheta)/2\right) \quad \text{for } \theta, \vartheta \in \Theta \text{ and } \lambda > 0.$$

In particular, this means that for fixed $t \in [0, T]$, any process $(X_t^\theta)_{\theta\in\Theta}$ is a *nearly sub-Gaussian random field* in the sense considered in the Appendix. In the case of $C = 1$, we end up with the notion of sub-Gaussian families of local martingales. The following result requires d_Θ to be totally bounded, and it relies on metric entropies with respect to d_Θ. These are the numbers $\{\ln N(\Theta, d_\Theta; \varepsilon) : \varepsilon > 0\}$, where $N(\Theta, d_\Theta; \varepsilon)$ denotes the minimal number of ε-balls needed to cover Θ with respect to d_Θ. In addition, we define

$$\mathcal{D}(\delta, d_\Theta) := \int_0^\delta \sqrt{\ln N(\Theta, d_\Theta; \varepsilon)}\, d\varepsilon.$$

Of special interest is $\mathcal{D}(\delta, d_\Theta)$ for $\delta = \Delta$, the diameter of Θ with respect to d_Θ.

Proposition 3.1 *Let $\mathcal{Q}$ be a parametric family of the form* (3.1) *such that the conditions* (2.3) *and* (2.1) *are fulfilled. Furthermore, let d_Θ be totally bounded and let there exist a nearly sub-Gaussian family of local martingales $X^\theta = (X_t^\theta)_{0\le t\le T}$, $\theta \in \Theta$, with associated family $[X^\theta] = ([X^\theta]_t)_{0\le t\le T}$, $\theta \in \Theta$, of quadratic variation processes such that the process $([X^\theta]_t)_{\theta\in\Theta}$ has d_Θ-uniformly continuous paths for every $t \in [0, T]$ and*

$$\left.\frac{d\mathrm{Q}_\theta}{d\mathrm{P}}\right|_{\mathcal{F}_t} = \exp(X_t^\theta - [X^\theta]_t/2) \quad \textit{pointwise in } \omega \textit{ for } t \in [0, T] \textit{ and } \theta \in \Theta.$$

If $\sup_{t\in[0,T]} \mathbb{E}[\exp(2X_t^{\overline{\theta}})] < \infty$ *for some* $\overline{\theta} \in \Theta$ *and* $\mathcal{D}(\Delta, d_\Theta) < \infty$, *then*

$$\sup_{\tau\in\mathcal{T}} \inf_{\mathrm{Q}\in\mathcal{Q}} \mathbb{E}_\mathrm{Q}[Y_\tau] = \sup_{\tau\in\mathcal{T}} \inf_{\mathrm{Q}\in\mathrm{co}(\mathcal{Q})} \mathbb{E}_\mathrm{Q}[Y_\tau] = \inf_{\mathrm{Q}\in\mathrm{co}(\mathcal{Q})} \sup_{\tau\in\mathcal{T}} \mathbb{E}_\mathrm{Q}[Y_\tau].$$

Moreover, $\mathcal{Q}$ is not time-consistent with respect to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathrm{P})$ *if each of its members is equivalent to* P, *$\mathcal{Q}$ has more than one element and* $L^1(\Omega, \mathcal{F}_t, \mathrm{P}|_{\mathcal{F}_t})$ *is weakly separable for every* $t > 0$.

The proof of Proposition 3.1 may be found in Sect. 6.6.

The following example illustrates how Proposition 3.1 can be applied to practically interesting cases of diffusion type families of martingales.

Example 3.2 Let $Z = (Z_s)_{s\geq 0}$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathrm{P})$ such that the process $(Z_t)_{0\leq t\leq T}$ is adapted to $(\mathcal{F}_t)_{0\leq t\leq T}$, and let $V = (V_t)_{0\leq t\leq T}$ be some $\mathbb{R}^d$-valued process (volatility) adapted to $(\mathcal{F}_t)_{0\leq t\leq T}$. Consider a class Ψ of Borel functions $\psi : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ such that for every $\psi \in \Psi$,

$$\sup_{x\in\mathbb{R}^d} \int_0^T \psi^2(u, x)\, du < \infty. \tag{3.2}$$

Then

$$d_\Psi : \Psi \times \Psi \to \mathbb{R},\ (\psi, \phi) \mapsto \sup_{x\in\mathbb{R}^d} \sqrt{\int_0^T (\psi - \phi)^2(u, x)\, du}$$

is a well-defined semimetric on Ψ. Assume that the process $(\psi(t, V_t))_{0\leq t\leq T}$ is progressively measurable for each $\psi \in \Psi$. Then the family of processes X^ψ, $\psi \in \Psi$, with

$$X_t^\psi := \int_0^t \psi(u, V_u)\, dZ_u, \qquad t \in [0, T],$$

is well-defined and the quadratic variation process of X^ψ is given by

$$[X^\psi]_t = \int_0^t \psi^2(u, V_u)\, du, \qquad t \in [0, T].$$

So by (3.2), each process $(\exp(X_t^\psi - [X^\psi]_t/2))_{t\in[0,T]}$ satisfies the Novikov condition. Hence it is a martingale and the density process of a probability measure on $\mathcal{F}$ which is absolutely continuous with respect to P.

For arbitrary $\overline{\psi}, \psi, \phi \in \Psi$, we may observe by the Cauchy–Schwarz inequality for every $t \in [0, T]$ that

$$|[X^\psi]_t - [X^\phi]_t| \leq \left(d_\Psi(\psi, \overline{\psi}) + d_\Psi(\phi, \overline{\psi}) + 2\sqrt{\sup_{x\in\mathbb{R}^d} \int_0^T \overline{\psi}^2(u, x)\, du}\right) d_\Psi(\psi, \phi).$$

Hence for fixed $t \in [0, T]$, the process $([(X^\psi)]_t)_{\psi\in\Psi}$ has d_Ψ-Lipschitz-continuous paths if d_Ψ is totally bounded. Moreover, it can be shown that X^ψ, $\psi \in \Psi$, is a nearly sub-Gaussian family of martingales with respect to d_Ψ with $C = 2$. The proof of this result can be found in Sect. 6.7.

4 Discussion

Let us discuss some related results in the literature. In [13] and [14], the minimax relationship (1.1) is studied for general convex sets $\mathcal{Q}$ of probability measures which are equivalent to P without explicitly imposing stability under pasting or time-consistency. However, it is implicitly assumed there (see [13, proof of Lemma B.1] and [14, proof of Proposition 3.1]) that one can find, for every $\tau \in \mathcal{T}$ and any $\overline{\mathrm{Q}} \in \mathcal{Q}$, a sequence $(\mathrm{Q}^k)_{k\in\mathbb{N}}$ of probability measures from $\mathcal{Q}$ which agree with $\overline{\mathrm{Q}}$ on $\mathcal{F}_\tau$ such that

$$\operatorname*{ess\,sup}_{\sigma\in\mathcal{T},\sigma\geq\tau} \mathbb{E}_{\mathrm{Q}^k}[Y_\tau|\mathcal{F}_\tau] \xrightarrow{k\to\infty} \operatorname*{ess\,inf}_{\mathrm{Q}\in\mathcal{Q}} \operatorname*{ess\,sup}_{\sigma\in\mathcal{T},\sigma\geq\tau} \mathbb{E}_{\mathrm{Q}}[Y_\sigma|\mathcal{F}_\tau] \qquad \text{P-a.s.}$$

It turns out that the above relation can hold in general only for time-consistent sets $\mathcal{Q}$.

Proposition 4.1 *Let each member of $\mathcal{Q}$ be equivalent to* P *and define the set $\mathcal{S}(\mathcal{Q})$ to consist of all uniformly bounded adapted càdlàg processes $Z = (Z_t)_{0\leq t\leq T}$ such that each of the single stopping problems*

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}_{\mathrm{Q}}[Z_\tau], \qquad \mathrm{Q}\in\mathcal{Q},$$

has a solution. Consider the following statements:

(1) *$\mathcal{Q}$ is time-consistent.*
(2) $\inf_{\mathrm{Q}\in\mathcal{Q}} \mathbb{E}_{\mathrm{Q}}[X] \leq \inf_{\mathrm{Q}\in\mathcal{Q}} \mathbb{E}_{\mathrm{Q}}[\operatorname{ess\,inf}_{\mathrm{Q}\in\mathcal{Q}} \mathbb{E}_{\mathrm{Q}}[X|\mathcal{F}_\tau]]$ *holds for every* P*-essentially bounded random variable X and every stopping time $\tau\in\mathcal{T}$.*
(3) $\widehat{\mathcal{Q}}^e := \{\mathrm{Q}\in\widehat{\mathcal{Q}} : \mathrm{Q}\approx \mathrm{P}\}$ *is stable under pasting and*

$$\operatorname*{ess\,inf}_{\mathrm{Q}\in\mathcal{Q}} \mathbb{E}_{\mathrm{Q}}[X|\mathcal{F}_\tau] = \operatorname*{ess\,inf}_{\mathrm{Q}\in\widehat{\mathcal{Q}}^e} \mathbb{E}_{\mathrm{Q}}[X|\mathcal{F}_\tau] \; \textit{for P-essentially bounded } X \textit{ and } \tau\in\mathcal{T},$$

where $\widehat{\mathcal{Q}}$ denotes the set of all probability measures on the σ-algebra $\mathcal{F}$ such that $\mathbb{E}_{\mathrm{Q}}[X] \geq \inf_{\mathrm{Q}'\in\mathcal{Q}} \mathbb{E}_{\mathrm{Q}'}[X]$ *for any* P*-essentially bounded random variable X.*
(4) *For an arbitrary process $Z = (Z_t)_{0\leq t\leq T} \in \mathcal{S}(\mathcal{Q})$ and for any $\tau\in\mathcal{T}$ as well as $\overline{\mathrm{Q}}\in\mathcal{Q}$, there is some sequence $(\mathrm{Q}^k)_{k\in\mathbb{N}}$ in $\mathcal{Q}$ whose elements agree with $\overline{\mathrm{Q}}$ on $\mathcal{F}_\tau$ such that*

$$\operatorname*{ess\,sup}_{\sigma\in\mathcal{T},\sigma\geq\tau} \mathbb{E}_{\mathrm{Q}_k}[Z_\sigma|\mathcal{F}_\tau] \xrightarrow{k\to\infty} \operatorname*{ess\,inf}_{\mathrm{Q}\in\mathcal{Q}} \operatorname*{ess\,sup}_{\sigma\in\mathcal{T},\sigma\geq\tau} \mathbb{E}_{\mathrm{Q}}[Z_\sigma|\mathcal{F}_\tau] \qquad \text{P-}a.s.$$

(5) *$\mathcal{Q}$ is stable under pasting.*

Then the statements (1)–(3) *are equivalent and* (4) *follows from* (5)*. Moreover, the implication* (4) $\Rightarrow$ (1) *holds.*

The proof of Proposition 4.1 is relegated to Sect. 6.8.

5 An abstract minimax result

Using the notation from (2.2), let us define the set $\mathcal{X}$ of all random variables X on $(\Omega, \mathcal{F}, \mathrm{P})$ satisfying

$$|X| \leq C(Y^* + 1) \qquad \text{P-a.s. for some } C > 0.$$

Note that $\mathcal{X}$ is a Stone vector lattice containing the set $\{Y_\tau : \tau \in \mathcal{T}\}$ and the space $L^\infty(\Omega, \mathcal{F}, \mathrm{P})$ of all P-essentially bounded random variables. Moreover, $\mathcal{X} \subseteq L^1(\mathcal{Q})$ is valid under (2.2), and in this case, we may introduce the mapping

$$\rho_{\mathcal{Q}} : \mathcal{X} \to \mathbb{R},\ X \mapsto \sup_{\mathrm{Q} \in \mathcal{Q}} \mathbb{E}_{\mathrm{Q}}[X].$$

We call $\rho_{\mathcal{Q}}$ *continuous from above at* 0 if $\rho_{\mathcal{Q}}(X_n) \searrow 0$ for $X_n \searrow 0$ P-a.s.

In this section, we want to present a general abstract minimax relation (1.1) which will be the starting point to derive the main result Theorem 2.1. It relies on the following key assumption.

(A) There exists some $\lambda \in (0, 1)$ such that for every $\tau_1, \tau_2 \in \mathcal{T}_f \setminus \{0\}$,

$$\inf_{A \in \mathcal{F}_{\tau_1 \wedge \tau_2}} \rho_{\mathcal{Q}}\big((\mathbb{1}_A - \lambda)(Y_{\tau_2} - Y_{\tau_1})\big) \leq 0,$$

where $\mathcal{T}_f$ denotes the set of stopping times from $\mathcal{T}$ with finitely many values.

Theorem 5.1 *If $Y = (Y_t)_{0 \leq t \leq T}$ fulfils* (2.2) *and if $\rho_{\mathcal{Q}}$ is continuous from above at* 0, *then under the assumption* (A),

$$\sup_{\tau \in \mathcal{T}} \inf_{\mathrm{Q} \in \mathcal{Q}} \mathbb{E}_{\mathrm{Q}}[Y_\tau] = \sup_{\tau \in \mathcal{T}} \inf_{\mathrm{Q} \in \mathrm{co}(\mathcal{Q})} \mathbb{E}_{\mathrm{Q}}[Y_\tau] = \inf_{\mathrm{Q} \in \mathrm{co}(\mathcal{Q})} \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathrm{Q}}[Y_\tau].$$

The proof of Theorem 5.1 is relegated to Sect. 6.2.

At this place, we may invoke the assumption of Theorem 2.1 that the range of the vector measure $\mu_{\mathcal{Q}}$ associated with $\mathcal{Q}$ is relatively compact with respect to the sup-norm $\|\cdot\|_\infty$. As the following result shows, this condition essentially implies assumption (A).

