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Consumption-investment problem with transaction costs for Lévy-driven price processes

Dimitri De Vallière[1] · Yuri Kabanov[1,2] · Emmanuel Lépinette[2,3]

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Abstract We consider an optimal control problem for a linear stochastic integro-differential equation with conic constraints on the phase variable and with the control of singular–regular type. Our setting includes consumption-investment problems for models of financial markets in the presence of proportional transaction costs, where the prices of the assets are given by a geometric Lévy process, and the investor is allowed to take short positions. We prove that the Bellman function of the problem is a viscosity solution of an HJB equation. A uniqueness theorem for the solution of the latter is established. Special attention is paid to the dynamic programming principle.

Keywords Consumption-investment problem · Lévy process · Transaction costs · Bellman function · Dynamic programming · HJB equation · Lyapunov function

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1 Introduction

In this paper, we study the classical consumption-investment model with infinite horizon in the presence of transaction costs in the case where the price evolution is given

✉ Y. Kabanov
Youri.Kabanov@univ-fcomte.fr

1 Laboratoire de Mathématiques, Université de Franche-Comté, 16 Route de Gray, 25030 Besançon, cedex, France

2 International Laboratory of Quantitative Finance, National Research University Higher School of Economics, Moscow, Russia

3 Ceremade, Université Paris Dauphine, Place du Maréchal De Lattre De Tassigny, 75775 Paris cedex 16, France

by a geometric Lévy process. There is a growing literature on investments into assets with such price dynamics; see e.g. [1, 2, 11, 14, 16], etc.; there are also a few papers combining a jump-diffusion setting with transaction costs; see [17, 10, 12].

Our aim is to extend the results of [22]. Namely, we show that the Bellman function is a viscosity solution of the corresponding Hamilton–Jacobi–Bellman equation. We also prove a uniqueness theorem for the latter.

Mathematically, the consumption-investment problem with transaction costs considered here is a regular–singular control problem for a linear stochastic equation in a cone. Its specificity is that the Bellman function need not be smooth, and therefore we cannot use a verification theorem (at least, in its traditional form) because the Itô formula cannot be applied. Nevertheless, we can show that the Bellman function is a solution of the HJB equation in the viscosity sense. Although the general line of arguments is familiar, we need to reexamine each step of the proof. In particular, for the considered jump-diffusion model, the HJB equation contains an integro-differential operator, and so the test functions involved in the definition of a viscosity solution must be "globally" defined. It seems that already in 1986, Soner [28, 29] noticed that control problems with jump parts can be considered in the framework of the theory of viscosity solutions.

There is a growing literature on extensions of the concept of viscosity solutions to equations with integro-differential operators; see e.g. [26, 4, 25, 9, 8, 5, 6, 18]. There are several variants of the definition of a viscosity solution. Our choice is intended to serve the model with a positive utility function. The definition can be viewed as a simplified version of that adopted in [19].

A rather detailed study of the HJB equation arising in consumption-investment problems for multiasset models of stock market when the prices follow exponential Lévy processes and the investor is constrained to keep long positions in all assets, money included, was undertaken by Benth et al. in [11] (frictionless market) and [10] (market with transaction costs). Of course, from the financial point of view, this is a serious constraint, which means that either the regulation of the market is so stringent that not only short selling of stocks, but also borrowing money is prohibited, or the investor is extremely risk averse and wants to avoid any possibility of bankruptcy. In the mentioned paper, the diffusion is assumed to be nondegenerate, i.e., the financially interesting models based on Lévy processes without Gaussian component are excluded.

Our geometric approach is more general than that of the mentioned papers where the authors considered a "parametric" version of the stock market, with transactions always passing through money (i.e., either "buy stock" or "sell stock"). A more important difference is that in our setting, the investor may take short positions as was always assumed in the classical papers [24, 15, 27]. If short positions are admitted, ruin may happen due to a jump of the price process. That is why the natural, "classical" setting considered here leads to a different HJB equation of more complicated structure. Following the ideas from the paper [22], we derive the dynamic programming principle (DPP) split into two separate assertions. Although it is the principal tool, which allows one to check that the Bellman function is a viscosity solution of the HJB equation, it is rarely discussed in the literature (and even taken for granted; see e.g. [3, 27, 10]).

For the models with jumps, there are new aspects of DPP, related to fundamental issues of financial modeling, needing special attention. The problem is that the Lévy process starts afresh at any stopping time, but the control dynamics does not. The instants of jumps of the Lévy process are totally inaccessible stopping times, whereas the instances of jumps of the predictable càdlàg processes representing the accumulated transfers are predictable stopping times and therefore cannot happen simultaneously; see Sect. 2. Thus, the jumps of the price process cannot be compensated by immediate control actions. To some extent, this feature, interpreted as a kind of inertia, agrees with financial intuition. On the other hand, this leads to the absence of an optimal control. In the present note, we investigate the traditional dynamics, leaving the version with làdlàg trajectories and the search of an optimal control (which is the most interesting part of the theory) for further studies.

As in [22], we work with a positive multivariate utility function for current consumption; see [12] and references therein for a detailed discussion of this object. But even in the case of a univariate utility function, e.g. $u(c) = c^\gamma/\gamma$, where $\gamma \in (0, 1)$, our paper presents some novelties with respect to the existing literature. In particular, we provide some sufficient conditions on the existence of Lyapunov functions and classical solutions of the HJB equation, ensuring in particular the finiteness of the Bellman function. It is worth noting that this utility function, considered just as an example, has the constant relative risk aversion coefficient $1 - \gamma \in (0, 1)$, whereas empirical studies show that in the real world, this coefficient is much greater; see e.g. [21]. On the other hand, the negative CRRA utility function with $\gamma < 0$ cannot be used to measure the current consumption rate because the immediate consumption of the whole wealth is optimal (this feature is often overlooked in the literature). The utility functions $u(c) = (c + a)^\gamma/\gamma - a^\gamma$, where $\gamma < 0$ and $a > 0$, with almost constant RRA fall in the scope of our model.

The main results of the paper are Theorem 10.1, claiming that if the Bellman function is continuous up to the boundary, then it is a viscosity solution of the HJB equation, and the uniqueness theorem for the Dirichlet problem arising in the model, Theorem 11.3. We formulate the latter in terms of a Lyapunov function, an object that is defined in terms of the truncated operator, in which the utility function is not involved. Its introduction allows us to disconnect the problems of the uniqueness of a solution and the existence of a classical supersolution.

Probably, the most important result of the paper is the uniqueness theorem for the HJB equation with a nonlocal operator. In contrast to the methods developed in [8] and [9], which are based on rather technical extensions of the Ishii lemma, we use the latter in its original and (very transparent) formulation.

Note that our choice of the definition of viscosity solution, namely, with the requirement of continuity up to the boundary, is adapted to the situation where the utility function is positive. However, it does not suit to treat the problem with logarithmic utility. In that case, it is not clear what kind of boundary condition should replace the Dirichlet one. This remains an interesting open problem.

The structure of the paper is the following. In Sects. 2 and 3, we introduce the model dynamics and describe the goal functional, providing comments on the concavity of the Bellman function W. In Sect. 4, we show that the Bellman function, if finite, is continuous in the interior of the solvency cone. In Sect. 5, we give a formal description of the HJB equation. Sections 6 and 7 contain a short account of basic facts on viscosity solutions for integro-differential equations. In Sect. 8, we explain the role of classical supersolutions to HJB equations. Section 9 is devoted to the dynamic programming principle. In Sect. 10, we use it to show that the Bellman function is the solution of our HJB equation. Section 11 contains a uniqueness theorem formulated in terms of a Lyapunov function. In Sect. 12, we provide examples of Lyapunov functions and classical supersolutions.

2 The model

Our setting is more general than that of the standard model of a financial market under constant proportional transaction costs. In particular, the cone K is not supposed to be polyhedral. We assume that the asset prices are geometric Lévy processes. Our framework appeals to a theory of viscosity solutions for nonlocal integro-differential equations.

Let $Y=(Y_t)$ be an $\mathbf{R}^d$-valued semimartingale on a filtered probability space $(\Omega,\mathcal{F},\mathbf{F},P)$ with the trivial initial σ-algebra. Let K and $\mathcal{C}$ be *proper* closed cones in $\mathbf{R}^d$ such that $\mathcal{C}\subseteq \operatorname{int} K\neq\emptyset$. Define the set $\mathcal{A}$ of controls $\pi=(B,C)$ as the set of *predictable* $\mathbf{R}^d$-valued càdlàg processes of bounded variation such that, up to an evanescent set,

$$\dot B\in -K,\qquad \dot C\in\mathcal{C}.$$

Here $\dot B$ denotes (a measurable version of) the Radon–Nikodým derivative of B with respect to the total-variation process $|B|$. The notation $\dot C$ has a similar meaning. Though models with arbitrary C are of interest, we restrict ourselves in the present paper by considering consumption processes admitting an intensity. To this end, we define $\mathcal{A}_a$ as the set of controls π with absolutely continuous components C such that $C_0=0$. For the elements of $\mathcal{A}_a$, we have $c:=dC/dt\in\mathcal{C}$.

The controlled process $V=V^{x,\pi}$ is the solution of the linear system

$$dV^i_t=V^i_{t-}dY^i_t+dB^i_t-dC^i_t,\qquad V^i_{0-}=x^i,\quad i=1,\dots,d. \tag{2.1}$$

In general, $\Delta V_0=\Delta B_0$ is not equal to zero: the investor may revise the portfolio when entering the market at time zero.

The solution of (2.1) can be expressed explicitly using the Doléans-Dade exponentials

$$\mathcal{E}_t(Y^i)=e^{Y^i_t-(1/2)\langle Y^{ic}\rangle_t}\prod_{s\le t}(1+\Delta Y^i_s)e^{-\Delta Y^i_s}.$$

Namely,

$$V^i_t=\mathcal{E}_t(Y^i)x^i+\mathcal{E}_t(Y^i)\int_{[0,t]}\mathcal{E}^{-1}_{s-}(Y^i)(dB^i_s-dC^i_s),\quad i=1,\dots,d. \tag{2.2}$$

The controls from $\mathcal{A}$ and $\mathcal{A}_a$ defines the dynamics of the process V for all $t\ge 0$. In the traditional setting of consumption-investment problems, everything stops when the process V leaves the interior of the solvency region. That is why it is sufficient to consider the subsets of controls stopped at the date of bankruptcy depending on the initial capital x. It is also natural to assume that the process V does not leave the interior of K due to a jump of B: the investor is reasonable enough not to ruin himself by making a too expensive portfolio revision. The formal description can be done as follows.

We introduce the stopping time

$$\theta=\theta^{x,\pi}:=\inf\{t: V_t^{x,\pi}\notin \operatorname{int} K\}. \tag{2.3}$$

For $x\in \operatorname{int} K$, we consider the subsets $\mathcal{A}^x\subset\mathcal{A}$ and $\mathcal{A}_a^x\subset\mathcal{A}_a$ of "admissible" controls for which $\pi=I_{[\![0,\theta^{x,\pi}]\!]}\pi$ and $\{V_-+\Delta B\in \operatorname{int} K\}=\{V_-\in \operatorname{int} K\}$. In financial terms, θ is the time of ruin. When $V^{x,\pi}$ leaves the interior of the solvency cone, the control of the portfolio and the consumption stop. The process V given by (2.1) continues to evolve after the time θ, but for us, only the stopped process $V^{x,\pi,\theta}$ has relevance.

The important hypothesis that the cone K is *proper*, i.e., $K\cap(-K)=\{0\}$ or, equivalently, $\operatorname{int} K^*\neq\emptyset$, corresponds to a model of a financial market with *efficient friction*. In a financial context, K (usually containing $\mathbf{R}_+^d$) is interpreted as the solvency region, and $C=(C_t)$ as the consumption process; the process $B=(B_t)$ describes accumulated fund transfers. In the "standard" model with proportional transaction costs (sometimes referred to as the model of a currency market),

$$K=\operatorname{cone}\{(1+\lambda^{ij})e_i-e_j,\ e_i,\ 1\le i,j\le d\},$$

where $\lambda^{ij}\ge 0$ are transaction cost coefficients. Note that our setting covers the model of a stock market and the model of an exchange where transactions charge only the bank account; see Sect. 3.1 in the book [23] for details.

The process Y represents the relative price movements. If S^i is the price process of the ith asset, then $dS_t^i=S_{t-}^i dY_t^i$ and $S_t^i=S_0^i\mathcal{E}_t(Y^i)$. Without loss of generality, we assume that $S_0^i=1$ for all i. In this case, Y^i is the so-called stochastic logarithm of S^i. Formula (2.2) can be rewritten as

$$V_t^i=S_t^ix^i+S_t^i\int_{[0,t]}\frac{1}{S_{s-}^i}(dB_s^i-dC_s^i),\quad i=1,\dots,d. \tag{2.4}$$

We work assuming that

$$Y_t=\mu t+\Xi w_t+\int_0^t\int z\big(p(dz,dt)-q(dz,dt)\big), \tag{2.5}$$

where $\mu\in\mathbf{R}^d$, w is an m-dimensional standard Wiener process, and $p(dz,dt)$ is a Poisson random measure with the compensator $q(dz,dt)=\Pi(dz)\,dt$ such that $\Pi(dz)$ is a measure concentrated on $(-1,\infty)^d$. Note that the latter property of the Lévy measure corresponds to the financially meaningful case where $S^i>0$. For the $m\times d$-dimensional matrix Ξ, we put $A=\Xi\,\Xi^*$. We assume that

$$\int(|z|^2\wedge|z|)\Pi(dz)<\infty,$$

and this assumption validates formula (2.5): by definition,

$$\int_0^t \int z\big(p(dz,dt)-q(dz,dt)\big) := \int_0^t \int_{\{|z|\le 1\}} z\big(p(dz,dt)-q(dz,dt)\big) + \int_0^t \int_{\{|z|>1\}} zp(dz,dt) - \int_0^t \int_{\{|z|>1\}} zq(dz,dt),$$

where the first integral is defined as a stochastic one, whereas the second and third are the usual Lebesgue integrals, both finite (a.s.).

