Finance Stoch (2012) 16:561–609
DOI 10.1007/s00780-012-0180-5

Continuous-time trading and the emergence of probability

Vladimir Vovk

Received: 9 August 2010 / Accepted: 2 April 2011 / Published online: 4 April 2012

Abstract This paper establishes a non-stochastic analog of the celebrated result by Dubins and Schwarz about reduction of continuous martingales to Brownian motion via time change. We consider an idealized financial security with continuous price paths, without making any stochastic assumptions. It is shown that typical price paths possess quadratic variation, where "typical" is understood in the following game-theoretic sense: there exists a trading strategy that earns infinite capital without risking more than one monetary unit if the process of quadratic variation does not exist. Replacing time by the quadratic variation process, we show that the price path becomes Brownian motion. This is essentially the same conclusion as in the Dubins–Schwarz result, except that the probabilities (constituting the Wiener measure) emerge instead of being postulated. We also give an elegant statement, inspired by Peter McCullagh's unpublished work, of this result in terms of game-theoretic probability theory.

Keywords Game-theoretic probability · Continuous time · Emergence of probability · Continuous price paths · Incomplete markets

Mathematics Subject Classification 91G99 · 60G17 · 60G05 · 60G44

JEL Classification C58 · G13 · G14

1 Introduction

This paper is a contribution to the game-theoretic approach to probability. This approach was explored (by e.g. von Mises, Wald and Ville) as a possible basis for prob-

V. Vovk (✉)
Department of Computer Science, Royal Holloway, University of London, Egham, Surrey TW20 0EX, England, UK
e-mail: v.vovk@rhul.ac.uk

ability theory at the same time as the now standard measure-theoretic approach (Kolmogorov), but then became dormant. The current revival of interest in it started with A.P. Dawid's prequential principle ([15], Sect. 5.1, [16], Sect. 3), and recent work on game-theoretic probability includes monographs [55, 59] and papers [32, 36–39, 61].

The treatment of continuous-time processes in game-theoretic probability often involves nonstandard analysis (see e.g. [55], Chaps. 11–14). The recent paper [60] suggested avoiding nonstandard analysis and introduced the key technique of "high-frequency limit order strategies," also used in this paper and its predecessors, [67] and [65].

An advantage of game-theoretic probability is that one does not have to start with a full-fledged probability measure from the outset to arrive at interesting conclusions, even in the case of continuous time. For example, ref. [67] shows that continuous price paths satisfy many standard properties of Brownian motion (such as the absence of isolated zeroes) and ref. [65] (developing [68] and [60]) shows that the variation index of a non-constant continuous price path is 2, as in the case of Brownian motion. The standard qualification "with probability one" is replaced with "unless a specific trading strategy increases manyfold the capital it risks" (the formal definitions, assuming zero interest rate, will be given in Sect. 2). This paper makes the next step, showing that the Wiener measure emerges in a natural way in the continuous trading protocol. Its main result contains all main results of [65, 67], together with several refinements, as special cases.

Other results about the emergence of the Wiener measure in game-theoretic probability can be found in [64] and [66]. However, the protocols of those papers are much more restrictive, involving an externally given quadratic variation (a game-theoretic analog of predictable quadratic variation, generally chosen by a player called Forecaster). In the present paper, the Wiener measure emerges in a situation with surprisingly little a priori structure, involving only two players: the market and a trader.

The reader will notice that not only our main result but also many of our definitions resemble those in Dubins and Schwarz's paper [20], which can be regarded as the measure-theoretic counterpart of this paper. The main difference of this paper is that we do not assume a given probability measure from the outset. A less important difference is that our main result will not assume that the price path is unbounded and nowhere constant (among other things, this generalization is important to include the main results of [65, 67] as special cases). A result similar to that of Dubins and Schwarz was almost simultaneously proved by Dambis [11]; however, Dambis, unlike Dubins and Schwarz, dealt with predictable quadratic variation, and his result can be regarded as the measure-theoretic counterpart of [64] and [66].

Another related result is the well-known observation (see e.g. [27], Theorem 5.39) that in the binomial model of a financial market, every contingent claim can be replicated by a self-financing portfolio whose initial price is the expected value (suitably discounted if the interest rate is not zero) of the payoff function with respect to the risk-neutral probability measure. This insight is, essentially, extended in this paper to the case of an incomplete market (the price for completeness in the binomial model is the artificial assumption that at each step the price can only go up or down by specified factors) and continuous time (continuous-time mathematical finance usually starts from an underlying probability measure, with some notable exceptions discussed in Sect. 12).

This paper's definitions and results have many connections with several other areas of finance and stochastics, including stochastic integration, the fundamental theorems of asset pricing, and model-free option pricing. These will be discussed in Sect. 12.