Proposition 5.2 *Let $Y = (Y_t)_{0 \leq t \leq T}$ satisfy* (2.2) *and let $\rho_{\mathcal{Q}}$ be continuous from above at* 0. *Suppose furthermore that $\{\mu_{\mathcal{Q}}(A) : A \in \mathcal{F}\}$ is relatively $\|\cdot\|_\infty$-compact. If for $\tau_1, \tau_2 \in \mathcal{T}_f \setminus \{0\}$ the probability space $(\Omega, \mathcal{F}_{\tau_1 \wedge \tau_2}, \mathrm{P}|_{\mathcal{F}_{\tau_1 \wedge \tau_2}})$ is atomless, then*

$$\inf_{A \in \mathcal{F}_{\tau_1 \wedge \tau_2}} \rho_{\mathcal{Q}}\big((\mathbb{1}_A - 1/2)(Y_{\tau_2} - Y_{\tau_1})\big) \leq 0.$$

The proof of Proposition 5.2 may be found in Sect. 6.3.

6 Proofs

Let (2.2) be fulfilled. Note that under (2.2),

$$Y_\tau \in \mathcal{L}^1(\mathcal{Q}) \qquad \text{for } \tau \in \mathcal{T}. \tag{6.1}$$

Condition (6.1) implies that its Radon–Nikodým derivative $d\mathrm{Q}/d\mathrm{P}$ satisfies for any $\mathrm{Q} \in \mathrm{co}(\mathcal{Q})$ that

$$Y_\tau \frac{d\mathrm{Q}}{d\mathrm{P}} \text{ is P-integrable for every } \tau \in \mathcal{T}.$$

Let the set $\mathcal{X}$ and the mapping $\rho_{\mathcal{Q}}$ be defined as at the beginning of Sect. 5.

6.1 A topological closure of $\mathcal{Q}$

Let $\mathcal{M}_1(\Omega, \mathcal{X})$ denote the set of all probability measures Q on $\mathcal{F}$ such that X is Q-integrable for every $X \in \mathcal{X}$. Set

$$\overline{\mathcal{Q}} := \left\{ \mathrm{Q} \in \mathcal{M}_1(\Omega, \mathcal{X}) : \sup_{X \in \mathcal{X}} \left(\mathbb{E}_{\mathrm{Q}}[X] - \rho_{\mathcal{Q}}(X) \right) \le 0 \right\}.$$

Obviously, $\mathrm{co}(\mathcal{Q}) \subseteq \overline{\mathcal{Q}}$ and

$$\overline{\mathcal{Q}} \text{ is convex} \quad \text{and} \quad \sup_{\mathrm{Q} \in \overline{\mathcal{Q}}} \mathbb{E}_{\mathrm{Q}}[X] = \rho_{\mathcal{Q}}(X) \text{ for all } X \in \mathcal{X}. \tag{6.2}$$

We endow $\overline{\mathcal{Q}}$ with the coarsest topology $\sigma(\overline{\mathcal{Q}}, \mathcal{X})$ such that the mappings

$$\varphi_X : \overline{\mathcal{Q}} \to \mathbb{R},\ \mathrm{Q} \mapsto \mathbb{E}_{\mathrm{Q}}[X], \qquad X \in \mathcal{X},$$

are continuous. In the next step, we investigate when $\overline{\mathcal{Q}}$ is compact with respect to $\sigma(\overline{\mathcal{Q}}, \mathcal{X})$, with $\mathrm{co}(\mathcal{Q})$ being a dense subset.

Lemma 6.1 *If* (2.2) *holds and if* $\rho_{\mathcal{Q}}$ *is continuous from above at* 0, *then* $\overline{\mathcal{Q}}$ *is compact with respect to* $\sigma(\overline{\mathcal{Q}}, \mathcal{X})$ *and Hausdorff. Moreover,* $\mathrm{co}(\mathcal{Q})$ *is a* $\sigma(\overline{\mathcal{Q}}, \mathcal{X})$*-dense subset of* $\overline{\mathcal{Q}}$, *and* $\overline{\mathcal{Q}}$ *is dominated by* P.

Proof The topology $\sigma(\overline{\mathcal{Q}}, \mathcal{X})$ is Hausdorff because the set $\{\varphi_X : X \in \mathcal{X}\}$ is separating points in $\overline{\mathcal{Q}}$. Let us now equip the algebraic dual $\mathcal{X}^*$ of $\mathcal{X}$ with the coarsest topology $\sigma(\mathcal{X}^*, \mathcal{X})$ such that the mappings

$$h_X : \mathcal{X}^* \to \mathbb{R}, \Lambda \mapsto \Lambda(X), \qquad X \in \mathcal{X},$$

are continuous. The functional $\rho_{\mathcal{Q}}$ is sublinear. Then by a version of the Banach–Alaoglu theorem (cf. [19, Theorem 1.6]), the set

$$\Delta_{\mathcal{Q}} := \left\{ \Lambda \in \mathcal{X}^* : \sup_{X \in \mathcal{X}} \left(\Lambda(X) - \rho_{\mathcal{Q}}(X) \right) \le 0 \right\}$$

is compact with respect to $\sigma(\mathcal{X}^*, \mathcal{X})$. Moreover, $\rho_{\mathcal{Q}}$ is assumed to be continuous from above at 0. This implies that every $\Lambda \in \Delta_{\mathcal{Q}}$ satisfies $\Lambda(X_n) \searrow 0$ whenever $X_n \searrow 0$. Since $\mathcal{X}$ is a Stone vector lattice containing the P-essentially bounded mappings on Ω, it generates the σ-algebra $\mathcal{F}$, and an application of the Daniell–Stone representation theorem yields that each $\Lambda \in \Delta_{\mathcal{Q}}$ is uniquely representable by a probability measure Q_Λ, namely

$$\mathrm{Q}_\Lambda : \mathcal{F} \to [0,1],\ A \mapsto \Lambda(\mathbb{1}_A).$$

Hence by the definition of $\overline{\mathcal{Q}}$, we obtain

$$\Delta_{\mathcal{Q}} = \{\Lambda_{\mathrm{Q}} : \mathrm{Q} \in \overline{\mathcal{Q}}\} \quad \text{and} \quad \Lambda_{\mathrm{Q}} \neq \Lambda_{\widetilde{\mathrm{Q}}} \text{ for } \mathrm{Q} \neq \widetilde{\mathrm{Q}}, \tag{6.3}$$

where

$$\Lambda_{\mathrm{Q}} : \mathcal{X} \to \mathbb{R},\ X \mapsto \mathbb{E}_{\mathrm{Q}}[X] \qquad \text{for } \mathrm{Q} \in \overline{\mathcal{Q}}.$$

Obviously, we may define a homeomorphism from $\Delta_{\mathcal{Q}}$ onto $\overline{\mathcal{Q}}$ with respect to the topologies $\sigma(\mathcal{X}^*, \mathcal{X})$ and $\sigma(\overline{\mathcal{Q}}, \mathcal{X})$. In particular, $\overline{\mathcal{Q}}$ is compact with respect to the topology $\sigma(\overline{\mathcal{Q}}, \mathcal{X})$.

Next, $\{\Lambda_{\mathrm{Q}} : \mathrm{Q} \in \mathrm{co}(\mathcal{Q})\}$ is a convex subset of $\mathcal{X}^*$. We may draw on a version of the bipolar theorem (cf. [19, Consequence 1.5]) to observe that the $\sigma(\mathcal{X}^*, \mathcal{X})$-closure $\mathrm{cl}(\{\Lambda_{\mathrm{Q}} : \mathrm{Q} \in \mathrm{co}(\mathcal{Q})\})$ of $\{\Lambda_{\mathrm{Q}} : \mathrm{Q} \in \mathrm{co}(\mathcal{Q})\}$ coincides with $\Delta_{\mathcal{Q}}$. Therefore (6.3) enables us to define a homeomorphism from $\mathrm{cl}(\{\Lambda_{\mathrm{Q}} : \mathrm{Q} \in \mathrm{co}(\mathcal{Q})\})$ onto $\overline{\mathcal{Q}}$ with respect to the topologies $\sigma(\mathcal{X}^*, \mathcal{X})$ and $\sigma(\overline{\mathcal{Q}}, \mathcal{X})$. Thus $\mathrm{co}(\mathcal{Q})$ is a $\sigma(\overline{\mathcal{Q}}, \mathcal{X})$-dense subset of $\overline{\mathcal{Q}}$, and by the definition of the topology $\sigma(\overline{\mathcal{Q}}, \mathcal{X})$, it may be verified easily that $\overline{\mathcal{Q}}$ is dominated by P. This completes the proof. □

Consider now the new optimisation problems

$$\text{maximise } \inf_{\mathrm{Q} \in \overline{\mathcal{Q}}} \mathbb{E}_{\mathrm{Q}}[Y_\tau] \qquad \text{over } \tau \in \mathcal{T} \tag{6.4}$$

and

$$\text{minimise } \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathrm{Q}}[Y_\tau] \qquad \text{over } \mathrm{Q} \in \overline{\mathcal{Q}}. \tag{6.5}$$

In view of (6.2), we obtain that (6.4) has the same optimal value as the corresponding one with respect to $\mathcal{Q}$ and $\mathrm{co}(\mathcal{Q})$ instead of $\overline{\mathcal{Q}}$.

Proposition 6.2 *Under assumption* (2.2), *we have*

$$\sup_{\tau \in \mathcal{T}} \inf_{\mathrm{Q} \in \overline{\mathcal{Q}}} \mathbb{E}_{\mathrm{Q}}[Y_\tau] = \sup_{\tau \in \mathcal{T}} \inf_{\mathrm{Q} \in \mathcal{Q}} \mathbb{E}_{\mathrm{Q}}[Y_\tau] = \sup_{\tau \in \mathcal{T}} \inf_{\mathrm{Q} \in \mathrm{co}(\mathcal{Q})} \mathbb{E}_{\mathrm{Q}}[Y_\tau].$$

The comparison of the optimal value of problem (6.5) with the corresponding one with respect to $\mathrm{co}(\mathcal{Q})$ instead of $\overline{\mathcal{Q}}$ is more difficult to handle. For preparation, let us introduce a sequence $(\overline{Y}^k)_{k \in \mathbb{N}}$ of stochastic processes $\overline{Y}^k := (\overline{Y}^k_t)_{0 \le t \le T}$ via $\overline{Y}^k_t := (Y_t \wedge k) \vee (-k)$. They all are adapted to $(\mathcal{F}_t)_{0 \le t \le T}$.

Lemma 6.3 *If* (2.2) *is satisfied and if* $\rho_{\mathcal{Q}}$ *is continuous from above at* 0, *then*

$$\lim_{k\to\infty}\sup_{\mathrm{Q}\in\overline{\mathcal{Q}}}\left|\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathrm{Q}}[Y_\tau]-\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathrm{Q}}\big[\overline{Y}_\tau^k\big]\right|=0.$$

Proof For $\tau\in\mathcal{T}$ and $k\in\mathbb{N}$, we may observe that $Y^*\in\mathcal{X}$ by (2.2) and

$$\big|Y_\tau-\overline{Y}_\tau^k\big|\le\mathbb{1}_{\{Y^*>k\}}(Y^*-k).$$

Then due to (6.2),

$$\begin{aligned}\sup_{\mathrm{Q}\in\overline{\mathcal{Q}}}\left|\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathrm{Q}}[Y_\tau]-\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathrm{Q}}\big[\overline{Y}_\tau^k\big]\right|&\le\sup_{\mathrm{Q}\in\overline{\mathcal{Q}}}\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathrm{Q}}\big[|Y_\tau-\overline{Y}_\tau^k|\big]\\&\le\sup_{\mathrm{Q}\in\overline{\mathcal{Q}}}\mathbb{E}_{\mathrm{Q}}\big[\mathbb{1}_{\{Y^*>k\}}(Y^*-k)\big]\\&=\rho_{\mathcal{Q}}\big(\mathbb{1}_{\{Y^*>k\}}(Y^*-k)\big)\end{aligned}$$

holds for any $k\in\mathbb{N}$. Finally, $\mathbb{1}_{\{Y^*>k\}}(Y^*-k)\searrow 0$, and thus the statement of Lemma 6.3 follows because $\rho_{\mathcal{Q}}$ is continuous from above at 0. □

In the next step, we replace in (6.5) the process Y with the processes $\overline{Y}^k$. We want to examine when the optimal value coincides with the optimal value of the corresponding problem with respect to $\mathrm{co}(\mathcal{Q})$ instead of $\overline{\mathcal{Q}}$.

Lemma 6.4 *If* (2.2) *holds and if* $\rho_{\mathcal{Q}}$ *is continuous from above at* 0, *then*

$$\inf_{\mathrm{Q}\in\overline{\mathcal{Q}}}\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathrm{Q}}\big[\overline{Y}_\tau^k\big]=\inf_{\mathrm{Q}\in\mathrm{co}(\mathcal{Q})}\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathrm{Q}}\big[\overline{Y}_\tau^k\big]\qquad\textit{for every } k\in\mathbb{N}.$$

Proof Let $k\in\mathbb{N}$ and fix $\mathrm{Q}_0\in\overline{\mathcal{Q}}$. In view of Lemma 6.1, its Radon–Nikodým derivative $d\mathrm{Q}_0/d\mathrm{P}$ is in the weak closure of $\{d\mathrm{Q}/d\mathrm{P}:\mathrm{Q}\in\mathrm{co}(\mathcal{Q})\}$, viewed as a subset of the L^1-space on $(\Omega,\mathcal{F},\mathrm{P})$. Moreover, by Lemma 6.1, the set $\{d\mathrm{Q}/d\mathrm{P}:\mathrm{Q}\in\mathrm{co}(\mathcal{Q})\}$ is a relatively weakly compact subset of the L^1-space on $(\Omega,\mathcal{F},\mathrm{P})$. Then by the Eberlein–Šmulian theorem (cf. e.g. [20]), we may select a sequence $(\mathrm{Q}_n)_{n\in\mathbb{N}}$ in $\mathrm{co}(\mathcal{Q})$ such that

$$\lim_{n\to\infty}\mathbb{E}_{\mathrm{Q}_n}[X]=\lim_{n\to\infty}\mathbb{E}\left[X\frac{d\mathrm{Q}_n}{d\mathrm{P}}\right]=\mathbb{E}\left[X\frac{d\mathrm{Q}_0}{d\mathrm{P}}\right]=\mathbb{E}_{\mathrm{Q}_0}[X]\tag{6.6}$$

for every P-essentially bounded random variable X.