Notation. For typographical reasons, we use the notation D_x instead of the common $\operatorname{diag} x$ for the diagonal operator (or matrix) generated by the vector $x=(x^1,\dots,x^d)$, i.e.,

$$D_x z = (x^1 z^d, \dots, x^d z^d).$$

System (2.1) can be written in integral vector form as

$$V_t = x + \int_0^t D_{V_{s-}}(\mu\,ds + \varXi\,dW_s) + \int_0^t \int D_{V_{s-}} z\big(p(dz,ds)-q(dz,ds)\big) + B_t - C_t. \tag{2.6}$$

It is important to note that the jumps of Y and B cannot occur simultaneously. More precisely, the process $|\Delta B||\Delta Y|$ is indistinguishable from zero. Indeed, for any $\varepsilon>0$, we have, using the predictability of the process $\Delta B = B - B_-$, that

$$E\sum_{s\ge 0} |\Delta B_s||\Delta Y_s| I_{\{|\Delta Y_s|>\varepsilon\}} = E\int_0^\infty \int |\Delta B_s| I_{\{|z|>\varepsilon\}}|z| p(dz,ds) = E\int_0^\infty \int |\Delta B_s||z| I_{\{|z|>\varepsilon\}} \varPi(dz)\,ds = 0$$

because for each ω, the set $\{s : \Delta B_s(\omega)\ne 0\}$ is at most countable and its Lebesgue measure is equal to zero. Thus, the process $|\Delta B||\Delta Y| I_{\{|\Delta Y|>\varepsilon\}}$ is indistinguishable from zero, and so is the process $|\Delta B||\Delta Y|$.

It follows that $\Delta B_\theta = 0$. Since the predictable process $I_{\{V_-\in\partial K\}} I_{[\![0,\theta]\!]}$ has at most a countable number of jumps, the same reasoning as before leads to the conclusion that $I_{\{V_-\in\partial K\}}|\Delta Y| I_{[\![0,\theta]\!]}$ is indistinguishable from zero. This means that θ is the first moment when either V or V_- leaves $\operatorname{int} K$. This property will be used in the proof that W is lower semicontinuous on $\operatorname{int} K$.

In our proof of the dynamic programming principle (needed to derive the HJB equation), we shall assume that the stochastic basis is the canonical one, that is, it consists of the space of càdlàg functions and a measure P under which the coordinate mapping is a Lévy process as before.

3 Goal functionals and concavity of the Bellman function

Let $U : \mathcal{C} \to \mathbf{R}_+$ be a concave function such that $U(0) = 0$ and $U(x)/|x| \to 0$ as $|x| \to \infty$. With every $\pi = (B, C) \in \mathcal{A}_a^x$, we associate the "utility process"

$$J_t^\pi := \int_0^{t\wedge\theta} e^{-\beta s} U(c_s)\, ds, \quad t \ge 0,$$

where $\beta > 0$, and $\theta = \theta^{x,\pi}$ is the exit time defined in (2.3). We consider the infinite horizon maximization problem with the *goal functional* EJ_∞^π and define its *Bellman function* W by

$$W(x) := \sup_{\pi\in\mathcal{A}_a^x} EJ_\infty^\pi, \quad x \in \operatorname{int} K.$$

Since $\mathcal{A}_a^{x_1} \subseteq \mathcal{A}_a^{x_2}$ when $x_2 - x_1 \in K$, the function W is *increasing* with respect to the partial ordering $\ge_K$ generated by the cone K.

If π_i, $i = 1, 2$, are admissible strategies for the initial points x_i, then their convex combination $\lambda\pi_1 + (1 - \lambda)\pi_2$ is an admissible strategy for the initial point $\lambda x_1 + (1 - \lambda)x_2$, $\lambda \in [0, 1]$, lying in the interval connecting x_1 and x_2. In the case where the relative price process Y is continuous, the corresponding ruin time for the process

$$V^{\lambda x_1+(1-\lambda)x_2,\lambda\pi_1+(1-\lambda)\pi_2} = \lambda V^{x_1,\pi_1} + (1 - \lambda)V^{x_2,\pi_2} \tag{3.1}$$

dominates the maximum of the ruin times for the processes V^{x_i,π_i}. The concavity of U then implies that

$$J_t^{\lambda\pi_1+(1-\lambda)\pi_2} \ge \lambda J_t^{\pi_1} + (1 - \lambda)J_t^{\pi_2}, \tag{3.2}$$

and hence the function W is concave on $\operatorname{int} K$.

Unfortunately, in our main case of interest where Y has jumps, the ruin times are not related in such a simple way since short positions are allowed. It is easy to give examples of trajectories such that

$$\theta^{x_1,\pi_1} = \theta^{\lambda x_1+(1-\lambda)x_2,\lambda\pi_1+(1-\lambda)\pi_2} < \infty,$$

whereas $\theta^{x_2,\pi_2} = \infty$ and relations (3.1) and (3.2) do not hold. Therefore, we cannot guarantee by the previous argument that the Bellman function is concave. Of course, these considerations show only that the concavity of W cannot be obtained in a straightforward way as for a model based on a continuous price process; but it is not excluded.

The concavity of the Bellman function W is not a property just interesting per se. The classical definition of a viscosity solution, as it was given by the famous "User's guide" [13], requires the continuity of W. On the other hand, a concave function is continuous in the interior of its domain (and even locally Lipschitz); see e.g. [7, Corollary 2.2]. Of course, the model must contain a provision that ensures that W is finite. But the latter property in the case of continuous price processes implies that W is continuous on $\operatorname{int} K$. In the case of processes with jumps, we need to analyze the continuity of W using other arguments.

In the next section, we show that the finiteness of W still guarantees its continuity on the interior of K. We do this using the following assertion.

Lemma 3.1 *Suppose that W is a finite function. Let $x \in \operatorname{int} K$. Then the function $\lambda \mapsto W(\lambda x)$ is right-continuous on $\mathbf{R}_+$.*

Proof Let $\lambda > 0$. Then $\lambda\pi \in \mathcal{A}_a^{\lambda x}$ if and only if $\pi \in \mathcal{A}_a^x$. For a concave function U with $U(0) = 0$, we have for any $\varepsilon > 0$ the inequality

$$U(c) \geq (1+\varepsilon)^{-1} U\big((1+\varepsilon)c\big).$$

Hence, for an arbitrary strategy $\pi \in \mathcal{A}_a^x$, we have for $\theta = \theta^{x,\pi} = \theta^{(1+\varepsilon)x,(1+\varepsilon)\pi}$ that

$$\begin{aligned} J_\infty^{(1+\varepsilon)\pi} - J_\infty^\pi &= E\int_0^\theta e^{-\beta t}\Big(U\big((1+\varepsilon)c_t\big) - U(c_t)\Big)\,dt \\ &\leq \varepsilon E\int_0^\theta e^{-\beta t} U(c_t)\,dt \leq \varepsilon W(x). \end{aligned}$$

It follows that $W((1+\varepsilon)x) \leq (1+\varepsilon)W(x)$. Since $W(x) \leq W((1+\varepsilon)x)$, we infer from here that $\lambda \mapsto W(\lambda x)$ is right-continuous at the point $\lambda = 1$. Replacing x by λx, we obtain the claim. □

If U is a homogeneous function of order $\gamma \in (0,1)$, i.e., $U(\lambda x) = \lambda^\gamma U(x)$ for all $\lambda > 0$, $x \in K$, then $W(\lambda x) = \lambda^\gamma W(x)$. Thus, the function $\lambda \mapsto W(\lambda x)$ is then concave and therefore continuous if finite.

Remark 3.2 In financial models, usually $\mathcal{C} = \mathbf{R}_+ e_1$, i.e., only the first asset is consumed. Correspondingly, we then have $U(c) = u(e_1 c) = u(c^1)$, where u is a utility function of a scalar argument. Our presentation is oriented to the power utility function $u_\gamma(x) = x^\gamma/\gamma$ with $\gamma \in (0,1)$. The case of $\gamma \leq 0$ where by convention $u_0(x) = \ln x$ is of interest but is not covered by the present study.

Remark 3.3 We consider here a model with mixed "regular–singular" controls. In fact, the assumption that the consumption process has an intensity $c = (c_t)$ and agent's utility depends only on this intensity is not very satisfactory from the economic point of view. We can consider models with intertemporal substitution and consumption by "gulps," i.e., dealing with "singular" controls of the class $\mathcal{A}^x$ and the goal functionals like

$$J_t^\pi := \int_0^t e^{-\beta s} U(\bar{C}_s)\,ds,$$

where

$$\bar{C}_s = \int_0^s K(s,r)\,dC_r$$

with a suitable kernel $K(s,r)$ (the exponential kernel $e^{-\gamma(s-r)}$ is the common choice); see [10].

4 Continuity of the Bellman function

Proposition 4.1 *Suppose that $W(x) < \infty$ for all $x \in \operatorname{int} K$. Then W is continuous on* $\operatorname{int} K$.

Proof First, we show that the function W is upper semicontinuous on $\operatorname{int} K$. Suppose that this is not the case and there is a sequence (x_n) converging to some $x_0 \in \operatorname{int} K$ such that $\limsup_n W(x_n) > W(x_0)$. Without loss of generality, we may assume that the sequence $(W(x_n))$ converges. The points $\tilde{x}_k = (1 + 1/k)x_0$, $k \geq 1$, belong to the ray $\mathbf{R}_+ x_0$ and converge to x_0. We find a subsequence (x_{n_k}) such that $\tilde{x}_k \geq_K x_{n_k}$ for all $k \geq 1$. Indeed, since

$$\tilde{x}_k = (1 + 1/k)x_0 \in x_0 + \operatorname{int} K,$$

there exists $\varepsilon_k > 0$ such that

$$\tilde{x}_k + \mathcal{O}_{\varepsilon_k}(0) \subseteq x_0 + \operatorname{int} K.$$

It follows that

$$\tilde{x}_k + (x_n - x_0) + \mathcal{O}_{\varepsilon_k}(0) \subseteq x_n + \operatorname{int} K,$$

and therefore $\tilde{x}_k \in x_n + \operatorname{int} K$ for all n such that $|x_n - x_0| < \varepsilon_k$. Any strictly increasing sequence of indices n_k with $|x_{n_k} - x_0| < \varepsilon_k$ gives us a subsequence of points x_{n_k} having the needed property. The function W is increasing with respect to the partial ordering $\geq_K$. Thus,

$$\lim_k W(\tilde{x}_k) \geq \lim_k W(x_{n_k}) > W(x_0).$$

On the other hand, the function $\lambda \mapsto W(\lambda x_0)$ is right-continuous at $\lambda = 1$, and hence $\lim_k W(\tilde{x}_k) = W(x_0)$. This contradiction shows that W is upper semicontinuous on $\operatorname{int} K$.

Let us show now that $\liminf_n W(x_n) \geq W(x_0)$ as $x_n \to x_0$, i.e., W is lower semicontinuous on $\operatorname{int} K$. Fix $\varepsilon > 0$. Due to the finiteness of the Bellman function, there are a strategy π and $T \in \mathbf{R}_+$ such that for $\theta = \theta^{x_0,\pi}$, we have the bound

$$E \int_0^{T \wedge \theta} e^{-\beta s} U(c_s)\, ds \geq W(x_0) - \varepsilon.$$

It remains to show that

$$\liminf_n (\theta_n \wedge T) \geq \theta \wedge T \quad \text{a.s.}, \tag{4.1}$$

where we use the abbreviation $\theta_n := \theta^{x_n,\pi}$. Indeed, with this bound, we get, using the Fatou lemma, that

$$\liminf_n W(x_n) \ge \liminf_n E \int_0^{\theta_n \wedge T} e^{-\beta s} U(c_s)\,ds \ge E \liminf_n \int_0^{\theta_n \wedge T} e^{-\beta s} U(c_s)\,ds$$
$$\ge E \int_0^{\theta \wedge T} e^{-\beta s} U(c_s)\,ds \ge W(x_0) - \varepsilon,$$

and the claim follows since ε is arbitrarily small.

To prove (4.1), we observe that by (2.4), on the interval $[\![0, \theta_n \wedge \theta \wedge T]\!]$, we have the representation

$$V_t^{x_n,\pi} - V_t^{x_0,\pi} = D_{x_n - x_0} S_t$$

implying that

$$\sup_{t \le \theta_n \wedge \theta \wedge T} |V_t^{x_n,\pi} - V_t^{x_0,\pi}| \le S_T^* |x_n - x_0|,$$

where $S_T^* := \sup_{t \le T} |S_t|$. Fix an arbitrary "small" $\delta > 0$. For almost all ω, the distance $\rho(\omega)$ of the trajectory $V^{x_0,\pi}(\omega)$ from the boundary ∂K on the interval $[0, \theta(\omega) \wedge T - \delta]$ is strictly positive. The above bound shows that for sufficiently large n, the trajectory $V^{x_n,\pi}(\omega)$ does not deviate from $V^{x_0,\pi}(\omega)$ more than on $\rho(\omega)/2$ on the interval $[0, \theta_n(\omega) \wedge \theta(\omega) \wedge T]$. It follows that $\theta_n(\omega) \ge \theta(\omega) \wedge T - \delta$. Thus,

$$\liminf_n (\theta_n \wedge T) \ge \theta \wedge T - \delta \quad \text{a.s.},$$

and (4.1) holds. □

5 The Hamilton–Jacobi–Bellman equation

Let $G := (-K) \cap \partial \mathcal{O}_1(0)$, where $\partial \mathcal{O}_r(y) := \{x \in \mathbf{R}^d : \ |x - y| = r\}$. The set G is compact, and $-K = \operatorname{cone} G$. We denote by Σ_G the *support function* of G, given by the relation $\Sigma_G(p) = \sup_{x \in G} px$. The convex function $U^*(\cdot)$ is the Fenchel dual of the convex function $-U(-\cdot)$ whose domain is $-\mathcal{C}$, i.e.,

$$U^*(p) = \sup_{x \in \mathcal{C}} \big(U(x) - px\big).$$

We denote by $C_1(K)$ the subspace of the space of continuous functions f on K such that $\sup_{x \in K} |f(x)|/(1 + |x|) < \infty$. In other words, $C_1(K)$ is the space of continuous functions on K of sublinear growth. The notation $f \in C^2(x)$ means that f is smooth (i.e. of the class C^2) in some neighborhood of x.

Let $f \in C_1(K) \cap C^2(\operatorname{int} K)$. Using the abbreviation

$$I(z, x) := I_{\{z:\ x + D_x z \in \operatorname{int} K\}} = I_{\operatorname{int} K}(x + D_x z),$$

we introduce the function

$$\mathcal{I}(f, x) := \int \Big(\big(f(x + D_x z) I(z, x) - f(x)\big) - D_x z f'(x)\Big) \Pi(dz), \quad x \in \operatorname{int} K.$$

It is well defined and continuous in x. Indeed, fix $x_0 \in \operatorname{int} K$. Let $\varepsilon \in (0,1]$ be such that $\mathcal{O}_{4\varepsilon}(x_0) \subset K$. With this choice, $x + D_x z \in \mathcal{O}_{2\varepsilon}(x_0)$ when $x \in \mathcal{O}_\varepsilon(x_0)$ and $|z| \le \delta := \varepsilon/(1+|x_0|)$. Using the Taylor formula for such a value of z and the sublinear growth of f for z with $|z| > \delta$, we obtain for $x \in \mathcal{O}_\varepsilon(x_0)$ the uniform bound

$$|f(x+D_x z)I(z,x) - f(x) - D_x z f'(x)| \le \kappa_1 |z|^2 I_{\mathcal{O}_\delta(x_0)}(z) + \kappa_2 |z| I_{\mathcal{O}^c_\delta(0)}(z).$$

This implies the needed integrability and the continuity of the integral in x.

