Finance Stoch (2016) 20:51–81
DOI 10.1007/s00780-015-0284-9

Model-independent superhedging under portfolio constraints

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Received: 11 February 2014 / Accepted: 30 May 2015 / Published online: 11 November 2015

Abstract In a discrete-time market, we study model-independent superhedging where the semi-static superhedging portfolio consists of *three* parts: static positions in liquidly traded vanilla calls, static positions in other tradable, yet possibly less liquid, exotic options, and a dynamic trading strategy in risky assets under certain constraints. By considering the limit order book of each tradable exotic option and employing the Monge–Kantorovich theory of optimal transport we establish a general superhedging duality, which admits a natural connection to convex risk measures. With the aid of this duality, we derive a model-independent version of the fundamental theorem of asset pricing. The notion "finite optimal arbitrage profit", weaker than no-arbitrage, is also introduced. It is worth noting that our method covers a large class of delta and gamma constraints.

Keywords Model-independent pricing · Robust superhedging · Limit order book · Fundamental theorem of asset pricing · Portfolio constraints · Monge–Kantorovich optimal transport

Mathematics Subject Classification (2010) 91G20 · 91G80

JEL Classification C61 · G13

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1 Introduction

To avoid model mis-specification, one may choose to consider only "must-be-true" implications from the market. The standard approach, suggested by Dupire [16], leverages on market prices of liquidly traded vanilla call options: one does not manage to specify a proper physical measure, but considers as plausible pricing measures all measures that are consistent with market prices of vanilla calls. These measures then provide model-independent bounds for prices of illiquid exotic options and motivate the practically useful semistatic hedging, which involves static holdings in vanilla calls and dynamic trading in risky assets. Pioneered by Hobson [25], this thread of research has drawn substantial attention; see e.g. [8, 6, 27, 26, 30, 11, 12, 5, 14]. In particular, Beiglböck, Henry-Labordère and Penkner [5] establish a general duality of model-independent superhedging in a discrete-time setting where market prices of vanilla calls with maturities *at or before* the terminal time $T > 0$ are all considered.

In reality, what we can rely on goes beyond vanilla calls. In the markets of commodities, for instance, Asian options and calendar spread options are largely traded, with their market or broker quotes easily accessible. In the New York Stock Exchange and the Chicago Board Options Exchange, standardized digital and barrier options have been introduced, mostly for equity indexes and exchange traded funds. What we can take advantage of, as a result, includes market prices of not only vanilla calls, but also certain tradable exotic options.

In this paper, we take up the model-independent framework in [5] and intend to establish a general superhedging duality under the consideration of additional tradable options besides vanilla calls and portfolio constraints on trading strategies in risky assets. More specifically, our semistatic superhedging portfolio consists of three parts: (1) static positions in liquidly traded vanilla calls, as in the literature of robust hedging; (2) static positions in additional tradable, yet possibly less liquid, exotic options; and (3) a dynamic trading strategy in risky assets under certain constraints.

While tradable, the additional exotic options may be very different from vanilla calls in terms of liquidity. Their limit order books are usually very shallow and admit large bid-ask spreads, compared to those of the underlying assets and the associated vanilla calls. It follows that we need to take into account the whole limit order book, instead of one single market quote, of each of the additional options in order to make trading possible. We formulate the limit order books in Sect. 2.1 and consider the corresponding nonconstant unit price functions. On the other hand, portfolio constraints on trading strategies in risky assets have been widely studied under the model-specific case; see [13] and [28] for deterministic convex constraints and [19, 9, 32], and [33], among others, for random and other more general constraints. Our goal is to place portfolio constraints in the current model-independent context and to investigate their implications for semistatic superhedging.

We particularly consider a general class of constraints that enjoys adaptive convexity and a continuous approximation property (Definition 2.7). This already covers a large collection of delta constraints, including adaptively convex constraints; see Remark 2.9. For the simpler case where no additional tradable option exists, we derive a superhedging duality in Proposition 3.10 by using the theory of optimal transport.

This in particular generalizes the duality in [5] to the multidimensional case with portfolio constraints; see Remarks 3.12 and 3.13. Then, on the strength of the convexity of the nonconstant unit price functions, we are able to extend the above duality to the general case where additional tradable options exist; see Theorem 3.14. Note that the result of Acciaio et al. [1] also applies to model-independent superhedging in the presence of tradable exotic options, while assuming implicitly that each option can be traded liquidly. Theorem 3.14 can therefore be seen as a generalization of [1] that deals with different levels of liquidity; see Remark 3.16.

The second part of the paper investigates the relation between the superhedging duality and the fundamental theorem of asset pricing (FTAP). It is well known in the classical model-specific case that the FTAP yields the superhedging duality. This relation has been carried over to the model-independent case by [1], where an appropriate notion of model-independent arbitrage was introduced. In the same spirit as in [1], we define model-independent arbitrage in Definition 4.1 under the current setting with additional tradable options and portfolio constraints. With the aid of the superhedging duality in Theorem 3.14, we are able to derive a model-independent FTAP; see Theorem 4.8. Whereas the theorem itself does not distinguish between arbitrage due to risky assets and arbitrage due to additional tradable options, Lemmas 4.4 and 4.6 can be used to differentiate one from the other. It is also worth noting that we derive the FTAP as a consequence of the superhedging duality. This was first observed in [15], as opposed to the standard argument of deriving the superhedging duality as a consequence of the FTAP, used in both the model-specific case and [1].

With the FTAP (Theorem 4.8) at hand, we observe from Theorem 3.14 and Proposition 3.17 that the problems of superhedging and risk-measuring can be well defined even when there is model-independent arbitrage to some extent. We relate this to optimal arbitrage under the formulation of [10] and show that superhedging and risk-measuring are well defined as long as "the optimal arbitrage profit is finite", a notion weaker than no-arbitrage; see Proposition 4.15. We also compare Theorem 4.8 with [21, Theorem 9.9], the classical model-specific FTAP under portfolio constraints and observe that a closedness condition in [21] is no longer needed under the current setting. An example given in Sect. 4.1 indicates that availability of vanilla calls obviates the need of the closedness condition.

Finally, we extend our scope to gamma constraints. Whereas gamma constraints do not satisfy the adaptive convexity in Definition 2.7(ii), they admit an additional boundedness property. Taking advantage of this, we are able to modify previous results to obtain the corresponding superhedging duality and FTAP in Propositions 6.3 and 6.6.

This paper is organized as follows. In Sect. 2, we describe the setup of our studies. In Sect. 3, we establish the superhedging duality and investigate its connection to other dualities in the literature and to convex risk measures. In Sect. 4, we define model-independent arbitrage under portfolio constraints with additional tradable options and derive the associated FTAP. The notion "finite optimal arbitrage profit", weaker than no-arbitrage, is also introduced. Section 5 presents concrete examples of portfolio constraints and the effect of additional tradable options. Section 6 deals with constraints that do not enjoy adaptive convexity but admit some boundedness property. Appendix contains a counterexample that emphasizes the necessity of the continuous approximation property required in Definition 2.7.

2 The setup

We consider a discrete-time market with finite horizon $T \in \mathbb{N}$. There are d risky assets $S = (S_t)_{t=0}^T = ((S_t^1, \dots, S_t^d))_{t=0}^T$ with given initial price $S_0 = x_0 \in \mathbb{R}_+^d$. There is also a risk-free asset $B = (B_t)_{t=0}^T$ which is normalized to $B_t \equiv 1$. Specifically, we take S as the canonical process $S_t(x_1, x_2, \dots, x_T) = x_t$ on the path space $\Omega := (\mathbb{R}_+^d)^T$ and denote by $\mathbb{F} = (\mathcal{F}_t)_{t=0}^T$ the natural filtration generated by S.

2.1 Vanilla calls and other tradable options

At time 0, we assume that the vanilla call option with payoff $(S_t^n - K)^+$ can be liquidly traded, at some price $C_n(t, K)$ given in the market, for all $n = 1, \dots, d$, $t = 1, \dots, T$, and $K \geq 0$. The collection of pricing measures consistent with market prices of vanilla calls is therefore

$$\Pi := \{\mathbb{Q} \in \mathcal{P}(\Omega) : \mathbb{E}^{\mathbb{Q}}[(S_t^n - K)^+] = C_n(t, K), \\ \forall n = 1, \dots, d, t = 1, \dots, T, \text{and } K \geq 0\}, \tag{2.1}$$

where $\mathcal{P}(\Omega)$ denotes the collection of all probability measures defined on Ω.

In view of [24, Proposition 2.1], for each $n = 1, \dots, d$ and $t = 1, \dots, T$, as long as $K \mapsto C_n(t, K)$ is nonnegative, convex and satisfies $\lim_{K \downarrow 0} \partial_K C_n(t, K) \geq -1$ and $\lim_{K \to \infty} C_n(t, K) = 0$, the relation $\mathbb{E}^{\mathbb{Q}}[(S_t^n - K)^+] = C_n(t, K)$ for all $K \geq 0$ already prescribes the distribution of S_t^n on $\mathbb{R}_+$, which will be denoted by μ_t^n. Thus, by setting $\mathbb{Q}_t^n$ as the law of S_t^n under $\mathbb{Q}$ we have

$$\Pi = \{\mathbb{Q} \in \mathcal{P}(\Omega) : \mathbb{Q}_t^n = \mu_t^n, \ \forall\, n = 1, \dots, d \text{ and } t = 1, \dots, T\}. \tag{2.2}$$

Remark 2.1 Given $\mathbb{Q} \in \Pi$, note that $\mathbb{E}^{\mathbb{Q}}[S_t^n] < \infty$ for all $n = 1, \dots, d$ and $t = 1, \dots, T$ (which can be seen by taking $K = 0$ in (2.1)).

Remark 2.2 In view of (2.2), Π is nonempty, convex, and weakly compact. This is a direct consequence of [29, Proposition 1.2] once we view $\Omega = (\mathbb{R}_+^d)^T$ as the product of $d \times T$ copies of $\mathbb{R}_+$.

Remark 2.3 We do not assume that $t \mapsto C_n(t, K)$ is increasing for any fixed n and K. This condition, normally required in the literature (see e.g. [5, p. 481]), implies that the set of martingale measures

$$\mathcal{M} := \{\mathbb{Q} \in \Pi : S = (S_t)_{t=0}^T \text{ is a martingale under } \mathbb{Q}\} \tag{2.3}$$

is nonempty, which underlies the superhedging duality in [5]. In contrast, the superhedging duality in Proposition 3.10 below hinges on a different collection $\mathcal{Q}_{\mathcal{S}}$, which contains $\mathcal{M}$ (see Definition 3.4). Since it is possible that our duality holds while $\mathcal{M} = \emptyset$, imposing "$t \mapsto C_n(t, K)$ is increasing" is not necessary.

Besides vanilla calls, there are other options tradable, while less liquid, at time 0. Let I be a (possibly uncountable) index set. For each $i \in I$, suppose that $\psi_i : \Omega \to \mathbb{R}$

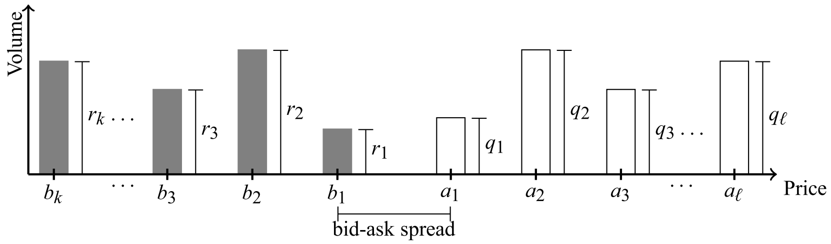

Fig. 1 A limit order book of ψ_i

is the payoff function of an option tradable at time 0. Let $\eta \in \mathbb{R}$ be the number of units of ψ_i being traded at time 0, with $\eta \geq 0$ denoting a purchase order and $\eta < 0$ a selling order. Let $c_i(\eta) \in \bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ denote the total cost of trading η units of ψ_i. **Throughout this paper, we impose the following condition**:

$$\text{for all } i \in I, \text{ the map } \eta \mapsto c_i(\eta) \text{ is convex with } c_i(0) = 0. \tag{2.4}$$

We can then define the unit price $p_i(\eta)$ for trading η units of ψ_i by

$$p_i(\eta) := \frac{c_i(\eta)}{\eta} \quad \text{for } \eta \in \mathbb{R} \setminus \{0\} \quad \text{and} \quad p_i(0) := c_i'(0+).$$

Remark 2.4 Condition (2.4) is motivated by the typical structure of a limit order book for a nonnegative option, as demonstrated in Fig. 1. That is, the option ψ_i can be purchased only at prices $0 \leq a_1 \leq a_2 \leq \cdots \leq a_\ell$ with numbers of units $q_1, q_2, \ldots, q_\ell > 0$, respectively, and sold only at prices $b_1 \geq b_2 \geq \cdots \geq b_k \geq 0$ with numbers of units $r_1, r_2, \ldots, r_k > 0$, respectively, where $b_1 \leq a_1$ reflects the bid-ask spread, and ℓ and k belong to $\mathbb{N} \cup \{+\infty\}$. The possibility of $\ell, k = \infty$ allows infinitely many buy/sell prices in the order book.

We keep track of $Q_m := \sum_{j=1}^m q_j$, the total number of units that can be bought at or below the price a_m for $m = 1, \ldots, \ell$. Similarly, $R_m := \sum_{j=1}^m r_j$ is the total number of units that can be sold at or above the price b_m for all $m = 1, \ldots, k$. The total cost $c_i(\eta)$ of trading η units of ψ_i is then given by

$$c_i(\eta) = \begin{cases} \sum_{m=1}^{u-1} a_m q_m + a_u(\eta - Q_{u-1}) \geq 0 & \text{if } \eta \in (Q_{u-1}, Q_u], \\ & u = 1, \ldots, \ell + 1, \\ 0 & \text{if } \eta = 0, \\ -\sum_{m=1}^{u-1} b_m r_m + b_u(\eta + R_{u-1}) \leq 0 & \text{if } \eta \in [-R_u, -R_{u-1}), \\ & u = 0, \ldots, k + 1, \end{cases}$$

where we set $Q_0 = R_0 = 0$, $Q_{\ell+1} = R_{k+1} = \infty$, $a_{\ell+1} = \infty$, $b_{k+1} = 0$, and use the convention that $0 \cdot \infty = 0$ and $\sum_{m=1}^0 = 0$. As shown in Fig. 2, $\eta \mapsto c_i(\eta)$ satisfies (2.4). In particular, c_i is linear on $\mathbb{R}$ if and only if $b_1 = a_1$ and $q_1 = r_1 = \infty$; this means that ψ_i can be traded liquidly at the price $a_1 = b_1$, which is the slope of c_i.

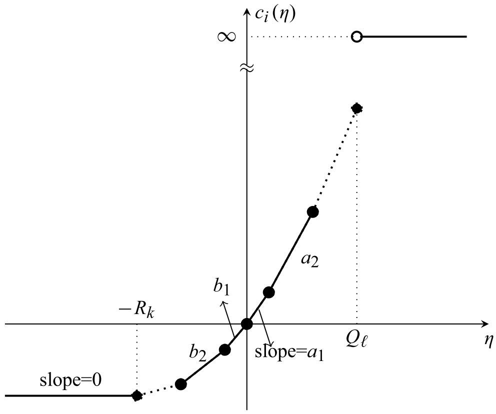

Fig. 2 Graph of $\eta \mapsto c_i(\eta)$: $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ placed on each segment indicate the slope of each segment and match the prices in the limit order book for volumes $q_1, q_2, \ldots$ and $r_1, r_2, \ldots$, respectively

Remark 2.5 Condition (2.4) captures two important features of the prices of ψ_i: (1) the bid-ask spread, formulated as $[c_i'(0-), c_i'(0+)]$; (2) the nonlinearity, i.e., the unit price $\eta \mapsto p_i(\eta)$ is nonconstant. This setting in particular allows a zero spread when $c_i'(0-) = c_i'(0+)$, whereas at the same time the limit order book may induce nonlinear pricing. This happens to a highly liquid asset for which the bid-ask spread is negligible, but transaction costs become significant for large trading volumes. Also, c_i is linear if and only if ψ_i can be traded liquidly, with whatever units, at one single price p_i (which is the slope of c_i).