Let us introduce the set $\mathcal{P}^e$ of probability measures on $\mathcal{F}$ which are equivalent to P. Then for any $\mathrm{Q}\in\mathcal{P}^e$, the σ-algebra $\mathcal{F}_0$ contains all Q-nullsets of $\mathcal{F}$ and $\mathrm{Q}[A]\in\{0,1\}$ for every $A\in\mathcal{F}_0$. In particular, the set $\mathcal{T}^{\mathrm{Q}}$ of all stopping times with respect to $(\Omega,\mathcal{F}_T,(\mathcal{F}_t)_{0\le t\le T},\mathrm{Q})$ coincides with $\mathcal{T}$ for $\mathrm{Q}\in\mathcal{P}^e$. Note also that $(\overline{Y}_t^k+k)_{0\le t\le T}$ is a nonnegative, bounded, right-continuous and $(\mathcal{F}_t)$-adapted process which is quasi-left-upper-semicontinuous with respect to every $\mathrm{Q}\in\mathcal{P}^e$. Here quasi-left-upper-semicontinuity with respect to Q is understood as we have defined it with respect to P. Hence by Fatou's lemma, $\limsup_{m\to\infty} \mathbb{E}_{\mathrm{Q}}[\overline{Y}^k_{\tau_m} + k] \le \mathbb{E}_{\mathrm{Q}}[\overline{Y}^k_{\tau} + k]$ for every $\mathrm{Q} \in \mathcal{P}^e$ whenever $(\tau_m)_{m\in\mathbb{N}}$ is a sequence in $\mathcal{T}$ satisfying $\tau_m \nearrow \tau$ for some $\tau \in \mathcal{T}$. Then we may draw on [16, Proposition B.6] to conclude that

$$\forall\, \mathrm{Q} \in \mathcal{P}^e\ \exists \tau \in \mathcal{T} : \mathbb{E}_{\mathrm{Q}}\big[\overline{Y}^k_{\tau} + k\big] = \sup_{\tau\in\mathcal{T}} \mathbb{E}_{\mathrm{Q}}\big[\overline{Y}^k_{\tau} + k\big]. \tag{6.7}$$

Let us define for $\mathrm{Q} \in \mathrm{co}(\mathcal{Q})$ and $\lambda \in (0,1)$ the probability measure Q^λ on $\mathcal{F}$ by $\mathrm{Q}^\lambda := \lambda \mathrm{Q} + (1-\lambda)\mathrm{P}$ and the sets

$$\mathcal{Q}^\lambda := \{\mathrm{Q}^\lambda : \mathrm{Q} \in \mathrm{co}(\mathcal{Q})\}, \qquad \lambda \in (0,1).$$

Obviously, these sets are contained in $\mathcal{P}^e$. Now define for $\lambda \in (0,1)$ the sequence $(f_{n,\lambda})_{n\in\mathbb{N}}$ of mappings

$$f_{n,\lambda} : \mathcal{T} \to \mathbb{R},\ \tau \mapsto \mathbb{E}_{\mathrm{Q}^\lambda_n}\big[\overline{Y}^k_{\tau} + k\big].$$

Notice that the sequence $(f_{n,\lambda})_{n\in\mathbb{N}}$ is uniformly bounded for $\lambda \in (0,1)$ because

$$|\overline{Y}^k_{\tau} + k| \le 2k \qquad \text{for every } \tau \in \mathcal{T}. \tag{6.8}$$

We want to apply Simons' lemma (cf. [22, Lemma 2]) to each sequence $(f_{n,\lambda})_{n\in\mathbb{N}}$. For this purpose, it remains to show for fixed $\lambda \in (0,1)$ that we may find for any countable convex combination of $(f_{n,\lambda})_{n\in\mathbb{N}}$ some maximiser. So let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[0,1]$ with $\sum_{n=1}^\infty \lambda_n = 1$. We may define by

$$\sum_{n=1}^\infty \lambda_n \mathrm{Q}^\lambda_n(A) =: \mathrm{Q}(A) \qquad \text{for every } A \in \mathcal{F}$$

a probability measure on $\mathcal{F}$ which belongs to $\mathcal{P}^e$. Then by monotone convergence,

$$\begin{aligned}\sum_{n=1}^\infty \lambda_n f_{n,\lambda}(\tau) &= \sum_{n=1}^\infty \lambda_n \int_0^\infty \mathrm{Q}^\lambda_n\big[\overline{Y}^k_{\tau} + k > x\big]\,dx \\ &= \int_0^\infty \mathrm{Q}\big[\overline{Y}^k_{\tau} + k > x\big]\,dx \\ &= \mathbb{E}_{\mathrm{Q}}\big[\overline{Y}^k_{\tau} + k\big] \qquad \text{for } \tau \in \mathcal{T}.\end{aligned}$$

Moreover, by (6.7), there exists some $\tau_* \in \mathcal{T}$ such that

$$\sum_{n=1}^\infty \lambda_n f_{n,\lambda}(\tau_*) = \mathbb{E}_{\mathrm{Q}}\big[\overline{Y}^k_{\tau_*} + k\big] = \sup_{\tau\in\mathcal{T}} \mathbb{E}_{\mathrm{Q}}\big[\overline{Y}^k_{\tau} + k\big] = \sup_{\tau\in\mathcal{T}} \sum_{n=1}^\infty \lambda_n f_{n,\lambda}(\tau).$$

Therefore, the assumptions of [22, Lemma 2] are satisfied, and we obtain

$$\sup_{\tau\in\mathcal{T}} \limsup_{n\to\infty} f_{n,\lambda}(\tau) \ge \inf_{f\in\mathrm{co}(\{f_{n,\lambda}:n\in\mathbb{N}\})} \sup_{\tau\in\mathcal{T}} f(\tau).$$

For any finite convex combination $f = \sum_{i=1}^r \lambda_i f_{n_i,\lambda}$, the probability measure $Q := \sum_{i=1}^r \lambda_i Q_{n_i}$ is a member of $\mathrm{co}(\mathcal{Q})$ and

$$f(\tau) = \mathbb{E}_{Q^\lambda}\big[\overline{Y}_\tau^k + k\big] \qquad \text{for } \tau \in \mathcal{T}.$$

Therefore, on the one hand,

$$\sup_{\tau\in\mathcal{T}} \limsup_{n\to\infty} \mathbb{E}_{Q_n^\lambda}\big[\overline{Y}_\tau^k + k\big] \geq \inf_{Q\in\mathcal{Q}^\lambda} \sup_{\tau\in\mathcal{T}} \mathbb{E}_Q\big[\overline{Y}_\tau^k + k\big].$$

On the other hand, by (6.6),

$$\sup_{\tau\in\mathcal{T}} \limsup_{n\to\infty} \mathbb{E}_{Q_n^\lambda}\big[\overline{Y}_\tau^k + k\big] = \sup_{\tau\in\mathcal{T}} \big(\lambda \mathbb{E}_{Q_0}\big[\overline{Y}_\tau^k + k\big] + (1-\lambda)\mathbb{E}\big[\overline{Y}_\tau^k + k\big]\big).$$

Hence by (6.8) and nonnegativity of $(\overline{Y}_t^k + k)_{0\leq t\leq T}$,

$$\begin{aligned}
\lambda \sup_{\tau\in\mathcal{T}} \mathbb{E}_{Q_0}\big[\overline{Y}_\tau^k + k\big] + (1-\lambda)2k &\geq \sup_{\tau\in\mathcal{T}} \big(\lambda \mathbb{E}_{Q_0}\big[\overline{Y}_\tau^k + k\big] + (1-\lambda)\mathbb{E}\big[\overline{Y}_\tau^k + k\big]\big) \\
&\geq \inf_{Q\in\mathcal{Q}^\lambda} \sup_{\tau\in\mathcal{T}} \mathbb{E}_Q\big[\overline{Y}_\tau^k + k\big] \\
&= \inf_{Q\in\mathrm{co}(\mathcal{Q})} \sup_{\tau\in\mathcal{T}} \big(\lambda \mathbb{E}_Q\big[\overline{Y}_\tau^k + k\big] + (1-\lambda)\mathbb{E}\big[\overline{Y}_\tau^k + k\big]\big) \\
&\geq \lambda \inf_{Q\in\mathrm{co}(\mathcal{Q})} \sup_{\tau\in\mathcal{T}} \mathbb{E}_Q\big[\overline{Y}_\tau^k + k\big].
\end{aligned}$$

Then by sending $\lambda \nearrow 1$,

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}_{Q_0}\big[\overline{Y}_\tau^k + k\big] \geq \inf_{Q\in\mathrm{co}(\mathcal{Q})} \sup_{\tau\in\mathcal{T}} \mathbb{E}_Q\big[\overline{Y}_\tau^k + k\big],$$

and thus

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}_{Q_0}\big[\overline{Y}_\tau^k\big] \geq \inf_{Q\in\mathrm{co}(\mathcal{Q})} \sup_{\tau\in\mathcal{T}} \mathbb{E}_Q\big[\overline{Y}_\tau^k\big].$$

This completes the proof because Q_0 was arbitrarily chosen from $\overline{\mathcal{Q}} \supseteq \mathrm{co}(\mathcal{Q})$. □

We are ready to provide the following criterion which ensures that the optimal value of (6.5) coincides with the optimal value of the corresponding problem with respect to $\mathrm{co}(\mathcal{Q})$ instead of $\overline{\mathcal{Q}}$.

Proposition 6.5 *Let* (2.2) *be fulfilled. If* $\rho_{\mathcal{Q}}$ *is continuous from above at* 0, *then*

$$\inf_{Q\in\overline{\mathcal{Q}}} \sup_{\tau\in\mathcal{T}} \mathbb{E}_Q[Y_\tau] = \inf_{Q\in\mathrm{co}(\mathcal{Q})} \sup_{\tau\in\mathcal{T}} \mathbb{E}_Q[Y_\tau].$$

Proof Firstly, we have $\inf_{\mathrm{Q}\in\overline{\mathcal{Q}}}\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathrm{Q}}[Y_\tau]\le\inf_{\mathrm{Q}\in\mathrm{co}(\mathcal{Q})}\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathrm{Q}}[Y_\tau]$ because $\mathrm{co}(\mathcal{Q})\subseteq\overline{\mathcal{Q}}$. Then we obtain by Lemma 6.4 for every $k\in\mathbb{N}$ that

$$
\begin{aligned}
0 &\le \inf_{\mathrm{Q}\in\mathrm{co}(\mathcal{Q})}\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathrm{Q}}[Y_\tau]-\inf_{\mathrm{Q}\in\overline{\mathcal{Q}}}\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathrm{Q}}[Y_\tau]\\
&=\inf_{\mathrm{Q}\in\mathrm{co}(\mathcal{Q})}\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathrm{Q}}[Y_\tau]-\inf_{\mathrm{Q}\in\mathrm{co}(\mathcal{Q})}\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathrm{Q}}\big[\overline{Y}^k_\tau\big]\\
&\quad+\inf_{\mathrm{Q}\in\overline{\mathcal{Q}}}\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathrm{Q}}\big[\overline{Y}^k_\tau\big]-\inf_{\mathrm{Q}\in\overline{\mathcal{Q}}}\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathrm{Q}}[Y_\tau]\\
&\le 2\sup_{\mathrm{Q}\in\overline{\mathcal{Q}}}\Big|\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathrm{Q}}[Y_\tau]-\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathrm{Q}}\big[\overline{Y}^k_\tau\big]\Big|.
\end{aligned}
$$

The statement of Proposition 6.5 follows now immediately from Lemma 6.3. □

6.2 Proof of Theorem 5.1

Let $\overline{\mathcal{Q}}$ be defined as in the previous subsection. The idea of the proof is to verify first duality of the problems (6.4) and (6.5), and then to apply Propositions 6.2 and 6.5. Concerning the minimax relationship of the problems (6.4) and (6.5), we may reduce considerations to stopping times with finite range if $\rho_{\mathcal{Q}}$ is continuous from above at 0.

Lemma 6.6 *If Y fulfils (2.2) and if $\rho_{\mathcal{Q}}$ is continuous from above at* 0, *then*

(i) $\sup_{\tau\in\mathcal{T}}\inf_{\mathrm{Q}\in\overline{\mathcal{Q}}}\mathbb{E}_{\mathrm{Q}}[Y_\tau]=\sup_{\tau\in\mathcal{T}_f}\inf_{\mathrm{Q}\in\overline{\mathcal{Q}}}\mathbb{E}_{\mathrm{Q}}[Y_\tau]$.
(ii) $\inf_{\mathrm{Q}\in\overline{\mathcal{Q}}}\sup_{\tau\in\mathcal{T}}\mathbb{E}_{\mathrm{Q}}[Y_\tau]=\inf_{\mathrm{Q}\in\overline{\mathcal{Q}}}\sup_{\tau\in\mathcal{T}_f}\mathbb{E}_{\mathrm{Q}}[Y_\tau]$.

Here $\mathcal{T}_f$ denotes the set of all stopping times from $\mathcal{T}$ with finite range.