The main part of the paper starts with the description of our continuous-time trading protocol and the definition of game-theoretic versions of the notion of probability (outer and inner content) in Sect. 2. In Sect. 3 we state our main result (Theorem 3.1), which becomes especially intuitive if we restrict our attention to the case of the initial price equal to 0 and price paths that do not converge to a finite value and are nowhere constant: the outer and the inner content of any event that is invariant with respect to time transformations then exist and coincide between themselves and with its Wiener measure (Corollary 3.7). This simple statement was made possible by Peter McCullagh's unpublished work on Fisher's fiducial probability: McCullagh's idea was that fiducial probability is only defined on the σ-algebra of events invariant with respect to a certain group of transformations. Section 4 presents several applications (connected with [67] and [65]) demonstrating the power of Theorem 3.1. The fact that typical price paths possess quadratic variation is proved in Sect. 8. It is, however, used earlier, in Sect. 5, where it allows us to state a constructive version of Theorem 3.1. The constructive version, Theorem 5.1, says that replacing time by the quadratic variation process turns the price path into Brownian motion. In Sect. 6 we state generalizations, from events to positive bounded measurable functions, of Theorem 3.1 and part of Theorem 5.1; these are Theorems 6.2 and 6.4, respectively. The easy directions in Theorems 6.2 and 6.4 are proved in the same section. Sections 7 and 9 prove part of Theorem 5.1 and prepare the ground for the proof of the remaining parts of Theorems 5.1 and 6.4 (in Sect. 10) and Theorem 6.2 (in Sect. 11). Section 12 continues the general discussion started in this section.

Words such as "positive," "negative," "before," "after," "increasing," and "decreasing" will be understood in the wide sense of $\geq$ or $\leq$, as appropriate; when necessary, we add the qualifier "strictly." As usual, $C(E)$ is the space of all continuous functions on a topological space E equipped with the sup norm. We often omit the parentheses around E in expressions such as $C[0,T] := C([0,T])$.

2 Outer content in a financial context

We consider a game between two players, Reality (a financial market) and Sceptic (a trader), over the time interval $[0,\infty)$. First Sceptic chooses his trading strategy and then Reality chooses a continuous function $\omega : [0,\infty) \to \mathbb{R}$ (the price path of a security).

Let Ω be the set of all continuous functions $\omega : [0,\infty) \to \mathbb{R}$. For each $t \in [0,\infty)$, $\mathcal{F}_t$ is defined to be the smallest σ-algebra that makes all functions $\omega \mapsto \omega(s)$, $s \in [0,t]$, measurable. A *process* $\mathfrak{S}$ is a family of functions $\mathfrak{S}_t : \Omega \to [-\infty,\infty]$, $t \in [0,\infty)$, each $\mathfrak{S}_t$ being $\mathcal{F}_t$-measurable; its *sample paths* are the functions $t \mapsto \mathfrak{S}_t(\omega)$. An *event* is an element of the σ-algebra $\mathcal{F}_\infty := \bigvee_t \mathcal{F}_t$, also denoted by $\mathcal{F}$. (We often consider arbitrary subsets of Ω as well.) Stopping times $\tau : \Omega \to [0,\infty]$ with respect to the filtration $(\mathcal{F}_t)$ and the corresponding σ-algebras $\mathcal{F}_\tau$ are defined as usual; $\omega(\tau(\omega))$ and $\mathfrak{S}_{\tau(\omega)}(\omega)$ will be simplified to $\omega(\tau)$ and $\mathfrak{S}_\tau(\omega)$, respectively (occasionally, the argument ω will be omitted in other cases as well).

The class of allowed strategies for Sceptic is defined in two steps. A *simple trading strategy* G consists of an increasing sequence of stopping times $\tau_1 \le \tau_2 \le \cdots$ and, for each $n = 1, 2, \ldots$, a bounded $\mathcal{F}_{\tau_n}$-measurable function h_n. It is required that for each $\omega \in \Omega$, $\lim_{n\to\infty} \tau_n(\omega) = \infty$. To such a G and an *initial capital* $c \in \mathbb{R}$ corresponds the *simple capital process*

$$\mathcal{K}_t^{G,c}(\omega) := c + \sum_{n=1}^{\infty} h_n(\omega)\big(\omega(\tau_{n+1} \wedge t) - \omega(\tau_n \wedge t)\big), \quad t \in [0, \infty) \tag{2.1}$$

(with the zero terms in the sum ignored, which makes the sum finite for each t); the value $h_n(\omega)$ will be called Sceptic's *bet* (or *bet on* ω, or *stake*) at time τ_n, and $\mathcal{K}_t^{G,c}(\omega)$ will be referred to as Sceptic's capital at time t.