Note that [3] has recently considered bid-ask spreads, but not nonlinear pricing, for hedging options under model uncertainty. In a model-independent setting, whereas a nonlinear pricing operator for hedging options has been used in [15], the nonlinearity does not reflect the nonconstant unit price of an option in its limit order book (see [15, (2.3)]); instead, it captures a market where the price of a portfolio of options may be lower than the sum of the respective prices of the options (see the second line in the proof of [15, Lemma 2.4]).

2.2 Constrained trading strategies

Definition 2.6 (Trading strategies) We call $\Delta = (\Delta_t)_{t=0}^{T-1}$ a *trading strategy* if $\Delta_0 \in \mathbb{R}^d$ is a constant and $\Delta_t : (\mathbb{R}_+^d)^t \to \mathbb{R}^d$ for $t = 1, \ldots, T-1$ is Borel-measurable. Moreover, the stochastic integral of Δ with respect to $x = (x_1, \ldots, x_T) \in (\mathbb{R}_+^d)^T$ is expressed as

$$(\Delta \cdot x)_t := \sum_{i=0}^{t-1} \Delta_i(x_1, \ldots, x_i) \cdot (x_{i+1} - x_i) \quad \text{for } t = 1, \ldots, T,$$

where in the right-hand side, $\Delta_i = (\Delta_i^1, \ldots, \Delta_i^d)$, $x_i = (x_i^1, \ldots, x_i^d)$, and $\cdot$ denotes the inner product in $\mathbb{R}^d$. We denote by $\mathcal{H}$ the collection of all trading strategies. Also, for any collection $\mathcal{J} \subseteq \mathcal{H}$, we introduce the subcollections

$$\begin{aligned} \mathcal{J}^\infty &:= \{\Delta \in \mathcal{J} : \ \Delta_t : (\mathbb{R}^d_+)^t \to \mathbb{R}^d \text{ is bounded } \forall\, t = 1, \dots, T-1\}, \\ \mathcal{J}^\infty_c &:= \{\Delta \in \mathcal{J}^\infty : \ \Delta_t : (\mathbb{R}^d_+)^t \to \mathbb{R}^d \text{ is continuous } \forall\, t = 1, \dots, T-1\}. \end{aligned} \tag{2.5}$$

In this paper, we require the trading strategies to lie in a subcollection $\mathcal{S}$ of $\mathcal{H}$, prescribed as below.

Definition 2.7 (Adaptively convex portfolio constraint) $\mathcal{S}$ is a set of trading strategies such that

(i) $0 \in \mathcal{S}$;
(ii) for any $\Delta, \Delta' \in \mathcal{S}$ and adapted process h with $h_t \in [0,1]$ for all $t = 0, \dots, T-1$,

$$\big(h_t \Delta_t + (1-h_t)\Delta'_t\big)_{t=0}^{T-1} \in \mathcal{S};$$

(iii) for any $\Delta \in \mathcal{S}^\infty$, $\mathbb{Q} \in \varPi$, and $\varepsilon > 0$, there exist a closed set $D_\varepsilon \subseteq (\mathbb{R}^d_+)^T$ and $\Delta^\varepsilon \in \mathcal{S}^\infty_c$ such that

$$\mathbb{Q}[D_\varepsilon] > 1 - \varepsilon \quad \text{and} \quad \Delta_t = \Delta^\varepsilon_t \text{ on } D_\varepsilon \text{ for } t = 0, \dots, T-1.$$

Remark 2.8 In Definition 2.7, (i) and (ii) are motivated by [21, Sect. 9.1], whereas (iii) is a technical assumption, which allows us to perform a continuous approximation in Lemma 3.3. This approximation in particular enables us to establish the superhedging duality in Proposition 3.10. In fact, if we only have conditions (i) and (ii), then the duality in Proposition 3.10 may fail in general, as demonstrated in Appendix.

As explained below, Definition 2.7(iii) covers a large class of convex constraints.

Remark 2.9 (Adaptively convex constraints) Let $(K_t)_{t=0}^{T-1}$ be an adapted set-valued process such that for each t, K_t maps $(x_1, \dots, x_t) \in (\mathbb{R}^d_+)^t$ to a closed convex set $K_t(x_1, \dots, x_t) \subseteq \mathbb{R}^d$ containing 0. Consider the collection of trading strategies

$$\begin{aligned} \mathcal{S} := \{\Delta \in \mathcal{H} : \forall\, t = 0, \dots, T-1,\ &\Delta_t(x_1, \dots, x_t) \in K_t(x_1, \dots, x_t) \\ &\forall\, (x_1, \dots, x_t) \in (\mathbb{R}^d_+)^t\}, \end{aligned}$$

which satisfies (i) and (ii) in Definition 2.7 trivially. To obtain (iii), we assume additionally that for each $t \geq 1$, the set-valued map $K_t : (\mathbb{R}^d_+)^t \to 2^{\mathbb{R}^d}$ is lower semicontinuous in the sense that

$$\text{for any open } V \text{ in } \mathbb{R}^d, \text{ the set } \{x \in (\mathbb{R}^d_+)^t : K_t(x) \cap V \neq \emptyset\} \text{ is open in } (\mathbb{R}^d_+)^t. \tag{2.6}$$

This is equivalent to the following condition:

$$\begin{aligned} &\forall\, y_0 \in K_t(x_0) \text{ and } (x_n) \subset (\mathbb{R}^d_+)^t \text{ such that } x_n \to x_0, \\ &\exists\, y_n \in K_t(x_n) \text{ such that } y_n \to y_0; \end{aligned} \tag{2.7}$$

see e.g. [2, Definition 1.4.2], the remark below it, and [22, Sect. 2.5].

To check (iii), let us fix $\Delta \in \mathcal{S}^\infty$, $\mathbb{Q} \in \Pi$, and $\varepsilon > 0$. For each $t = 1, \dots, T-1$, by Lusin's theorem, there exists a closed set $D_{\varepsilon,t} \subset (\mathbb{R}^d_+)^t$ such that $\Delta_t|_{D_{\varepsilon,t}} : D_{\varepsilon,t} \to K_t$ is continuous and $\mathbb{Q}[D_{\varepsilon,t} \times (\mathbb{R}^d_+)^{T-t}] > 1 - \frac{\varepsilon}{T-1}$. Under (2.6), we can apply the theory of continuous selection (see e.g. [31, Theorem 3.2″]) to find a bounded continuous function $\Delta^\varepsilon_t : (\mathbb{R}^d_+)^t \to K_t$ such that $\Delta^\varepsilon_t = \Delta_t$ on $D_{\varepsilon,t}$. Now set $\Delta^\varepsilon_0 = \Delta_0$ and define $D_\varepsilon := \bigcap_t D_{\varepsilon,t} \times (\mathbb{R}^d_+)^{T-t}$, which is by definition closed in $\mathbb{R}^T_+$. We see that $\Delta^\varepsilon \in \mathcal{S}^\infty_c$, $\mathbb{Q}[D_\varepsilon] > 1 - \varepsilon$ and $\Delta_t = \Delta^\varepsilon_t$ on D_ε for all $t = 0, \dots, T-1$. This already satisfies Definition 2.7(iii).

Note that for the special case where $(K_t)_{t=1}^T$ is deterministic, K_t is a fixed subset of $\mathbb{R}^d$ for each t and thus (2.7) is trivially satisfied. See Examples 5.2 and 5.3 for a concrete illustration of deterministic and adaptively convex constraints.

3 The superhedging duality

For a path-dependent exotic option with payoff function $\Phi : (\mathbb{R}^d_+)^T \to \mathbb{R}$, we intend to construct a semistatic superhedging portfolio, which consists of three parts: static positions in vanilla calls, static positions in $(\psi_i)_{i\in I}$, and a dynamic trading strategy $\Delta \in \mathcal{S}$. More precisely, consider

$$\mathcal{C} := \left\{\varphi : \mathbb{R} \mapsto \mathbb{R} :\ \varphi(x) = a + \sum_{i=1}^n b_i (x - K_i)^+ \right.$$
$$\left. \text{with } a \in \mathbb{R},\ n \in \mathbb{N},\ b_i \in \mathbb{R},\ K_i > 0 \right\},$$
$$\mathcal{R}^I := \{\eta = (\eta_i)_{i\in I} \in \mathbb{R} : \eta_i \neq 0 \text{ for finitely many } i\}.$$

We intend to find $u = \{u^n_t \in \mathcal{C} : n = 1, \dots, d,\ t = 1, \dots, T\}$, $\eta \in \mathcal{R}^I$, and $\Delta \in \mathcal{S}$ such that

$$\begin{aligned}\Psi_{u,\eta,\Delta}(x) &:= \sum_{t=1}^T \sum_{n=1}^d u^n_t(x^n_t) + \sum_{i\in I} \big(\eta_i \psi_i - c_i(\eta_i)\big) + (\Delta \cdot x)_T \\ &\geq \Phi(x) \quad \forall x \in (\mathbb{R}^d_+)^T,\end{aligned} \tag{3.1}$$

where we set $0 \cdot \infty = 0$. In the definition of $\mathcal{C}$, we specifically require K_i to be strictly positive. This is because $K_i = 0$ corresponds to trading the risky assets, which is already incorporated into $\Delta \in \mathcal{S}$ and should not be treated as part of the static positions. By setting $\mathcal{U}$ as the collection of all $u = \{u^n_t \in \mathcal{C} : n = 1, \dots, d,\ t = 1, \dots, T\}$ we define the superhedging price of Φ by

$$D(\Phi) := \inf \left\{ \sum_{t=1}^T \sum_{n=1}^d \int_{\mathbb{R}_+} u^n_t d\mu^n_t : u \in \mathcal{U},\ \eta \in \mathcal{R}^I \text{ and } \Delta \in \mathcal{S} \right.$$
$$\left. \text{satisfy } \Psi_{u,\eta,\Delta} \geq \Phi,\ \forall x \in (\mathbb{R}^d_+)^T \right\}. \tag{3.2}$$

By introducing $\mathcal{U}_0 := \{u \in \mathcal{U} : \sum_{t=1}^T \sum_{n=1}^d \int_{\mathbb{R}_+} u_t^n d\mu_t^n = 0\}$ we may express (3.2) as

$$D(\Phi) = \inf\big\{a \in \mathbb{R} : u \in \mathcal{U}_0, \eta \in \mathcal{R}^I \text{and } \Delta \in \mathcal{S} \\ \text{satisfy } a + \Psi_{u,\eta,\Delta} \ge \Phi \ \forall x \in (\mathbb{R}_+^d)^T\big\}.$$

Our goal in this section is to derive a superhedging duality associated with $D(\Phi)$.

3.1 Upper variation process

In order to deal with the portfolio constraint $\mathcal{S}$, we introduce an auxiliary process for each $\mathbb{Q} \in \Pi$, as suggested in [21, Sect. 9.2].

Definition 3.1 Given $\mathbb{Q} \in \Pi$, the *upper variation process* $A^{\mathbb{Q}}$ for $\mathcal{S}$ is defined by $A_0^{\mathbb{Q}} := 0$ and

$$A_{t+1}^{\mathbb{Q}} - A_t^{\mathbb{Q}} := \operatorname*{ess\,sup}^{\mathbb{Q}}_{\Delta \in \mathcal{S}} \Delta_t \cdot (\mathbb{E}^{\mathbb{Q}}[S_{t+1}|\mathcal{F}_t] - S_t), \quad t = 0, \dots, T-1.$$

First, note that the conditional expectation in the definition of $A^{\mathbb{Q}}$ is well defined thanks to Remark 2.1. Next, since Definition 2.7(i)–(ii) implies that $1_{\{|\Delta| \le n\}}\Delta \in \mathcal{S}$ whenever $\Delta \in \mathcal{S}$, we may replace $\mathcal{S}$ by $\mathcal{S}^\infty$ in the above definition. It follows that

$$A_{t+1}^{\mathbb{Q}} - A_t^{\mathbb{Q}} = \operatorname*{ess\,sup}^{\mathbb{Q}}_{\Delta \in \mathcal{S}^\infty} \mathbb{E}^{\mathbb{Q}}[\Delta_t \cdot (S_{t+1} - S_t)|\mathcal{F}_t], \quad t = 0, \dots, T-1. \tag{3.3}$$

Therefore,

$$A_t^{\mathbb{Q}} = \sum_{i=1}^t \operatorname*{ess\,sup}^{\mathbb{Q}}_{\Delta \in \mathcal{S}^\infty} \mathbb{E}^{\mathbb{Q}}[\Delta_{i-1} \cdot (S_i - S_{i-1})|\mathcal{F}_{i-1}], \quad t = 1, \dots, T. \tag{3.4}$$

Lemma 3.2 *For any $\mathbb{Q} \in \Pi$ and $t = 1, \dots, T$, we have*

$$\mathbb{E}^{\mathbb{Q}}\left[\operatorname*{ess\,sup}^{\mathbb{Q}}_{\Delta \in \mathcal{S}^\infty} \mathbb{E}^{\mathbb{Q}}[\Delta_t \cdot (S_{t+1} - S_t)|\mathcal{F}_t]\right] = \sup_{\Delta \in \mathcal{S}^\infty} \mathbb{E}^{\mathbb{Q}}[\Delta_t \cdot (S_{t+1} - S_t)].$$

This in particular implies that

$$\mathbb{E}^{\mathbb{Q}}\big[A_t^{\mathbb{Q}}\big] = \sup_{\Delta \in \mathcal{S}^\infty} \mathbb{E}^{\mathbb{Q}}[(\Delta \bullet S)_t]. \tag{3.5}$$

Proof First, note that $\{\mathbb{E}^{\mathbb{Q}}[\Delta_t \cdot (S_{t+1} - S_t)|\mathcal{F}_t] : \Delta \in \mathcal{S}^\infty\}$ is directed upward. Indeed, given $\Delta, \Delta' \in \mathcal{S}^\infty$, define $\tilde{\Delta}_s := \Delta_s 1_{\{s \ne t\}} + (\Delta_s 1_A + \Delta_s' 1_{A^c}) 1_{\{s=t\}}$, where

$$A := \big\{\mathbb{E}^{\mathbb{Q}}[\Delta_t \cdot (S_{t+1} - S_t)|\mathcal{F}_t] \ge \mathbb{E}^{\mathbb{Q}}[\Delta_t' \cdot (S_{t+1} - S_t)|\mathcal{F}_t]\big\} \in \mathcal{F}_t.$$

Then $\tilde{\Delta} \in \mathcal{S}^\infty$ by Definition 2.7(ii), and

$$E^{\mathbb{Q}}[\tilde{\Delta}_t \cdot (S_{t+1} - S_t)|\mathcal{F}_t] = \max\big(\mathbb{E}^{\mathbb{Q}}[\Delta_t \cdot (S_{t+1} - S_t)|\mathcal{F}_t], \mathbb{E}^{\mathbb{Q}}[\Delta_t' \cdot (S_{t+1} - S_t)|\mathcal{F}_t]\big).$$

We can therefore apply [21, Theorem A.33] and get

$$\mathbb{E}^{\mathbb{Q}}\left[\operatorname*{ess\,sup}^{\mathbb{Q}}_{\Delta\in\mathcal{S}^\infty}\mathbb{E}^{\mathbb{Q}}[\Delta_t\cdot(S_{t+1}-S_t)|\mathcal{F}_t]\right]=\sup_{\Delta\in\mathcal{S}^\infty}\mathbb{E}^{\mathbb{Q}}[\Delta_t\cdot(S_{t+1}-S_t)].$$

Now, in view of (3.4), we have

$$\mathbb{E}^{\mathbb{Q}}\big[A_t^{\mathbb{Q}}\big]=\sum_{i=1}^{t}\sup_{\Delta\in\mathcal{S}^\infty}\mathbb{E}^{\mathbb{Q}}[\Delta_{i-1}\cdot(S_i-S_{i-1})]=\sup_{\Delta\in\mathcal{S}^\infty}\mathbb{E}^{\mathbb{Q}}[(\Delta\bullet S)_t],$$

where the last equality follows from Definition 2.7(ii). □

On the strength of Definition 2.7(iii), we can actually replace $\mathcal{S}^\infty$ by $\mathcal{S}^\infty_c$ in (3.5).