We introduce the function of five variables

$$F\big(X, p, \mathcal{I}(f,x), W, x\big) := \max\big\{F_0\big(X, p, \mathcal{I}(f,x), W, x\big) + U^*(p), \Sigma_G(p)\big\},$$

where X belongs to $\mathcal{S}_d$, the set of $d \times d$ symmetric matrices, $p, x \in \mathbf{R}^d$, $W \in \mathbf{R}$, $f \in C_1(K) \cap C^2(x)$, and the function F_0 is given by

$$F_0\big(X, p, \mathcal{I}(f,x), W, x\big) := \frac{1}{2}\operatorname{tr} A(x)X + \mu(x)p + \mathcal{I}(f,x) - \beta W,$$

where $A(x)$ is the matrix with $A^{ij}(x) := a^{ij}x^i x^j$, and $\mu(x)$ is the vector with components $\mu^i(x) := \mu^i x^i$, $1 \le i, j \le d$. In a more detailed form, we have that

$$F_0\big(X, p, \mathcal{I}(f,x), W, x\big) = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}x^i x^j X^{ij} + \sum_{i=1}^{d} \mu^i x^i p^i + \mathcal{I}(f,x) - \beta W.$$

Note that F_0 is increasing in the argument f in the same sense as $\mathcal{I}$. If ϕ is a smooth function, then we put

$$\mathcal{L}\phi(x) := F\big(\phi''(x), \phi'(x), \mathcal{I}(\phi,x), \phi(x), x\big).$$

In a similar way, $\mathcal{L}_0$ corresponds to the function F_0.

In Sects. 6–10, we show under mild hypotheses that W is a viscosity solution of the Dirichlet problem for the HJB equation

$$F\big(W''(x), W'(x), \mathcal{I}(W,x), W(x), x\big) = 0, \quad x \in \operatorname{int} K, \tag{5.1}$$

$$W(x) = 0, \quad x \in \partial K, \tag{5.2}$$

with the boundary condition understood in the usual classical sense, and we establish a uniqueness result for this problem.

6 Viscosity solutions for integro-differential equations

Since in general, W may have no derivatives at some points $x \in \operatorname{int} K$ (and this is indeed the case for the model considered here), the notation (5.1) needs to be interpreted. The idea of viscosity solutions is to substitute W in F by suitable test functions. Formal definitions (adapted to the case we are interested in) are as follows.

A function $v \in C(K)$ is called *viscosity supersolution* of (5.1) if for every $x \in \operatorname{int} K$ and every $f \in C_1(K) \cap C^2(x)$ such that $v(x) = f(x)$ and $v \ge f$, the inequality $\mathcal{L}f(x) \le 0$ holds.

A function $v \in C(K)$ is called a *viscosity subsolution* of (5.1) if for every $x \in \operatorname{int} K$ and every $f \in C_1(K) \cap C^2(x)$ such that $v(x) = f(x)$ and $v \le f$, the inequality $\mathcal{L}f(x) \ge 0$ holds.

A function $v \in C(K)$ is a *viscosity solution* of (5.1) if v is simultaneously a viscosity super- and subsolution.

Finally, a function $v \in C_1(K) \cap C^2(\operatorname{int} K)$ is called a *classical supersolution* of (5.1) if $\mathcal{L}v \le 0$ on $\operatorname{int} K$. We add the adjective *strict* when $\mathcal{L}v < 0$ on the set $\operatorname{int} K$.

For simplicity and having in mind the specific case we shall work on, we have incorporated in the definitions the requirement that the viscosity super- and subsolutions are continuous on K including the boundary. For other cases, this might be too restrictive, and more general and flexible formulations can be used.

Lemma 6.1 *Suppose that the function v is a viscosity solution of* (5.1). *If v is twice differentiable at $x_0 \in \operatorname{int} K$, then it satisfies* (5.1) *at this point in the classical sense.*

Proof We need to be more precise with definitions since it is not assumed that v' is defined at every point of a neighborhood of x_0. "Twice differentiable" means here that the Taylor formula at x_0 holds, i.e.,

$$v(x) = P_2(x - x_0) + (x - x_0)^2 h(|x - x_0|),$$

where

$$P_2(x - x_0) := v(x_0) + \langle v'(x_0), x - x_0\rangle + \frac{1}{2}\langle v''(x_0)(x - x_0), x - x_0\rangle$$

and $h(r) \to 0$ as $r \downarrow 0$. We introduce the notation

$$\Gamma_r := \{z \in \mathbf{R}^d : |D_{x_0} z| \le r\}, \quad r > 0.$$

Note that $\mathcal{O}_{r/|x_0|} \subset \Gamma_r$. Hence, $\Pi(\Gamma_r^c) < \infty$.

Let $\varepsilon \in (0, 1]$. We choose a number $\delta_0 \in (0, 1)$ such that $x_0 + \mathcal{O}_{\delta_0}(0) \subset \operatorname{int} K$ and $|h(s)| \le \varepsilon$ for $s \le \delta_0$. Put $\delta := \delta_0/(1 + |x_0|)$. Take $\Delta \in (\delta, \delta_0)$ sufficiently close to δ to ensure that $x_0 + \mathcal{O}_\Delta(0) \subset \operatorname{int} K$ and $\Pi(\Gamma_\Delta \setminus \Gamma_\delta) \le \varepsilon$.

We define the function $f_\varepsilon \in C_1(K) \cap C^2(x_0)$ by the formula

$$f_\varepsilon(x) = \begin{cases} P_2(x - x_0) + \varepsilon(x - x_0)^2, & x \in x_0 + \mathcal{O}_\delta(0), \\ g(x) \vee v(x), & x \in x_0 + \mathcal{O}_\Delta(0) \setminus \mathcal{O}_\delta(0), \\ v(x), & x \in x_0 + \mathcal{O}_\Delta^c(0), \end{cases}$$

where

$$g(x) := P_2\left(\delta \frac{x - x_0}{|x - x_0|}\right) + \varepsilon\delta^2 + \frac{\delta - |x - x_0|}{\Delta - |x - x_0|}.$$

Clearly, $f_\varepsilon(x_0) = v(x_0)$ and $f_\varepsilon \ge v$. Since v is a viscosity subsolution, we have the inequality $\mathcal{L}f_\varepsilon(x_0) \ge 0$. Note that

$$|\mathcal{L}f_\varepsilon(x_0) - \mathcal{L}v(x_0)| \le \varepsilon \sum_{i=1}^n a^{ii}(x_0^i)^2 + \mathcal{I}(f_\varepsilon - v, x_0)$$

with

$$\mathcal{I}(f_\varepsilon - v, x_0) = \int (f_\varepsilon - v)(x_0 + D_{x_0} z) I_{\{x_0 + D_{x_0} z \in \operatorname{int} K\}} \Pi(dz).$$

Let us check that $\mathcal{I}(f_\varepsilon - v, x_0) \le \kappa\varepsilon$. Indeed,

$$\begin{aligned}(f_\varepsilon - v)(x_0 + D_{x_0} z) &\le \varepsilon (D_{x_0} z)^2 I_{\Gamma_\delta} + M I_{\Gamma_\Delta \setminus \Gamma_\delta} \\ &\le \varepsilon \min\{|x_0|^2 |z|^2, \delta^2\} I_{\Gamma_\delta} + M I_{\Gamma_\Delta \setminus \Gamma_\delta},\end{aligned}$$

where $M = 1 + \sup_{y \in \mathcal{O}_1(0)} |P_2(y)|$. It follows that

$$\begin{aligned}\mathcal{I}\big((f_\varepsilon - v), x_0\big) &= \mathcal{I}\big((f_\varepsilon - v) I_{x_0 + \mathcal{O}^c_\Delta(0)}, x_0\big) \\ &\le \varepsilon (1 + |x_0|)^2 \int (|z|^2 \wedge |z|) \Pi(dz) + M\varepsilon.\end{aligned}$$

Letting ε tend to zero, we obtain that $\mathcal{L}v(x_0) \ge 0$. Arguing similarly with $\varepsilon < 0$, we get the opposite inequality. □

7 Jets

Let f and g be functions defined in a neighborhood of zero. We shall write $f(\cdot) \lessapprox g(\cdot)$ if $f(h) \le g(h) + o(|h|^2)$ as $|h| \to 0$. The notations $f(\cdot) \gtrapprox g(\cdot)$ and $f(\cdot) \approx g(\cdot)$ have the obvious meaning.

For $p \in \mathbf{R}^d$ and $X \in \mathcal{S}_d$, we consider the quadratic function

$$Q_{p,X}(z) := pz + (1/2)\langle Xz, z\rangle, \quad z \in \mathbf{R}^d,$$

and define the *super*- and *subjets* of a function v at the point x as

$$\begin{aligned}J^+ v(x) &:= \{(p, X) \colon v(x + \cdot) \lessapprox v(x) + Q_{p,X}(\cdot)\}, \\ J^- v(x) &:= \{(p, X) \colon v(x + \cdot) \gtrapprox v(x) + Q_{p,X}(\cdot)\}.\end{aligned}$$

In other words, $J^+ v(x)$ (resp. $J^- v(x)$) is the family of coefficients of quadratic functions $v(x) + Q_{p,X}(y - \cdot)$ dominating the function $v(\cdot)$ (resp., dominated by this function) in a neighborhood of the point x with precision up to the second order included, and coinciding with $v(\cdot)$ at this point.

In the classical theory developed for differential equations, the notion of a viscosity solution admits an equivalent formulation in terms of super- and subjets. Since the latter are "local" concepts, such a characterization is not possible for integro-differential equations. Nevertheless, we can construct from jets test functions with useful properties.

The following lemma claims for $v \in C_1(K)$ that with any element (p, X) from $J^+ v(x)$, $x \in \operatorname{int} K$, we can relate a test function dominating v, arbitrarily close to v in the uniform metric, touching v at the point x, smooth in a neighborhood of x, and having at this point the first and second derivatives coinciding with p and X.

Lemma 7.1 *Let* $v \in C_1(K)$ *and* $\alpha > 0$. *Let* $x \in \operatorname{int} K$ *and* $(p, X) \in J^+ v(x)$. *Then there exist a number* $a_0 \in (0,1)$ *and a* C^2*-function* $r: \mathbf{R}^d \to \mathbf{R}$ *with compact support such that*

$$\lim_{|h| \to 0} |h|^{-2} r(h) = 0,$$

and the function $f_0 : K \to \mathbf{R}$ *given by the formula*

$$f_0(x+h) := \Big(\big(v(x) + Q_{p,X}(h) + r(h) \big) \vee v(x+h) \Big) \wedge \big(v(x+h) + \alpha \big), \quad x + h \in K,$$

has the following properties:

$$f_0(x+h) = v(x) + Q_{p,X}(h) + r(h), \quad h \in \mathcal{O}_{a_0}(0),$$

$v \le f_0 \le v + \alpha$ *on* K, $f_0(x) = v(x)$, $f_0'(x) = p$, $f_0''(x) = X$.

In particular, if v *is a subsolution of the HJB equation, then* $\mathcal{L} f \le 0$ *on* $\operatorname{int} K$.

Proof Take $a_0 \in (0,1)$ such that the ball $\mathcal{O}_{2a_0}(x) = \{y \in \mathbf{R}^d : |y - x| \le 2a_0\}$ lies in the interior of K. By definition,

$$v(x+h) - v(x) - Q_{p,X}(h) \le |h|^2 \varphi(|h|),$$

where $\varphi(u) \to 0$ as $u \downarrow 0$. We consider on $(0, a_0)$ the function

$$\delta(u) := \sup_{\{h:\, |h| \le u\}} \frac{1}{|h|^2} \big(v(x+h) - v(x) - Q_{p,X}(h) \big)^+ \le \sup_{\{y:\, 0 \le y \le u\}} \varphi^+(y).$$

Obviously, δ is continuous, increasing and $\delta(u) \to 0$ as $u \downarrow 0$. We extend δ to a continuous function on $\mathbf{R}_+$ with $\delta(u) = 0$ for $u \ge 1$.

The function

$$\Delta(u) := \frac{2}{3} \int_u^{2u} \int_\eta^{2\eta} \delta(\xi)\, d\xi\, d\eta$$

vanishes at zero with its two right derivatives, and $u^2 \delta(u) \le \Delta(u) \le u^2 \delta(4u)$. It follows that the function $r: h \mapsto \Delta(|h|)$ has compact support, belongs to $C^2(\mathcal{O}_{a_0}(0))$, its Hessian vanishes at zero, and

$$v(x+h) - v(x) - Q_{p,X}(h) \le |h|^2 \delta(|h|) \le \Delta(|h|) = r(h), \quad h \in \mathcal{O}_{a_0}(0).$$

Thus, the function $y \mapsto v(x) + Q_{p,X}(y - x) + r(y - x)$ dominates v on the ball $\mathcal{O}_{a_0}(x)$. Without loss of generality, diminishing a_0 if necessary, we may assume that it is dominated by $v + \alpha$ on this ball. Now the assertion of the lemma is obvious. □

The corresponding assertion for $J^- v(x)$ also holds, with obvious changes in the formulation.

For the proof of the uniqueness theorem, we need specific families of test functions coinciding with sub- and supersolutions outside a neighborhood of x. To this end, we introduce the following definitions.

Let $0 < a < a'$. We say that a continuous mapping $\xi_{a,a'} : \mathbf{R}^d \to [0,1]$ is an (a,a')-*cutoff function* if $\xi_{a,a'} = 1$ on $\mathcal{O}_a(0)$ and $\xi_{a,a'} = 0$ outside $\mathcal{O}_{a'}(0)$. If L is a linear subspace of $\mathbf{R}^d$, then we define the *cylindrical* (a,a')-*cutoff function* $\xi^L_{a,a'}$ by putting $\xi^L_{a,a'}(x) = \xi_{a,a'}(P_L x)$, where P_L is the projection of x onto L.