Proof For $\tau\in\mathcal{T}$, we may define by

$$\tau[j](\omega):=\min\{k/2^j : k\in\mathbb{N},\tau(\omega)\le k/2^j\}\wedge T$$

a sequence $(\tau^r[j])_{j\in\mathbb{N}}$ in $\mathcal{T}_f$ satisfying $\tau[j]\searrow\tau$ pointwise, and by right-continuity of the paths of Y,

$$\lim_{j\to\infty}Y_{\tau[j](\omega)}(\omega)=Y_{\tau(\omega)}(\omega)\qquad\text{for any }\omega\in\Omega. \tag{6.9}$$

For the proof of (i), fix any $\tau\in\mathcal{T}$. Then $|Y_\tau-Y_{\tau[j]}|\to 0$ pointwise for $j\to\infty$ due to (6.9). Set

$$\widehat{Y}_k:=\sup_{j\ge k}|Y_\tau-Y_{\tau[j]}|\qquad\text{for }k\in\mathbb{N}.$$

This defines a sequence $(\widehat{Y}_k)_{k\in\mathbb{N}}$ of random variables $\widehat{Y}_k$ on $(\Omega,\mathcal{F},\mathrm{P})$ which satisfy $|\widehat{Y}_k|\le 2\sup_{t\in[0,T]}|Y_t|$ so that they belong to $\mathcal{X}$. Since $\widehat{Y}_k\searrow 0$ for $k\to\infty$ and since $\rho_{\mathcal{Q}}$ is continuous from above at 0, we obtain

$$0\le\rho_{\mathcal{Q}}(|Y_\tau-Y_{\tau[j]}|)\le\rho_{\mathcal{Q}}(\widehat{Y}_j)\xrightarrow{j\to\infty}0.$$

Hence by (6.2),

$$0 \le \left| \inf_{Q \in \overline{\mathcal{Q}}} \mathbb{E}_Q[Y_\tau] - \inf_{Q \in \overline{\mathcal{Q}}} \mathbb{E}_Q[Y_{\tau[j]}] \right| \le \rho_{\mathcal{Q}}(|Y_\tau - Y_{\tau[j]}|) \xrightarrow{j \to \infty} 0,$$

and thus

$$\sup_{\tau \in \mathcal{T}_f} \inf_{Q \in \overline{\mathcal{Q}}} \mathbb{E}_Q[Y_\tau] \ge \lim_{j \to \infty} \inf_{Q \in \overline{\mathcal{Q}}} \mathbb{E}_Q[Y_{\tau[j]}] = \inf_{Q \in \overline{\mathcal{Q}}} \mathbb{E}_Q[Y_\tau].$$

Since τ was arbitrary, we may conclude that

$$\sup_{\tau \in \mathcal{T}_f} \inf_{Q \in \overline{\mathcal{Q}}} \mathbb{E}_Q[Y_\tau] \ge \sup_{\tau \in \mathcal{T}} \inf_{Q \in \overline{\mathcal{Q}}} \mathbb{E}_Q[Y_\tau] \ge \sup_{\tau \in \mathcal{T}_f} \inf_{Q \in \overline{\mathcal{Q}}} \mathbb{E}_Q[Y_\tau],$$

where the last inequality is obvious due to $\mathcal{T}_f \subseteq \mathcal{T}$. So (i) is shown.

In order to prove (ii), fix any $\varepsilon > 0$. Then for arbitrary $Q \in \overline{\mathcal{Q}}$, we may find some $\tau_0 \in \mathcal{T}$ such that

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}_Q[Y_\tau] - \varepsilon < \mathbb{E}_Q[Y_{\tau_0}]. \tag{6.10}$$

We have $Y_{\tau_0[j]} \to Y_{\tau_0}$ pointwise for $j \to \infty$ and $|Y_{\tau_0[j]}| \le \sup_{t \in [0,T]} |Y_t|$ for every $j \in \mathbb{N}$. So in view of (2.2), we may apply the dominated convergence theorem to conclude that

$$\lim_{j \to \infty} \mathbb{E}_Q[Y_{\tau_0[j]}] = \mathbb{E}_Q[Y_{\tau_0}],$$

and thus by (6.10),

$$\sup_{\tau \in \mathcal{T}_f} \mathbb{E}_Q[Y_\tau] \ge \lim_{j \to \infty} \mathbb{E}_Q[Y_{\tau_0[j]}] = \mathbb{E}_Q[Y_{\tau_0}] > \sup_{\tau \in \mathcal{T}} \mathbb{E}_Q[Y_\tau] - \varepsilon.$$

Letting $\varepsilon \searrow 0$, we obtain

$$\sup_{\tau \in \mathcal{T}_f} \mathbb{E}_Q[Y_\tau] \ge \sup_{\tau \in \mathcal{T}} \mathbb{E}_Q[Y_\tau] \ge \sup_{\tau \in \mathcal{T}_f} \mathbb{E}_Q[Y_\tau],$$

where the last inequality is trivial due to $\mathcal{T}_f \subseteq \mathcal{T}$. Since Q was arbitrary, (ii) follows immediately. The proof is complete. □

In the next step, we show $\sup_{\tau \in \mathcal{T}_f} \inf_{Q \in \overline{\mathcal{Q}}} \mathbb{E}_Q[Y_\tau] = \inf_{Q \in \overline{\mathcal{Q}}} \sup_{\tau \in \mathcal{T}_f} \mathbb{E}_Q[Y_\tau]$.

Proposition 6.7 *Let* (2.2) *be fullfilled. If* $\rho_{\mathcal{Q}}$ *is continuous from above at* 0, *then under assumption* (A) *from Sect.* 5,

$$\sup_{\tau \in \mathcal{T}_f} \inf_{Q \in \overline{\mathcal{Q}}} \mathbb{E}_Q[Y_\tau] = \inf_{Q \in \overline{\mathcal{Q}}} \sup_{\tau \in \mathcal{T}_f} \mathbb{E}_Q[Y_\tau].$$

Proof By assumption,

$$Y_{1/k} \to Y_0 \qquad \text{pointwise for } k \to \infty.$$

Then

$$\sup_{\ell\geq k}|Y_{1/\ell}-Y_0|\searrow 0 \qquad \text{pointwise for } k\to\infty,$$

and $\sup_{\ell\geq k}|Y_{1/\ell}-Y_0|\in\mathcal{X}$ due to $\sup_{\ell\geq k}|Y_{1/\ell}-Y_0|\leq 2\sup_{t\in[0,T]}|Y_t|$. Since $\rho_{\mathcal{Q}}$ is continuous from above at 0, we may conclude from (6.2) that

$$\begin{aligned}0\leq\Big|\inf_{\mathrm{Q}\in\overline{\mathcal{Q}}}\mathbb{E}_{\mathrm{Q}}[Y_{1/k}]-\inf_{\mathrm{Q}\in\overline{\mathcal{Q}}}\mathbb{E}_{\mathrm{Q}}[Y_0]\Big|&\leq\sup_{\mathrm{Q}\in\overline{\mathcal{Q}}}\mathbb{E}_{\mathrm{Q}}[|Y_{1/k}-Y_0|]\\&\leq\rho_{\mathcal{Q}}\Big(\sup_{\ell\geq k}|Y_{1/\ell}-Y_0|\Big)\xrightarrow{k\to\infty}0.\end{aligned}$$

In particular,

$$\sup_{\tau\in\mathcal{T}_f}\inf_{\mathrm{Q}\in\overline{\mathcal{Q}}}\mathbb{E}_{\mathrm{Q}}[Y_\tau]=\sup_{\tau\in\mathcal{T}_f\setminus\{0\}}\inf_{\mathrm{Q}\in\overline{\mathcal{Q}}}\mathbb{E}_{\mathrm{Q}}[Y_\tau] \tag{6.11}$$

and

$$\inf_{\mathrm{Q}\in\overline{\mathcal{Q}}}\sup_{\tau\in\mathcal{T}_f}\mathbb{E}_{\mathrm{Q}}[Y_\tau]=\inf_{\mathrm{Q}\in\overline{\mathcal{Q}}}\sup_{\tau\in\mathcal{T}_f\setminus\{0\}}\mathbb{E}_{\mathrm{Q}}[Y_\tau]. \tag{6.12}$$

We want to apply König's minimax theorem (cf. [17, Theorem 4.9]) to the mapping

$$h:\overline{\mathcal{Q}}\times\mathcal{T}_f\setminus\{0\}\to\mathbb{R},\ (\mathrm{Q},\tau)\mapsto\mathbb{E}_{\mathrm{Q}}[-Y_\tau].$$

For preparation, we endow $\overline{\mathcal{Q}}$ with the topology $\sigma(\overline{\mathcal{Q}},\mathcal{X})$ as defined in Sect. 6.1. Then by the definitions of $\sigma(\overline{\mathcal{Q}},\mathcal{X})$ and $\mathcal{X}$ along with (6.1), we may observe that

$$h(\cdot,\tau)\text{ is continuous with respect to }\sigma(\overline{\mathcal{Q}},\mathcal{X})\text{ for }\tau\in\mathcal{T}_f\setminus\{0\}. \tag{6.13}$$

By convexity of $\overline{\mathcal{Q}}$ (see (6.2)), we also get for $\mathrm{Q}_1,\mathrm{Q}_2\in\overline{\mathcal{Q}}$, $\lambda\in[0,1]$, $\tau\in\mathcal{T}_f$ that

$$h\big(\lambda\mathrm{Q}_1+(1-\lambda)\mathrm{Q}_2,\tau\big)=\lambda h(\mathrm{Q}_1,\tau)+(1-\lambda)h(\mathrm{Q}_2,\tau). \tag{6.14}$$

In view of König's minimax result along with (6.13), (6.14) and Lemma 6.1, it remains to investigate when the following property is satisfied:

There exists some $\lambda\in(0,1)$ such that for every $\tau_1,\tau_2\in\mathcal{T}_f\setminus\{0\}$,

$$\inf_{\tau\in\mathcal{T}_f\setminus\{0\}}\sup_{\mathrm{Q}\in\overline{\mathcal{Q}}}\big(h(\mathrm{Q},\tau)-\lambda h(\mathrm{Q},\tau_1)-(1-\lambda)h(\mathrm{Q},\tau_2)\big)\leq 0. \tag{6.15}$$

By assumption (A), there exists some $\lambda\in(0,1)$ such that for $\tau_1,\tau_2\in\mathcal{T}_f\setminus\{0\}$,

$$\inf_{A\in\mathcal{F}_{\tau_1\wedge\tau_2}}\rho_{\mathcal{Q}}\big((\mathbb{1}_A-\lambda)(Y_{\tau_2}-Y_{\tau_1})\big)\leq 0. \tag{6.16}$$

Next, define for arbitrary $\tau_1,\tau_2\in\mathcal{T}_f\setminus\{0\}$ and $A\in\mathcal{F}_{\tau_1\wedge\tau_2}$ the stopping time $\tau_A:=\mathbb{1}_A\tau_1+\mathbb{1}_{\Omega\setminus A}\tau_2\in\mathcal{T}_f\setminus\{0\}$. In view of (6.2) and (6.16), we then get

$$\inf_{\tau\in\mathcal{T}_f\setminus\{0\}} \sup_{\mathrm{Q}\in\overline{\mathcal{Q}}} \big(h(\mathrm{Q},\tau) - \lambda h(\mathrm{Q},\tau_1) - (1-\lambda)h(\mathrm{Q},\tau_2)\big)$$
$$\leq \inf_{A\in\mathcal{F}_{\tau_1\wedge\tau_2}} \sup_{\mathrm{Q}\in\overline{\mathcal{Q}}} \big(h(\mathrm{Q},\tau_A) - \lambda h(\mathrm{Q},\tau_1) - (1-\lambda)h(\mathrm{Q},\tau_2)\big)$$
$$= \inf_{A\in\mathcal{F}_{\tau_1\wedge\tau_2}} \rho_{\mathcal{Q}}\big((\mathbb{1}_A - \lambda)(Y_{\tau_2} - Y_{\tau_1})\big) \leq 0.$$

This shows (6.15), and by König's minimax theorem, we obtain

$$\inf_{\tau\in\mathcal{T}_f\setminus\{0\}} \sup_{\mathrm{Q}\in\overline{\mathcal{Q}}} h(Q,\tau) = \sup_{\mathrm{Q}\in\overline{\mathcal{Q}}} \inf_{\tau\in\mathcal{T}_f\setminus\{0\}} h(\mathrm{Q},\tau).$$

In view of (6.11) along with (6.12), this completes the proof of Proposition 6.7. □

Now we are ready to show Theorem 5.1.