Lemma 3.3 *For each $t=1,\dots,T$,*

$$\mathbb{E}^{\mathbb{Q}}\big[A_t^{\mathbb{Q}}\big]=\sup_{\Delta\in\mathcal{S}^\infty_c}\mathbb{E}^{\mathbb{Q}}[(\Delta\bullet S)_t].$$

Proof In view of (3.5), it suffices to show that for each fixed $\Delta\in\mathcal{S}^\infty$, there exists $(\Delta^\varepsilon)_\varepsilon\subset\mathcal{S}^\infty_c$ such that $\mathbb{E}^{\mathbb{Q}}[(\Delta\bullet S)_t]=\lim_{\varepsilon\to0}\mathbb{E}^{\mathbb{Q}}[(\Delta^\varepsilon\bullet S)_t]$. Take $M>0$ such that $|\Delta_t|\le M$ for all $t=1,\dots,T-1$. By Definition 2.7(iii), for any $\varepsilon>0$, there exist D_ε closed in $\mathbb{R}^T_+$ and $\Delta^\varepsilon\in\mathcal{S}^\infty_c$ such that $\mathbb{Q}[D_\varepsilon]>1-\varepsilon$, $\Delta^\varepsilon=\Delta$ on D_ε, and $|\Delta^\varepsilon_t|\le M$ for all $t=1,\dots,T-1$. It follows that

$$\begin{aligned}\big|\mathbb{E}^{\mathbb{Q}}[(\Delta\bullet S)_t]-\mathbb{E}^{\mathbb{Q}}[(\Delta^\varepsilon\bullet S)_t]\big|&\le\mathbb{E}^{\mathbb{Q}}\left[\big|\big((\Delta-\Delta^\varepsilon)\bullet S\big)_t\big|1_{D^c_\varepsilon}\right]\\&\le\mathbb{E}^{\mathbb{Q}}\left[2M\sum_{i=0}^{t-1}|S_{i+1}-S_i|1_{D^c_\varepsilon}\right].\end{aligned}$$

Thanks to Remark 2.1, the random variable $2M\sum_{i=0}^{T-1}|S_{i+1}-S_i|$ is $\mathbb{Q}$-integrable. We can then conclude from the above inequality that $\mathbb{E}^{\mathbb{Q}}[(\Delta^\varepsilon\bullet S)_t]\to\mathbb{E}^{\mathbb{Q}}[(\Delta\bullet S)_t]$. □

Definition 3.4 Let $\mathcal{Q}_{\mathcal{S}}$ be the collection of $\mathbb{Q}\in\Pi$ such that $\mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}]<\infty$.

Remark 3.5 If the strategies in $\mathcal{S}$ are uniformly bounded, i.e., $\exists\ c>0$ with $|\Delta|\le c$ for all $\Delta\in\mathcal{S}$, then we deduce from (3.5) and Remark 2.1 that $\mathcal{Q}_{\mathcal{S}}=\Pi$.

Lemma 3.6 *Given any $\Delta\in\mathcal{S}$, the process $(\Delta\bullet S)_t-A_t^{\mathbb{Q}}$ is a local $\mathbb{Q}$-supermartingale for all $\mathbb{Q}\in\mathcal{Q}_{\mathcal{S}}$.*

Proof This result follows from the argument in [21, Proposition 9.18]. We present the proof here for completeness. Consider the stopping time

$$\tau_n:=\inf\{t\ge0:|\Delta_t|>n\text{ or }\mathbb{E}^{\mathbb{Q}}[|S_{t+1}-S_t||\mathcal{F}_t]>n\}\wedge T,$$

where the conditional expectation is well defined thanks to Remark 2.1. Given a process V, let us denote by V^n the stopped process $V^{\tau_n} = V_{\cdot\wedge\tau^n}$. Observe that

$$|(\Delta \cdot S)^n_{t+1} - (\Delta \cdot S)^n_t| \le 1_{\{\tau_n \ge t+1\}} |\Delta_t|\ |S_{t+1} - S_t|.$$

Thanks again to Remark 2.1, this implies that $(\Delta \cdot S)^n_t$ is $\mathbb{Q}$-integrable. Moreover,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[(\Delta \cdot S)^n_{t+1} - (\Delta \cdot S)^n_t|\mathcal{F}_t] &= 1_{\{\tau_n \ge t+1\}} \Delta_t \cdot (\mathbb{E}^{\mathbb{Q}}[S_{t+1}|\mathcal{F}_t] - S_t)\\ &\le (A^{\mathbb{Q}})^n_{t+1} - (A^{\mathbb{Q}})^n_t,\end{aligned}$$

where the inequality follows from (3.3). Since $\mathbb{E}^{\mathbb{Q}}[A^{\mathbb{Q}}_T] < \infty$, the above inequality shows that $(\Delta \cdot S)^n_t - (A^{\mathbb{Q}})^n_t$ is a $\mathbb{Q}$-supermartingale. □

With some integrability at the terminal time T, the local supermartingale in the above result becomes a true supermartingale.

Lemma 3.7 *Fix $\Delta \in \mathcal{S}$ and $\mathbb{Q} \in \mathcal{Q}_{\mathcal{S}}$. If there exists a $\mathbb{Q}$-integrable random variable φ such that $(\Delta \cdot S)_T \ge \varphi$ $\mathbb{Q}$-a.s., then $(\Delta \cdot S)_t - A^{\mathbb{Q}}_t \ge \mathbb{E}^{\mathbb{Q}}[\varphi - A^{\mathbb{Q}}_T|\mathcal{F}_t]$ $\mathbb{Q}$-a.s. for all $t = 1, \ldots, T$. This in particular implies that $(\Delta \cdot S)_t - A^{\mathbb{Q}}_t$ is a true $\mathbb{Q}$-supermartingale.*

Proof Using the same notation as in the proof of Lemma 3.6, we know that there exists a sequence (τ_n) of stopping times such that $\tau^n \uparrow \infty$ $\mathbb{Q}$-a.s. and the stopped process $(\Delta \cdot S)^n - (A^{\mathbb{Q}})^n$ is a $\mathbb{Q}$-supermartingale for each $n \in \mathbb{N}$. We prove this lemma by induction. Given any $t = 2, \ldots, T$ such that $(\Delta \cdot S)_t - A^{\mathbb{Q}}_t \ge \mathbb{E}^{\mathbb{Q}}[\varphi - A^{\mathbb{Q}}_T|\mathcal{F}_t]$, we obtain from the supermartingale property that

$$\begin{aligned}0 &\ge 1_{\{t-1<\tau_n\}} \mathbb{E}^{\mathbb{Q}}\big[\big((\Delta \cdot S)^n_t - (A^{\mathbb{Q}})^n_t\big) - \big((\Delta \cdot S)^n_{t-1} - (A^{\mathbb{Q}})^n_{t-1}\big)\big|\mathcal{F}_{t-1}\big]\\ &= \mathbb{E}^{\mathbb{Q}}\Big[1_{\{t-1<\tau_n\}}\Big(\big((\Delta \cdot S)_t - A^{\mathbb{Q}}_t\big) - \big((\Delta \cdot S)_{t-1} - A^{\mathbb{Q}}_{t-1}\big)\Big)\Big|\mathcal{F}_{t-1}\Big]\\ &\ge \mathbb{E}^{\mathbb{Q}}\Big[1_{\{t-1<\tau_n\}}\Big(\mathbb{E}^{\mathbb{Q}}[\varphi - A^{\mathbb{Q}}_T|\mathcal{F}_t] - \big((\Delta \cdot S)_{t-1} - A^{\mathbb{Q}}_{t-1}\big)\Big)\Big|\mathcal{F}_{t-1}\Big]\\ &= 1_{\{t-1<\tau_n\}} \mathbb{E}^{\mathbb{Q}}[\varphi - A^{\mathbb{Q}}_T|\mathcal{F}_{t-1}] - 1_{\{t-1<\tau_n\}}\big((\Delta \cdot S)_{t-1} - A^{\mathbb{Q}}_{t-1}\big).\end{aligned}$$

Sending $n \to \infty$, we conclude that $(\Delta \cdot S)_{t-1} - A^{\mathbb{Q}}_{t-1} \ge \mathbb{E}^{\mathbb{Q}}[\varphi - A^{\mathbb{Q}}_T|\mathcal{F}_{t-1}]$ $\mathbb{Q}$-a.s. Now by Lemma 3.6, $(\Delta \cdot S) - A^{\mathbb{Q}}$ is a local $\mathbb{Q}$-supermartingale bounded from below by a martingale and thus a true $\mathbb{Q}$-supermartingale (see e.g. [21, Proposition 9.6]). □

3.2 Derivation of the superhedging duality

In view of the static holdings of $(\psi_i)_{i\in I}$ in (3.1), we introduce

$$\mathcal{E}^{\mathbb{Q}}_I := \sup_{\eta \in \mathcal{R}^I} \sum_{i\in I} \big(\eta_i \mathbb{E}^{\mathbb{Q}}[\psi_i] - c_i(\eta_i)\big) \ge 0 \quad \text{for } \mathbb{Q} \in \Pi. \tag{3.6}$$

Set $F(I) := \{J \subseteq I : J \text{ is a finite set}\}$. We observe that

$$\mathcal{E}_I^{\mathbb{Q}} = \sup_{J\in F(I)} \sup_{\eta\in\mathbb{R}^{|J|}} \sum_{i\in J} \left(\eta_i \mathbb{E}^{\mathbb{Q}}[\psi_i] - c_i(\eta_i)\right)$$
$$= \sup_{J\in F(I)} \sum_{i\in J} \sup_{\eta\in\mathbb{R}} \left(\eta \mathbb{E}^{\mathbb{Q}}[\psi_i] - c_i(\eta)\right). \tag{3.7}$$

Consider the collection of measures

$$\mathcal{Q}_{\mathcal{S},I} := \{\mathbb{Q} \in \mathcal{Q}_{\mathcal{S}} : \mathcal{E}_I^{\mathbb{Q}} < \infty\}. \tag{3.8}$$

Remark 3.8 Fix $\mathbb{Q} \in \Pi$. For any $i \in I$, suppose that the following two conditions hold.

(i) $c_i(\eta) = \infty$ for some $\eta > 0$ or $\mathbb{E}^{\mathbb{Q}}[\psi_i] < c_i'(\infty)$;
(ii) $c_i(\eta) = \infty$ for some $\eta < 0$ or $\mathbb{E}^{\mathbb{Q}}[\psi_i] > c_i'(-\infty)$.

By the convexity of $\eta \mapsto c_i(\eta)$ we have $\sup_{\eta\in\mathbb{R}}(\eta\mathbb{E}^{\mathbb{Q}}[\psi_i] - c_i(\eta)) < \infty$. Thus, in view of (3.7), if I is a finite set and (i)–(ii) are satisfied for all $i \in I$, then $\mathcal{E}_I^{\mathbb{Q}} < \infty$.

We now work on deriving a duality between $D(\Phi)$ defined in (3.2) and

$$P(\Phi) := \sup_{\mathbb{Q}\in\mathcal{Q}_{\mathcal{S},I}} \left(\mathbb{E}^{\mathbb{Q}}[\Phi - A_T^{\mathbb{Q}}] - \mathcal{E}_I^{\mathbb{Q}}\right). \tag{3.9}$$

The following minimax result, taken from [34, Corollary 2], will be useful.

Lemma 3.9 *Let X be a compact convex subset of a topological vector space, Y a convex subset of a vector space, and $f : X \times Y \to \mathbb{R}$ a function satisfying*

(i) *for each $x \in X$, the map $y \mapsto f(x, y)$ is convex on Y;*
(ii) *for each $y \in Y$, the map $x \mapsto f(x, y)$ is upper semicontinuous and concave on X.*

Then $\inf_{y\in Y} \sup_{x\in X} f(x, y) = \sup_{x\in X} \inf_{y\in Y} f(x, y)$.

Let us first derive a superhedging duality for the case where $I = \emptyset$, i.e., no option is tradable at time 0 except vanilla calls. The pathwise relation in (3.1) reduces to

$$\Psi_{u,\Delta}(x) := \sum_{t=1}^{T} \sum_{n=1}^{d} u_t^n(x_t^n) + (\Delta \cdot x)_T \geq \Phi(x) \quad \forall x \in (\mathbb{R}_+^d)^T.$$

By the convention that the sum over an empty set is 0, $D(\Phi)$ in (3.2) becomes

$$D_\emptyset(\Phi) := \inf\left\{ \sum_{t=1}^{T} \sum_{n=1}^{d} \int_{\mathbb{R}_+} u_t^n d\mu_t^n : u \in \mathcal{U} \text{ and } \Delta \in \mathcal{S} \text{ such that} \right.$$
$$\left. \Psi_{u,\Delta}(x) \geq \Phi(x)\ \forall x \in (\mathbb{R}_+^d)^T \right\}.$$

Also, since $I=\emptyset$ implies that $F(I)=\{\emptyset\}$, we deduce from (3.7) that $\mathcal{E}_I^{\mathbb{Q}}=0$ as it is a summation over an empty set. It follows that $P(\Phi)$ in (3.9) reduces to

$$P_\emptyset(\Phi) := \sup_{\mathbb{Q}\in\mathcal{Q}_\mathcal{S}} \mathbb{E}^{\mathbb{Q}}[\Phi - A_T^{\mathbb{Q}}]. \tag{3.10}$$

Proposition 3.10 *Let $I=\emptyset$. Suppose that $\Phi:(\mathbb{R}_+^d)^T\to\mathbb{R}$ is measurable and there exists $K>0$ such that*

$$\Phi(x_1,\dots,x_T)\le K\left(1+\sum_{t=1}^T\sum_{n=1}^d x_t^n\right) \quad \textit{for all } x\in(\mathbb{R}_+^d)^T. \tag{3.11}$$

(i) *We have $P_\emptyset(\Phi)\le D_\emptyset(\Phi)$.*
(ii) *If Φ is upper semicontinuous, then $P_\emptyset(\Phi)=D_\emptyset(\Phi)$.*
(iii) *If Φ is upper semicontinuous and $\mathcal{Q}_\mathcal{S}\neq\emptyset$, then there exists $\mathbb{Q}^*\in\mathcal{Q}_\mathcal{S}$ such that $P_\emptyset(\Phi)=\mathbb{E}^{\mathbb{Q}^*}[\Phi-A_T^{\mathbb{Q}^*}]$.*

Proof First, by Remark 2.1, (3.11), and Definition 3.4, $P_\emptyset$ is indeed well defined.