It is clear that in the notation of the above lemma, for any $a' \in (0, a_0)$, the functions $f : K \to \mathbf{R}$ given by the formulae

$$f(x+h) := \big(v(x) + Q_{p,X}(h) + r(h)\big)\xi_{a,a'}(h) + v(x+h)\big(1 - \xi_{a,a'}(h)\big),$$

$$f(x+h) := f_0(x+h)\xi^L_{a,a'}(h) + v(x+h)\big(1 - \xi^K_{a,a'}(h)\big), \quad x+h \in K, \tag{7.1}$$

satisfy all the properties claimed for f_0.

The following lemma will be used in the specific case where

$$D = D_x = \operatorname{diag} x, \qquad \tilde{D} = D_y = \operatorname{diag} y,$$

and x, y have no zero components.

Lemma 7.2 *Let D, $\tilde{D}$ be two invertible linear operators on $\mathbf{R}^d$, and let $\xi_{a,a'}$ be an (a,a')-cutoff function. Then there is an $(\tilde{a},\tilde{a}')$-cutoff function $\tilde{\xi}_{\tilde{a},\tilde{a}'}$ with arbitrarily small $\tilde{a} \le a\|D\tilde{D}^{-1}\|^{-1}$ and arbitrary $\tilde{a}' \ge a'\|\tilde{D}D^{-1}\|$ such that*

$$\tilde{\xi}_{\tilde{a},\tilde{a}'}(\tilde{D}z) = \xi_{a,a'}(Dz), \quad \forall z \in \mathbf{R}^d.$$

Proof Put

$$\tilde{\xi}_{\tilde{a},\tilde{a}'}(u) := \xi_{a,a'}(D\tilde{D}^{-1}u), \quad u \in \mathbf{R}^d.$$

Then $\xi_{\tilde{a},\tilde{a}'}(u) = 1$ if $|u| \le a\|\tilde{D}D^{-1}\|^{-1}$ and $\tilde{\xi}_{\tilde{a},\tilde{a}'}(u) = 0$ if $|D\tilde{D}^{-1}u| \ge a'$. The last inequality holds when $|u| \ge a'\|(D\tilde{D}^{-1})^{-1}\| = a'\|\tilde{D}D^{-1}\|$. □

Remark 7.3 The assertion of the lemma is not completely satisfactory, but it remains true for cylindrical cutoff functions in the situation where D and $\tilde{D}$ are two symmetric operators with the common image space $L = \operatorname{Im} D = \operatorname{Im} \tilde{D}$. The norm in the formulation is then the norm in L of their restrictions.

8 Supersolutions and properties of the Bellman function

8.1 When is the Bellman function W finite on K?

First, we present sufficient conditions ensuring that the Bellman function W of the considered maximization problem is finite.

The functions we are interested in are defined on the solvency cone K, whereas the process V may jump out of the latter. In order to be able to apply later the Itô formula, we stop $V = V^{x,\pi}$ at the moment immediately preceding the ruin and define the process

$$\tilde{V} = V^{\theta-} = V I_{[\![0,\theta[\![} + V_{\theta-} I_{[\![\theta,\infty[\![},$$

where θ is the exit time of V from the interior of the solvency cone K. This process coincides with V on $[\![0,\theta[\![$, but in contrast to the latter either always remains in K (due to the stopping at θ if $V_{\theta-} \in \operatorname{int} K$) or exits to the boundary in a continuous way and stays on it at the exit point.

Since $\tilde{V}_t = V_{t\wedge\theta} - \Delta V_\theta I_{[\![\theta,\infty[\![}(t)$, we obtain from (2.6) the representation

$$\tilde{V}_t = x + \int_0^{t\wedge\theta} D_{\tilde{V}_{s-}}(\mu\, ds + \Xi\, dw_s) + \int_0^{t\wedge\theta} \int D_{\tilde{V}_{s-}} z\big(p(dz,ds) - q(dz,ds)\big) \\ - \Delta V_\theta I_{[\![\theta,\infty[\![}(t) + B_t - C_t.$$

Let Φ be the set of continuous functions $f : K \to \mathbf{R}_+$ increasing with respect to the partial ordering $\geq_K$ and such that for all $x \in \operatorname{int} K$ and $\pi \in \mathcal{A}_a^x$, the positive process $X^f = X^{f,x,\pi}$ given by the formula

$$X_t^f := e^{-\beta t} f(\tilde{V}_t) I_{[\![0,\theta[\![}(t) + J_t^\pi$$

is a supermartingale. The set Φ of f with this property is convex and stable under the operation $\wedge$ (recall that the minimum of two supermartingales is a supermartingale). Any continuous function that is a monotone limit (increasing or decreasing) of functions from Φ also belongs to Φ.

The interest in the processes X^f with $f \in \Phi$ is explained by the following result.

Lemma 8.1 (a) *If $f \in \Phi$, then $W \leq f$.*

(b) *Let $y \in \partial K$. Suppose that for every $\varepsilon > 0$, there exists $f_\varepsilon \in \Phi$ such that $f_\varepsilon(y) \leq \varepsilon$. Then W is continuous at y, and $W(y) = 0$.*

Proof (a) On the boundary ∂K, the inequality is trivial. Using the positivity of f, the supermartingale property of X^f, and finally the monotonicity of f, we get for $x \in \operatorname{int} K$ the following chain of inequalities leading to the required property:

$$EJ_t^\pi \leq EX_t^f \leq f(\tilde{V}_0) = f(V_0) \leq f(V_{0-}) = f(x).$$

(b) The continuity of the function W at the point $y \in \partial K$ follows from the inequalities $0 \leq W \leq f_\varepsilon$. □

Remark 8.2 Recall that Proposition 4.1 asserts that the function W, if finite, is continuous on the interior of K. Thus, Lemma 8.1 implies that W is continuous on $\operatorname{int} K$ if Φ is not empty. If Φ is rich enough to apply (b) at every point of the boundary, then W is continuous on K and vanishes on the boundary.

Lemma 8.3 *Let $f : K \to \mathbf{R}_+$ be a function in $C_1(K) \cap C^2(\operatorname{int} K)$. If f is a classical supersolution of (5.1), then $f \in \Phi$, i.e., f is increasing with respect to the partial ordering $\geq_K$, and X^f is a supermartingale.*

Proof First, notice that a classical supersolution is increasing with respect to the partial ordering $\geq_K$. Indeed, by the finite increments formula, we have, for any $x, h \in \operatorname{int} K$,

$$f(x+h) - f(x) = f'(x + \vartheta h)h$$

for some $\vartheta \in [0, 1]$. The right-hand side is greater than or equal to zero because for the supersolution f, we have the inequality $\Sigma_G(f'(y)) \leq 0$ whatever is $y \in \operatorname{int} K$, or equivalently $f'(y)h \geq 0$ for every $h \in K$, just by the definition of the support function Σ_G and the choice of G as a generator of the cone $-K$. By continuity, $f(x+h) - f(x) \geq 0$ for all $x, h \in K$.

Let $\theta_n := \inf\{t : \operatorname{dist}(\tilde{V}_t, \partial K) \leq 1/n\}$. The stopped processes $\tilde{V}^{\theta_n}$ evolve in $\operatorname{int} K$. Thus, we can apply the "standard" Itô formula to $e^{-\beta t} f(\tilde{V}_t)$ and obtain for $t \leq \theta$ that

$$\begin{aligned} e^{-\beta t} f(\tilde{V}_t) = f(x) &+ \int_0^t e^{-\beta s} f'(\tilde{V}_{s-})\, d\tilde{V}_s - \beta \int_0^t e^{-\beta s} f(\tilde{V}_{s-})\, ds \\ &+ \frac{1}{2} \int_0^t e^{-\beta s} \operatorname{tr} A(\tilde{V}_{s-}) f''(\tilde{V}_{s-})\, ds \\ &+ \sum_{s \leq t} e^{-\beta s} \big(f(\tilde{V}_{s-} + \Delta \tilde{V}_s) - f(\tilde{V}_{s-}) - f'(\tilde{V}_{s-}) \Delta \tilde{V}_s \big). \end{aligned}$$

Taking into account that the processes Y and B do not jump simultaneously and that ruin cannot happen due to a jump of B, we get that

$$\begin{aligned} &\sum_{s \leq t} e^{-\beta s} \big(f(\tilde{V}_{s-} + \Delta \tilde{V}_s) - f(\tilde{V}_{s-}) - f'(\tilde{V}_{s-}) \Delta \tilde{V}_s \big) \\ &\quad - e^{-\beta \theta} f'(V_{\theta-}) \Delta V_\theta I_{\{\theta\}}(t) - e^{-\beta \theta} f(V_{\theta-}) I_{\{\theta\}}(t) \\ &= \sum_{s \leq t} e^{-\beta s} \big(f(V_{s-} + \Delta V_s) I_{\operatorname{int} K}(V_{s-} + \Delta V_s) - f(V_{s-}) - f'(V_{s-}) \Delta V_s \big) \\ &= \int_0^t \int e^{-\beta s} \big(f(V_{s-} + D_{V_{s-}} z) I(V_{s-}, z) - f(V_{s-}) - f'(V_{s-}) D_{V_{s-}} z) \big) \\ &\quad \times I_{\{\Delta B_s = 0\}} p(ds, dz) \\ &\quad + \sum_{s \leq t} e^{-\beta s} \big(f(V_{s-} + \Delta B_s) - f(V_{s-}) - f'(V_{s-}) \Delta B_s \big) \\ &= \int_0^t \int e^{-\beta s} (\ldots) I_{\{\Delta B_s = 0\}} \big(p(ds, dz) - \Pi(dz)\, ds \big) + \int_0^t \int e^{-\beta s} (\ldots) \Pi(dz)\, ds \\ &\quad + \sum_{s \leq t} e^{-\beta s} \big(f(V_{s-} + \Delta B_s) - f(V_{s-}) - f'(V_{s-}) \Delta B_s \big), \end{aligned}$$

where we replace in the integrals by dots ... the lengthy expression

$$f(\tilde{V}_{s-} + D_{\tilde{V}_{s-}} z) I(V_{s-}, z) - f(\tilde{V}_{s-}) - f'(\tilde{V}_{s-}) D_{\tilde{V}_{s-}} z.$$

Noting that

$$X_t^f = e^{-\beta t} f(\tilde{V}_t) - e^{-\beta\theta} f(V_{\theta-}) I_{\{\theta\}}(t) + J_t^\pi$$

and using (2.6) and the above formulae, we obtain after regrouping terms the representation

$$X_t^f = f(x) + \int_0^{t\wedge\theta} e^{-\beta s}\big(\mathcal{L}_0 f(\tilde{V}_s) - c_s f'(\tilde{V}_s) + U(c_s)\big)\, ds + R_t + m_t, \tag{8.1}$$

where

$$R_t := \int_0^{t\wedge\theta} e^{-\beta s} f'(V_{s-})\, dB_s^c + \sum_{s\le t} e^{-\beta s}\big(f(\tilde{V}_{s-} + \Delta B_s) - f(\tilde{V}_{s-})\big), \tag{8.2}$$

and m is the local martingale

$$\begin{aligned} m_t = &\int_0^{t\wedge\theta} e^{-\beta s} f'(\tilde{V}_{s-}) D_{\tilde{V}_{s-}} \Xi\, dw_s \\ &+ \int_0^{t\wedge\theta}\int e^{-\beta s}\big(f(\tilde{V}_{s-} + D_{\tilde{V}_{s-}} z) I(\tilde{V}_{s-}, z) - f(\tilde{V}_{s-})\big) \\ &\qquad \big(p(dz, ds) - \Pi(dz)\, ds\big). \end{aligned} \tag{8.3}$$

By the definition of a supersolution, for any $x \in \operatorname{int} K$,

$$\mathcal{L}_0 f(x) \le -U^*\big(f'(x)\big) \le c f'(x) - U(c), \quad \forall c \in \mathcal{C}.$$

Thus, the integral in (8.1) is a decreasing process. The process R is also decreasing. Indeed, the terms of the sum in (8.2) are less than or equal to zero by the monotonicity of f and

$$f'(V_{s-})\, dB_s^c = I_{\{\Delta B_s = 0\}} f'(V_{s-}) \dot{B}_s d|B|_s,$$

where $f'(V_{s-})\dot{B}_s \le 0$ since $\dot{B}$ takes values in $-K$. Let (σ_n) be a localizing sequence for m. Taking into account that $X^f \ge 0$, we obtain from (8.1) that, for each n, the negative decreasing process $(R_{t\wedge\sigma_n})$ dominates an integrable process, and so it is integrable. The same conclusion holds for the stopped integral. Being a sum of an integrable decreasing process and a martingale, the process $(X^f_{t\wedge\sigma_n})$ is a positive supermartingale, and hence, by the Fatou lemma, X^f is a supermartingale as well. □

Lemma 8.3 implies that the existence of a smooth positive supersolution f of (5.1) ensures the finiteness of W on K. We discuss a method how to construct supersolutions in Sect. 12.

Remark 8.4 Let $\bar{\mathcal{O}}$ be the closure of an open subset $\mathcal{O}$ of K, and $f : \bar{\mathcal{O}} \to \mathbf{R}_+$ a classical supersolution in $\bar{\mathcal{O}}$ increasing with respect to the partial ordering $\ge_K$. Let $x \in \mathcal{O}$, and let τ be the exit time of the process $V^{x,\pi}$ from $\bar{\mathcal{O}}$. The above arguments imply that the process $(X^f_{t\wedge\tau})$ is a supermartingale, and therefore

$$E\big(e^{-\beta(t\wedge\tau)} f(\tilde{V}_{t\wedge\tau}) I_{[\![0,\theta[\![}(t\wedge\tau) + J^\pi_{t\wedge\tau}\big) \le f(x).$$

8.2 Strict local supersolutions

For a strict supersolution, we can get a more precise result, which will play a crucial role in deducing from the dynamic programming principle the property of W to be a subsolution of the HJB equation.