A *positive capital process* is any process $\mathfrak{S}$ that can be represented in the form

$$\mathfrak{S}_t(\omega) := \sum_{n=1}^{\infty} \mathcal{K}_t^{G_n,c_n}(\omega), \tag{2.2}$$

where the simple capital processes $\mathcal{K}_t^{G_n,c_n}(\omega)$ are required to be positive, for all t and ω, and the positive series $\sum_{n=1}^{\infty} c_n$ is required to converge. The sum (2.2) is always positive but allowed to take value ∞. Since $\mathcal{K}_0^{G_n,c_n}(\omega) = c_n$ does not depend on ω, $\mathfrak{S}_0(\omega)$ also does not depend on ω and will sometimes be abbreviated to $\mathfrak{S}_0$.

Remark 2.1 The financial interpretation of a positive capital process (2.2) is that it represents the total capital of a trader who splits his initial capital into a countable number of accounts, and on each account runs a simple trading strategy making sure that this account never goes into debit.

The *outer content* of a set $E \subseteq \Omega$ (not necessarily $E \in \mathcal{F}$) is defined as

$$\overline{\mathbb{P}}(E) := \inf\Big\{\mathfrak{S}_0 \;\Big|\; \forall\omega \in \Omega : \liminf_{t\to\infty} \mathfrak{S}_t(\omega) \ge \mathbf{1}_E(\omega)\Big\}, \tag{2.3}$$

where $\mathfrak{S}$ ranges over the positive capital processes and $\mathbf{1}_E$ stands for the indicator function of E. In the financial terminology (and ignoring the fact that the inf in (2.3) need not be attained), $\overline{\mathbb{P}}(E)$ is the price of the cheapest superhedge for the European contingent claim paying $\mathbf{1}_E$ at time ∞. It is easy to see that the $\liminf_{t\to\infty}$ in (2.3) can be replaced by $\sup_t$ (and, therefore, by $\limsup_{t\to\infty}$): we can always stop (i.e., set all bets to 0) when $\mathfrak{S}$ reaches the level 1 (or a level arbitrarily close to 1).

We say that a set $E \subseteq \Omega$ is *null* if $\overline{\mathbb{P}}(E) = 0$. If E is null, there is a positive capital process $\mathfrak{S}$ such that $\mathfrak{S}_0 = 1$ and $\lim_{t\to\infty} \mathfrak{S}_t(\omega) = \infty$ for all $\omega \in E$ (it suffices to sum over $\epsilon = 1/2, 1/4, \ldots$ positive capital processes $\mathfrak{S}^\epsilon$ satisfying $\mathfrak{S}_0^\epsilon = \epsilon$ and $\liminf_{t\to\infty} \mathfrak{S}_t^\epsilon \ge \mathbf{1}_E$). A property of $\omega \in \Omega$ will be said to hold *for typical* ω if the set of ω where it fails is null. Correspondingly, a set $E \subseteq \Omega$ is *full* if $\overline{\mathbb{P}}(E^c) = 0$, where $E^c := \Omega \setminus E$ stands for the complement of E.

We can also define *inner content* by

$$\underline{\mathbb{P}}(E) := 1 - \overline{\mathbb{P}}\big(E^c\big)$$

(intuitively, this is the price of the most expensive subhedge of $\mathbf{1}_E$). This notion of inner content will not be useful in this paper (but a simple modification will be).

Remark 2.2 Another natural setting is where Ω is defined as the set of all continuous functions $\omega : [0, T] \to \mathbb{R}$ for a given constant T (the time horizon). In this case the definition of outer content simplifies: instead of $\liminf_{t\to\infty} \mathfrak{S}_t(\omega)$, we have simply $\mathfrak{S}_T(\omega)$ in (2.3).

Remark 2.3 Alternative names (used in e.g. [55]) for outer and inner content are upper and lower probability in the case of sets and upper and lower expectation in the case of functions (the latter case will be considered in Sect. 6). Our terminology essentially follows refs. [31] and [56], but we drop "probability" in outer/inner probability content. We also avoid expressions such as "for almost all" and "almost surely." Hopefully, this terminology will remind the reader that we do not start from a probability measure on Ω. For terminology used in the finance literature, see Sect. 12.