Proof of Theorem 5.1 Under the assumptions of Theorem 5.1, we may apply Propositions 6.2 and 6.5 to obtain

$$\sup_{\tau\in\mathcal{T}} \inf_{\mathrm{Q}\in\overline{\mathcal{Q}}} \mathbb{E}_{\mathrm{Q}}[Y_\tau] = \sup_{\tau\in\mathcal{T}} \inf_{\mathrm{Q}\in\mathrm{co}(\mathcal{Q})} \mathbb{E}_{\mathrm{Q}}[Y_\tau]$$

and

$$\inf_{\mathrm{Q}\in\overline{\mathcal{Q}}} \sup_{\tau\in\mathcal{T}} \mathbb{E}_{\mathrm{Q}}[Y_\tau] = \inf_{\mathrm{Q}\in\mathrm{co}(\mathcal{Q})} \sup_{\tau\in\mathcal{T}} \mathbb{E}_{\mathrm{Q}}[Y_\tau].$$

Moreover, in view of Lemma 6.6 along with Proposition 6.7, we have

$$\sup_{\tau\in\mathcal{T}} \inf_{\mathrm{Q}\in\overline{\mathcal{Q}}} \mathbb{E}_{\mathrm{Q}}[Y_\tau] = \sup_{\tau\in\mathcal{T}_f} \inf_{\mathrm{Q}\in\overline{\mathcal{Q}}} \mathbb{E}_{\mathrm{Q}}[Y_\tau] = \inf_{\mathrm{Q}\in\overline{\mathcal{Q}}} \sup_{\tau\in\mathcal{T}_f} \mathbb{E}_{\mathrm{Q}}[Y_\tau] = \inf_{\mathrm{Q}\in\overline{\mathcal{Q}}} \sup_{\tau\in\mathcal{T}} \mathbb{E}_{\mathrm{Q}}[Y_\tau].$$

Now the statement of Theorem 5.1 follows immediately. □

6.3 Proof of Proposition 5.2

Let the assumptions of Proposition 5.2 be fulfilled. Fix $\varepsilon > 0$. Observe that we have $|Y_{\tau_1} - Y_{\tau_2}|\mathbb{1}_{\{|Y_{\tau_1}-Y_{\tau_2}|>k\}} \searrow 0$ for $k\to\infty$. Since $\rho_{\mathcal{Q}}$ is continuous from above at 0, we may select some $k_0\in\mathbb{N}$ such that

$$\rho_{\mathcal{Q}}\big(|Y_{\tau_1} - Y_{\tau_2}|\mathbb{1}_{\{|Y_{\tau_1}-Y_{\tau_2}|>k_0\}}\big) \leq \varepsilon/3. \tag{6.17}$$

The random variable $|Y_{\tau_1} - Y_{\tau_2}|\mathbb{1}_{\{|Y_{\tau_1}-Y_{\tau_2}|\leq k_0\}}$ is bounded so that we may find some random variable X on $(\Omega,\mathcal{F},\mathrm{P})$ with finite range satisfying

$$\sup_{\omega\in\Omega} \big|\big(Y_{\tau_2}(\omega) - Y_{\tau_1}(\omega)\big)\mathbb{1}_{\{|Y_{\tau_1}-Y_{\tau_2}|\leq k_0\}}(\omega) - X(\omega)\big| \leq \varepsilon/3$$

(cf. e.g. [18, Proposition 22.1]). In particular with $\widetilde{Y} := Y_{\tau_2} - Y_{\tau_1}$,

$$\rho_{\mathcal{Q}}\big(\big|\widetilde{Y}\mathbb{1}_{\{|\widetilde{Y}|\leq k_0\}} - X\big|\big) \leq \sup_{\omega\in\Omega} \big|\widetilde{Y}(\omega)\mathbb{1}_{\{|\widetilde{Y}|\leq k_0\}}(\omega) - X(\omega)\big| \leq \varepsilon/3. \tag{6.18}$$

Since X has finite range, there exist pairwise disjoint $B_1, \dots, B_r \in \mathcal{F}$ and in addition $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ such that $X = \sum_{i=1}^r \lambda_i \mathbb{1}_{B_i}$. Now let $(A_k)_{k\in\mathbb{N}}$ be any sequence in $\mathcal{F}$. We may observe by assumption that any sequence $(\mu_{\mathcal{Q}}(A_k \cap B_i))_{k\in\mathbb{N}}$ is relatively $\|\cdot\|_\infty$-compact for $i = 1, \dots, r$ so that there exist a subsequence $(A_{\varphi(k)})_{k\in\mathbb{N}}$ and $f_1, \dots, f_r \in \ell^\infty(\mathcal{Q})$ such that

$$\|\mu_{\mathcal{Q}}(A_{\varphi(k)} \cap B_i) - f_i\|_\infty \xrightarrow{k\to\infty} 0 \qquad \text{for every } i \in \{1, \dots, r\}.$$

Then

$$\sup_{Q\in\mathcal{Q}} \left| \mathbb{E}_Q[\mathbb{1}_{A_{\varphi(k)}} X] - \sum_{i=1}^r \lambda_i f_i(Q) \right| \le \sum_{i=1}^r |\lambda_i| \|\mu_{\mathcal{Q}}(A_{\varphi(k)} \cap B_i) - f_i\|_\infty \xrightarrow{k\to\infty} 0.$$

This means that

$$\left\{ (\mathbb{E}_Q[\mathbb{1}_A X])_{Q\in\mathcal{Q}} : A \in \mathcal{F} \right\} \text{ is relatively} \|\cdot\|_\infty\text{-compact.} \tag{6.19}$$

Next, let $L^1(\Omega, \mathcal{F}_{\tau_1\wedge\tau_2}, P|_{\mathcal{F}_{\tau_1\wedge\tau_2}})$ denote the L^1-space on $(\Omega, \mathcal{F}_{\tau_1\wedge\tau_2}, P|_{\mathcal{F}_{\tau_1\wedge\tau_2}})$ and write $L^\infty(\Omega, \mathcal{F}_{\tau_1\wedge\tau_2}, P|_{\mathcal{F}_{\tau_1\wedge\tau_2}})$ for the space of all $P|_{\mathcal{F}_{\tau_1\wedge\tau_2}}$-essentially bounded random variables. The latter is equipped with the weak* topology $\sigma(L^\infty_{\tau_1\wedge\tau_2}, L^1_{\tau_1\wedge\tau_2})$. Since the probability space $(\Omega, \mathcal{F}_{\tau_1\wedge\tau_2}, P|_{\mathcal{F}_{\tau_1\wedge\tau_2}})$ is assumed to be atomless, we already know from [15, Lemma 3] that $\{\mathbb{1}_A : A \in \mathcal{F}_{\tau_1\wedge\tau_2}\}$ is a $\sigma(L^\infty_{\tau_1\wedge\tau_2}, L^1_{\tau_1\wedge\tau_2})$-dense subset of the set Δ defined to consist of all $Z \in L^\infty(\Omega, \mathcal{F}_{\tau_1\wedge\tau_2}, P|_{\mathcal{F}_{\tau_1\wedge\tau_2}})$ satisfying $0 \le Z \le 1$ P-a.s. In particular, we may find a net $(A_i)_{i\in I}$ such that $(\mathbb{1}_{A_i})_{i\in I}$ converges to $1/2$ with respect to $\sigma(L^\infty_{\tau_1\wedge\tau_2}, L^1_{\tau_1\wedge\tau_2})$. In view of (6.19), there is a subnet $(\mathbb{1}_{A_{i(j)}})_{j\in J}$ such that

$$\lim_{j\to\infty} \sup_{Q\in\mathcal{Q}} |\mathbb{E}_Q[\mathbb{1}_{A_{i(j)}} X] - f(Q)| = 0 \qquad \text{for some } f \in \ell^\infty(\mathcal{Q}).$$

Notice further that $\mathbb{E}[X \frac{dQ}{dP} | \mathcal{F}_{\tau_1\wedge\tau_2}]$ is in $L^1(\Omega, \mathcal{F}_{\tau_1\wedge\tau_2}, P|_{\mathcal{F}_{\tau_1\wedge\tau_2}})$ for any $Q \in \mathcal{Q}$. This implies for every $Q \in \mathcal{Q}$ that

$$\begin{aligned} f(Q) = \lim_{j\to\infty} \mathbb{E}_Q[\mathbb{1}_{A_{i(j)}} X] &= \lim_{j\to\infty} \mathbb{E}\left[\mathbb{1}_{A_{i(j)}} X \frac{dQ}{dP}\right] \\ &= \lim_{j\to\infty} \mathbb{E}\left[\mathbb{1}_{A_{i(j)}} \mathbb{E}\left[X \frac{dQ}{dP} \middle| \mathcal{F}_{\tau_1\wedge\tau_2}\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[X \frac{dQ}{dP} \middle| \mathcal{F}_{\tau_1\wedge\tau_2}\right]/2\right] = \mathbb{E}_Q[X/2]. \end{aligned}$$

Hence

$$\sup_{Q\in\mathcal{Q}} |\mathbb{E}_Q[\mathbb{1}_{A_{i(j_0)}} X] - \mathbb{E}_Q[X/2]| < \varepsilon/3 \qquad \text{for some } j_0 \in J. \tag{6.20}$$

We may observe directly by the sublinearity of $\rho_{\mathcal{Q}}$ along with (6.17), (6.18) and (6.20) that

$$\begin{aligned}
&\rho_{\mathcal{Q}}\big((\mathbb{1}_{A_{i(j_0)}} - 1/2)(Y_{\tau_2} - Y_{\tau_1})\big)\\
&\le \rho_{\mathcal{Q}}\big((\mathbb{1}_{A_{i(j_0)}} - 1/2)\widetilde{Y}_{>k_0}\big) + \rho_{\mathcal{Q}}\big((\mathbb{1}_{A_{i(j_0)}} - 1/2)(\widetilde{Y}_{\le k_0} - X)\big)\\
&\quad + \rho_{\mathcal{Q}}\big((\mathbb{1}_{A_{i(j_0)}} - 1/2)X\big)\\
&\le \rho_{\mathcal{Q}}\big(|\widetilde{Y}_{>k_0}|\big) + \rho_{\mathcal{Q}}\big(|\widetilde{Y}_{\le k_0} - X|\big) + \rho_{\mathcal{Q}}\big((\mathbb{1}_{A_{i(j_0)}} - 1/2)X\big)\\
&\le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,
\end{aligned}$$

where $\widetilde{Y}_{>k_0} := \widetilde{Y}\mathbb{1}_{\{|\widetilde{Y}|>k_0\}}$ and $\widetilde{Y}_{\le k_0} := \widetilde{Y}\mathbb{1}_{\{|\widetilde{Y}|\le k_0\}}$. Hence we have shown that

$$\inf_{A\in\mathcal{F}_{\tau_1\wedge\tau_2}} \rho_{\mathcal{Q}}\big((\mathbb{1}_A - 1/2)(Y_{\tau_2} - Y_{\tau_1})\big) \le \varepsilon$$

which completes the proof by sending $\varepsilon \searrow 0$.

6.4 Proof of Theorem 2.1

Note first that for $\tau_1, \tau_2 \in \mathcal{T}_f \setminus \{0\}$, there is some $t > 0$ such that $\mathcal{F}_t \subseteq \mathcal{F}_{\tau_1\wedge\tau_2}$. Therefore by assumption (2.1), the probability space $(\Omega, \mathcal{F}_{\tau_1\wedge\tau_2}, \mathrm{P}|_{\mathcal{F}_{\tau_1\wedge\tau_2}})$ is atomless for $\tau_1, \tau_2 \in \mathcal{T}_f \setminus \{0\}$. Then in view of Proposition 5.2 along with Theorem 5.1, it remains to show the following auxiliary result.

Lemma 6.8 *Let (2.2) be fulfilled and let the range of $\mu_{\mathcal{Q}}$ be relatively $\|\cdot\|_\infty$-compact. If $\rho_{\mathcal{Q}}(Y^*\mathbb{1}_{\{Y^*>a\}}) \to 0$ for $a \to \infty$, then $\rho_{\mathcal{Q}}$ is continuous from above at 0.*

Proof Let $(X_n)_{n\in\mathbb{N}}$ be any nonincreasing sequence in $\mathcal{X}$ with $X_n \searrow 0$ P-a.s. and let $\varepsilon > 0$. Then by the definition of $\mathcal{X}$, we may find some $C > 0$ such that

$$0 \le X_n \le X_1 \le C(Y^* + 1) \qquad \text{P-a.s. for } n \in \mathbb{N}.$$

Then for any $n \in \mathbb{N}$ and every $k \in \mathbb{N}$, we may observe by sublinearity of $\rho_{\mathcal{Q}}$ that

$$\begin{aligned}
0 \le \rho_{\mathcal{Q}}(X_n) &\le \rho_{\mathcal{Q}}(X_n\mathbb{1}_{\{Y^*\le k\}}) + \rho_{\mathcal{Q}}(X_n\mathbb{1}_{\{Y^*>k\}})\\
&\le \rho_{\mathcal{Q}}(X_n\mathbb{1}_{\{Y^*\le k\}}) + \rho_{\mathcal{Q}}(C(Y^*+1)\mathbb{1}_{\{Y^*>k\}})\\
&\le \rho_{\mathcal{Q}}(X_n\mathbb{1}_{\{Y^*\le k\}}) + 2C\rho_{\mathcal{Q}}(Y^*\mathbb{1}_{\{Y^*>k\}}). \qquad (6.21)
\end{aligned}$$

Next, observe that $(X_n\mathbb{1}_{\{Y^*\le k\}})_{n\in\mathbb{N}}$ is uniformly bounded by some constant, say C_k, for any $k \in \mathbb{N}$. Then with $\overline{X}_{k,n} := X_n\mathbb{1}_{\{Y^*\le k\}}$, we obtain for $k, n \in \mathbb{N}$ that

$$0 \le \rho_{\mathcal{Q}}\big(\overline{X}_{k,n}\big) = \sup_{\mathrm{Q}\in\mathcal{Q}} \int_0^\infty \mathrm{Q}\big[\overline{X}_{k,n} > x\big]\,dx \le \int_0^{C_k} \sup_{\mathrm{Q}\in\mathcal{Q}} \mathrm{Q}\big[\overline{X}_{k,n} > x\big]\,dx. \qquad (6.22)$$

Now fix $k \in \mathbb{N}$ and $x \in (0, C_k)$. Since the range of $\mu_{\mathcal{Q}}$ is relatively compact with respect to $\|\cdot\|_\infty$, we may find for any subsequence $(\mu_{\mathcal{Q}}(\{\overline{X}_{k,i(n)} > x\}))_{n\in\mathbb{N}}$ a further subsequence $(\mu_{\mathcal{Q}}(\{\overline{X}_{k,j(i(n))} > x\}))_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \sup_{\mathrm{Q}\in\mathcal{Q}} \big|\mu_{\mathcal{Q}}(\{\overline{X}_{k,j(i(n))} > x\})(\mathrm{Q}) - f_k(\mathrm{Q})\big| = 0$$

for some $f_k \in \ell^\infty(\mathcal{Q})$. Furthermore, $\mathrm{Q}[\overline{X}_{k,j(i(n))} > x] \searrow 0$ for $n \to \infty$ if $\mathrm{Q} \in \mathcal{Q}$. This implies $f_k \equiv 0$, and thus $\sup_{\mathrm{Q}\in\mathcal{Q}} \mathrm{Q}[\overline{X}_{k,n} > x] \searrow 0$ for $n \to \infty$. Hence in view of (6.22), an application of the dominated convergence theorem yields

$$\rho_{\mathcal{Q}}(X_n \mathbb{1}_{\{Y^* \le k\}}) \xrightarrow{n\to\infty} 0 \qquad \text{for } k \in \mathbb{N}.$$

Then by (6.21),

$$0 \le \limsup_{n\to\infty} \rho_{\mathcal{Q}}(X_n) \le 2C\rho_{\mathcal{Q}}(Y^* \mathbb{1}_{\{Y^* > k\}}) \qquad \text{for } k \in \mathbb{N}.$$

Finally, by assumption, $C\rho_{\mathcal{Q}}(Y^* \mathbb{1}_{\{Y^* > k\}}) \to 0$ for $k \to \infty$ so that

$$0 \le \limsup_{n\to\infty} \rho_{\mathcal{Q}}(X_n) \le 0.$$

This completes the proof. □

Proof of Theorem 2.1 As discussed before Lemma 6.8, the statement of Theorem 2.1 follows directly by combining Proposition 5.2 and Theorem 5.1 with Lemma 6.8. □

6.5 Proof of Theorem 2.4

Since d is totally bounded, the completion $(\check{\mathcal{Q}}, \check{d})$ of $(\mathcal{Q}, d)$ is compact (see [26, Sect. 9.2, Problem 2]). Since $(d\mathrm{Q}/d\mathrm{P})_{\mathrm{Q}\in\mathcal{Q}}$ has P-almost surely d-uniformly continuous paths, we may find some $A \in \mathcal{F}$ with $\mathrm{P}[A] = 1$ such that we may define a nonnegative stochastic process $(Z_{\check{\mathrm{Q}}})_{\check{\mathrm{Q}}\in\check{\mathcal{Q}}}$ such that $Z := Z_\bullet(\omega)$ is continuous for any $\omega \in A$ and $Z_{\mathrm{Q}} = d\mathrm{Q}/d\mathrm{P}$ holds for $\mathrm{Q} \in \mathcal{Q}$ (see [26, Theorem 11.3.4]).