Fix $x \in \operatorname{int} K$ and a ball $\bar{\mathcal{O}}_r(x) \subseteq \operatorname{int} K$ such that $\bar{\mathcal{O}}_{2r}(x) \subseteq \operatorname{int} K$. We define $\tau^\pi = \tau^\pi_r$ as the exit time of $V^{\pi,x}$ from $\mathcal{O}_r(x)$, i.e.,

$$\tau^\pi := \inf\{t \ge 0 \colon |V^{\pi,x}_t - x| \ge r\}.$$

Lemma 8.5 *Let $f \in C_1(K) \cap C^2(\mathcal{O}_{2r}(x))$ be such that $\mathcal{L}f \le -\varepsilon < 0$ on $\bar{\mathcal{O}}_r(x)$. Then there exist a constant $\eta = \eta_\varepsilon > 0$ and an interval $(0, t_0]$ such that*

$$\sup_{\pi \in \mathcal{A}^x_a} E X^{f,x,\pi}_{t\wedge\tau^\pi} \le f(x) - \eta t, \quad \forall t \in (0, t_0].$$

Proof We fix a strategy π and omit its symbol in the notations below. In what follows, only the behavior of the processes on $[\![0,\tau]\!]$ does matter. Note that $|V_\tau - x| \ge r$ on the set $\{\tau < \infty\}$ and $\tau \le \theta$. As in the proof of Lemma 8.3, we apply the Itô formula and obtain, with the same notations (8.2) and (8.3), the representation

$$\begin{aligned} X^f_{t\wedge\tau} &:= e^{-\beta(t\wedge\tau)} f(\tilde{V}_{t\wedge\tau}) I_{[\![0,\theta[\![}(t\wedge\tau) + J^\pi_{t\wedge\tau} \\ &= f(x) + \int_0^{t\wedge\tau} e^{-\beta s}(\mathcal{L}_0 f + U^*)(\tilde{V}_s)\,ds \\ &\quad - \int_0^{t\wedge\tau} e^{-\beta s}\big(U^*(\tilde{V}_s) + c_s f'(\tilde{V}_s) - U(c_s)\big)\,ds + R_{t\wedge\tau} + m_{t\wedge\tau}. \end{aligned}$$

Due to the monotonicity of f, we may assume without loss of generality that on the interval $[0,\tau(\omega)]$, the increment ΔB_t does not exceed the distance of V_{s-} to the boundary of $\mathcal{O}_r(x)$. In other words, if the exit from the ball is due to an action (and not because of a jump of the price process), then we can replace this action by a less expensive one, with the jump of the process $\tilde{V}$ in the same direction but smaller, ending on the boundary of the ball. So, $|\Delta B_t| \le 2r$ for $t \le \tau$.

By assumption, for $y \in \bar{\mathcal{O}}_r(x)$, we have the bounds $(\mathcal{L}_0 f + U^*)(y) \le -\varepsilon$ (implying that the first integral on the right-hand side above is dominated by $-\varepsilon\,(t\wedge\tau)$) and $\Sigma_G(f'(y)) \le -\varepsilon$. The latter inequality means that the scalar product $kf'(y) \le -\varepsilon|k|$ for every $k \in -K$ (therefore, we have the inclusion $f'(\bar{\mathcal{O}}_r(x)) \subset \operatorname{int} K^*$). In particular, for $s \in [0,\tau(\omega)]$,

$$f'(V_{s-})\dot{B}_s \le -\varepsilon|\dot{B}_s|, \qquad \big(f(\tilde{V}_{s-} + \Delta B_s) - f(\tilde{V}_{s-})\big) \le -\varepsilon|\Delta B_s|.$$

Since $|\tilde{V}_{s-} - x| \le r$ for $s \in [0, \tau(\omega)]$, we obtain, using the finite increment formula and the linear growth of f, the bounds

$$\big(f(\tilde{V}_{s-} + D_{\tilde{V}_{s-}} z) - f(\tilde{V}_{s-})\big)^2 I(\tilde{V}_{s-}, z) I_{\{|z| \le 1/2\}} \le \kappa |z|^2 I_{\{|z| \le 1/2\}},$$
$$\big(f(\tilde{V}_{s-} + D_{\tilde{V}_{s-}} z) - f(\tilde{V}_{s-})\big) I(\tilde{V}_{s-}, z) I_{\{|z| > 1/2\}} \le \kappa (1 + |z|) I_{\{|z| > 1/2\}},$$

and since $I(\tilde{V}_{s-}, z) = 1$ when $|z| < r/(|x| + r)$,

$$f(\tilde{V}_{s-})\big(1 - I(\tilde{V}_{s-}, z)\big) \le \kappa I_{\{|z| \ge r/(|x|+r)\}}$$

for some constant κ independent of the strategy. Thus, the integrand in the stochastic integral with respect to the centered Poisson measure in (8.3) for $t \le \tau$ is bounded by the function $|z|^2 \wedge |z|$ multiplied by a constant, whereas the integrand in the integral with respect to the Wiener process is bounded. It follows that the local martingale $(m_{t\wedge\tau})_{t\ge 0}$ is a martingale and $Em_{t\wedge\tau} = 0$.

The above observations imply the inequality

$$EX^{f,x}_{t\wedge\tau} \le f(x) - e^{-\beta t} EN_t,$$

where

$$N_t := \varepsilon\,(t \wedge \tau) + \int_0^{t\wedge\tau} H\big(c_s, f'(V_s)\big)\, ds + \varepsilon \int_0^{t\wedge\tau} |\dot{B}_s|\, d|B|_s$$

with $H(c, p) := U^*(p) + pc - U(c) \ge 0$. It remains to verify that EN_t dominates, on a certain interval $(0, t_0]$, a strictly increasing linear function, which is independent of π.

The process N looks a bit complicated, but we can replace it by another one of a simpler structure. To this end, note that there is a constant κ ("large"; for convenience, $\kappa \ge 1$) such that

$$\inf_{p \in f'(\bar{\mathcal{O}}_r(x))} H(c, p) \ge \frac{\varepsilon}{2}|c|, \quad \forall c \in \mathcal{C}, |c| \ge \kappa.$$

Indeed, being the image of a closed ball under a continuous mapping, the set $f'(\bar{\mathcal{O}}_r(x))$ is compact in $\operatorname{int} K^*$. The lower bound of the continuous function U^* on $f'(\bar{\mathcal{O}}_r(x))$ is finite. For any p from $f'(\bar{\mathcal{O}}_r(x))$ and $c \in \mathcal{C} \subseteq K$, we have the inequality $pc/|c| \ge \varepsilon$. Finally, $U(c)/|c| \to 0$ as $c \to \infty$. Combining these facts, we infer the claimed inequality. Thus, for the first integral in the definition of N_t, we have the bound

$$\int_0^{t\wedge\tau} H\big(c_s, f'(V_s)\big)\, ds \ge \frac{\varepsilon}{2} \int_0^{t\wedge\tau} I_{\{|c_s| \ge \kappa\}} |c_s|\, ds.$$

The second integral in the definition dominates $\kappa_1 |B|_{t\wedge\tau}$ for some $\kappa_1 > 0$. To see this, let us consider the absolute norm $|\cdot|_1$ in $\mathbf{R}^d$. In contrast with the total variation $|B|$ calculated with respect to the Euclidean norm $|\cdot|$, the total variation of B with respect to the absolute norm admits the simpler expression $\sum_i \operatorname{Var} B^i$, where $\operatorname{Var} B^i$ is the total variation of the scalar process B^i. Obviously,

$$|\dot{B}|_1 = \sum_i |\dot{B}^i| = \sum_i \left|\frac{dB^i}{d|B|}\right| = \sum_i \left|\frac{dB^i}{d\operatorname{Var} B^i}\right| \frac{d\operatorname{Var} B^i}{d|B|} = \frac{d\sum_i \operatorname{Var} B^i}{d|B|}.$$

But all norms in $\mathbf{R}^d$ are equivalent, i.e., $\tilde{\kappa}^{-1}|\cdot| \le |\cdot|_1 \le \tilde{\kappa}|\cdot|$ for some strictly positive constant $\tilde{\kappa}$. The same inequalities relate the corresponding total-variation processes. The claimed property follows from here with the constant $\kappa_1 = \tilde{\kappa}^{-2}$.

Summarizing, we conclude that it is sufficient to check the domination property for $E\tilde{N}_t$ with

$$\tilde{N}_t := t \wedge \tau + \int_0^{t\wedge\tau} I_{\{|c_s| \ge \kappa\}} |c_s|\, ds + |B|_{t\wedge\tau}.$$

These processes $\tilde{N} = \tilde{N}^\pi$ have a transparent dependence on the control. The idea of the concluding reasoning is very simple: on a certain set of strictly positive probability, where we may neglect the random fluctuations, either τ is "large", or the total variation of the control is "large": we can accelerate exit only by an intensive trading or consumption.

The formal arguments are as follows. Using the stochastic Cauchy formula (2.2) and the fact that $\mathcal{E}_{0+}(Y^i) = \mathcal{E}_0(Y^i) = 1$, we get immediately that there exist a number $t_0 > 0$ and a measurable set Γ with $P[\Gamma] > 0$ on which

$$|V^{x,\pi} - x| \le r/2 + 2(|B| + |C|) \quad \text{on } [0, t_0]$$

whatever is the control $\pi = (B, C)$. Of course, diminishing t_0, we may assume without loss of generality that $\kappa t_0 \le r/8$. For any $t \le t_0$, we have on the set $\Gamma \cap \{\tau \le t\}$ the inequality $|B|_\tau + |C|_\tau \ge r/4$, and hence

$$\tilde{N}_t \ge |B|_\tau + |C|_\tau - \int_0^\tau I_{\{|c_s| < \kappa\}} |c_s|\, ds \ge \frac{r}{4} - \kappa t_0 \ge \kappa t_0 \ge t_0 \ge t.$$

On the set $\Gamma \cap \{\tau > t\}$, the inequality $\tilde{N}_t \ge t$ is obvious. Thus, $E\tilde{N}_t \ge tP[\Gamma]$ on $[0, t_0]$, and the result is proved. □

9 Dynamic programming principle

The aim of this section is to establish the following two assertions, which will serve to derive the HJB equation for the Bellman function. For the considered model, they constitute an analogue of the classical dynamic programming principle. The latter is usually written in the form of a single identity (see the remark at the end of the section), but for our purposes, we need a more precise form.

Lemma 9.1 *Let $\mathcal{T}_f$ be the set of finite stopping times. Then*

$$W(x) \le \sup_{\pi \in \mathcal{A}_a^x} \inf_{\tau \in \mathcal{T}_f} E\big(J_\tau^\pi + e^{-\beta\tau} W(V_\tau^{x,\pi}) I_{\{\tau < \theta\}}\big). \tag{9.1}$$

Lemma 9.2 *Suppose that W is continuous on* $\operatorname{int} K$. *Then, for any $\tau \in \mathcal{T}_f$,*

$$W(x) \geq \sup_{\pi \in \mathcal{A}_a^x} E\big(J_\tau^\pi + e^{-\beta\tau} W(V_\tau^{x,\pi}) I_{\{\tau<\theta\}}\big). \tag{9.2}$$

We work on the canonical filtered space of càdlàg functions equipped with the measure P that is the distribution of the driving Lévy process. The generic point $\omega = \omega_{.}$ of this space is a d-dimensional càdlàg function on $\mathbf{R}_+$, zero at the origin. Let $\mathcal{F}_t^\circ := \sigma(\omega_s, s \leq t)$ and $\mathcal{F}_t := \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}^\circ$. We add the superscript P to denote σ-algebras augmented by all P-null sets from Ω. Recall that $\mathcal{F}_t^{\circ,P}$ coincides with $\mathcal{F}_t^P$ (this assertion follows easily from the predictable representation theorem). The Skorokhod metric makes Ω a Polish space, and its Borel σ-algebra coincides with $\mathcal{F}_\infty$; see [20, Chapter VI] for this and other relevant information.

Since elements of Ω are paths, we can define operators such as the stopping $\omega_{.} \mapsto \omega_{.}^s$, $s \geq 0$, where $\omega_{.}^s = \omega_{s\wedge .}$, and the translation $\omega_{.} \mapsto \omega_{s+.} - \omega_s$. Taking Doob's theorem into account, we can describe $\mathcal{F}_s^\circ$-measurable random variables as those of the form $h(\omega_{.}) = h(\omega_{.}^s)$, where h is a measurable function on Ω.

We define also the "concatenation" operator as the measurable function

$$g : \mathbf{R}_+ \times \Omega \times \Omega \to \Omega$$

for which the image of $(s, \omega_{.}, \tilde{\omega}_{.})$ is the trajectory $(g_t)_{t\geq 0}$ with

$$g_t(s, \omega_{.}, \tilde{\omega}_{.}) = \omega_t I_{[0,s]}(t) + (\tilde{\omega}_{t-s} + \omega_s) I_{(s,\infty)}(t).$$

Notice that

$$g_t(s, \omega_{.}^s, \omega_{.+s} - \omega_s) = \omega_t.$$

Thus, $\pi(\omega) = \pi(g(s, \omega_{.}^s, \omega_{.+s} - \omega_s))$.

Let π be a fixed strategy from $\mathcal{A}_a^x$, and let $\theta = \theta^{x,\pi}$ be the exit time from $\operatorname{int} K$ for the process $V^{x,\pi}$.

Recall the following general fact on regular conditional distributions. Let ξ and η be two random variables taking values in Polish spaces X and Y equipped with their Borel σ-algebras $\mathcal{X}$ and $\mathcal{Y}$. Then ξ admits a regular conditional distribution given $\eta = y$, which we denote by $p_{\xi|\eta}(\Gamma, y)$. This means that $p_{\xi|\eta}(\cdot, y)$ is a probability measure on $\mathcal{X}$, $p_{\xi|\eta}(\Gamma, \cdot)$ is a $\mathcal{Y}$-measurable function, and

$$E[f(\xi,\eta)|\eta] = \int f(x,y) p_{\xi|\eta}(dx, y)\Big|_{y=\eta} \quad \text{(a.s.)}$$

for any $\mathcal{X} \times \mathcal{Y}$-measurable function $f(x, y) \geq 0$.

We apply the above relation to the random variables $\xi = (\omega_{.+\tau} - \omega_\tau)$ and $\eta = (\tau, \omega^\tau)$. It is well known that a Lévy process starts afresh at stopping times, i.e., the measure $P[\cdot]$ itself (not depending on y) is the regular conditional distribution

$p_{\xi|\eta}(\cdot, y)$ of ξ given $\eta = y$. For fixed s and w, we define the shifted control as the mapping

$$(\tilde{\omega}, t) \mapsto \widehat{\pi}_t(\tilde{\omega}) := \pi_{t+s}\big(g(s, \omega^s_{\cdot}, \tilde{\omega}_{\cdot})\big) - \pi_s\big(g(s, \omega^s_{\cdot}, \tilde{\omega}_{\cdot})\big)$$

defined on $\Omega \times \mathbf{R}_+$, where $\tilde{\omega}_{\cdot}$ is a generic point of the canonical space.