(i) Take $u\in\mathcal{U}$ and $\Delta\in\mathcal{S}$ such that $\Psi_{u,\Delta}\ge\Phi$. For any $\mathbb{Q}\in\mathcal{Q}_\mathcal{S}$, note that

$$(\Delta\cdot S)_T\ge\Phi(x)-\sum_{t=1}^T\sum_{n=1}^d u_t^n(x_t^n).$$

If $\mathbb{E}^{\mathbb{Q}}[\Phi^-]<\infty$, then $\Phi(x)-\sum_{t=1}^T\sum_{i=1}^d u_t^i(x_t)$ is $\mathbb{Q}$-integrable thanks to (3.11). We then conclude from Lemma 3.7 that $(\Delta\cdot S)_t-A_t^{\mathbb{Q}}$ is a true $\mathbb{Q}$-supermartingale. Hence,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[\Phi-A_T^{\mathbb{Q}}]&\le\mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1}^T\sum_{n=1}^d u_t^n(S_t^n)+(\Delta\cdot S)_T-A_T^{\mathbb{Q}}\right]\\&\le\sum_{t=1}^T\sum_{n=1}^d\int_{\mathbb{R}_+}u_t^n\,d\mu_t^n.\end{aligned}\tag{3.12}$$

If $\mathbb{E}^{\mathbb{Q}}[\Phi^-]=\infty$, then (3.12) trivially holds. By taking the supremum over $\mathbb{Q}\in\mathcal{Q}_\mathcal{S}$ and using the arbitrariness of u we obtain from (3.12) the desired inequality.

(ii) We use an argument similar to [5, (3.1)–(3.4)]. First, observe that

$$\begin{aligned}D_\emptyset(\Phi)&\le\inf\left\{\sum_{t=1}^T\sum_{n=1}^d\int_{\mathbb{R}_+}u_t^n\,d\mu_t^i : u\in\mathcal{U}\text{ and }\Delta\in\mathcal{S}_c^\infty\text{such that }\Psi_{u,\Delta}(x)\ge\Phi(x)\right\}\\&=\inf_{\Delta\in\mathcal{S}_c^\infty}\inf\left\{\sum_{t=1}^T\sum_{n=1}^d\int_{\mathbb{R}_+}u_t^n\,d\mu_t^n : u\in\mathcal{U}\text{ such that}\right.\\&\qquad\qquad\left.\sum_{t=1}^T\sum_{n=1}^d u_t^n(x_t^n)\ge\Phi(x)-(\Delta\cdot x)_T\right\}\\&=\inf_{\Delta\in\mathcal{S}_c^\infty}\sup_{\mathbb{Q}\in\Pi}\mathbb{E}^{\mathbb{Q}}[\Phi(x)-(\Delta\cdot x)_T].\end{aligned}\tag{3.13}$$

Here, (3.13) follows from the theory of optimal transport (see e.g. [5, Proposition 2.1]), which requires the upper semicontinuity of Φ. Now we intend to apply Lemma 3.9 to (3.13) with $X = \Pi$, $Y = \mathcal{S}_c^\infty$, and $f(\mathbb{Q}, \Delta) = \mathbb{E}^{\mathbb{Q}}[\Phi(x) - (\Delta \cdot x)_T]$. The only condition in Lemma 3.9 that is not obvious is the upper semicontinuity of $\mathbb{Q} \mapsto f(\mathbb{Q}, \Delta)$. For each $\Delta \in \mathcal{S}_c^\infty$, thanks to (3.11), the upper semicontinuous function $x \mapsto \Phi(x) - (\Delta \cdot x)_T$ is bounded from above by the continuous function

$$\ell(x) := K\left(1 + \sum_{t=1}^{T}\sum_{n=1}^{d} x_t^n\right) + |\Delta|_\infty \sum_{n=1}^{d}\left(x_0^n + 2(x_1^n + \cdots + x_{T-1}^n) + x_T^n\right), \quad (3.14)$$

where $|\Delta|_\infty := |\Delta_0| \vee \max\{\sup_{z\in(\mathbb{R}_+^d)^t} |\Delta_t(z)| : t = 1, \ldots, T-1\} < \infty$. Take any sequence $(\mathbb{Q}_n)_{n\in\mathbb{N}}$ in Π that converges weakly to some $\mathbb{Q}^* \in \Pi$. Observing that $\mathbb{Q} \mapsto \mathbb{E}^{\mathbb{Q}}[\ell]$ is a constant function on Π, we conclude from [35, Lemma 4.3] that

$$\limsup_{n\to\infty} \mathbb{E}^{\mathbb{Q}_n}[\Phi(x) - (\Delta \cdot x)_T] \le \mathbb{E}^{\mathbb{Q}^*}[\Phi(x) - (\Delta \cdot x)_T],$$

which shows the upper semicontinuity of $\mathbb{Q} \mapsto f(\mathbb{Q}, \Delta)$. Now, applying Lemma 3.9 to (3.13) yields

$$\begin{aligned} D_\emptyset(\Phi) &\le \sup_{\mathbb{Q}\in\Pi} \inf_{\Delta\in\mathcal{S}_c^\infty} \mathbb{E}^{\mathbb{Q}}[\Phi(x) - (\Delta \cdot x)_T] = \sup_{\mathbb{Q}\in\Pi}\left(\mathbb{E}^{\mathbb{Q}}[\Phi] - \sup_{\Delta\in\mathcal{S}_c^\infty} \mathbb{E}^{\mathbb{Q}}[(\Delta \cdot S)_T]\right) \\ &= \sup_{\mathbb{Q}\in\Pi}\left(\mathbb{E}^{\mathbb{Q}}[\Phi] - \mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}]\right) = \sup_{\mathbb{Q}\in\mathcal{Q}_\mathcal{S}}\left(\mathbb{E}^{\mathbb{Q}}[\Phi] - \mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}]\right) = P_\emptyset(\Phi), \end{aligned}$$

where the second line follows from Lemma 3.3.

(iii) In view of Definition 3.4, we can write $P_\emptyset(\Phi) = \sup_{\mathbb{Q}\in\Pi} \mathbb{E}^{\mathbb{Q}}[\Phi - A_T^{\mathbb{Q}}]$ by replacing $\mathcal{Q}_\mathcal{S}$ by Π in (3.10). Since Π is compact under the topology of weak convergence (Remark 2.2), it suffices to show that $\mathbb{Q} \mapsto f(\mathbb{Q}) := \mathbb{E}^{\mathbb{Q}}[\Phi - A_T^{\mathbb{Q}}]$ is upper semicontinuous. Since the argument in part (ii) already implies that $\mathbb{Q} \mapsto \mathbb{E}^{\mathbb{Q}}[\Phi]$ is upper semicontinuous, it remains to show that $\mathbb{Q} \mapsto g(\mathbb{Q}) := \mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}]$ is lower semicontinuous. Similarly to (3.14), for each $\Delta \in \mathcal{S}_c^\infty$, we have $|(\Delta \cdot x)_T| \le h(x)$ with h defined by $h(x) := |\Delta|_\infty \sum_{n=1}^d (x_0^n + 2(x_1^n + \cdots + x_{T-1}^n) + x_T^n)$. For any sequence $(\mathbb{Q}_n)_{n\in\mathbb{N}}$ in Π that converges weakly to some $\mathbb{Q}^* \in \Pi$, applying [35, Lemma 4.3] to the functions $(\Delta \cdot x)_T$ and $-(\Delta \cdot x)_T$ gives

$$\liminf_{n\to\infty} \mathbb{E}^{\mathbb{Q}_n}[(\Delta \cdot x)_T] \ge \mathbb{E}^{\mathbb{Q}^*}[(\Delta \cdot x)_T] \ge \limsup_{n\to\infty} \mathbb{E}^{\mathbb{Q}_n}[(\Delta \cdot x)_T].$$

It follows that $\mathbb{Q} \mapsto g_\Delta(\mathbb{Q}) := \mathbb{E}^{\mathbb{Q}}[(\Delta \cdot S)_T]$ is continuous. Thanks to Lemma 3.3, we have that $g(\mathbb{Q}) = \sup_{\Delta\in\mathcal{S}_c^\infty} g_\Delta(\mathbb{Q})$ is lower semicontinuous as a supremum of continuous functions. □

Remark 3.11 The condition $\mathcal{Q}_\mathcal{S} \ne \emptyset$ is not needed for Proposition 3.10(i) and (ii). Indeed, if $\mathcal{Q}_\mathcal{S} = \emptyset$, then $P(\Phi) = -\infty$, and thus part (i) trivially holds; also, the arguments in part (ii) hold as long as $\Pi \ne \emptyset$, which is guaranteed by Remark 2.2.

Remark 3.12 Proposition 3.10 extends [5, Theorem 1.1] to the case with portfolio constraints. To see this, consider the no-constraint case, i.e., $\mathcal{S} = \mathcal{H}$. Observe that $\mathcal{S} = \mathcal{H}$ implies $\mathcal{Q}_{\mathcal{S}} = \mathcal{M}$ with $\mathcal{M}$ defined as in (2.3). Whereas $\mathcal{M} \subseteq \mathcal{Q}_{\mathcal{S}}$ is obvious, the other inclusion follows from Definition 3.1. Indeed, given $\mathbb{Q} \in \mathcal{Q}_{\mathcal{S}} \setminus \mathcal{M}$, there must exist $t \in \{0, \ldots, T-1\}$ such that $\mathbb{E}^{\mathbb{Q}}[S_{t+1}|\mathcal{F}_t] \neq S_t$. Since $\mathcal{S} = \mathcal{H}$, we have $A^{\mathbb{Q}}_{t+1} - A^{\mathbb{Q}}_t = \infty$, contradicting $\mathbb{Q} \in \mathcal{Q}_{\mathcal{S}}$. The duality in Proposition 3.10 reduces to

$$D_\emptyset(\Phi) = P_\emptyset(\Phi) = \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\Phi],$$

which recovers [5, Theorem 1.1].

Remark 3.13 Proposition 3.10 also extends [5, Theorem 1.1] to the case with multidimensional S. Since [5, Theorem 1.1] relies on the one-dimensional Monge–Kantorovich duality, which works on the product of T copies of $\mathbb{R}_+$ (i.e., [5, Proposition 2.1]), one may expect to prove Proposition 3.10 via the multidimensional Monge–Kantorovich duality, which works on the product of T copies of $\mathbb{R}^d_+$. Whereas such a duality does exist (e.g. [29, Theorem 2.14]), applying it requires the knowledge of the joint distribution of $(S^1_t, \ldots, S^d_t)$ for each $t = 1, \ldots, T$. This is not practically feasible since vanilla calls only specify the distribution of S^n_t for each n and t.

As a result, in Proposition 3.10, we still rely on the one-dimensional result [5, Proposition 2.1]. By treating $\Omega = (\mathbb{R}^d_+)^T$ as the product of $(d \times T)$ copies of $\mathbb{R}_+$ (as in Remark 2.2), [5, Proposition 2.1] is indeed applicable since the distributions μ^n_t of S^n_t for all $n = 1, \ldots, d$ and $t = 1, \ldots, T$ are known. Note that it was first mentioned in [23, Theorem 2.1] that [5, Theorem 1.1] can be generalized to higher dimensions.

By the convexity of c_i Proposition 3.10 extends to the general case where $I \neq \emptyset$.

Theorem 3.14 *Suppose that ψ_i is continuous and $|\psi_i|$ satisfies (3.11) for all $i \in I$. Then for any upper semicontinuous function $\Phi : (\mathbb{R}^d_+)^T \to \mathbb{R}$ satisfying (3.11), we have $D(\Phi) = P(\Phi)$ with D and P defined as in (3.2) and (3.9). Moreover, if $\mathcal{Q}_{\mathcal{S},I} \neq \emptyset$, then the supremum in (3.9) is attained at some $\mathbb{Q}^* \in \mathcal{Q}_{\mathcal{S},I}$.*

Proof Observe from (3.2) and Proposition 3.10 that

$$\begin{aligned} D(\Phi) &= \inf_{\eta \in \mathcal{R}^I} D_\emptyset\Big(\Phi - \sum_{i \in I}\big(\eta_i \psi_i - c_i(\eta_i)\big)\Big) \\ &= \inf_{\eta \in \mathcal{R}^I} \sup_{\mathbb{Q} \in \Pi} \mathbb{E}^{\mathbb{Q}}\Big[\Phi - \sum_{i \in I}\big(\eta_i \psi_i - c_i(\eta_i)\big) - A^{\mathbb{Q}}_T\Big]. \end{aligned} \tag{3.15}$$

Consider the function $f(\mathbb{Q}, \eta) := \mathbb{E}^{\mathbb{Q}}[\Phi - \sum_{i \in I}(\eta_i \psi_i - c_i(\eta_i)) - A^{\mathbb{Q}}_T]$ for $\mathbb{Q} \in \Pi$ and $\eta \in \mathcal{R}^I$. By the upper semicontinuity of Φ, the continuity of ψ_i, and (3.11) we may argue as in Proposition 3.10(i) and (ii) that f is upper semicontinuous in $\mathbb{Q} \in \Pi$. Moreover, by the convexity of $\eta \mapsto c_i(\eta)$ for all $i \in I$, f is convex in $\eta \in \mathcal{R}^I$. Thus, we may apply Lemma 3.9 to (3.15) and get

$$D(\Phi) = \sup_{\mathbb{Q}\in\Pi} \inf_{\eta\in\mathcal{R}^I} \mathbb{E}^{\mathbb{Q}}\Big[\Phi - \sum_{i\in I}\big(\eta_i\psi_i - c_i(\eta_i)\big) - A_T^{\mathbb{Q}}\Big]$$
$$= \sup_{\mathbb{Q}\in\mathcal{Q}_{\mathcal{S},I}} \big(\mathbb{E}^{\mathbb{Q}}[\Phi - A_T^{\mathbb{Q}}] - \mathcal{E}_I^{\mathbb{Q}}\big) = P(\Phi).$$

In view of the argument in Proposition 3.10(iii) and the continuity of ψ_i, we obtain that $\mathbb{Q} \mapsto \mathbb{E}^{\mathbb{Q}}[\Phi - A_T^{\mathbb{Q}}] - \mathcal{E}_I^{\mathbb{Q}}$ is upper semicontinuous on the compact set Π. Thus, the supremum in (3.9) is attained if $\mathcal{Q}_{\mathcal{S},I} \neq \emptyset$. □

3.3 Connection to the model-free duality in [1]

Consider the case where every option ψ_i can actually be liquidly traded at time 0, just as vanilla calls. That is, for each $i \in I$, c_i is linear (see Remark 2.5), and we let $p_i \in \mathbb{R}$ be the slope of c_i. By (3.6), $\mathcal{E}_I^{\mathbb{Q}}$ equals 0 iff $\mathbb{E}^{\mathbb{Q}}[\psi_i] = p_i$ for all $i \in I$ and ∞ otherwise. It follows from (3.8) that

$$\mathcal{Q}_{\mathcal{S},I} = \mathcal{Q}_{\mathcal{S},(p_i)_{i\in I}} := \{\mathbb{Q} \in \mathcal{Q}_{\mathcal{S}} : \mathbb{E}^{\mathbb{Q}}[\psi_i] = p_i \ \forall i \in I\}. \tag{3.16}$$

Recall $\mathcal{M}$ defined in (2.3). Let us also consider

$$\mathcal{M}_I := \{\mathbb{Q} \in \mathcal{M} : c_i'(0-) \le \mathbb{E}^{\mathbb{Q}}[\psi_i] \le c_i'(0+) \ \forall i \in I\}.$$

Under the current setting, this becomes

$$\mathcal{M}_I = \mathcal{M}_{(p_i)_{i\in I}} := \{\mathbb{Q} \in \mathcal{M} : \mathbb{E}^{\mathbb{Q}}[\psi_i] = p_i \ \forall i \in I\}.$$

Corollary 3.15 *For each $i \in I$, suppose that ψ_i is continuous, $|\psi_i|$ satisfies* (3.11), *and ψ_i can be traded liquidly at the price $p_i \in \mathbb{R}$. Let $\Phi : (\mathbb{R}_+^d)^T \to \mathbb{R}$ be upper semicontinuous and satisfy* (3.11).