Now let $(\mathrm{Q}_n)_{n\in\mathbb{N}}$ be any sequence in $\mathcal{Q}$. By compactness, we may select a subsequence $(\mathrm{Q}_{i(n)})_{n\in\mathbb{N}}$ which converges to some $\check{\mathrm{Q}} \in \check{\mathcal{Q}}$ with respect to $\check{d}$ (see [26, Theorem 7.2.1]). Since the process Z has $\check{d}$-continuous paths on A, we obtain

$$\frac{d\mathrm{Q}_{i(n)}}{d\mathrm{P}}(\omega) = Z_{\mathrm{Q}_{i(n)}}(\omega) \xrightarrow{n\to\infty} Z_{\check{\mathrm{Q}}}(\omega) \qquad \text{for all } \omega \in A.$$

Moreover, by assumption, $(d\mathrm{Q}_{i(n)}/d\mathrm{P})_{n\in\mathbb{N}}$ is dominated by some P-integrable random variable U. Then an application of the dominated convergence theorem yields

$$\mathbb{E}\left[\left|\frac{d\mathrm{Q}_{i(n)}}{d\mathrm{P}} - Z_{\check{\mathrm{Q}}}\right|\right] \xrightarrow{n\to\infty} 0.$$

Thus we have shown that $\mathcal{Q}$ is relatively compact with respect to the topology of total variation as defined in Remark 2.3, and Theorem 2.4 may be concluded by combining Remark 2.3 with Theorem 2.1.

6.6 Proof of Proposition 3.1

Fix $t \in [0, T]$. By assumption, $(X^\theta)_{\theta\in\Theta}$ is a nearly sub-Gaussian random field in the sense of the Appendix. Then by Proposition A.2, we may fix some separable version $(\widehat{X}_t^\theta)_{\theta\in\Theta}$ of $(X_t^\theta)_{\theta\in\Theta}$. It is also assumed that there is some $\overline{\theta}\in\Theta$ such that $\mathbb{E}[\exp(2\widehat{X}_t^{\overline{\theta}})]=\mathbb{E}[\exp(2X_t^{\overline{\theta}})]<\infty$. In addition, by Proposition A.2 again, we may find a nonnegative random variable $U_t^{\overline{\theta}}$ and some $A_t\in\mathcal{F}$ with $\mathrm{P}[A_t]=1$ such that

$$\mathbb{E}[\exp(pU_t^{\overline{\theta}})]<\infty \qquad \text{for every } p\in(0,\infty), \tag{6.23}$$

$$\sup_{\theta\in\Theta}\exp\big(\widehat{X}_t^\theta(\omega)\big)\le\exp\big(U_t^{\overline{\theta}}(\omega)\big)\exp\big(\widehat{X}_t^{\overline{\theta}}(\omega)\big) \qquad \text{for } \omega\in A_t. \tag{6.24}$$

By assumption and since $A_t\in\mathcal{F}_t$, for every $\theta\in\Theta$,

$$M_t^\theta:=\exp\big(\widehat{X}_t^\theta-[X^\theta]_t/2\big)\mathbb{1}_{A_t}$$

defines a Radon–Nikodým derivative of $\mathrm{Q}_\theta|_{\mathcal{F}_t}$. Then due to the nonnegativity of the process $([X^\theta]_t)_{\theta\in\Theta}$, (6.24) yields

$$\sup_{\theta\in\Theta}M_t^\theta\le\exp(U_t^{\overline{\theta}})\exp(\widehat{X}_t^{\overline{\theta}}) \qquad \text{pointwise.}$$

By (6.23) along with the assumptions on $X_t^{\overline{\theta}}$, the random variables $\exp(2U_t^{\overline{\theta}})$ and $\exp(\widehat{X}_t^{\overline{\theta}})$ are square-integrable. By the Cauchy–Schwarz inequality, $\exp(U_t^{\overline{\theta}})\exp(\widehat{X}_t^{\overline{\theta}})$ is therefore integrable so that $(M_t^\theta)_{\theta\in\Theta}$ is dominated by some P-integrable random variable. Thus by Theorem 2.4, it remains to show that $(M_t^\theta)_{\theta\in\Theta}$ has d_Θ-uniformly continuous paths. For $\theta,\vartheta\in\Theta$, we may conclude from (6.24) and the nonnegativity of the process $([X^\theta]_t)_{\theta\in\Theta}$ that

$$\begin{aligned}|M_t^\theta-M_t^\vartheta|&\le\big(\exp(\widehat{X}_t^\theta)+\exp(\widehat{X}_t^\vartheta)\big)\big(|\widehat{X}_t^\theta-\widehat{X}_t^\vartheta|+|[X^\theta]_t/2-[X^\vartheta]_t/2|\big)\mathbb{1}_{A_t}\\&\le 2\sup_{\theta\in\Theta}\exp(\widehat{X}_t^\theta)\Big(|\widehat{X}_t^\theta-\widehat{X}_t^\vartheta|+\frac{1}{2}|[X^\theta]_t-[X^\vartheta]_t|\Big)\mathbb{1}_{A_t}\\&\le 2\exp(U_t^{\overline{\theta}})\exp(\widehat{X}_t^{\overline{\theta}})\Big(|\widehat{X}_t^\theta-\widehat{X}_t^\vartheta|+\frac{1}{2}|[X^\theta]_t-[X^\vartheta]_t|\Big)\mathbb{1}_{A_t}.\end{aligned} \tag{6.25}$$

In view of Proposition A.2, the process $(\widehat{X}_t^\theta)_{\theta\in\Theta}$ has d_Θ-uniformly continuous paths and $([X^\theta]_t)_{\theta\in\Theta}$ satisfies this property by assumption. Thus by (6.25), $(M_t^\theta)_{\theta\in\Theta}$ has d_Θ-uniformly continuous paths.

6.7 Proof for Example 3.2

Firstly, each X^ψ is a centered martingale. Secondly, by a time change, we may construct an enlargement $(\overline{\Omega},\overline{\mathcal{F}}_T,(\overline{\mathcal{F}}_t)_{0\le t\le T},\overline{\mathrm{P}})$ of $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\le t\le T},\mathrm{P})$ with $\overline{\Omega}=\Omega\times\widetilde{\Omega}$ for some set $\widetilde{\Omega}$ such that for every fixed pair $\psi,\phi\in\Psi$, there exists a Brownian motion $\overline{Z}^{\psi,\phi}$ with $(\overline{Z}_t^{\psi,\phi})_{0\le t\le T}$ adapted to $(\overline{\mathcal{F}}_t)_{0\le t\le T}$ and such that for every $t\in[0,T]$,

$$\overline{X}_t^\psi-\overline{X}_t^\phi=\overline{Z}^{\psi,\phi}_{\int_0^t(\psi-\phi)^2(u,V_u)\,du}$$

(see e.g. [21, proof of Theorem V.1.7]). Here we set for each $\overline{\omega} = (\omega, \tilde{\omega}) \in \overline{\Omega}$

$$\overline{X}_t^{\psi}(\overline{\omega}) - \overline{X}_t^{\phi}(\overline{\omega}) = \overline{X}_t^{\psi}(\omega, \tilde{\omega}) - \overline{X}_t^{\phi}(\omega, \tilde{\omega}) := X_t^{\psi}(\omega) - X_t^{\phi}(\omega).$$

Then for fixed $\lambda > 0$, $t \in [0, T]$ and $\psi, \phi \in \Psi$, we obtain

$$\begin{aligned}\mathbb{E}\big[\exp\big(\lambda(X_t^{\psi} - X_t^{\phi})\big)\big] &= \mathbb{E}_{\overline{\mathrm{P}}}\Big[\exp\Big(\lambda \overline{Z}^{\psi,\phi}_{\int_0^t (\psi-\phi)^2(u,V_u)\,du}\Big)\Big] \\ &\le \mathbb{E}_{\overline{\mathrm{P}}}\Big[\exp\Big(\lambda \max_{0\le s\le d(\psi,\phi)^2} \overline{Z}_s^{\psi,\phi}\Big)\Big].\end{aligned}$$

Now we derive by the reflection principle for Brownian motion that

$$\mathbb{E}_{\overline{\mathrm{P}}}\Big[\exp\Big(\lambda \max_{0\le s\le d(\psi,\phi)^2} \overline{Z}_s^{\psi,\phi}\Big)\Big] \le 2\mathbb{E}_{\overline{\mathrm{P}}}\big[\exp\big(\lambda \overline{Z}^{\psi,\phi}_{d(\psi,\phi)^2}\big)\big] = 2\exp\big(\lambda^2 d(\psi,\phi)^2/2\big).$$

Hence $(X_t^{\psi}, \psi \in \Psi)$ is a nearly sub-Gaussian family of local martingales with constant $C = 2$.

6.8 Proof of Proposition 4.1

Let $\widehat{\mathcal{Q}}$ and $\widehat{\mathcal{Q}}^e$ be defined as in Proposition 4.1. Furthermore, let $L^p(\Omega, \mathcal{F}, \mathrm{P})$ denote the classical L^p-space on $(\Omega, \mathcal{F}, \mathrm{P})$ for $p \in [1, \infty]$. We need the following auxiliary result for preparation.

Lemma 6.9 *The set $\mathbb{F}_{\widehat{\mathcal{Q}}} := \{d\mathrm{Q}/d\mathrm{P} : \mathrm{Q} \in \widehat{\mathcal{Q}}\}$ is closed with respect to the L^1-norm. It is even compact with respect to the L^1-norm if $\mathcal{Q}$ is relatively compact with respect to the topology of total variation. In this case, $\mathbb{F}_{\widehat{\mathcal{Q}}^e} := \{d\mathrm{Q}/d\mathrm{P} : \mathrm{Q} \in \widehat{\mathcal{Q}}^e\}$ is relatively compact with respect to the L^1-norm.*

Proof The set $\mathbb{F}_{\widehat{\mathcal{Q}}}$ is obviously convex, and it is also known to be the topological closure of the convex hull $\mathrm{co}(\mathbb{F}_{\mathcal{Q}})$ of $\mathbb{F}_{\mathcal{Q}}$ with respect to the weak topology on $L^1(\Omega, \mathcal{F}, \mathrm{P})$ (see [19, Theorem 1.4]). Thus by convexity, $\mathbb{F}_{\widehat{\mathcal{Q}}}$ is also the closed convex hull of $\mathbb{F}_{\mathcal{Q}}$ with respect to the L^1-norm topology. Moreover, if $\mathcal{Q}$ is relatively compact with respect to the topology of total variation, the set $\mathbb{F}_{\mathcal{Q}}$ is relatively L^1-norm-compact so that its L^1-norm-closed convex hull $\mathbb{F}_{\widehat{\mathcal{Q}}}$ is L^1-norm-compact (see e.g. [1, Theorem 5.35]). This completes the proof because $\mathbb{F}_{\widehat{\mathcal{Q}}^e} \subseteq \mathbb{F}_{\widehat{\mathcal{Q}}}$. □

Proof of Proposition 4.1 The implication (5) ⇒ (4) is already known (see [23, Lemma 5.3], and [11, Lemma 6.48] for the discrete-time case). Concerning the implication (1) ⇒ (2), let $\tau \in \mathcal{T}$ and $X \in L^{\infty}(\Omega, \mathcal{F}, \mathrm{P})$. Then $Z := \operatorname{ess\,inf}_{\mathrm{Q}\in\mathcal{Q}} \mathbb{E}_{\mathrm{Q}}[X|\mathcal{F}_\tau]$ is in $L^{\infty}(\Omega, \mathcal{F}, \mathrm{P})$ and $\mathcal{F}_\tau$-measurable so that

$$\operatorname*{ess\,inf}_{\mathrm{Q}\in\mathcal{Q}} \mathbb{E}_{\mathrm{Q}}[X|\mathcal{F}_\tau] = \operatorname*{ess\,inf}_{\mathrm{Q}\in\mathcal{Q}} \mathbb{E}_{\mathrm{Q}}[Z|\mathcal{F}_\tau].$$

Then by the time-consistency in (1), we get

$$\inf_{Q\in\mathcal{Q}} \mathbb{E}_Q[X] = \inf_{Q\in\mathcal{Q}} \mathbb{E}_Q[Z]$$

which shows (2).