For a finite stopping time τ, we put $\tilde{Y}_t = Y_{t+\tau} - Y_\tau$, $t \ge 0$. Then

$$V^{i,x}_{t+\tau} = \mathcal{E}_t(\tilde{Y}^i)V^{i,x}_\tau + \mathcal{E}_t(\tilde{Y}^i)\int_{(0,t]} \mathcal{E}_{s-}(\tilde{Y}^i)\, d(B^i_{\tau+s} - C^i_{\tau+s}).$$

This equation (sometimes referred to as the flow property) has an obvious meaning: after the stopping time τ, the solution of the equation starting from zero coincides with the solution starting at the time τ from the value attained at τ. On the other hand, this shows that the conditional distribution of the process $(V^{x,\pi}_{t+\tau}(\omega))_{t\ge 0}$ given (τ, ω^τ) coincides with the distribution of the process $(V^{v,\hat{\pi}}_t(\tilde{\omega}))_{t\ge 0}$ with $v = V^{x,\pi}_\tau(\omega)$.

Proof of Lemma 9.1 For arbitrary $\pi \in \mathcal{A}^x_a$ and $\tau \in \mathcal{T}_f$, we have that

$$\begin{aligned} EJ^\pi_\infty &= EJ^\pi_\tau + Ee^{-\beta\tau} I_{\{\tau<\theta\}} \int_0^\infty e^{-\beta r} U(c_{r+\tau})\, dr \\ &= EJ^\pi_\tau + Ee^{-\beta\tau} I_{\{\tau<\theta\}} E\left[\int_0^\infty e^{-\beta r} U(c_{r+\tau})\, dr \,\middle|\, (\tau, \omega^\tau)\right]. \end{aligned}$$

According to the above discussion, we can rewrite the second term of the right-hand side as

$$Ee^{-\beta\tau} I_{\{\tau<\theta\}} \int \left(\int_0^\infty e^{-\beta r} U\Big(c_{r+\tau}\big(g(\tau, \omega^\tau, \tilde{\omega})\big)\Big)\, dr\right) P(d\tilde{\omega})$$

and dominate it by $Ee^{-\beta\tau} I_{\{\tau<\theta\}} W(V^{x,\pi}_\tau)$. Thus,

$$EJ^\pi_\infty \le EJ^\pi_\tau + Ee^{-\beta\tau} I_{\{\tau<\theta\}} W(V^{x,\pi}_\tau).$$

This bound leads directly to the announced inequality. □

Proof of Lemma 9.2 Fix $\varepsilon > 0$. By hypothesis the function W is continuous on $\operatorname{int} K$. For each $x \in \operatorname{int} K$, we can find an open ball $\mathcal{O}_r(x) = x + \mathcal{O}_r(0)$ with $r = r(\varepsilon, x) < \varepsilon$ contained in the open set $\{y \in \operatorname{int} K : |W(y) - W(x)| < \varepsilon\}$. Moreover, we can find a smaller ball $\mathcal{O}_{\tilde{r}}(x)$ contained in the set $y(x) + K$ with some $y(x) \in \mathcal{O}_r(x)$. Indeed, take an arbitrary $x_0 \in \operatorname{int} K$. Then, for some $\delta > 0$, we have $x_0 + \mathcal{O}_\delta(0) \subset K$. Since K is a cone, $\lambda x_0 + \mathcal{O}_{\lambda\delta}(0) \subset K$ for every $\lambda > 0$, and this inclusion implies that

$$x + \mathcal{O}_{\lambda\delta}(0) \subset x - \lambda x_0 + K.$$

Clearly, the requirement is met for $y(x) = x - \lambda x_0$ and $\tilde{r} = \lambda\delta$ when $\lambda|x_0| < r$ and $\lambda\delta < r$. The family of sets $\mathcal{O}_{\tilde{r}(x)/2}(x)$, $x \in \operatorname{int} K$, is an open covering of $\operatorname{int} K$.

But any open covering of a separable metric space contains a countable subcovering (this is the Lindelöf property; in our case, where $\operatorname{int} K$ is a countable union of compacts, it is obvious). Take a countable subcovering indexed by points x_n. For notational simplicity, we denote the open balls $\mathcal{O}_{\tilde{r}(x_n)/2}(x_n)$ by $\mathcal{O}_n$ and $y(x_n)$ by y_n.

Let $\pi^n = (B^n, C^n) \in \mathcal{A}_a^{y_n}$ be an ε-optimal strategy for the initial point y_n, i.e., such that

$$EJ_\infty^{\pi_n} \geq W(y_n) - \varepsilon.$$

Let $\pi \in \mathcal{A}_a^x$ be an arbitrary strategy. Put

$$\rho := \inf\{j \geq 1 : V_\tau^{x,\pi} \in \mathcal{O}_j\}.$$

Let us introduce the strategy

$$\pi' := \pi I_{[\![0,\tau]\!]} + (0,0) I_{]\!]\tau,\infty[\![}$$

and the predictable stopping times $\tau_k := \tau + 1/k$. Finally, put

$$\tilde{\pi} := \pi I_{[\![0,\tau]\!]} + \sum_{n=1}^{\infty} \big((y_n - V_{\tau_k}^{x,\pi'}, 0) + \bar{\pi}^{n,k}\big) I_{[\![\tau_k,\infty[\![} I_{\{\rho=n\}} I_{\{V_{\tau_k}^{x,\pi'} - y_n \in K\}} I_{\{\tau_k < \theta\}},$$

where $\bar{\pi}^{n,k}$ is the translation of the strategy π^n: namely, for a point $\omega_.$ on which $\tau(\omega_.) = s < \infty$, we have

$$\bar{\pi}_t^{n,k}(\omega_.) := \pi_{t-s-1/k}^n(\omega_{.+s+1/k} - \omega_{s+1/k}), \quad t \geq s + 1/k.$$

In other words, the strategy $\tilde{\pi}$ coincides with π on $[\![0,\tau[\![$, is zero on the interval $[\![\tau,\tau_k[\![$, and coincides with the shift of π^n on $[\![\tau_k,\infty[\![$ when $V_\tau^{x,\pi}$ is in $\mathcal{O}_n$ and $V_{\tau_k}^{x,\pi} - y_n \in K$; the correction term guarantees that in the latter case, the trajectory of the control system corresponding to the control $\tilde{\pi}$ passes at time τ_k through the point y_n. We can check that $\tilde{\pi} I_{[\![0,\theta^{x,\tilde{\pi}}]\!]} \in \mathcal{A}_a^x$.

Now, using the same considerations as in the previous lemma, we have

$$\begin{aligned} W(x) \geq EJ_\infty^{\tilde{\pi}} &= EJ_\tau^\pi + \sum_{n=1}^{\infty} EI_{\{\rho=n\}} I_{\{\tau<\theta\}} I_{\{V_{\tau_k}^{x,\pi'} - y_n \in K\}} \int_{\tau_k}^{\infty} e^{-\beta s} U(\bar{c}_s^n)\, ds \\ &= EJ_\tau^\pi + \sum_{n=1}^{\infty} EI_{\{\rho=n\}} I_{\{\tau<\theta\}} I_{\{V_{\tau_k}^{x,\pi'} - y_n \in K\}} e^{-\beta\tau_k} \int_0^{\infty} e^{-\beta s} U(c_s^n)\, ds. \end{aligned}$$

Note that $I_{\{V_{\tau_k}^{x,\pi'} - y_n \in K\}} \to 1$ as $k \to \infty$, and by the Fatou lemma, we obtain that

$$
\begin{aligned}
W(x) \geq EJ_{\infty}^{\tilde{\pi}} &= EJ_{\tau}^{\pi} + \sum_{n=1}^{\infty} EI_{\{\rho=n\}} I_{\{\tau<\theta\}} e^{-\beta\tau} \int_0^{\infty} e^{-\beta s} U(c_s^n)\, ds \\
&\geq EJ_{\tau}^{\pi} + \sum_{n=1}^{\infty} EI_{\{\rho=n\}} I_{\{\tau<\theta\}} e^{-\beta\tau} \big(W(y_n) - \varepsilon\big) \\
&\geq EJ_{\tau}^{\pi} + Ee^{-\beta\tau} W(V_{\tau}^{x,\pi}) I_{\{\tau<\theta\}} - 2\varepsilon.
\end{aligned}
$$

Since π and ε are arbitrary, the result follows. □

Remark 9.3 The previous lemmas imply that for any $\tau \in \mathcal{T}_f$, we have the identity

$$
W(x) = \sup_{\pi \in \mathcal{A}_a^x} E\big(J_{\tau}^{\pi} + e^{-\beta\tau} W(V_{\tau}^{x,\pi}) I_{\{\tau<\theta\}}\big).
$$

This can be considered as a form of the dynamic programming principle, but seemingly, it is not sufficient for our derivation of the HJB equation.

10 The Bellman function and the HJB equation

Theorem 10.1 *Assume that the Bellman function W is in $C(K)$. Then W is a viscosity solution of* (5.1).

Proof The claim follows from the following two lemmas. □

Lemma 10.2 *If W is in $C(\text{int}\, K)$, then $W \geq 0$ is a viscosity supersolution of* (5.1).

Proof Let $x \in \text{int}\, K$, and let $\phi \in C^1(K) \cap C^2(x)$ be a function such that $\phi(x) = W(x)$ and $W \geq \phi$ on K.

Fix an arbitrary point $m \in K$. Let $\varepsilon > 0$ be sufficiently small to guarantee that $x - \varepsilon m \in \mathcal{O}_r(x)$. The function W is increasing with respect to the partial ordering generated by K. Thus,

$$
\phi(x) = W(x) \geq W(x - \varepsilon m) \geq \phi(x - \varepsilon m).
$$

Taking the limit as $\varepsilon \to 0$ in the inequality $\varepsilon^{-1}(\phi(x - \varepsilon m) - \phi(x)) \leq 0$, we obtain that $-m\phi'(x) \leq 0$, and hence $\Sigma_G(\phi'(x)) \leq 0$.

Take now π with $B_t = 0$ and $c_t = c \in \mathcal{C}$ for all t. Let $\tau_r = \tau_r^{\pi} \leq \theta$ be the exit time of the process $V = V^{x,\pi}$ from the ball $\mathcal{O}_r(x)$; obviously, $\tau_r \leq \theta$. The properties of the test function and inequality (9.2) imply that

$$
\begin{aligned}
\phi(x) = W(x) &\geq E\big(J_{t\wedge\tau_r}^{\pi} + e^{-\beta(t\wedge\tau_r)} W(V_{t\wedge\tau_r}) I_{\{t\wedge\tau_r<\theta\}}\big) \\
&\geq E\big(J_{t\wedge\tau_r}^{\pi} + e^{-\beta(t\wedge\tau_r)} \phi(V_{t\wedge\tau_r}) I_{\{t\wedge\tau_r<\theta\}}\big).
\end{aligned}
$$

We get from here, using the Itô formula (8.1), that

$$
\begin{aligned}
0 &\geq E\left(\int_0^{t\wedge\tau_r} e^{-\beta s}U(c_s)\,ds + e^{-\beta(t\wedge\tau_r)}\phi(V_{t\wedge\tau_r})I_{\{t\wedge\tau_r<\theta\}}\right) - \phi(x) \\
&\geq EI_{\{t\wedge\tau_r<\theta\}}\int_0^{t\wedge\tau_r} e^{-\beta s}\big(\mathcal{L}_0\phi(V_s) - c\phi'(V_s) + U(c)\big)\,ds \\
&\geq \min_{y\in\bar{\mathcal{O}}_r(x)}\big(\mathcal{L}_0\phi(y) - c\phi'(y) + U(c)\big)EI_{\{t\wedge\tau_r<\theta\}}\left(\frac{1}{\beta}\big(1-e^{-\beta(t\wedge\tau_r)}\big)\right).
\end{aligned}
$$

Dividing the resulting inequality by t and taking successively the limits as t and r converge to zero, we infer that $\mathcal{L}_0\phi(x) - c\phi'(x) + U(c) \leq 0$. Maximizing over $c \in \mathcal{C}$ yields the bound $\mathcal{L}_0\phi(x) + U^*(\phi'(x)) \leq 0$, and therefore W is a supersolution of the HJB equation. □

Lemma 10.3 *If* (9.1) *holds, then* $W \geq 0$ *is a viscosity subsolution of* (5.1).

Proof Let $x \in \operatorname{int} K$, and let $\phi \in C^1(K)\cap C^2(x)$ be a function such that $\phi(x) = W(x)$ and $W \leq \phi$ on K. Suppose that the subsolution inequality for ϕ fails at x. Thus, there exists $\varepsilon > 0$ such that $\mathcal{L}\phi \leq -\varepsilon$ on some ball $\bar{\mathcal{O}}_r(x) \subset \operatorname{int} K$. By Lemma 8.5 (applied to the function ϕ) there are $t_0 > 0$ and $\eta > 0$ such that on the interval $(0, t_0]$, for any strategy $\pi \in \mathcal{A}_a^x$,

$$
E\big(J^\pi_{t\wedge\tau^\pi} + e^{-\beta\tau^\pi}\phi(V^{x,\pi}_{t\wedge\tau^\pi})I_{\{t\wedge\tau^\pi<\theta\}}\big) \leq \phi(x) - \eta t,
$$

where τ^π is the exit time of the process $V^{x,\pi}$ from the ball $\mathcal{O}_r(x)$. Fix an arbitrary $t \in (0, t_0]$. By the second claim of Lemma 9.1 there exists $\pi \in \mathcal{A}_a^x$ such that

$$
W(x) \leq E\big(J^\pi_{t\wedge\tau} + e^{-\beta\tau}W(V^{x,\pi}_{t\wedge\tau})I_{\{t\wedge\tau<\theta\}}\big) + \frac{1}{2}\eta t
$$

for every stopping time τ and, in particular, for τ^π.

Using the inequality $W \leq \phi$ and applying Lemma 8.5, we obtain from the above relations that $W(x) \leq \phi(x) - (1/2)\eta t$. This is a contradiction because at the point x, the values of W and ϕ are the same. □

11 Uniqueness theorem

Before formulating the uniqueness theorem, we recall the Ishii lemma.

Lemma 11.1 *Let v and $\tilde{v}$ be two continuous functions on an open subset $\mathcal{O} \subseteq \mathbf{R}^d$. Consider the function $\Delta(x, y) := v(x) - \tilde{v}(y) - \frac{1}{2}n|x-y|^2$ with $n > 0$. Suppose that Δ attains a local maximum at $(\widehat{x}, \widehat{y})$. Then there are symmetric matrices X and Y such that*

$$
\big(n(\widehat{x}-\widehat{y}), X\big) \in \bar{J}^+v(\widehat{x}), \qquad \big(n(\widehat{x}-\widehat{y}), Y\big) \in \bar{J}^-\tilde{v}(\widehat{y}),
$$

and

$$
\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3n\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{11.1}
$$

In this statement, I is the identity matrix, and $\bar{J}^+ v$ and $\bar{J}^- \tilde{v}$ are the values of the set-valued mappings whose graphs are the closures of the graphs of the set-valued mappings $J^+ v$ and $J^- \tilde{v}$, respectively. Inequality (11.1) means that for any vectors x and y from $\mathbf{R}^d$,

$$(Xx, x) - (Yy, y) \le 3n|x - y|^2.$$

Of course, if v and $\tilde{v}$ are smooth, then the claim follows directly from the necessary conditions of a local maximum (with $X = v''(\widehat{x})$, $Y = \tilde{v}''(\widehat{y})$, and the constant 1 instead of 3 in (11.1)).