(i) *We have*

$$D(\Phi) = P(\Phi) = \sup_{\mathbb{Q}\in\mathcal{Q}_{\mathcal{S},(P_i)_{i\in I}}} \mathbb{E}^{\mathbb{Q}}\big[\Phi - A_T^{\mathbb{Q}}\big].$$

(ii) *Furthermore, if there is no portfolio constraint, i.e., $\mathcal{S} = \mathcal{H}$, then*

$$D(\Phi) = P(\Phi) = \sup_{\mathbb{Q}\in\mathcal{M}_{(p_i)_{i\in I}}} \mathbb{E}^{\mathbb{Q}}[\Phi].$$

Proof (i) simply follows from Theorem 3.14 and (3.16). For (ii), recalling from Remark 3.12 that $\mathcal{S} = \mathcal{H}$ implies $\mathcal{Q}_{\mathcal{S}} = \mathcal{M}$, we have $\mathcal{Q}_{\mathcal{S},(p_i)_{i\in I}} = \mathcal{M}_{(p_i)_{i\in I}}$. Then part (i) just becomes the desired result. □

Remark 3.16 Corollary 3.15(ii) states that to find the superhedging price of Φ, one needs to consider expectations of Φ under martingale measures that are consistent with market prices of both vanilla calls and other options $(\psi_i)_{i\in I}$. This in particular recovers [1, Theorem 1.4] for the case where tradable options at time 0 include at least vanilla calls with all maturities and strikes.

3.4 Connection to convex risk measures

Let $\mathcal{X}$ be the collection of measurable functions $\Phi : (\mathbb{R}_+^d)^T \to \mathbb{R}$ satisfying the linear growth condition (3.11). We say that $\rho : \mathcal{X} \to \mathbb{R}$ is a *convex risk measure* if for all Φ, $\Phi' \in \mathcal{X}$, the following conditions hold:

- *Monotonicity*: If $\Phi \le \Phi'$, then $\rho(\Phi) \ge \rho(\Phi')$.
- *Translation invariance*: If $m \in \mathbb{R}$, then $\rho(\Phi + m) = \rho(\Phi) - m$.
- *Convexity*: If $0 \le \lambda \le 1$, then $\rho(\lambda\Phi + (1-\lambda)\Phi') \le \lambda\rho(\Phi) + (1-\lambda)\rho(\Phi')$.

Consider the *acceptance set*

$$\mathcal{A}_{\mathcal{S}} := \big\{\Phi \in \mathcal{X} : u \in \mathcal{U}_0,\ \eta \in \mathcal{R}^I,\ \text{and } \Delta \in \mathcal{S} \\ \text{such that } \Phi(x) + \Psi_{u,\eta,\Delta}(x) \ge 0\ \forall x \in (\mathbb{R}^d)^T\big\}.$$

Then define the function $\rho_{\mathcal{S}} : \mathcal{X} \to \mathbb{R}$ by

$$\rho_{\mathcal{S}}(\Phi) := \inf\{m \in \mathbb{R} : m + \Phi \in \mathcal{A}_{\mathcal{S}}\} = D(-\Phi).$$

Proposition 3.17 *If $\mathcal{Q}_{\mathcal{S},I} \neq \emptyset$, then $\rho_{\mathcal{S}}$ is a convex risk measure and admits the dual formulation*

$$\rho_{\mathcal{S}}(\Phi) = \sup_{\mathbb{Q}\in\Pi} \big(\mathbb{E}^{\mathbb{Q}}[-\Phi] - \alpha^*(\mathbb{Q})\big), \tag{3.17}$$

where the penalty function α^* is given by

$$\alpha^*(\mathbb{Q}) := \begin{cases} \mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] + \mathcal{E}_I^{\mathbb{Q}} & \text{if } \mathbb{Q} \in \mathcal{Q}_{\mathcal{S},I}, \\ \infty & \text{otherwise.} \end{cases}$$

Moreover, for any $\alpha : \Pi \to \mathbb{R} \cup \{\infty\}$ such that (3.17) holds (with α^* replaced by α), we have $\alpha^*(\mathbb{Q}) \le \alpha(\mathbb{Q})$ for all $\mathbb{Q} \in \Pi$.

Proof Monotonicity and translation invariance can be easily verified, whereas the convexity of $\rho_{\mathcal{S}}$ follows from the convexity of $\mathcal{U}_0$, $\mathcal{R}^I$, and $\mathcal{S}$. Now the duality (3.17) is a direct consequence of Theorem 3.14. Since $\mathcal{Q}_{\mathcal{S},I} \neq \emptyset$, (3.17) shows that $\rho_{\mathcal{S}}$ is real-valued and thus a convex risk measure. To show that α^* is the minimal penalty function, observe that for any $\alpha : \Pi \mapsto \mathbb{R} \cup \{\infty\}$ satisfying (3.17), we have

$$\alpha(\mathbb{Q}) \ge \sup_{\Phi\in\mathcal{X}} \big(\mathbb{E}^{\mathbb{Q}}[-\Phi] - \rho_{\mathcal{S}}(\Phi)\big) \\ \ge \sup_{\Phi\in\mathcal{A}_{\mathcal{S}}} \big(\mathbb{E}^{\mathbb{Q}}[-\Phi] - \rho_{\mathcal{S}}(\Phi)\big) \ge \sup_{\Phi\in\mathcal{A}_{\mathcal{S}}} \mathbb{E}^{\mathbb{Q}}[-\Phi] \quad \forall \mathbb{Q} \in \Pi. \tag{3.18}$$

By Lemma 3.3 and (3.6),

$$\alpha^*(\mathbb{Q}) = \sup\left\{\mathbb{E}^{\mathbb{Q}}\left[(\Delta \cdot S)_T + \sum_{i\in I}\big(\eta_i\psi_i - c_i(\eta_i)\big)\right] : \Delta \in \mathcal{S}^\infty, \eta \in \mathcal{R}^I\right\}$$

for all $\mathbb{Q} \in \Pi$. Since $-(\Delta \cdot S)_T - \sum_{i \in I}(\eta_i \psi_i - c_i(\eta_i)) \in \mathcal{A}_\mathcal{S}$ for all $\Delta \in \mathcal{S}^\infty$ and $\eta \in \mathcal{R}^I$, we conclude from (3.18) that $\alpha^*(\mathbb{Q}) \le \alpha(\mathbb{Q})$. □

Remark 3.18 Proposition 3.17 generalizes Proposition 16 and Theorem 17 in [20] to a model-independent setting. Note that a no-arbitrage condition (under a given physical measure $\mathbb{P}$) is imposed in [20]. Here, we require only $\mathcal{Q}_{\mathcal{S},I} \neq \emptyset$, which is weaker than the model-independent no-arbitrage condition; see Sect. 4 for details.

4 Fundamental theorem of asset pricing via duality

Following the formulation in [1], we introduce the notion of arbitrage in the strong pathwise sense.

Definition 4.1 (Model-independent arbitrage) We say there is *model-independent arbitrage under the constraint* $\mathcal{S}$ if there exist $u \in \mathcal{U}_0$, $\eta \in \mathcal{R}^I$, and $\Delta \in \mathcal{S}$ such that

$$\sum_{t=1}^{T}\sum_{n=1}^{d} u_t^n(x_t^n) + \sum_{i \in I}\big(\eta_i \psi_i(x) - c_i(\eta_i)\big) + (\Delta \cdot x)_T > 0 \quad \text{for all } x \in (\mathbb{R}_+^d)^T.$$

Remark 4.2 It is immediate from this definition that if a model-independent arbitrage exists, then it is arbitrage under any probability measure $\mathbb{P}$ defined on Ω.

Note that instead of using the pathwise formulation, the authors in [7] introduce a weaker notion of arbitrage under model uncertainty via quasi-sure analysis. They include more strategies in the definition of arbitrage and provide different characterization of no-arbitrage condition and superhedging duality. We do not pursue this direction in this paper.

Consider the collection of measures given by

$$\mathcal{P}_\mathcal{S} := \big\{\mathbb{Q} \in \Pi : \big((\Delta \cdot S)_t\big)_{t=0}^T \text{ is a local } \mathbb{Q}\text{-supermartingale for all } \Delta \in \mathcal{S}\big\}.$$

Remark 4.3 ($\mathcal{M}$ and $\mathcal{P}_\mathcal{S}$) By definition we see that $\mathcal{M} \subset \mathcal{P}_\mathcal{S}$. Given $\alpha > 0$, if $\bar{\alpha} := (\Delta_t^n \equiv \alpha)_{t,n}$ and $-\bar{\alpha} := (\Delta_t^n \equiv -\alpha)_{t,n}$ both belong to $\mathcal{S}$, then $\mathcal{P}_\mathcal{S} = \mathcal{M}$. Indeed, given $\mathbb{Q} \in \mathcal{P}_\mathcal{S}$, since $\bar{\alpha}, -\bar{\alpha} \in \mathcal{S}^\infty$, $\alpha S = \bar{\alpha} \cdot S$ and $-\alpha S = -\bar{\alpha} \cdot S$ are both supermartingales under $\mathbb{Q}$. We thus conclude that $\mathbb{Q} \in \mathcal{M}$.

The following lemma provides a characterization of $\mathcal{P}_\mathcal{S}$.

Lemma 4.4 *Fix* $\mathbb{Q} \in \Pi$. *Then,* $\mathbb{Q} \in \mathcal{P}_\mathcal{S} \iff A_T^\mathbb{Q} = 0$ $\mathbb{Q}$-*a.s.*

Proof This is a consequence of [21, Proposition 9.6] and Lemma 3.6. □

Remark 4.5 ($\mathcal{P}_\mathcal{S}$ and $\mathcal{Q}_\mathcal{S}$) Lemma 4.4 in particular implies that $\mathcal{P}_\mathcal{S} \subseteq \mathcal{Q}_\mathcal{S}$. Observe that if $\mathcal{S}^\infty$ is composed of all nonnegative bounded trading strategies in $\mathcal{H}$, then $\mathcal{P}_\mathcal{S} = \mathcal{Q}_\mathcal{S}$. Indeed, for any $\mathbb{Q} \in \mathcal{Q}_\mathcal{S}$, we see from (3.5) that $A_T^\mathbb{Q} = 0$ $\mathbb{Q}$-a.s. Then $\mathbb{Q} \in \mathcal{P}_\mathcal{S}$ by Lemma 4.4.

To state an equivalent condition for no arbitrage, we consider

$$\mathcal{P}_{\mathcal{S},I} := \{\mathbb{Q} \in \mathcal{P}_{\mathcal{S}} : c_i'(0-) \le \mathbb{E}^{\mathbb{Q}}[\psi_i] \le c_i'(0+) \text{ for all } i \in I\}.$$

Recall $\mathcal{E}_I^{\mathbb{Q}}$ in (3.6). It is easy to verify the following characterization for $\mathcal{E}_I^{\mathbb{Q}} = 0$.

Lemma 4.6 *Given $\mathbb{Q} \in \Pi$, we have $c_i'(0-) \le \mathbb{E}^{\mathbb{Q}}[\psi_i] \le c_i'(0+)$ for all $i \in I$ if and only if $\mathcal{E}_I^{\mathbb{Q}} = 0$.*

To derive a model-independent FTAP, we need the following lemma.

Lemma 4.7 *Suppose that ψ_i is continuous and $|\psi_i|$ satisfies (3.11) for all $i \in I$. Then,*

$$\mathcal{P}_{\mathcal{S},I} = \emptyset \quad \Longrightarrow \quad \inf_{\mathbb{Q}\in\Pi} \left(\mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] + \mathcal{E}_I^{\mathbb{Q}}\right) > 0.$$

Proof Assume to the contrary that $\inf_{\mathbb{Q}\in\Pi}(\mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] + \mathcal{E}_I^{\mathbb{Q}}) = 0$. For any $\varepsilon > 0$, there exists $\mathbb{Q}_\varepsilon \in \Pi$ such that $0 \le \mathbb{E}^{\mathbb{Q}_\varepsilon}[A_T^{\mathbb{Q}_\varepsilon}] + \mathcal{E}_I^{\mathbb{Q}_\varepsilon} < \varepsilon$. Since Π is weakly compact (Remark 2.2), $(\mathbb{Q}_\varepsilon)$ must converge weakly to some $\mathbb{Q}^* \in \Pi$. For each $\Delta \in \mathcal{S}_c^\infty$, we can argue as in Proposition 3.10(iii) to show that $\mathbb{Q} \mapsto \mathbb{E}^{\mathbb{Q}}[(\Delta \cdot x)_T]$ is continuous on Π under the topology of weak convergence. Also, for each $i \in I$, since ψ_i is continuous and $|\psi_i|$ satisfies (3.11), we may argue as in Proposition 3.10(ii) to show that $\mathbb{Q} \mapsto \mathbb{E}^{\mathbb{Q}}[\psi_i]$ is continuous on Π. Now, by using Lemma 3.3,

$$\begin{aligned}
0 &= \lim_{\varepsilon\to 0} \mathbb{E}^{\mathbb{Q}_\varepsilon}\left[A_T^{\mathbb{Q}_\varepsilon}\right] + \mathcal{E}_I^{\mathbb{Q}_\varepsilon} \\
&= \lim_{\varepsilon\to 0} \sup_{\Delta\in\mathcal{S}_c^\infty} \mathbb{E}^{\mathbb{Q}_\varepsilon}[(\Delta \cdot S)_T] + \sup_{\eta\in\mathcal{R}^I} \sum_{i\in I} \left(\eta_i \mathbb{E}^{\mathbb{Q}_\varepsilon}[\psi_i] - c_i(\eta_i)\right) \\
&\ge \sup_{\Delta\in\mathcal{S}_c^\infty} \lim_{\varepsilon\to 0} \mathbb{E}^{\mathbb{Q}_\varepsilon}[(\Delta \cdot S)_T] + \sup_{\eta\in\mathcal{R}^I} \lim_{\varepsilon\to 0} \sum_{i\in I} \left(\eta_i \mathbb{E}^{\mathbb{Q}_\varepsilon}[\psi_i] - c_i(\eta_i)\right) \\
&= \sup_{\Delta\in\mathcal{S}_c^\infty} \mathbb{E}^{\mathbb{Q}^*}[(\Delta \cdot S)_T] + \sup_{\eta\in\mathcal{R}^I} \sum_{i\in I} \left(\eta_i \mathbb{E}^{\mathbb{Q}^*}[\psi_i] - c_i(\eta_i)\right) \\
&= \mathbb{E}^{\mathbb{Q}^*}\left[A_T^{\mathbb{Q}^*}\right] + \mathcal{E}_I^{\mathbb{Q}^*}.
\end{aligned}$$

Thus, $\mathbb{E}^{\mathbb{Q}^*}[A_T^{\mathbb{Q}^*}] = \mathcal{E}_I^{\mathbb{Q}^*} = 0$. By Lemmas 4.4 and 4.6 we have $\mathbb{Q}^* \in \mathcal{P}_{\mathcal{S},I}$, contradicting $\mathcal{P}_{\mathcal{S},I} = \emptyset$. □

Now we are ready to present the main result of this section.