Next, we want to show (2) $\Rightarrow$ (3). Firstly, (2) obviously implies

$$\inf_{Q\in\mathcal{Q}} \mathbb{E}_Q[X] = \inf_{Q\in\mathcal{Q}} \mathbb{E}_Q\left[\operatorname*{ess\,inf}_{Q\in\mathcal{Q}} \mathbb{E}_Q[X|\mathcal{F}_\tau]\right] \qquad \text{for } \tau \in \mathcal{T} \text{ and } X \in L^\infty(\Omega, \mathcal{F}, P),$$

which may be rewritten as

$$\rho_0(X) = \rho_0\big(-\rho_\tau(X)\big) \qquad \text{for } \tau \in \mathcal{T} \text{ and } X \in L^\infty(\Omega, \mathcal{F}, P), \tag{6.26}$$

where

$$\rho_s(X) := \operatorname*{ess\,sup}_{Q\in\mathcal{Q}} \mathbb{E}_Q[-X|\mathcal{F}_s] \qquad \text{for } X \in L^\infty(\Omega, \mathcal{F}, P),\, s \in \{0, \tau\},\, \tau \in \mathcal{T}.$$

Since each member of $\mathcal{Q}$ is equivalent to P, we may observe from (6.26) that for every $\tau \in \mathcal{T} \setminus \{0\}$, the functions ρ_0, ρ_τ fulfil the assumptions and statement (a) from [11, Theorem 11.22]. Then in the proof of this theorem, it is shown that

$$\rho_\tau(X) = \operatorname*{ess\,sup}_{Q\in\widehat{\mathcal{Q}}^e} \mathbb{E}_Q[-X|\mathcal{F}_\tau] \qquad \text{for } X \in L^\infty(\Omega, \mathcal{F}, P)$$

so that

$$\operatorname*{ess\,inf}_{Q\in\mathcal{Q}} \mathbb{E}_Q[X|\mathcal{F}_\tau] = \operatorname*{ess\,inf}_{Q\in\widehat{\mathcal{Q}}^e} \mathbb{E}_Q[X|\mathcal{F}_\tau] \qquad \text{for } \tau \in \mathcal{T} \text{ and } X \in L^\infty(\Omega, \mathcal{F}, P). \tag{6.27}$$

In order to verify (3), it remains to show that $\widehat{\mathcal{Q}}^e$ is stable under pasting. So take $Q_1, Q_2 \in \widehat{\mathcal{Q}}^e$, $\tau \in \mathcal{T}$, and let $\overline{Q}$ denote the pasting of Q_1, Q_2 in τ. Then for any P-essentially bounded random variable X, (6.27) yields

$$\begin{aligned}
\mathbb{E}_{\overline{Q}}[X] = \mathbb{E}_{Q_1}\Big[\mathbb{E}_{Q_2}[X|\mathcal{F}_\tau]\Big] &\geq \mathbb{E}_{Q_1}\Big[\operatorname*{ess\,inf}_{Q\in\widehat{\mathcal{Q}}^e} \mathbb{E}_Q[X|\mathcal{F}_\tau]\Big] \\
&= \mathbb{E}_{Q_1}\Big[\operatorname*{ess\,inf}_{Q\in\mathcal{Q}} \mathbb{E}_Q[X|\mathcal{F}_\tau]\Big] \\
&\geq \inf_{Q\in\widehat{\mathcal{Q}}^e} \mathbb{E}_Q\Big[\operatorname*{ess\,inf}_{Q\in\mathcal{Q}} \mathbb{E}_Q[X|\mathcal{F}_\tau]\Big] \\
&= \inf_{Q\in\mathcal{Q}} \mathbb{E}_Q\Big[\operatorname*{ess\,inf}_{Q\in\mathcal{Q}} \mathbb{E}_Q[X|\mathcal{F}_\tau]\Big].
\end{aligned}$$

Hence $\mathbb{E}_{\overline{Q}}[X] \geq \inf_{Q\in\mathcal{Q}} \mathbb{E}_Q[X]$ holds due to (2). Therefore $\overline{Q}$ belongs to $\widehat{\mathcal{Q}}$, and thus also to $\widehat{\mathcal{Q}}^e$.

Let us now turn to the implication (3) $\Rightarrow$ (1). So take $\overline{X}, X \in L^\infty(\Omega, \mathcal{F}, P)$ and $\sigma, \tau \in \mathcal{T}$ with $\sigma \leq \tau$ such that

$$\operatorname*{ess\,inf}_{Q\in\mathcal{Q}} \mathbb{E}_Q[\overline{X}|\mathcal{F}_\tau] \leq \operatorname*{ess\,inf}_{Q\in\mathcal{Q}} \mathbb{E}_Q[X|\mathcal{F}_\tau].$$

In view of (6.27), this means that

$$\operatorname*{ess\,inf}_{Q\in\widehat{\mathcal{Q}}^e}\mathbb{E}_Q[\overline{X}|\mathcal{F}_\tau]\le\operatorname*{ess\,inf}_{Q\in\widehat{\mathcal{Q}}^e}\mathbb{E}_Q[X|\mathcal{F}_\tau]. \tag{6.28}$$

Let $\varepsilon>0$ with $|\overline{X}|\le\varepsilon$ P-a.s. and define the uniformly bounded, nonnegative càdlàg process $H=(H_t)_{0\le t\le T}$ via $H_t:=\mathbb{1}_{\{T\}}(t)(\varepsilon-\overline{X})$. Since $\widehat{\mathcal{Q}}^e$ is stable under pasting by (3), applying [23, Lemma 4.17] to H yields

$$\operatorname*{ess\,sup}_{Q\in\widehat{\mathcal{Q}}^e}\mathbb{E}_Q[\varepsilon-\overline{X}|\mathcal{F}_\sigma]=\operatorname*{ess\,sup}_{Q\in\widehat{\mathcal{Q}}^e}\mathbb{E}_Q\bigg[\operatorname*{ess\,sup}_{Q\in\widehat{\mathcal{Q}}^e}\mathbb{E}_Q[\varepsilon-\overline{X}|\mathcal{F}_\tau]\bigg|\mathcal{F}_\sigma\bigg].$$

In particular, we obtain

$$\operatorname*{ess\,inf}_{Q\in\widehat{\mathcal{Q}}^e}\mathbb{E}_Q[\overline{X}|\mathcal{F}_\sigma]=\operatorname*{ess\,inf}_{Q\in\widehat{\mathcal{Q}}^e}\mathbb{E}_Q\bigg[\operatorname*{ess\,inf}_{Q\in\widehat{\mathcal{Q}}^e}\mathbb{E}_Q[\overline{X}|\mathcal{F}_\tau]\bigg|\mathcal{F}_\sigma\bigg].$$

In view of (6.27) along with (6.28), this implies

$$\operatorname*{ess\,inf}_{Q\in\mathcal{Q}}\mathbb{E}_Q[\overline{X}|\mathcal{F}_\sigma]\le\operatorname*{ess\,inf}_{Q\in\mathcal{Q}}\mathbb{E}_Q[X|\mathcal{F}_\sigma].$$

For the implication (4) $\Rightarrow$ (2), let $X\in L^\infty(\Omega,\mathcal{F},P)$. There is some $C>0$ such that $X+C\ge1$ P-a.s. Then $Z_t:=\mathbb{1}_{\{T\}}(t)(X+C)$ defines a uniformly bounded, nonnegative adapted càdlàg process $Z=(Z_t)_{0\le t\le T}$ from $\mathcal{S}(\mathcal{Q})$. Furthermore, fix $\overline{Q}\in\mathcal{Q}$ and $\tau\in\mathcal{T}$. By (4), we can find some sequence $(Q^k)_{k\in\mathbb{N}}$ in $\mathcal{Q}$ whose members coincide with $\overline{Q}$ on $\mathcal{F}_\tau$ such that

$$\mathbb{E}_{Q_k}[X+C|\mathcal{F}_\tau]=\operatorname*{ess\,sup}_{\sigma\in\mathcal{T},\sigma\ge\tau}\mathbb{E}_{Q_k}[Z_\sigma|\mathcal{F}_\tau]\xrightarrow{k\to\infty}\operatorname*{ess\,inf}_{Q\in\mathcal{Q}}\operatorname*{ess\,sup}_{\sigma\in\mathcal{T},\sigma\ge\tau}\mathbb{E}_Q[Z_\sigma|\mathcal{F}_\tau]\quad\text{P-a.s.}$$

Since in addition $\operatorname{ess\,sup}_{\sigma\in\mathcal{T},\sigma\ge\tau}\mathbb{E}_Q[Z_\sigma|\mathcal{F}_\tau]=\mathbb{E}_Q[X+C|\mathcal{F}_\tau]$ for every $Q\in\mathcal{Q}$, we obtain by dominated convergence that

$$\begin{aligned}\mathbb{E}_{\overline{Q}}\bigg[\operatorname*{ess\,inf}_{Q\in\mathcal{Q}}\mathbb{E}_Q[X+C|\mathcal{F}_\tau]\bigg]&=\lim_{k\to\infty}\mathbb{E}_{\overline{Q}}\big[\mathbb{E}_{Q_k}[X+C|\mathcal{F}_\tau]\big]\\&=\lim_{k\to\infty}\mathbb{E}_{Q^k}\big[\mathbb{E}_{Q^k}[X+C|\mathcal{F}_\tau]\big]\\&=\lim_{k\to\infty}\mathbb{E}_{Q^k}[X+C]\\&\ge\inf_{Q\in\mathcal{Q}}\mathbb{E}_Q[X+C].\end{aligned}$$

For the second equality, we have invoked that $Q^k|_{\mathcal{F}_\tau}=\overline{Q}|_{\mathcal{F}_\tau}$ for every $k\in\mathbb{N}$. Then (2) is obvious, and the proof is complete. □

6.9 Proof for Remark 2.3

If the sets $\widehat{\mathcal{Q}}$ and $\widehat{\mathcal{Q}}^e$ are defined as in Proposition 4.1, then in view of Proposition 4.1, it remains to show that under the assumptions of Remark 2.3, the set $\widehat{\mathcal{Q}}^e$ is not stable under pasting with respect to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathrm{P})$. Since the set of all probability measures on $\mathcal{F}$ which are equivalent to P is stable under pasting with respect to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathrm{P})$ and contains $\widehat{\mathcal{Q}}^e$, we may define the minimal set $\widehat{\mathcal{Q}}^{\mathrm{st}}$ which is stable under pasting and contains $\widehat{\mathcal{Q}}^e$. We want to show that $\widehat{\mathcal{Q}}^e$ is a proper subset of $\widehat{\mathcal{Q}}^{\mathrm{st}}$ within the setting of Remark 2.3. The argumentation is based on the following observation.

Lemma 6.10 *Let $L^p(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathrm{P}})$ denote the classical L^p-space on a probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathrm{P}})$ for $p \in [0, \infty]$ and let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\overline{\mathcal{F}}$ satisfying*

$$\lim_{n\to\infty} \mathbb{E}[\mathbb{1}_{A_n} Z] = \frac{1}{2}\mathbb{E}[Z] \qquad \text{for every } Z \in L^1(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathrm{P}}). \tag{6.29}$$

Then for any $Z \in L^1(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathrm{P}}) \setminus \{0\}$, the sequence $(\mathbb{1}_{A_n} Z)_{n\in\mathbb{N}}$ does not have any accumulation point in $L^1(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathrm{P}})$ with respect to the L^1-norm.

Proof Assume that there is some $Z \in L^1(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathrm{P}}) \setminus \{0\}$ such that the sequence $(\mathbb{1}_{A_n} Z)_{n\in\mathbb{N}}$ has an accumulation point $X \in L^1(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathrm{P}})$ with respect to the L^1-norm. By passing to a subsequence, we can assume that

$$\mathbb{E}\big[|\mathbb{1}_{A_n} Z - X|\big] \xrightarrow{n\to\infty} 0. \tag{6.30}$$

Therefore by Hölder's inequality,

$$\lim_{n\to\infty} \mathbb{E}\big[|\mathbb{1}_{A_n} ZW - XW|\big] = 0 \qquad \text{for any } W \in L^\infty(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathrm{P}}). \tag{6.31}$$

Applying (6.29) and (6.31), we get

$$\mathbb{E}\left[\left(\frac{1}{2}Z - X\right)W\right] = 0 \qquad \text{for any } W \in L^\infty(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathrm{P}}). \tag{6.32}$$

Setting $W := ((\frac{1}{2}Z - X) \wedge 1) \vee (-1)$ in (6.32), we arrive at $X = \frac{1}{2}Z$. Hence, by (6.30),

$$\frac{1}{2}\mathbb{E}[|Z|] = \mathbb{E}\left[\left|\left(\mathbb{1}_{A_n} - \frac{1}{2}\right)Z\right|\right] \xrightarrow{n\to\infty} 0.$$

This contradicts $\overline{\mathrm{P}}[Z \neq 0] > 0$ and completes the proof. □

Proof for Remark 2.3 Let us fix different $\mathrm{Q}_1, \mathrm{Q}_2 \in \mathcal{Q} \subseteq \widehat{\mathcal{Q}}^e$. Since $\widehat{\mathcal{Q}}^{\mathrm{st}}$ is stable under pasting, we may define for every $\tau \in \mathcal{T}$ by

$$\frac{d\mathrm{Q}^\tau}{d\mathrm{P}} := \frac{\mathbb{E}[\frac{d\mathrm{Q}_1}{d\mathrm{P}}|\mathcal{F}_\tau]}{\mathbb{E}[\frac{d\mathrm{Q}_2}{d\mathrm{P}}|\mathcal{F}_\tau]}\frac{d\mathrm{Q}_2}{d\mathrm{P}}$$

the Radon–Nikodým derivative with respect to P of some probability measure Q^τ which is in $\widehat{\mathcal{Q}}^{\mathrm{st}}$. In particular, $\mathrm{Q}^0 = \mathrm{Q}_2$ and $\mathrm{Q}^T = \mathrm{Q}_1$, and using the càdlàg modifications of the density processes

$$\left(\mathbb{E}\left[\frac{d\mathrm{Q}_i}{d\mathrm{P}}\middle|\mathcal{F}_t\right]\right)_{0\le t\le T}, \qquad i = 1, 2,$$

we derive

$$\frac{d\mathrm{Q}^t}{d\mathrm{P}} \longrightarrow \frac{d\mathrm{Q}_2}{d\mathrm{P}} \qquad \text{for } t \searrow 0.$$