Inequality (11.1) implies the bound

$$\operatorname{tr}\big(A(x)X - A(y)Y\big) \le 3n|A^{1/2}|^2|x - y|^2, \tag{11.2}$$

which will be used in the sequel (for the proof, see e.g. Sect. 4.2 in [23]).

The following concept plays a crucial role in the proof of the purely analytic result on the uniqueness of the viscosity solution, which we establish by the classical method of doubling variables using the Ishii lemma.

Definition 11.2 We say that a positive function $\ell \in C_1(K) \cap C^2(\operatorname{int} K)$ is a *Lyapunov function* if the following properties are satisfied:

1) $\ell'(x) \in \operatorname{int} K^*$ and $\mathcal{L}_0 \ell(x) \le 0$ for all $x \in \operatorname{int} K$.

2) $\ell(x) \to \infty$ as $|x| \to \infty$.

In other words, ℓ is a classical strict supersolution of the *truncated* equation (without the term U^*), continuous up to the boundary, and increasing to infinity at infinity.

Theorem 11.3 *Suppose that there exists a Lyapunov function ℓ and the Lévy measure Π is such that*

$$\Pi(\{z : \hat{x} + D_{\hat{x}} z \in \partial K\}) = 0, \quad \forall \hat{x} \in \operatorname{int} K.$$

Then the Dirichlet problem (5.1), (5.2) *has at most one viscosity solution in the class of continuous functions satisfying the growth condition*

$$W(x)/\ell(x) \to 0, \quad |x| \to \infty.$$

Proof Let W and $\tilde{W}$ be two viscosity solutions of (5.1) coinciding on the boundary ∂K. Suppose that $W(z) > \tilde{W}(z)$ for some $z \in K$. Take $\varepsilon > 0$ such that

$$W(z) - \tilde{W}(z) - 2\varepsilon \ell(z) > 0.$$

We introduce a family of continuous functions $\Delta_n : K \times K \to \mathbf{R}$ by putting

$$\Delta_n(x, y) := W(x) - \tilde{W}(y) - \frac{1}{2} n|x - y|^2 - \varepsilon\big(\ell(x) + \ell(y)\big), \quad n \ge 0.$$

Note that $\Delta_n(x,x)=\Delta_0(x,x)$ for all $x\in K$ and $\Delta_0(x,x)\le 0$ when $x\in\partial K$. From the assumption that the function ℓ has a higher growth rate than W, we deduce that $\Delta_n(x,y)\to-\infty$ as $|x|+|y|\to\infty$. It follows that the level sets $\{\Delta_n\ge a\}$ are compact and the function Δ_n attains its maximum on a compact subset of $K\times K$ that does not depend on n. That is, there exists $(x_n,y_n)\in K\times K$ such that

$$\Delta_n(x_n,y_n)=\bar\Delta_n:=\sup_{(x,y)\in K\times K}\Delta_n(x,y)\ge\bar\Delta:=\sup_{x\in K}\Delta_0(x,x)>0.$$

All (x_n,y_n) belong to the compact set $\{(x,y):\ \Delta_0(x,y)\ge 0\}$. It follows that the sequence $n|x_n-y_n|^2$ is bounded. We continue to argue (without introducing new notations) with a subsequence along which (x_n,y_n) converges to some limit $(\widehat{x},\widehat{x})$. Necessarily, $n|x_n-y_n|^2\to 0$ (otherwise, we should have $\Delta_0(\widehat{x},\widehat{x})>\bar\Delta$). It is easily seen that $\bar\Delta_n\to\Delta_0(\widehat{x},\widehat{x})=\bar\Delta$. Thus, $\widehat{x}$ is an interior point of K, and so are x_n and y_n for sufficiently large n.

By the Ishii lemma applied to the functions $v:=W-\varepsilon\ell$ and $\tilde v:=\tilde W+\varepsilon\ell$ at the point (x_n,y_n), there exist matrices X^n and Y^n satisfying (11.1) such that

$$\big(n(x_n-y_n),X^n\big)\in\bar J^+v(x_n),\qquad\big(n(x_n-y_n),Y^n\big)\in\bar J^-\tilde v(y_n).\tag{11.3}$$

To make ideas clearer, we suppose first that

$$\big(n(x_n-y_n),X^n\big)\in J^+v(x_n),\qquad\big(n(x_n-y_n),Y^n\big)\in J^-\tilde v(y_n).\tag{11.4}$$

Using the notations $p_n:=n(x_n-y_n)+\varepsilon\ell'(x_n)$, $q_n:=n(x_n-y_n)-\varepsilon\ell'(y_n)$ and putting $X_n:=X^n+\varepsilon\ell''(x_n)$, $Y_n:=Y^n-\varepsilon\ell''(y_n)$, we may rewrite the last relations in the equivalent form

$$(p_n,X_n)\in J^+W(x_n),\qquad(q_n,Y_n)\in J^-\tilde W(y_n).$$

Now because W is a viscosity subsolution, by Lemma 7.1 there exists a function $f_n\in C_1(K)\cap C^2(x_n)$ with $f_n'(x_n)=p_n$, $f_n''(x_n)=X_n$, $f_n(x_n)=W(x_n)$, and $W\le f_n\le W+1/n$ on K. Since $\tilde W$ is a viscosity supersolution, we conclude in the same way that there exists a function $\tilde f_n\in C_1(K)\cap C^2(y_n)$ such that $\tilde f_n'(y_n)=q_n$, $\tilde f_n''(y_n)=Y_n$, $\tilde f_n(y_n)=\tilde W(y_n)$, and $\tilde W-1/n\le\tilde f_n\le\tilde W$ on K. To deal with the nonlocal integral operator, we take f_n and $\tilde f_n$ having the structure given in (7.1) with an appropriate choice of the cylindrical cutoff functions. We discuss details of this choice later.

By the definitions of sub- and supersolutions we have that

$$F\big(X_n,p_n,\mathcal{I}(f_n,x_n),W(x_n),x_n\big)\ge 0\ge F\big(Y_n,q_n,\mathcal{I}(\tilde f_n,y_n),\tilde W(y_n),y_n\big).$$

The second inequality implies that $mq_n\le 0$ for each $m\in G=(-K)\cap\partial\mathcal{O}_1(0)$. But for the Lyapunov function, $\ell'(x)\in\operatorname{int}K^*$ when $x\in\operatorname{int}K$, and therefore

$$mp_n = mq_n + \varepsilon m\big(\ell'(x_n) + \ell'(y_n)\big) < 0.$$

Since G is compact, $\Sigma_G(p_n) < 0$. It follows that

$$F_0\big(X_n, p_n, \mathcal{I}(f_n, x_n), W(x_n), x_n\big) + U^*(p_n) \geq 0,$$
$$F_0\big(Y_n, q_n, \mathcal{I}(\tilde{f}_n, y_n), \tilde{W}(y_n), y_n\big) + U^*(q_n) \leq 0.$$

Recall that U^* is decreasing with respect to the partial ordering generated by $\mathcal{C}^*$ and hence also by K^*. Thus, $U^*(p_n) \leq U^*(q_n)$, and we obtain the inequality

$$b_n := F_0\big(X_n, p_n, \mathcal{I}(f_n, x_n), W(x_n), x_n\big) - F_0\big(Y_n, q_n, \mathcal{I}(\tilde{f}_n, y_n), \tilde{W}(y_n), y_n\big) \geq 0.$$

Clearly,

$$\begin{aligned} b_n = {} & \frac{1}{2}\sum_{i,j=1}^{d}(a^{ij}x_n^i x_n^j X_{ij}^n - a^{ij}y_n^i y_n^j Y_{ij}^n) + n\sum_{i=1}^{d}\mu^i(x_n^i - y_n^i)^2 \\ & - \frac{1}{2}\beta n|x_n - y_n|^2 - \beta\Delta_n(x_n, y_n) + \mathcal{I}(f_n - \varepsilon\ell, x_n) - \mathcal{I}(\tilde{f}_n + \varepsilon\ell, y_n) \\ & + \varepsilon\big(\mathcal{L}_0\ell(x_n) + \mathcal{L}_0\ell(y_n)\big). \end{aligned}$$

By (11.2) the first term on the right-hand side is dominated by a constant multiplied by $n|x_n - y_n|^2$; a similar bound for the second sum is obvious; the last term is negative according to the definition of a Lyapunov function. To complete the proof, it is sufficient to show that

$$\limsup_n \big(\mathcal{I}(f_n - \varepsilon\ell, x_n) - \mathcal{I}(\tilde{f}_n + \varepsilon\ell, y_n)\big) \leq 0. \tag{11.5}$$

Indeed, with this, we have that $\limsup b_n \leq -\beta\bar{\Delta} < 0$, i.e., a contradiction arising from the assumption $W(z) > \tilde{W}(z)$.

In general, we cannot guarantee that (11.5) holds for arbitrary test functions f_n and $\tilde{f}_n$. That is why we choose them in accordance with the expressions given by Lemma 7.1 with $\alpha = 1/n$, i.e.,

$$f_n(x_n + h) := f_n^0(x_n + h)\xi_{a_n,a_n'}^L(h) + W(x_n + h)\big(1 - \xi_{a_n,a_n'}^L(h)\big), \quad x_n + h \in K,$$
$$\tilde{f}_n(y_n + h) := \tilde{f}_n^0(y_n + h)\tilde{\xi}_{\tilde{a}_n,\tilde{a}_n'}^L(h) + \tilde{W}(y_n + h)\big(1 - \tilde{\xi}_{\tilde{a}_n,\tilde{a}_n'}^L(h)\big), \quad y_n + h \in K,$$

where L is the linear space $L := \operatorname{Im} D_{\widehat{x}} = (\operatorname{Ker} D_{\widehat{x}})^{\perp}$, and

$$\begin{aligned} f_n^0(x_n + h) &:= \Big(\big(W(x_n) + Q_{p_n,X_n}(h) + r_n(h)\big) \vee W_n(x_n + h)\Big) \\ &\qquad \wedge \big(W(x_n + h) + 1/n\big), \\ \tilde{f}_n^0(y_n + h) &:= \Big(\big(\tilde{W}(y_n) + Q_{q_n,Y_n}(h) - \tilde{r}_n(h)\big) \wedge \tilde{W}_n(y_n + h)\Big) \\ &\qquad \vee \big(\tilde{W}(y_n + h) - 1/n\big). \end{aligned}$$

Let $\delta := (1/2)R/(|\widehat{x}| + R)$, where $R = d(\widehat{x}, \partial K)$ is the distance of the point $\widehat{x}$ from the boundary ∂K. Then $I(z, x_n) = 1$ and $I(z, y_n) = 1$ for $z \in \mathcal{O}_\delta(0)$ when n is sufficiently large. Indeed, when $|x_n - \widehat{x}| \le R/2$ and $|y_n - \widehat{x}| \le R/2$, we have that

$$|x_n + D_{x_n} z - \widehat{x}| \le R/2 + |x_n||z| \le R/2 + (|\widehat{x}| + R/2)|z| < R$$

when $|z| \le \delta$, and a similar estimate holds for y_n.

We have

$$\mathcal{I}(f_n - \varepsilon\ell, x_n) - \mathcal{I}(\tilde{f}_n + \varepsilon\ell, y_n) = \int_{\{|z|\le\delta\}} H_n(z)\Pi(dz) + \int_{\{|z|>\delta\}} H_n(z)\Pi(dz),$$

where $H_n(z) := F_n(z) - \tilde{F}_n(z)$ with

$$\begin{aligned} F_n(z) &:= (f_n - \varepsilon\ell)(x_n + D_{x_n} z)I(z, x_n) - (W - \varepsilon\ell)(x_n) - (f_n' - \varepsilon\ell')(x_n)D_{x_n} z \\ &= (f_n - \varepsilon\ell)(x_n + D_{x_n} z)I(z, x_n) - (W - \varepsilon\ell)(x_n) - n(x_n - y_n)D_{x_n} z, \\ \tilde{F}_n(z) &:= (\tilde{f}_n + \varepsilon\ell)(y_n + D_{y_n} z)I(z, y_n) - (\tilde{W} + \varepsilon\ell)(y_n) - (\tilde{f}_n' + \varepsilon\ell')(y_n)D_{y_n} z \\ &= (\tilde{f}_n + \varepsilon\ell)(y_n + D_{y_n} z)I(z, y_n) - (\tilde{W} + \varepsilon\ell)(y_n) - n(x_n - y_n)D_{y_n} z. \end{aligned}$$

Define also the functions

$$\begin{aligned} G_n(z) &:= (W - \varepsilon\ell)(x_n + D_{x_n} z)I(z, x_n) - (W - \varepsilon\ell)(x_n) - n(x_n - y_n)D_{x_n} z, \\ \tilde{G}_n(z) &:= (\tilde{W} + \varepsilon\ell)(y_n + D_{y_n} z)I(z, y_n) - (\tilde{W} + \varepsilon\ell)(y_n) - n(x_n - y_n)D_{y_n} z. \end{aligned}$$

Put $x_n' := P_L x_n$, $y_n' := P_L y_n$. The restrictions to L of D_{x_n} and $D_{x_n'}$ and of D_{y_n} and $D_{y_n'}$, coincide. Considering only sufficiently large n, we may assume without loss of generality that $(1/2)\widehat{x}^i \le x_n^i,\ y_n^i \le \widehat{x}^i + 1$ for the nonzero coordinates of $\widehat{x}$, and therefore the norms of the restrictions of the diagonal operators $D_{x_n'}$, $D_{y_n'}$ to L and their inverses are bounded (even uniformly in n). Therefore, we may apply Lemma 7.2 if $\widehat{x}$ has no zero components or its extension given by the accompanying remark and argue further supposing that

$$\xi^L_{a_n, a_n'}(D_{x_n} z) = \xi^L_{a_n, a_n'}(D_{x_n'} z) = \xi^L_{\tilde{a}_n, \tilde{a}_n'}(D_{y_n'} z) = \xi^L_{\tilde{a}_n, \tilde{a}_n'}(D_{y_n} z), \quad \forall z \in \mathbf{R}^d.$$