Theorem 4.8 *Suppose that ψ_i is continuous and $|\psi_i|$ satisfies (3.11) for all $i \in I$. Then there is no model-independent arbitrage under constraint $\mathcal{S}$ if and only if $\mathcal{P}_{\mathcal{S},I} \neq \emptyset$.*

Proof To prove "⇐", suppose there is model-independent arbitrage. That is, there exist $u \in \mathcal{U}_0$, $\eta \in \mathcal{R}^I$, and $\Delta \in \mathcal{S}$ such that

$$\sum_{t=1}^{T}\sum_{n=1}^{d} u_t^n(x_t^n) + \sum_{i\in I}\big(\eta_i\psi_i(x) - c_i(\eta_i)\big) + (\Delta\cdot x)_T > 0 \quad \text{for all } x \in \mathbb{R}_+^T.$$

It follows that for any $\mathbb{Q} \in \mathcal{Q}_{\mathcal{S},I}$,

$$\sum_{t=1}^{T}\sum_{n=1}^{d} u_t^n(S_t^n) + \sum_{i\in I}\big(\eta_i\psi_i(S) - c_i(\eta_i)\big) + (\Delta\cdot S)_T - A_T^{\mathbb{Q}} > -A_T^{\mathbb{Q}} \quad \mathbb{Q}\text{-a.s.}$$

By taking expectations on both sides we obtain from Lemma 3.7 that

$$\mathcal{E}_I^{\mathbb{Q}} \ge \sum_{i\in I}\big(\eta_i\mathbb{E}^{\mathbb{Q}}[\psi_i] - c_i(\eta_i)\big) > -\mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] \quad \text{for all } \mathbb{Q} \in \mathcal{Q}_{\mathcal{S},I}.$$

If $\mathbb{Q} \in \mathcal{P}_{\mathcal{S},I}$, then this inequality becomes $\mathcal{E}_I^{\mathbb{Q}} > 0$, thanks to Lemma 4.4. However, in view of Lemma 4.6, this implies $\mathbb{E}^{\mathbb{Q}}[\psi_i] \notin [c_i'(0-), c_i'(0+)]$ for some $i \in I$ and thus $\mathbb{Q} \notin \mathcal{P}_{\mathcal{S},I}$, a contradiction. Hence, we conclude that $\mathcal{P}_{\mathcal{S},I} = \emptyset$.

To show "⇒", we assume to the contrary that $\mathcal{P}_{\mathcal{S},I} = \emptyset$ and intend to find model-independent arbitrage. By Theorem 3.14 and Lemma 4.7 we have

$$D(0) = \sup_{\mathbb{Q}\in\mathcal{Q}_{\mathcal{S},I}}\big(\mathbb{E}^{\mathbb{Q}}[-A_T^{\mathbb{Q}}] - \mathcal{E}_I^{\mathbb{Q}}\big) = -\inf_{\mathbb{Q}\in\mathcal{Q}_{\mathcal{S},I}}\big(\mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] + \mathcal{E}_I^{\mathbb{Q}}\big) < 0,$$

which already induces model-independent arbitrage. □

Let us recall the setup in Sect. 3.3: Every option ψ_i can actually be liquidly traded at time 0, that is, for each $i \in I$, we have $c_i'(\eta)$ being a constant $p_i \in \mathbb{R}$. Hence,

$$\mathcal{P}_{\mathcal{S},I} = \mathcal{P}_{\mathcal{S},(p_i)_{i\in I}} := \{\mathbb{Q} \in \mathcal{P}_{\mathcal{S}} : \mathbb{E}^{\mathbb{Q}}[\psi_i] = p_i \ \forall i \in I\}.$$

Corollary 4.9 *Suppose that ψ_i is continuous, $|\psi_i|$ satisfies (3.11), and ψ_i can be liquidly traded at the price $p_i \in \mathbb{R}$ for all $i \in I$.*

(i) *There is no model-independent arbitrage under constraint $\mathcal{S}$ if and only if $\mathcal{P}_{\mathcal{S},(p_i)_{i\in I}} \neq \emptyset$.*
(ii) *Furthermore, suppose that there is no portfolio constraint, i.e., $\mathcal{S} = \mathcal{H}$. Then there is no model-independent arbitrage if and only if $\mathcal{M}_{(p_i)_{i\in I}} \neq \emptyset$.*

Remark 4.10 Corollary 4.9(ii) recovers [1, Theorem 1.3] for the case where tradable options at time 0 include at least vanilla calls of all maturities and strikes.

Remark 4.11 Among different model-independent versions of the fundamental theorem of asset pricing (FTAP), Theorem 4.8 and [1, Theorem 1.3] are unique in their ability to accommodate general collections of tradable options. Whereas our framework deals with a wide range of tradable options beyond liquidly traded vanilla calls,

[1] does not even assume that vanilla calls have to be tradable. On the other hand, whereas [1] implicitly assume that any tradable option is traded liquidly, we allow less liquid options by taking into account their limit order books.

Also notice that our method differs from that in [1]. Techniques in functional analysis, which involve the use of the Hahn–Banach theorem, are directly used to establish [1, Theorem 1.3]; see [1, Proposition 2.3]. In our case, we first derive a superhedging duality in Theorem 3.14 via optimal transport and the minimax theorem, which does not bring the technical analysis to the surface of the arguments. Leveraging on this duality, we obtain the desired FTAP in Theorem 4.8.

4.1 Comparison with the classical theory

In the classical theory, a physical measure $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ is a priori given. We say there is no arbitrage under $\mathbb{P}$ with the constraint $\mathcal{S}$ if for any $\Delta \in \mathcal{S}$, $(\Delta \cdot S)_T \geq 0$ $\mathbb{P}$-a.s. implies $(\Delta \cdot S)_T = 0$ $\mathbb{P}$-a.s.

Consider the positive cone $\mathcal{K} := \{\lambda\Delta : \Delta \in \mathcal{S},\ \lambda \geq 0\}$ generated by $\mathcal{S}$. For all $t = 1, \ldots, T$, we define $\mathcal{S}_t := \{\Delta_t : \Delta \in \mathcal{S}\}$, $\mathcal{K}_t := \{\Delta_t : \Delta \in \mathcal{K}\}$ and introduce

$$N_t := \{\eta \in L^0(\Omega, \mathcal{F}_{t-1}, \mathbb{P}; \mathbb{R}^d) : \eta \cdot (S_t - S_{t-1}) = 0\ \mathbb{P}\text{-a.s.}\},$$
$$N_t^{\perp} := \{\xi \in L^0(\Omega, \mathcal{F}_{t-1}, \mathbb{P}; \mathbb{R}^d) : \xi \cdot \eta = 0\ \mathbb{P}\text{-a.s. for all } \eta \in N_t\}.$$

By [21, Lemma 1.66] every $\xi \in L^0(\Omega, \mathcal{F}_{t-1}, \mathbb{P}; \mathbb{R}^d)$ has a unique decomposition $\xi = \eta + \xi^{\perp}$ with $\eta \in N_t$ and $\xi^{\perp} \in N_t^{\perp}$. We denote by $\hat{\mathcal{S}}_t$ and $\hat{\mathcal{K}}_t$ the closures of $\mathcal{S}_t$ and $\mathcal{K}_t$, respectively, in $L^0(\Omega, \mathcal{F}_{t-1}, \mathbb{P}; \mathbb{R}^d)$. The following characterization of no arbitrage under $\mathbb{P}$ is taken from [21, Theorem 9.9].

Proposition 4.12 *Suppose that for all $t = 1, \ldots, T$,*

$$\mathcal{S}_t = \hat{\mathcal{S}}_t, \quad \hat{\mathcal{K}}_t \cap L^\infty(\Omega, \mathcal{F}_{t-1}, \mathbb{P}; \mathbb{R}^d) \subseteq \mathcal{K}_t, \quad \textit{and} \quad \xi^{\perp} \in \mathcal{S}_t \textit{ for any } \xi \in \mathcal{S}_t. \tag{4.1}$$

Then there is no arbitrage under $\mathbb{P}$ with the constraint $\mathcal{S}$ if and only if

$$\mathcal{P}_{\mathcal{S}}(\mathbb{P}) := \Big\{\mathbb{Q} \approx \mathbb{P} : S_t \in L^1(\mathbb{Q}) \textit{ and } \big((\Delta \cdot S)_t\big)_{t=0}^T \textit{ is}$$
$$\textit{a local } \mathbb{Q}\textit{-supermartingale } \forall \Delta \in \mathcal{S}\Big\} \neq \emptyset.$$

Theorem 4.8 can be viewed as a generalization of Proposition 4.12 to a model-independent setting. There is, however, a notable discrepancy: the closedness assumption (4.1) is no longer needed in Theorem 4.8. In the following, we provide a detailed illustration of this discrepancy in a simple example.

A typical example showing that condition (4.1) is indispensable for Proposition 4.12 is a one-period model containing two risky assets (S^1, S^2) with the collection of constrained strategies

$$\mathcal{S} := \{(\Delta^1, \Delta^2) \in \mathbb{R}^2 : (\Delta^1)^2 + (\Delta^2 - 1)^2 \leq 1\}.$$

One easily sees that (4.1) is not satisfied since

$$\hat{\mathcal{K}}_1 \cap L^\infty(\Omega, \mathcal{F}_0, \mathbb{P}) = \bar{\mathcal{K}} = \{\Delta^2 \geq 0\}$$

is not contained in $\mathcal{K}_1 = \mathcal{K} = \{(0,0)\} \cup \{\Delta^2 > 0\}$. At time $t = 0$, suppose $(S_0^1, S_0^2) = (1, 1)$ and assume that by analyzing market quotes on call option prices, we obtain that a reasonable pricing measure $\mathbb{Q}$ satisfies

$$S_1^1 \text{ is uniformly distributed on } [1, 2], \text{ and } S_1^2 \text{ is concentrated solely on } \{0\}. \quad (4.2)$$

Under the classical framework, the physical measure $\mathbb{P}$ should be compatible with market quotes on call option price so that any pricing measure $\mathbb{Q} \approx \mathbb{P}$ satisfies (4.2). Given $\Delta \in \mathcal{S}$, it can be checked that if $(\Delta \cdot S)_T = \Delta^1(S_1^1 - 1) - \Delta^2 \geq 0$ $\mathbb{P}$-a.s., then $\Delta^1 = \Delta^2 = 0$. There is therefore no arbitrage under $\mathbb{P}$ with the constraint $\mathcal{S}$. However, as observed in [4, Example 2.1], $\mathcal{P}_{\mathcal{S}}(\mathbb{P})$ is empty. Indeed, given $\mathbb{Q} \approx \mathbb{P}$, since $\mathbb{E}^{\mathbb{Q}}[S_1^1 - 1] > 0$, by taking $\Delta \in \mathcal{S}$ with $\Delta^2/\Delta^1 < \mathbb{E}^{\mathbb{Q}}[S_1^1 - 1]$ we get $\mathbb{E}^{\mathbb{Q}}[(\Delta \cdot S)_T] = \Delta^1 \mathbb{E}^{\mathbb{Q}}[S_1^1 - 1] - \Delta^2 > 0$.

Under our model-independent framework, (4.2) is reflected through market prices of vanilla calls $(S_1^1 - K)^+$ and $(S_1^2 - K)^+$ for all $K \geq 0$. That is, Π in (2.2) is the collection $\{\mathbb{Q} \in \mathcal{P}(\Omega) : (4.2) \text{ is satisfied}\}$. Given $\mathbb{Q} \in \Pi$, since $\mathbb{E}^{\mathbb{Q}}[S_1^1] = 3/2$, by taking $\Delta \in \mathcal{S}$ with $\Delta^2/\Delta^1 < 1/2$ we get $\mathbb{E}^{\mathbb{Q}}[(\Delta \cdot S)_T] = \Delta^1 \mathbb{E}^{\mathbb{Q}}[S_1^1 - 1] - \Delta^2 > 0$. This shows that $\mathcal{P}_{\mathcal{S}} = \emptyset$. Note that this does not violate Theorem 4.8 since there *is* model-independent arbitrage. To see this, consider trading dynamically with $\Delta \in \mathcal{S}$ satisfying $\Delta^2 < \varepsilon \Delta^1$ for some $\varepsilon \in (0, 1)$, and holding a static position u_1^1 given by $-\Delta^1(S_1^1 - \varepsilon)^+ + (1 + \varepsilon)\Delta^1 \in \mathcal{C}$. Observe that the initial wealth required is

$$\int_{\mathbb{R}_+} u_1^1(x) d\mu_1^1(x) = -\Delta^1(3/2 - \varepsilon) + (1 + \varepsilon)\Delta^1 = (2\varepsilon - 1/2)\Delta^1,$$

whereas the terminal wealth is always strictly positive: for any $(S_1^1, S_1^2) \in \mathbb{R}_+^2$,

$$u_1^1 + (\Delta \cdot S)_1 = \begin{cases} 2\varepsilon\Delta^1 + \Delta^2(S_1^2 - 1) > \varepsilon\Delta^1 - \Delta^2 > 0 & \text{if } S_1^1 \geq \varepsilon, \\ (\varepsilon\Delta^1 - \Delta^2) + (\Delta^1 S_1^1 + \Delta^2 S_1^2) > 0 & \text{if } S_1^2 < \varepsilon. \end{cases} \quad (4.3)$$

By taking $\varepsilon \in (0, 1/4]$ we have the required initial wealth no greater than 0 and thus obtain model-independent arbitrage.

Remark 4.13 In the model-independent setting, we may hold static positions in vanilla calls $(S_1^i - K)^+$ for $i \in \{1, 2\}$ and all $K > 0$ in addition to trading $S = (S^1, S^2)$ dynamically. This additional flexibility, unavailable under the classical framework, allows us to construct the arbitrage in (4.3).

It is of interest to see if (4.1) can be relaxed in the classical case where enough tradable options are available at time 0. Recently, with additional tradable options, [4] obtained a result similar to Proposition 4.12 with a collection $\mathcal{P}$ of possible physical measures a priori given. However, since their method allows only *finitely many* tradable options, a closedness assumption similar to (4.1) is still assumed.

4.2 Optimal arbitrage under a model-independent framework

In view of Theorem 3.14 and Proposition 3.17, the problems of superhedging and risk-measuring are well defined as long as $\mathcal{Q}_{\mathcal{S},I} \neq \emptyset$, which is weaker than the no-arbitrage condition $\mathcal{P}_{\mathcal{S},I} \neq \emptyset$. It is therefore of interest to provide characterizations for the condition $\mathcal{Q}_{\mathcal{S},I} \neq \emptyset$.

Definition 4.14 Consider

$$G_{\mathcal{S},I} := \sup \big\{ a \in \mathbb{R} : u \in \mathcal{U}_0, \eta \in \mathcal{R}^I \text{and } \Delta \in \mathcal{S} \\ \text{such that } \Psi_{u,\eta,\Delta}(x) > a \ \forall x \in (\mathbb{R}^d_+)^T \big\}. \tag{4.4}$$

By definition, $G_{\mathcal{S},I} \geq 0$. If $G_{\mathcal{S},I} > 0$, then we say that it is the (model-independent) *optimal arbitrage profit*.

The notion of optimal arbitrage goes back at least to [17], where the authors studied the highest return one can achieve relative to the market capitalization in a diffusion setting. Generalizations to semimartingale models and model uncertainty settings have been done in [10] and [18], respectively. Our definition above is similar to the formulation in [10, Sect. 3]. It is straightforward from the definitions of $D(0)$ and $G_{\mathcal{S},I}$ that

$$G_{\mathcal{S},I} = -D(0) = \inf_{\mathbb{Q} \in \mathcal{Q}_{\mathcal{S},I}} \big(\mathbb{E}^{\mathbb{Q}}[A^{\mathbb{Q}}_T] + \mathcal{E}^{\mathbb{Q}}_I \big).$$

This immediately yields the following result.

Proposition 4.15 (i) $G_{\mathcal{S},I} = 0 \iff \mathcal{P}_{\mathcal{S},I} \neq \emptyset$.
(ii) $G_{\mathcal{S},I} < \infty \iff \mathcal{Q}_{\mathcal{S},I} \neq \emptyset$.