Therefore, we may find some $t_0 \in (0, T)$ such that $\mathrm{Q}^{t_0} \neq \mathrm{Q}^T$. Since by assumption $(\Omega, \mathcal{F}_{t_0}, \mathrm{P}|_{\mathcal{F}_{t_0}})$ is atomless with $L^1(\Omega, \mathcal{F}_{t_0}, \mathrm{P}|_{\mathcal{F}_{t_0}})$ being weakly separable, we may draw on [15, Lemma 3] (or [6, Corollary C.4]) along with [6, Lemma C.1 and Proposition B.1] to find some sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{F}_{t_0}$ such that

$$\lim_{n\to\infty} \mathbb{E}[\mathbb{1}_{A_n} Z] = \frac{1}{2}\mathbb{E}[Z] \qquad \text{for any } \mathcal{F}_{t_0}\text{-measurable } Z \text{ which is } P\text{-integrable.}$$

In particular,

$$\lim_{n\to\infty} \mathbb{E}[\mathbb{1}_{A_n} Z] = \lim_{n\to\infty} \mathbb{E}\big[\mathbb{1}_{A_n}\mathbb{E}[Z|\mathcal{F}_{t_0}]\big] = \frac{1}{2}\mathbb{E}\big[\mathbb{E}[Z|\mathcal{F}_{t_0}]\big] = \frac{1}{2}\mathbb{E}[Z] \qquad (6.33)$$

holds for every $Z \in L^1(\Omega, \mathcal{F}, \mathrm{P})$. Moreover, $\tau_n := t_0\mathbb{1}_{A_n} + T\mathbb{1}_{\Omega\setminus A_n}$ defines a sequence $(\tau_n)_{n\in\mathbb{N}}$ in $\mathcal{T}$ which induces the sequence $(\mathrm{Q}^{\tau_n})_{n\in\mathbb{N}}$ in $\widehat{\mathcal{Q}}^{\mathrm{st}}$ whose Radon–Nikodým derivatives with respect to P satisfy, by the optional stopping theorem,

$$\begin{aligned}
\frac{d\mathrm{Q}^{\tau_n}}{d\mathrm{P}} &= \frac{\mathbb{E}[\frac{d\mathrm{Q}_1}{d\mathrm{P}}|\mathcal{F}_{t_0}]\mathbb{1}_{\{\tau_n=t_0\}} + \mathbb{E}[\frac{d\mathrm{Q}_1}{d\mathrm{P}}|\mathcal{F}_T]\mathbb{1}_{\{\tau_n=T\}}}{\mathbb{E}[\frac{d\mathrm{Q}_2}{d\mathrm{P}}|\mathcal{F}_{t_0}]\mathbb{1}_{\{\tau_n=t_0\}} + \mathbb{E}[\frac{d\mathrm{Q}_2}{d\mathrm{P}}|\mathcal{F}_T]\mathbb{1}_{\{\tau_n=T\}}}\,\frac{d\mathrm{Q}_2}{d\mathrm{P}} \\
&= \mathbb{1}_{A_n}\frac{d\mathrm{Q}^{t_0}}{d\mathrm{P}} + \mathbb{1}_{\Omega\setminus A_n}\frac{d\mathrm{Q}^T}{d\mathrm{P}} \\
&= \mathbb{1}_{A_n}\left(\frac{d\mathrm{Q}^{t_0}}{d\mathrm{P}} - \frac{d\mathrm{Q}^T}{d\mathrm{P}}\right) + \frac{d\mathrm{Q}^T}{d\mathrm{P}}.
\end{aligned}$$

Since $Q^{t_0} \neq Q^T$, we have $d\mathrm{Q}^{t_0}/d\mathrm{P} - d\mathrm{Q}^T/d\mathrm{P} \in L^1(\Omega, \mathcal{F}, \mathrm{P}) \setminus \{0\}$. So in view of Lemma 6.10 along with (6.33), we may observe that the sequence

$$\left(\mathbb{1}_{A_n}\Big(\frac{d\mathrm{Q}^{t_0}}{d\mathrm{P}} - \frac{d\mathrm{Q}^T}{d\mathrm{P}}\Big)\right)_{n\in\mathbb{N}}$$

does not have any accumulation point in $L^1(\Omega, \mathcal{F}, \mathrm{P})$ with respect to the L^1-norm, and thus the sequence $(d\mathrm{Q}^{\tau_n}/d\mathrm{P})_{n\in\mathbb{N}}$ also has no accumulation point. Hence we have found a sequence in $\mathbb{F}_{\widehat{\mathcal{Q}}^{\mathrm{st}}} := \{d\mathrm{Q}/d\mathrm{P} : \mathrm{Q} \in \widehat{\mathcal{Q}}^{\mathrm{st}}\}$ without any accumulation point with respect to the L^1-norm. This means that $\mathbb{F}_{\widehat{\mathcal{Q}}^{\mathrm{st}}}$ is not relatively compact with respect to the L^1-norm. However, the set $\mathbb{F}_{\widehat{\mathcal{Q}}^e}$ from Lemma 6.9 has been shown there to be relatively compact with respect to the L^1-norm. Hence $\mathbb{F}_{\widehat{\mathcal{Q}}^e} \neq \mathbb{F}_{\widehat{\mathcal{Q}}^{\mathrm{st}}}$, and thus $\widehat{\mathcal{Q}}^e$ is a proper subset of $\widehat{\mathcal{Q}}^{\mathrm{st}}$. So by the construction of $\widehat{\mathcal{Q}}^{\mathrm{st}}$, the set $\widehat{\mathcal{Q}}^e$ is not stable under pasting with respect to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathrm{P})$ so that $\mathcal{Q}$ is not time-consistent with respect to $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathrm{P})$ by Proposition 4.1. The proof for Remark 2.3 is complete. □

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Appendix A: Paths of nearly sub-Gaussian random fields

Let (Θ, d) be some totally bounded semimetric space with diameter Δ. For $\delta, \varepsilon > 0$, the symbols $\mathcal{D}(\delta, d)$ and $N(\Theta, d; \varepsilon)$ are used in an analogous manner as the notations $\mathcal{D}(\delta, d_\Theta)$ and $N(\Theta, d_\Theta; \varepsilon)$ from Sect. 3. We call a centered stochastic process $(X^\theta)_{\theta\in\Theta}$ a *nearly sub-Gaussian random field with respect to* d if there is some $C \geq 1$ with

$$\mathbb{E}\big[\exp\big(\lambda(X^\theta - X^\vartheta)\big)\big] \leq C \exp\big(\lambda^2\, d(\theta, \vartheta)^2/2\big) \qquad \text{for } \theta, \vartheta \in \Theta \text{ and } \lambda > 0. \quad \text{(A.1)}$$

Note that by symmetry, condition (A.1) also holds for arbitrary $\lambda \in \mathbb{R}$. For $C = 1$, this definition reduces to the ordinary notion of sub-Gaussian random fields. For further information on sub-Gaussian random fields, see e.g. [12, Sect. 2.3]. By a suitable change of the semimetric, we may describe any nearly sub-Gaussian random field as a sub-Gaussian random field.

Lemma A.1 *If $(X^\theta)_{\theta\in\Theta}$ is a nearly sub-Gaussian random field with respect to d, then it is a sub-Gaussian random field with respect to $\overline{d} := \varepsilon d$ for some $\varepsilon > 1$.*

Proof Let $C > 1$ be such that $(X^\theta)_{\theta\in\Theta}$ satisfies (A.1). Then $\varepsilon := \sqrt{12(2C+1)}$ is as required (cf. [12, Lemma 2.3.2]). □

The following properties of sub-Gaussian random fields are fundamental.

Proposition A.2 *Let $X = (X^\theta)_{\theta\in\Theta}$ be a nearly sub-Gaussian random field on some probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathrm{P}})$ with respect to d. If $\mathcal{D}(\Delta, d) < \infty$, then X admits a separable version, and each separable version of X has $\overline{\mathrm{P}}$-almost surely bounded and d-uniformly continuous paths. In particular, for any separable version $\widehat{X}$ and for every $\overline{\theta} \in \Theta$, there is some random variable $U^{\overline{\theta}}$ on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathrm{P}})$ such that*

$$\sup_{\theta\in\Theta} \widehat{X}^\theta \leq U^{\overline{\theta}} + \widehat{X}^{\overline{\theta}} \quad \overline{\mathrm{P}}\text{-a.s.} \qquad \textit{and} \qquad \mathbb{E}_{\overline{\mathrm{P}}}[\exp(pU^{\overline{\theta}})] < \infty \quad \textit{for every } p \in (0, \infty).$$

Proof In view of Lemma A.1, we may assume without loss of generality that X is a sub-Gaussian random field with respect to d. It is already known (see [12, Theorem 2.3.7]) that X admits a separable version, and that each such version has $\overline{\mathrm{P}}$-almost surely bounded and d-uniformly continuous paths. Now fix any separable version $\widehat{X}$ of X and an arbitrary $\overline{\theta} \in \Theta$. We have

$$\sup_{\theta\in\Theta} \widehat{X}^\theta \leq \widehat{X}^{\overline{\theta}} + \sup_{\theta\in\Theta} |\widehat{X}^\theta - \widehat{X}^{\overline{\theta}}|,$$

and the process $(|\widehat{X}^\theta - \widehat{X}^{\overline{\theta}}|)_{\theta\in\Theta}$ is separable due to the separability of $\widehat{X}$. Then we may find some at most countable subset $\Theta_0 \subseteq \Theta$ such that

$$\sup_{\theta\in\Theta} |\widehat{X}^\theta - \widehat{X}^{\overline{\theta}}| = \sup_{\theta\in\Theta_0} |\widehat{X}^\theta - \widehat{X}^{\overline{\theta}}| \qquad \overline{\mathrm{P}}\text{-a.s.}$$

Hence $U^{\overline{\theta}} := \sup_{\theta\in\Theta_0} |X^\theta - X^{\overline{\theta}}|$ defines a random variable on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathrm{P}})$ satisfying

$$U^{\overline{\theta}} = \sup_{\theta\in\Theta} |\widehat{X}^\theta - \widehat{X}^{\overline{\theta}}| \qquad \overline{\mathrm{P}}\text{-a.s.}$$

It remains to show that $\mathbb{E}_{\overline{\mathrm{P}}}[\exp(pU^{\overline{\theta}})] < \infty$ for $p \in (0, \infty)$. So fix $p \in (0, \infty)$. First, observe that $\widehat{X}$ is again a sub-Gaussian random field. Thus by [12, Lemma 2.3.1], we have

$$\mathbb{E}_{\overline{\mathrm{P}}}\bigg[\exp\bigg(\Big(\frac{X^\theta - X^\vartheta}{\sqrt{6}\, d(\theta,\vartheta)}\Big)^2\bigg)\bigg] \le 2 \qquad \text{for } \theta,\vartheta \in \Theta,\ d(\theta,\vartheta) \neq 0.$$

Hence we may apply the results from [25] with respect to the totally bounded semi-metric $\overline{d} := \sqrt{6}d$. Note that $(\widehat{X}^\theta)_{\theta\in\Theta}$ is also separable with respect to $\overline{d}$, and that $\overline{\Delta} = \sqrt{6}\Delta$ for the diameter $\overline{\Delta}$ with respect to $\overline{d}$. Since $N(\Theta, \overline{d}; \varepsilon) \le N(\Theta, d; \varepsilon/\sqrt{6})$ for every $\varepsilon > 0$, we obtain for every $\delta > 0$ that

$$\int_0^\delta \sqrt{\ln N(\Theta, \overline{d}; \varepsilon)}\, d\varepsilon \le \int_0^\delta \sqrt{\ln N(\Theta, d; \varepsilon/\sqrt{6})}\, d\varepsilon = \sqrt{6}\mathcal{D}(\delta/\sqrt{6}, d).$$

Then in view of [25, Corollary 3.2], we may find some constant $C > 0$ such that

$$\overline{\mathrm{P}}\big[U^{\overline{\theta}} > xC\sqrt{6}\mathcal{D}(\Delta, d)\big] \le 2\exp(-x^2/2) \qquad \text{for } x \ge 1.$$

Furthermore, setting $\widehat{C} := C\sqrt{6}\mathcal{D}(\Delta, d)$, we may observe that

$$\begin{aligned}\int_1^\infty \overline{\mathrm{P}}\big[U^{\overline{\theta}} > x\widehat{C}\big]\exp(xp\widehat{C})\, dx &\le \int_1^\infty 2\exp(-x^2/2)\exp(xp\widehat{C})\, dx \\ &\le 2\sqrt{2\pi}\exp(p^2\widehat{C}^2/2).\end{aligned}$$

Then applying the change of variables formula several times, we obtain

$$\int_{\exp(p\widehat{C})}^\infty \overline{\mathrm{P}}\big[\exp(pU^{\overline{\theta}}) > y\big]\, dy = p\widehat{C}\int_1^\infty \overline{\mathrm{P}}\big[U^{\overline{\theta}} > \widehat{C}u\big]\exp(p\widehat{C}u)\, du < \infty.$$

Hence

$$\mathbb{E}_{\overline{\mathrm{P}}}[\exp(pU^{\overline{\theta}})] = \int_0^\infty \overline{\mathrm{P}}\big[\exp(pU^{\overline{\theta}}) > y\big]\, dy < \infty$$

which completes the proof. □

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