According to our choice of δ, for $z \in \mathcal{O}_\delta(0)$, we have $I(z, x_n) = I(z, y_n) = 1$ for sufficiently large n, and as a consequence, the easily verified identity

$$G_n(z) - \tilde{G}_n(z) = \Delta_n(x_n + D_{x_n} z, y_n + D_{y_n} z) - \Delta_n(x_n, y_n) + \frac{1}{2} n|(D_{x_n} - D_{y_n})z|^2.$$

Recalling that the function $\Delta_n(x, y)$ attains its maximum at (x_n, y_n), we get from here the bound

$$G_n(z) - \tilde{G}_n(z) \le \frac{1}{2} n|x_n - y_n|^2|z|^2.$$

Thus, on the set $\{z\colon |D_{x_n'}z|\ge a_n'\}\cap\mathcal{O}_\delta(0)$, we have that

$$H_n(z)=G_n(z)-\tilde G_n(z)\le\frac{1}{2}n|x_n-y_n|^2|z|^2.$$

On the set $\{z\colon |D_{x_n'}z|\le a_n\}\cap\mathcal{O}_\delta(0)$,

$$\begin{aligned}F_n(z)&=\frac{1}{2}(X^nD_{x_n}z,D_{x_n}z)+r_n(D_{x_n}z)\\&\quad+\varepsilon\Big(\ell(x_n)+\ell'(x_n)D_{x_n}z+\frac{1}{2}\big(\ell''(x_n)D_{x_n}z,D_{x_n}z\big)-\ell(x_n+D_{x_n}z)\Big),\\\tilde F_n(z)&=\frac{1}{2}(Y^nD_{y_n}z,D_{y_n}z)+\tilde r_n(D_{y_n}z)\\&\quad+\varepsilon\Big(\ell(y_n)+\ell'(y_n)D_{y_n}z+\frac{1}{2}\big(\ell''(y_n)D_{y_n}z,D_{y_n}z\big)-\ell(y_n+D_{y_n}z)\Big).\end{aligned}$$

By the Ishii lemma,

$$(X^nD_{x_n}z,D_{x_n}z)-(Y^nD_{y_n}z,D_{y_n}z)\le 3n|D_{x_n}z-D_{y_n}z|^2\le 3n|x_n-y_n|^2|z|^2.$$

We take a_n small enough to ensure that

$$\int_{\{|D_{x_n'}z|\le a_n\}}\big(r_n(D_{x_n}z)+r_n(D_{y_n}z)\big)\Pi(dz)\le 1/n.$$

On the set $\{z\colon |D_{x_n'}z|\ge a_n\}\cap\mathcal{O}_\delta(0)$, the sequence of functions $|H_n|$ is bounded by a constant. We choose a_n' sufficiently close to a_n to guarantee that

$$\int_{\{a_n\le|D_{x_n'}z|\le a_n'\}}|H_n(z)|\Pi(dz)\le\frac{1}{n}.$$

The expressions in parentheses in the above formulae for $F_n(z)$ and $\tilde F_n(z)$ (residual terms in the Taylor formula for the smooth function ℓ) are bounded by a constant times $|z|^2$ and converge to the same limit as $n\to\infty$.

Summarizing the above facts, we conclude that

$$\limsup_n\int_{\{|z|\le\delta\}}H_n(z)\Pi(dz)=0.$$

Since the continuous functions W and ℓ are of sublinear growth and the sequences (x_n) and $(n(x_n-y_n))$ are converging (hence bounded), the absolute value of F_n is dominated by a function $c(1+|z|)$. The arguments for $-\tilde F_n(z)$ are similar. So, the function H_n is dominated by a function of sublinear growth.

Put $\mathcal{Z}:=\{z:\hat x+D_{\hat x}z\in\partial K\}$. By assumption, $\mathcal{Z}$ is Π-null. If $z\notin\mathcal{Z}$ and n is large enough, then we have $I(x_n,z)=I(y_n,z)=I(\hat x,z)$. It follows that

$$
\begin{aligned}
H_n(z) &= \big(G_n(z) - \tilde{G}_n(z)\big) I(\hat{x}, z) - n(x_n - y_n)(D_{x_n} - D_{y_n}) z \big(1 - I(\hat{x}, z)\big) \\
&\quad - (W - f_n)(x_n + D_{x_n} z) I(\hat{x}, z) + (\tilde{W} - \tilde{f}_n)(y_n + D_{y_n} z) I(\hat{x}, z) \\
&\quad + \big((\tilde{W} + \varepsilon\ell)(y_n) - (W - \varepsilon\ell)(x_n)\big)\big(1 - I(\hat{x}, z)\big) \\
&\le \big(\Delta_n(x_n + D_{x_n} z, y_n + D_{y_n} z) - \Delta_n(x_n, y_n)\big) I(\hat{x}, z) \\
&\quad + \left(\frac{1}{2} n |x_n - y_n + (D_{x_n} - D_{y_n}) z|^2 - \frac{1}{2} n |x_n - y_n|^2\right) I(\hat{x}, z) \\
&\quad - n(x_n - y_n)(D_{x_n} - D_{y_n}) z + 2/n \\
&\quad + \big((\tilde{W} + \varepsilon\ell)(y_n) - (W - \varepsilon\ell)(x_n)\big)\big(1 - I(\hat{x}, z)\big).
\end{aligned}
$$

The first term on the right-hand side of this inequality is negative since $\Delta_n(x, y)$ attains its maximum at (x_n, y_n). We only need to consider the second term when $I(\hat{x}, z) \neq 0$. In this case, we may combine it with the third term and conclude that the sum is less than $(1/2)n|x_n - y_n|^2 \to 0$. The last term converges to zero because of continuity. Thus,

$$
\limsup_n \int_{\{|z|>\delta\}} H_n(z) \Pi(dz) = 0.
$$

Our reasoning above is based on the assumption (11.4), whereas we know only (11.3). Fortunately, using the definitions of $\bar{J}^+ v(x_n)$ and $\bar{J}^- \tilde{v}(y_n)$, we can replace the objects x_n, y_n, X_n, Y_n by their approximations $\widehat{x}_n, \widehat{y}_n, \widehat{X}_n, \widehat{Y}_n$ approaching rapidly the initial ones, and for those, (11.4) holds. Repeating the arguments and controlling the approximation errors, we get the same contradiction. □

Remark 11.4 Note that the definition of a Lyapunov function does not depend on U, and hence the uniqueness holds for any utility function U for which U^* is decreasing with respect to the partial ordering induced by K^*. However, to apply the uniqueness theorem, we need to determine the growth rate of W and provide a Lyapunov function with a faster growth.

12 Existence of Lyapunov functions and classical supersolutions

In this section, we extend results of [22] on the existence of Lyapunov functions and classical supersolutions to the considered case of nonlocal equations.

12.1 Construction of Lyapunov functions

Let $u \in C(\mathbf{R}_+) \cap C^2(\mathbf{R}_+ \setminus \{0\})$ be an increasing strictly concave function with $u(0) = 0$ and $u(\infty) = \infty$. Introduce the function $R := -u'^2/(u''u)$. Assume that $\bar{R} := \sup_{z>0} R(z) < \infty$.

For $p \in K^*$, we define on K the positive function $f \in C_1(K) \cap C^2(\operatorname{int} K)$ by putting $f(x) = f_p(x) := u(px)$. If $y \in K$, then $yf'(x) = (py)u'(px) \ge 0$.

If $p \in \operatorname{int} K^*$, then for any $x, y \in K \setminus \{0\}$, we have the strict inequality $yf'(x) > 0$ implying that $f'(x) \in \operatorname{int} K^*$. Thus, for $p \in \operatorname{int} K^*$, the function f is a Lyapunov function, provided that the inequality $\mathcal{L}_0 f(x) \le 0$ is satisfied for $\mathcal{L}_0$ defined in Sect. 5. We show that under some mild conditions, this inequality holds for sufficiently large β.

Put

$$\kappa_p := \sup_{x \in \operatorname{int} K} \frac{u'(px)}{u(px)} |p||x|, \qquad \eta_p := \kappa_p \int_{\{|z| > \kappa_p^{-1}\}} |z| \Pi(dz).$$

Define also

$$\tilde{\eta}_p := \frac{1}{2} \sup_{x \in \operatorname{int} K} \frac{\langle \mu(x), p \rangle^2}{\langle A(x)p, p \rangle} I_{\{\langle A(x)p, p \rangle \neq 0\}}.$$

Recall that $A(x)$ is the matrix with $A^{ij}(x) = a^{ij} x^i x^j$ and the vector $\mu(x)$ has the components $\mu^i x^i$.

Note that if $\kappa_p < \infty$, then $\eta_p < \infty$ (we assume that $\int (|z|^2 \wedge |z|) \Pi(dz) < \infty$).

Example 12.1 Let $u(z) := z^\rho / \rho$, where $\rho \in (0, 1)$. Then $\bar{R} = R(z) = \rho/(1-\rho)$, and for $p \in \operatorname{int} K^*$,

$$\kappa_p \le \rho \sup_{x \in K \setminus \{0\}} \frac{|p||x|}{px} < \infty$$

(the strictly positive function $y \mapsto py$ on the compact $K \cap \{y : |y| = 1\}$ attains its minimum).

Proposition 12.2 *Let $p \in \operatorname{int} K$. If $\kappa_p < \infty$ and $\beta \ge \tilde{\eta}_p \bar{R} + \eta_p + \max_i |\mu_i| \kappa_p$, then f_p is a Lyapunov function.*

Proof Let $x \in \operatorname{int} K$. Recall that

$$\mathcal{I}(f, x) := \int \big(f(x + D_x z) I_{\operatorname{int} K}(x + D_x z) - f(x) - D_x z f'(x) \big) \Pi(dz).$$

If $x + D_x z \in \operatorname{int} K$, then the integrand defining $\mathcal{I}(f, x)$ has three nontrivial terms, and we have by the Taylor formula (in which $\vartheta \in [0, 1]$) that

$$f(x + D_x z) - f(x) - D_x z f'(x) = \frac{1}{2} u''(px + \vartheta p D_x z)(p D_x z)^2 \le 0.$$

If $x + D_x z \notin \operatorname{int} K$, then the integrand is reduced to two terms. Moreover, for $|z| \le 1/\kappa_p$, we have the bound

$$|D_x z p u'(px)| \le |z||p||x| u'(px) \le u(px),$$

implying that

$$-f(x) - D_x z f'(x) = -u(px) - D_x z p u'(px) \le 0.$$

We obtain from here, taking into account that $u(px) \geq 0$, the bound

$$\mathcal{I}(f, x) \leq u'(px)|p||x| \int_{\{|z|>1/\kappa_p\}} |z|\big(1 - I_{\text{int}\, K}(x + D_x z)\big)\Pi(dz) \leq \eta_p u(px).$$

Suppose that $\langle A(x)p, p\rangle \neq 0$. Isolating the full square, we obtain that

$$\begin{aligned}\mathcal{L}_0 f(x) &= \frac{1}{2}\left(\langle A(x)p, p\rangle u''(px) + 2\langle \mu(x), p\rangle u'(px) + \frac{\langle \mu(x), p\rangle^2}{\langle A(x)p, p\rangle}\frac{u'^2(px)}{u''(px)}\right)\\ &\quad + \frac{1}{2}\frac{\langle \mu(x), p\rangle^2}{\langle A(x)p, p\rangle} R(px)u(px) + \mathcal{I}(f, x) - \beta u(px)\\ &\leq \frac{1}{2}\frac{\langle \mu(x), p\rangle^2}{\langle A(x)p, p\rangle} R(px)u(px) + \eta_p u(px) - \beta u(px).\end{aligned}$$

It follows that $\mathcal{L}_0 f(x) \leq 0$ if $\beta \geq \tilde{\eta}_p \bar{R} + \eta_p$.

Of course, if $\langle A(x)p, p\rangle = 0$, then we cannot argue as before. In this case,

$$\mathcal{L}_0 f(x) \leq \langle \mu(x), p\rangle u'(px) + \eta_p u(px) - \beta u(px).$$

Taking into account that

$$\sup_{x\in \text{int}\, K} \frac{\langle \mu(x), p\rangle u'(px)}{u(px)} \leq \max_i |\mu_i| \sup_{x\in \text{int}\, K} \frac{u'(px)}{u(px)}|p||x| = \max_i |\mu_i|\kappa_p,$$

we get that $\mathcal{L}_0 f(x) \leq 0$ if

$$\beta \geq \eta_p + \max_i |\mu_i|\kappa_p.$$

Combining these two cases, we get the result. □

Remark 12.3 An inspection of the above arguments shows that we can get that f_p is a Lyapunov function for

$$\begin{aligned}\beta \geq \sup_{x\in \text{int}\, K}&\left(\frac{1}{2}\frac{\langle \mu(x), p\rangle^2}{\langle A(x)p, p\rangle} R(px) I_{\{\langle A(x)p,p\rangle\neq 0\}} + \frac{\langle \mu(x), p\rangle u'(px)}{u(px)} I_{\{\langle A(x)p,p\rangle = 0\}}\right)\\ &+ \eta_p.\end{aligned}$$

Of course, such a bound is less tractable than that given before.

12.2 Construction of classical supersolutions

Similar arguments are useful in the search of classical supersolutions for the equation associated to the operator $\mathcal{L}$. Since $\mathcal{L}f = \mathcal{L}_0 f + U^*(f')$, it is natural to choose u related to U. For the particular case where $\mathcal{C} = \mathbf{R}^d_+$ and $U(c) = u(e_1 c)$, with u satisfying the postulated properties (except, maybe, unboundedness), and assuming moreover that the inequality

$$u^*\big(au'(z)\big) \le g(a)u(z) \tag{12.1}$$

holds, we get, using the homogeneity of $\mathcal{L}_0$, the following result.

Proposition 12.4 *Let* $p \in \operatorname{int} K$. *Suppose that* (12.1) *holds for all* $a, z > 0$ *with* $g(a) = o(a)$ *as* $a \to \infty$. *If* $\kappa_p < \infty$ *and* $\beta > \tilde{\eta}_p \bar{R} + \eta_p + \max_i |\mu_i|\kappa_p$, *then there exists* a_0 *such that for every* $a \in (0, a_0]$, *the function* af_p *is a classical strict supersolution of* (5.1).

For the power utility function $u(z) = z^\gamma/\gamma$, $\gamma \in (0, 1)$, we have

$$u^*\big(au'(z)\big) = (1-\gamma)a^{\gamma/(\gamma-1)}u(z).$$

Therefore, inequality (12.1) holds with $g(a) = o(a)$, $a \to 0$.

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