5 Examples

In this section, we provide several concrete examples of the collection $\mathcal{S}$ of constrained trading strategies. An example illustrating the effect of an additional tradable, yet less liquid, option is also given. It will be convenient to keep in mind the relation $\mathcal{M} \subseteq \mathcal{P}_{\mathcal{S}} \subseteq \mathcal{Q}_{\mathcal{S}} \subseteq \Pi$, obtained from Remarks 4.3 and 4.5. Let us start with analyzing $\mathcal{Q}_{\mathcal{S}}$ further.

Proposition 5.1 *Let $\mathcal{S}^\infty$ contain all nonnegative bounded trading strategies in $\mathcal{H}$.*

(i) *For any $\mathbb{Q} \in \mathcal{Q}_{\mathcal{S}}$, $(S_t)_{t=0}^T$ is a $\mathbb{Q}$-supermartingale.*
(ii) *Furthermore, if the trading strategies in $\mathcal{S}^\infty$ are uniformly bounded from below, that is,*

$$\sup_{\Delta \in \mathcal{S}} \sup_{x \in (\mathbb{R}^T_+)^d} |\Delta^-(x)| \leq C \quad \textit{for some } C > 0,$$

then $\mathcal{Q}_{\mathcal{S}} = \{\mathbb{Q} \in \Pi : (S_t)_{t=0}^T \text{ is a } \mathbb{Q}\text{-supermartingale}\}$.

Proof (i) Given $\mathbb{Q} \in \mathcal{Q}_{\mathcal{S}}$, if $(S_t)_{t=0}^T$ is not a $\mathbb{Q}$-supermartingale, there must exist $n^* \in \{1, \ldots, d\}$ and $t^* \in \{0, \ldots, T-1\}$ such that $\mathbb{Q}[\mathbb{E}^{\mathbb{Q}}[S_{t^*+1}^{n^*} | \mathcal{F}_{t^*}] - S_{t^*}^{n^*} > 0] > 0$. We then deduce from (3.3) that $\mathbb{Q}[A_{t^*+1}^{\mathbb{Q}} - A_{t^*}^{\mathbb{Q}} = \infty] > 0$. This implies $\mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] = \infty$, a contradiction to $\mathbb{Q} \in \mathcal{Q}_{\mathcal{S}}$.

(ii) Let $\mathbb{Q} \in \Pi$ be such that $(S_t)_{t=0}^T$ is a $\mathbb{Q}$-supermartingale. It can be easily checked that $((\Delta \cdot S)_t)_{t=0}^T$ is a $\mathbb{Q}$-supermartingale for any nonnegative bounded trading strategy $\Delta \in \mathcal{H}$. By (3.5),

$$\mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] = \sup_{\Delta \in \mathcal{S}^\infty} \left(\mathbb{E}^{\mathbb{Q}}[(\Delta^+ \cdot S)_T] - \mathbb{E}^{\mathbb{Q}}[(\Delta^- \cdot S)_T]\right)$$

$$\leq \sup_{\Delta \in \mathcal{S}^\infty} \mathbb{E}^{\mathbb{Q}}[|(\Delta^- \cdot S)_T|] \leq 2C \sum_{t=1}^T \sum_{n=1}^d \left(\mathbb{E}^{\mathbb{Q}}[S_{t+1}^n] + \mathbb{E}^{\mathbb{Q}}[S_t^n]\right) < +\infty,$$

which implies that $\mathbb{Q} \in \mathcal{Q}_{\mathcal{S}}$. □

Example 5.2 (Shortselling constraint) Given $c_t^n \geq 0$ for each $t = 0, \ldots, T-1$ and $n = 1, \ldots, d$, we see from Remark 2.9 (with $K_t = \prod_n [-c_t^n, \infty)$ for all t) that

$$\mathcal{S} := \{\Delta \in \mathcal{H} : \Delta_t^n \geq -c_t^n,\ \forall t = 0, \ldots, T-1 \text{ and } n = 1, \ldots, d\}$$

satisfies Definition 2.7. By Proposition 5.1 we have

$$\mathcal{Q}_{\mathcal{S}} = \{\mathbb{Q} \in \Pi : (S_t)_{t=0}^T \text{ is a } \mathbb{Q}\text{-supermartingale}\}.$$

Furthermore, if $c_t^n > 0$ for all t and n, then $\mathcal{M} = \mathcal{P}_{\mathcal{S}}$ by virtue of Remark 4.3. If there exists $n \in \{1, \ldots, d\}$ such that $c_t^n = 0$ for all t, then $(S_t^n)_{t=0}^T$ is a $\mathbb{Q}$-supermartingale for all $\mathbb{Q} \in \mathcal{P}_{\mathcal{S}}$. Thus, if $c_t^n = 0$ for all t and n, then $\mathcal{P}_{\mathcal{S}} = \mathcal{Q}_{\mathcal{S}}$.

Example 5.3 (Relative drawdown constraint) Let $x_0^n > 0$ for $n = 1, \ldots, d$. For all $x \in (\mathbb{R}_+^d)^T$, $t = 1, \ldots, T$, and $n \in \{1, \ldots, d\}$, consider the running maximum $x_t^{*,n}$ given by $\max\{x_0^n, x_1^n, \ldots, x_t^n\}$. Then, define the relative drawdown process $\{\tilde{x}_t : t = 0, \ldots, T\}$ by $\tilde{x}_t := (x_t^1 / x_t^{*,1}, \ldots, x_t^d / x_t^{*,d})$. For each $n = 1, \ldots, d$, take two continuous functions $a^n : [0,1]^d \to (-\infty, 0]$ and $b^n : [0,1]^d \to [0, \infty)$ and introduce

$$\mathcal{S} := \{\Delta \in \mathcal{H} : a^n(\tilde{x}_t) \leq \Delta_t^n(x_t) \leq b^n(\tilde{x}_t),\ \forall t = 0, \ldots, T-1,\ n = 1, \ldots, d\}$$

$$= \{\Delta \in \mathcal{H} : \Delta_t \in K_t\ \forall t = 0, \ldots, T-1\} \quad \text{with } K_t := \prod_{i=1}^d [a^n(\tilde{x}_t), b^n(\tilde{x}_t)].$$

Since K_t satisfies (2.7), Remark 2.9 shows that $\mathcal{S}$ satisfies Definition 2.7. Thanks to Remark 3.5, we have $\mathcal{Q}_{\mathcal{S}} = \Pi$.

Example 5.4 (Nontradable assets) Suppose that certain risky assets are not tradable. In markets of electricity and foreign exchange rates, for example, people trade options written on a nontradable underlying. More precisely, let $d' \in \{1, \ldots, d\}$ and set

$$\mathcal{S} := \{\Delta \in \mathcal{H} : \Delta_t^n \equiv 0 \text{ for all } t = 1, \ldots, T \text{ and } n = 1, \ldots, d'\}.$$

By a similar argument as in Proposition 5.1 one can show that

$$\mathcal{P}_{\mathcal{S}} = \mathcal{Q}_{\mathcal{S}} = \{\mathbb{Q} \in \Pi : (S_t^n)_{t=1}^T \text{ is a } \mathbb{Q}\text{-martingale for all } n = d' + 1, \dots, d\}.$$

By Theorem 4.8 there is no model-independent arbitrage if and only if there exists $\mathbb{Q} \in \Pi$ under which all tradable assets are martingales. We can also modify this example by imposing additional constraint on the tradable assets satisfying Definition 2.7. In this case, Theorem 4.8 suggests that there is no arbitrage if and only if there is no arbitrage in the market consisting of tradable assets only.

Example 5.5 (Less liquid option) Consider a two-period model with one risky asset starting from $S_0 = 2$. We assume as in [5, Sect. 4.2] that the marginal distributions for S_1 and S_2 are given by

$$d\mu_1(x) = \frac{1}{2} 1_{[1,3]}(x)\, dx,$$
$$d\mu_2(x) = \frac{x}{3} 1_{[0,1]}(x)\, dx + \frac{1}{3} 1_{[1,3]}(x)\, dx + \frac{4-x}{3} 1_{[3,4]}(x)\, dx.$$

In addition to vanilla calls, we assume that a forward-start straddle with payoff $\psi(S) = |S_2 - S_1|$ is also tradable at time 0, whose unit price for trading η units is given by $p(\eta) := \infty 1_{\{\eta>1\}} + a 1_{\{0\le\eta\le1\}} + b 1_{\{-1\le\eta<0\}}$, where $0 \le b \le a$ and $0 \cdot \infty = 0$. We assume that the portfolio constraint $\mathcal{S}$ satisfies $\mathcal{Q}_{\mathcal{S}} = \mathcal{M}$. This readily covers the no-constraint case, as explained in Remark 3.12. Moreover, it also includes the shortselling constraint as in Example 5.2. To see this, note from Example 5.2 that $\mathcal{Q}_{\mathcal{S}} = \{\mathbb{Q} \in \Pi : (S_t)_{t=0}^T \text{ is a } \mathbb{Q}\text{-supermartingale}\}$. But since $\mathbb{Q} \in \Pi$ implies that $\mathbb{E}^{\mathbb{Q}}[S_1] = \mathbb{E}^{\mathbb{Q}}[S_2] = 2$ (computed from μ_1 and μ_2), every $\mathbb{Q} \in \mathcal{Q}_{\mathcal{S}}$ is actually a martingale. We thus obtain $\mathcal{Q}_{\mathcal{S}} = \mathcal{M}$.

We intend to price an exotic option with payoff $\Phi(x_1, x_2) = (x_2 - x_1)^2$. Our goal is to see how using the additional option ψ in static hedging affects the superhedging price of Φ. First, for any $\mathbb{Q} \in \mathcal{M}$, the martingale property of S implies $\mathbb{E}^{\mathbb{Q}}[(S_2 - S_1)^2] = \mathbb{E}^{\mathbb{Q}}[S_2^2] - \mathbb{E}^{\mathbb{Q}}[S_1^2] = \frac{1}{2}$, which is obtained solely from μ_1 and μ_2. Since $\mathcal{Q}_{\mathcal{S}} = \mathcal{M}$, Proposition 3.10 gives $D_\emptyset(\Phi) = \frac{1}{2}$. On the other hand, $\mathcal{Q}_{\mathcal{S}} = \mathcal{M}$ and $\mathcal{E}^{\mathbb{Q}}_{\{\psi\}} < \infty$ for all $\mathbb{Q} \in \mathcal{M}$ (see Remark 3.8) imply that $\mathcal{Q}_{\mathcal{S},I} = \mathcal{M}$. Theorem 3.14 thus yields

$$\begin{aligned} D(\Phi) &= \sup_{\mathbb{Q}\in\mathcal{M}} \big(\mathbb{E}^{\mathbb{Q}}[(S_2 - S_1)^2] - \mathcal{E}_I^{\mathbb{Q}}\big) \\ &= \frac{1}{2} - \inf_{\mathbb{Q}\in\mathcal{M}} \sup_{\eta\in[-1,1]} \eta\big(\mathbb{E}^{\mathbb{Q}}[|S_2 - S_1|] - p(\eta)\big) \\ &= \frac{1}{2} - \sup_{\eta\in[-1,1]} \eta\Big(\inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[|S_2 - S_1|] - p(\eta)\Big), \end{aligned}$$

where in the second line we used Lemma 3.9. Recalling from [5, Sect. 4.2] that $\inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[|S_2 - S_1|] = \frac{1}{3}$, we get $D(\Phi) = \frac{1}{2} - \max\{(\frac{1}{3} - a)^+, (b - \frac{1}{3})^+\}$.

6 Bounded constraints without adaptive convexity

In this section, we extend the main results of this paper, Theorems 3.14 and 4.8, to a class of constraints which does not satisfy adaptive convexity (Definition 2.7(ii)), but instead admits an additional boundedness property. Motivations behind this include gamma constraints, which will be discussed in Sect. 6.1.

Definition 6.1 $\mathcal{S}$ is a collection of trading strategies satisfying Definition 2.7(i) and (iii), whereas condition (ii) is replaced by

(ii)$'$ (Boundedness) For any $\Delta \in \mathcal{S}$, there exists $c > 0$ such that $|\Delta_t(x)| \leq c$ for all $x \in (\mathbb{R}^d_+)^t$ and $t \in \{0, \dots, T-1\}$ (i.e., $\mathcal{S} = \mathcal{S}^\infty$).

Under the current setting, Lemma 3.2 does not hold anymore, and thus the upper variation process $A^{\mathbb{Q}}$ is no longer useful. We adjust the definitions of $\mathcal{Q}_{\mathcal{S}}$ and $\mathcal{P}_{\mathcal{S}}$ accordingly.

Definition 6.2 For any $\mathbb{Q} \in \Pi$, we define

$$C^{\mathbb{Q}} := \sup_{\Delta \in \mathcal{S}} \mathbb{E}^{\mathbb{Q}}[(\Delta \cdot S)_T]. \tag{6.1}$$

In analogy to $\mathcal{Q}_{\mathcal{S}}$ in Definition 3.4 and the characterization of $\mathcal{P}_{\mathcal{S}}$ in Lemma 4.4, we define

$$\mathcal{Q}'_{\mathcal{S}} := \{\mathbb{Q} \in \Pi \ : \ C^{\mathbb{Q}} < +\infty\} \quad \text{and} \quad \mathcal{P}'_{\mathcal{S}} := \{\mathbb{Q} \in \Pi \ : \ C^{\mathbb{Q}} = 0\}. \tag{6.2}$$

Recall from (2.5) that $\mathcal{S}_c$ denotes the collection of $\Delta \in \mathcal{S}$ with $\Delta_t : (\mathbb{R}^d_+)^t \to \mathbb{R}^d$ continuous for all $t = 1, \dots, T$. Using the arguments in Lemma 3.3 gives

$$C^{\mathbb{Q}} = \sup_{\Delta \in \mathcal{S}_c} \mathbb{E}^{\mathbb{Q}}[(\Delta \cdot S)_T], \quad \forall \mathbb{Q} \in \Pi. \tag{6.3}$$

By (6.1), (6.3), and similar arguments as in Proposition 3.10 (with $\mathbb{E}^{\mathbb{Q}}[A^{\mathbb{Q}}_T]$ replaced by $C^{\mathbb{Q}}$), we obtain the following:

Proposition 6.3 *Suppose that $\mathcal{S}$ satisfies Definition 6.1. Let $\Phi : (\mathbb{R}^d_+)^T \to \mathbb{R}$ be a measurable function for which there exists $K > 0$ such that (3.11) holds.*

(i) *We have*

$$P'_\emptyset(\Phi) := \sup_{\mathbb{Q} \in \mathcal{Q}'_{\mathcal{S}}} \mathbb{E}^{\mathbb{Q}}[\Phi] - C^{\mathbb{Q}} \leq D_\emptyset(\Phi).$$

(ii) *Furthermore, if Φ is upper semicontinuous, then $P'_\emptyset(\Phi) = D_\emptyset(\Phi)$.*

(iii) *If Φ is upper semicontinuous and $\mathcal{Q}'_{\mathcal{S}} \neq \emptyset$, then there exists $\mathbb{Q}^* \in \mathcal{Q}'_{\mathcal{S}}$ such that $P'_\emptyset(\Phi) = \mathbb{E}^{\mathbb{Q}^*}[\Phi] - C^{\mathbb{Q}^*}$.*

Remark 6.4 Under adaptive convexity (Definition 2.7(ii)), Lemma 3.2 asserts that $\mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] = C^{\mathbb{Q}}$. This need not be true in general. For any $\mathbb{Q} \in \Pi$, we observe from Lemma 3.2 that in general $\mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}] \geq \sup_{\Delta \in \mathcal{S}^\infty} \mathbb{E}^{\mathbb{Q}}[(\Delta \cdot S)_T] = C^{\mathbb{Q}}$. This in particular implies that $P'_\emptyset(\Phi) \geq P_\emptyset(\Phi)$.

We now include the collection of options $(\psi_i)_{i\in I}$ in the superhedging strategy. Recalling the definition of $\mathcal{E}_I^{\mathbb{Q}}$ from (3.6), we consider the collection of measures $\mathcal{Q}'_{\mathcal{S},I} := \{\mathbb{Q} \in \mathcal{Q}'_{\mathcal{S}} : \mathcal{E}_I^{\mathbb{Q}} < \infty\}$ and define

$$P'(\Phi) := \sup_{\mathbb{Q}\in\mathcal{Q}'_{\mathcal{S}}} \big(\mathbb{E}^{\mathbb{Q}}[\Phi] - C^{\mathbb{Q}} - \mathcal{E}_I^{\mathbb{Q}}\big). \tag{6.4}$$

The following result follows from a straightforward adjustment of Theorem 3.14.

Proposition 6.5 *Suppose that $\mathcal{S}$ satisfies Definition 6.1, ψ_i is continuous, and $|\psi_i|$ satisfies (3.11) for all $i \in I$. Then for any upper semicontinuous function $\Phi : (\mathbb{R}_+^d)^T \to \mathbb{R}$ satisfying (3.11), we have $D(\Phi) = P'(\Phi)$ with D and P' defined as in (3.2) and (6.4). Moreover, if $\mathcal{Q}'_{\mathcal{S},I} \neq \emptyset$, then the supremum in (6.4) is attained at some $\mathbb{Q}^* \in \mathcal{Q}'_{\mathcal{S},I}$.*

To derive the FTAP, we consider the collection of measures

$$\mathcal{P}'_{\mathcal{S},I} := \{\mathbb{Q} \in \mathcal{P}'_{\mathcal{S}} : c'_i(0-) \leq \mathbb{E}^{\mathbb{Q}}[\psi_i] \leq c'_i(0+) \text{ for all } i \in I\}.$$

By (6.3) the same arguments as in Lemma 4.7 (with $\mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}]$ replaced by $C^{\mathbb{Q}}$) yield

$$\mathcal{P}'_{\mathcal{S},I} = \emptyset \implies \inf_{\mathbb{Q}\in\mathcal{Q}'_{\mathcal{S}}} \big(C^{\mathbb{Q}} + \mathcal{E}_I^{\mathbb{Q}}\big) > 0. \tag{6.5}$$

On the strength of (6.5) and Proposition 6.5, we may argue as in Theorem 4.8 and Proposition 4.15 (with $\mathbb{E}^{\mathbb{Q}}[A_T^{\mathbb{Q}}]$ replaced by $C^{\mathbb{Q}}$) to establish the following:

Proposition 6.6 *Suppose that $\mathcal{S}$ satisfies Definition 6.1. Then:*

(i) *There exists no model-independent arbitrage under the constraint $\mathcal{S}$ if and only if $\mathcal{P}'_{\mathcal{S},I} \neq \emptyset$.*
(ii) *The optimal arbitrage profit is finite under the constraint $\mathcal{S}$ (i.e., $G_{\mathcal{S},I} < \infty$) if and only if $\mathcal{Q}'_{\mathcal{S},I} \neq \emptyset$.*

6.1 Gamma constraints

Given $\Gamma = (\Gamma_1, \dots, \Gamma_d) \in \mathbb{R}_+^d$, we consider the collection of trading strategies

$$\mathcal{S}_\Gamma := \{\Delta \in \mathcal{H} : |\Delta_t^n - \Delta_{t-1}^n| \leq \Gamma_n,\ \forall t = 0, \dots, T-1,\ n = 1, \dots, d\},$$

where we set $\Delta_{-1} \equiv 0 \in \mathbb{R}^d$. Observe that $\mathcal{S}_\Gamma$ does not admit adaptive convexity (Definition 2.7(ii)). Indeed, consider $\Delta \equiv 0$ and $\Delta' := (1_{\{t=0\}}\Gamma + 1_{\{t>0\}}2\Gamma)_{t=0}^{T-1}$, both of which trivially lie in $\mathcal{S}_\Gamma$. Given a fixed $s \in \{1, \dots, T-1\}$, the trading strategy $\tilde{\Delta} := (\Delta_t 1_{\{t<s\}} + \Delta'_t 1_{\{t\geq s\}})_{t=0}^{T-1}$ does not belong to $\mathcal{S}_\Gamma$ since $\tilde{\Delta}_s - \tilde{\Delta}_{s-1} = 2\Gamma$. The constrained collection $\mathcal{S}_\Gamma$ instead satisfies Definition 6.1.

Lemma 6.7 *$\mathcal{S}_\Gamma$ satisfies Definition 6.1.*

Proof It is trivial that $0 \in \mathcal{S}_\Gamma$. For each $\Delta \in \mathcal{S}_\Gamma$, since $\Delta_t = \sum_{j=0}^{t}(\Delta_j - \Delta_{j-1})$, we have $|\Delta_t| \leq (t+1)|\Gamma|$, which shows that (ii)$'$ in Definition 6.1 is satisfied. It remains to prove (iii).

In view of Remark 2.9, it follows from Lusin's theorem that for any $\mathbb{Q} \in \Pi$ and $\varepsilon > 0$, there exist a closed set $D_\varepsilon \subseteq \Omega$ and a sequence of continuous functions $\Delta^\varepsilon(x) = (\Delta_t^\varepsilon(x_1, \dots, x_t))_{t=0}^{T-1}$ such that $\mathbb{Q}[D_\varepsilon] > 1 - \varepsilon$ and $\Delta = \Delta^\varepsilon$ on D_ε. That is, for all $t = 1, \dots, T-1$, Δ_t is a continuous function when it is restricted to the domain

$$\mathrm{proj}_{(\mathbb{R}_+^d)^t} D_\varepsilon := \{x \in (\mathbb{R}_+^d)^t : \exists\, y \in (\mathbb{R}_+^d)^{T-t} \text{ such that } (x, y) \in D_\varepsilon\}.$$

In the following, by induction over time t, we construct a continuous strategy $\bar{\Delta}^\varepsilon \in \mathcal{S}_{\Gamma,c}$. At time $t = 0$, $\bar{\Delta}_0^\varepsilon := \Delta_0$ is a constant in $\prod_{n=1}^{d}[-\Gamma_n, \Gamma_n]$ and therefore continuous. Fix $t \geq 1$. We assume that we have constructed continuous functions $(\bar{\Delta}_s^\varepsilon : (\mathbb{R}_+^d)^s \to \mathbb{R}^d)_{s=0}^{t-1}$ such that $\bar{\Delta}_s^\varepsilon = \Delta_s$ on $\mathrm{proj}_{(\mathbb{R}_+^d)^s} D_\varepsilon$ and $|\bar{\Delta}_s^\varepsilon - \bar{\Delta}_{s-1}^\varepsilon| \leq \Gamma$ on $(\mathbb{R}_+^d)^s \setminus \mathrm{proj}_{(\mathbb{R}_+^d)^s} D_\varepsilon$, for any $s < t$. By the continuity of $\bar{\Delta}_{t-1}^\varepsilon$ the set-valued function defined by

$$K_t(x_1, \dots, x_t) := \begin{cases} \{\Delta_t(x_1, \dots, x_t)\} & \text{on } \mathrm{proj}_{(\mathbb{R}_+^d)^t} D_\varepsilon, \\ \Delta_{t-1}(x_1, \dots, x_{t-1}) & \\ \quad + \prod_{n=1}^{d}[-\Gamma_n, \Gamma_n] & \text{on } (\mathbb{R}_+^d)^t \setminus \mathrm{proj}_{(\mathbb{R}_+^d)^t} D_\varepsilon \end{cases}$$

satisfies (2.7) and thus admits a continuous selection ([31, Theorem 3.2$''$]), that is, there is a continuous function $\bar{\Delta}_t^\varepsilon : (\mathbb{R}_+^d)^t \to \mathbb{R}^d$ such that we have $\bar{\Delta}_t^\varepsilon(x_1, \dots, x_t) \in K_t(x_1, \dots, x_t)$ for all $x \in \mathbb{R}_+^t$. Thus, we can construct $\bar{\Delta}^\varepsilon \in \mathcal{S}_{\Gamma,c}$ as required by Definition 2.7(iii). □

Proposition 6.8 *$\mathcal{Q}'_{\mathcal{S}_\Gamma} = \Pi$ and $\mathcal{P}'_{\mathcal{S}_\Gamma} = \mathcal{M}$.*

Proof From the proof of Lemma 6.7, every $\Delta \in \mathcal{S}_\Gamma$ is bounded by $c := (T+1)|\Gamma|$. This gives $\mathcal{Q}'_{\mathcal{S}_\Gamma} = \Pi$ by Remark 3.5. For any $t \in \{0, \dots, T-1\}$ and $A \in \mathcal{F}_t$, observe that $\Delta^{(+)} = (\Delta_s^+)_{s=0}^{T-1} := +\Gamma 1_A 1_{\{s=t\}}$ and $\Delta^{(-)} = (\Delta_s^-)_{s=0}^{T-1} := -\Gamma 1_A 1_{\{s=t\}}$ both belong to $\mathcal{S}_\Gamma$. Given $\mathbb{Q} \in \mathcal{P}'_{\mathcal{S}_\Gamma}$, the definition of $\mathcal{P}'_{\mathcal{S}_\Gamma}$ in (6.2) implies that $\mathbb{E}^\mathbb{Q}[\Gamma 1_A(S_{t+1} - S_t)] = 0$. This readily implies $\mathbb{E}^\mathbb{Q}[S_{t+1}|\mathcal{F}_t] = S_t$, and thus $\mathbb{Q} \in \mathcal{M}$. □

By Proposition 6.8, the following is a direct consequence of Proposition 6.6.

Corollary 6.9 *$\mathcal{S}_\Gamma$ satisfies the following*:

(i) *There is no model-independent arbitrage under $\mathcal{S}_\Gamma$ if and only if $\mathcal{M}_I \neq \emptyset$.*
(ii) *The optimal arbitrage profit is finite under $\mathcal{S}_\Gamma$ (i.e., $G_{\mathcal{S}_\Gamma,I} < \infty$) if and only if $\mathcal{E}_I^{\mathbb{Q}} < \infty$ for some $\mathbb{Q} \in \Pi$.*

Remark 6.10 By Theorem 4.8, Remark 3.12, and Corollary 6.9 we have equivalence between:

(i) There is model-independent arbitrage with $\Delta \in \mathcal{H}$ (i.e., the no-constraint case).
(ii) There is model-independent arbitrage with $\Delta \in \mathcal{S}_\Gamma$.

Although these two arbitrage opportunities coexist, they are very different in terms of the optimal arbitrage profit defined in (4.4). By Proposition 4.15 we see that $G_{\mathcal{H}} < \infty$ if and only if $\mathcal{Q}_{\mathcal{H},I} = \{\mathbb{Q} \in \mathcal{M} : \mathcal{E}_I^{\mathbb{Q}} < \infty\} \neq \emptyset$, whereas $G_{\mathcal{S}_\Gamma} < \infty$ if and only if $\mathcal{Q}'_{\mathcal{S}_\Gamma,I} = \{\mathbb{Q} \in \Pi : \mathcal{E}_I^{\mathbb{Q}} < \infty\} \neq \emptyset$.

Acknowledgements We thank Pierre Henry-Labordère, Constantinos Kardaras and Jan Obłój for their thoughtful suggestions. We are also thankful to the anonymous referees for their elaborate comments which contribute to the quality of this work.

A. Fahim is partially supported by Florida State University CRC FYAP (315-81000-2424) and the NSF (DMS-1209519).

Y.-J. Huang is partially supported by SFI (07/MI/008 and 08/SRC/FMC1389) and the ERC (278295).

Appendix: An example related to Definition 2.7(iii)

In this appendix, we provide an example showing that if Definition 2.7(iii) is not satisfied, then the duality in Proposition 3.10 may fail. Let $d = 1$, $T = 2$, and $x_0 = 1$. Assume that $\mu_1(dx) = \frac{1}{2}\delta_1(dx) + \frac{1}{2}\delta_2(dx)$ and $\mu_2(dx) = \delta_2(dx)$, where δ_x is the Dirac measure at $x \in \mathbb{R}$. Thus, $\Pi = \{\mathbb{Q}\}$ with $\mathbb{Q}[S_1 = 1, S_2 = 2] = \mathbb{Q}[S_1 = 2, S_2 = 2] = \frac{1}{2}$. Consider the collection of trading strategies

$$\mathcal{S} = \{\Delta = (\Delta_0, \Delta_1) : \Delta_0 \equiv 0,\ \Delta_1(x) = \alpha 1_{\{x=1\}}(x) \text{ for some } \alpha \in [0,1]\}.$$

While $\mathcal{S}$ trivially satisfies (i) and (ii) in Definition 2.7, (iii) does not hold. To see this, note that $\mathcal{S}_c^\infty = \{(0,0)\}$, and thus for any $\Delta \in \mathcal{S}$ with $\alpha > 0$, we have $\mathbb{Q}[\Delta \neq (0,0)] = 1/2$. In order to superhedge the claim $\Phi(x_1,x_2) \equiv 0$, we need to find $n, m \in \mathbb{N}$, $a, b_i, c_j \in \mathbb{R}$, $K_i^1, K_j^2 \geq 0$, and $\Delta \in \mathcal{S}$ such that for all $(x_1,x_2) \in \mathbb{R}_+^2$,

$$0 \leq a + \sum_{i=1}^n b_i(x_1 - K_i^1)^+ + \sum_{j=1}^m c_j(x_2 - K_j^2)^+ + \Delta_0(x_1 - x_0) + \Delta_1(x_1)(x_2 - x_1).$$

Since $\Delta_0 \equiv 0$ and $\Delta(x_1) = \alpha 1_{\{x_1=1\}}$, this inequality reduces to

$$\begin{aligned} f_\alpha(x_1,x_2) &:= -\alpha 1_{\{x_1=1\}}(x_1)(x_2 - x_1) \\ &\leq a + \sum_{i=1}^n b_i(x_1 - K_i^1)^+ + \sum_{j=1}^m c_j(x_2 - K_j^2)^+ \end{aligned} \tag{A.1}$$

for all $(x_1, x_2) \in \mathbb{R}^2_+$. Let f^*_α denote the upper semicontinuous envelope of f_α. We observe that (A.1) holds for f_α if and only if it also holds for f^*_α. It follows that

$$D_\emptyset(0) = \inf_{0\le\alpha\le 1} D_\emptyset(f_\alpha) = \inf_{0\le\alpha\le 1} D_\emptyset(f^*_\alpha) = \inf_{0\le\alpha\le 1} P_\emptyset(f^*_\alpha)$$
$$= \inf_{0\le\alpha\le 1} \alpha \mathbb{E}^{\mathbb{Q}}\left[1_{\{S_1=1\}}(S_1)(S_2 - S_1)^-\right] = 0,$$

where the third equality follows from Proposition 3.10, and the fourth is due to $f^*_\alpha = \alpha 1_{\{x_1=1\}}(x_1)(x_2 - x_1)^-$. On the other hand, since

$$A^{\mathbb{Q}}_2 = \sup_{\alpha\in[0,1]} \alpha\mathbb{Q}[S_1 = 1] = \frac{1}{2},$$

we have $P_\emptyset(0) = -\mathbb{E}^{\mathbb{Q}}[A^{\mathbb{Q}}_2] = -\frac{1}{2}$, which indicates a duality gap.

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