3 Main result: abstract version

A *time transformation* is defined to be a continuous increasing (not necessarily strictly increasing) function $f : [0, \infty) \to [0, \infty)$ satisfying $f(0) = 0$. Equipped with the binary operation of composition, $(f \circ g)(t) := f(g(t))$, $t \in [0, \infty)$, the time transformations form a (non-commutative) monoid, with the identity time transformation $t \mapsto t$ as the unit. The *action* of a time transformation f on $\omega \in \Omega$ is defined to be the composition $\omega^f := \omega \circ f \in \Omega$, $(\omega \circ f)(t) := \omega(f(t))$. The *trail* of $\omega \in \Omega$ is the set of all $\psi \in \Omega$ such that $\psi^f = \omega$ for some time transformation f. (These notions are often defined for groups rather than monoids: see e.g. [47]; in this case the trail is called the orbit. In their "time-free" considerations, Dubins and Schwarz [20, 53, 54] make simplifying assumptions that make the monoid of time transformations a group; we make similar assumptions in Corollary 3.7.) A subset E of Ω is *time-superinvariant* if together with any $\omega \in \Omega$, it contains the whole trail of ω; in other words, if for each $\omega \in \Omega$ and each time transformation f, it is true that

$$\omega^f \in E \Longrightarrow \omega \in E. \tag{3.1}$$

The *time-superinvariant class* $\mathcal{I}$ is defined to be the family of those events (elements of $\mathcal{F}$) that are time-superinvariant.

Let $c \in \mathbb{R}$. The probability measure $\mathcal{W}_c$ on Ω is defined by the conditions that $\omega(0) = c$ with probability one and, for all $0 \le s < t$, $\omega(t) - \omega(s)$ is independent of $\mathcal{F}_s$ and has the Gaussian distribution with mean 0 and variance $t - s$. (In other words, $\mathcal{W}_c$ is the distribution of Brownian motion started at c.) In this paper, we rely on the classical arguments for the existence of $\mathcal{W}_c$ (see e.g. [35], Chap. 2).

Theorem 3.1 *Let $c \in \mathbb{R}$. Each event $E \in \mathcal{I}$ such that $\omega(0) = c$ for all $\omega \in E$ satisfies*

$$\overline{\mathbb{P}}(E) = \mathcal{W}_c(E). \tag{3.2}$$

The main part of (3.2) is the inequality $\leq$, whose proof will occupy us in Sects. 7–11. The easy part $\geq$ will be established in Sect. 6.

Remark 3.2 The time-superinvariant class $\mathcal{I}$ is closed under countable unions and intersections; in particular, it is a monotone class. However, it is not closed under complementation, and so is not a σ-algebra (unlike McCullagh's invariant σ-algebras). An example of a time-superinvariant event E such that E^c is not time-superinvariant is the set of all increasing (not necessarily strictly increasing) $\omega \in \Omega$ satisfying $\lim_{t\to\infty} \omega(t) = \infty$: the implication (3.1) is violated when ω is the identity function (i.e., $\omega(t) = t$ for all t), $f = 0$, and we have E^c in place of E.

Remark 3.3 This remark explains the intuitive meaning of time-superinvariance. Let f be a time transformation. Transforming ω into ω^f is either trivial (ω is replaced by the constant $\omega(0)$, if $f = 0$) or can be split into three steps: (a) remove $[T, \infty)$ from the domain of ω, i.e., transform ω into $\omega' := \omega|_{[0,T)}$, for some $T \in (0, \infty]$ (namely, $T := \lim_{t\to\infty} f(t)$); (b) continuously deform the time interval $[0, T)$ into $[0, T')$ for some $T' \in (0, \infty]$, i.e., transform ω' into $\omega'' \in C[0, T')$ defined by $\omega''(t) := \omega'(g(t))$ for some increasing homeomorphism $g : [0, T') \to [0, T)$ (e.g., the graph of g can be obtained from the graph of f by removing all horizontal pieces); (c) insert countably many (perhaps a finite number of, perhaps zero) horizontal pieces into the graph of ω'' making sure to obtain an element of Ω (inserting a horizontal piece means replacing $\psi \in \Omega$ with

$$\psi'(t) := \begin{cases} \psi(t) & \text{if } t < a \\ \psi(a) & \text{if } a \leq t < b \\ \psi(t + a - b) & \text{if } t \geq b, \end{cases}$$

for some a and b, $a < b$, in the domain of ψ, or

$$\psi'(t) := \begin{cases} \psi(t) & \text{if } t < c \\ \lim_{s\to c} \psi(s) & \text{if } t \geq c \end{cases}$$

if the domain of ψ is $[0, c)$ for some $c < \infty$ and $\lim_{s\to c} \psi(s)$ exists in $\mathbb{R}$). Therefore, the trail of $\omega \in \Omega$ consists of all elements of Ω that can be obtained from ω by an application of the following steps: (a) remove any number of horizontal pieces from the graph of ω; let $[0, T)$ be the domain of the resulting function ω' (it is possible that $T < \infty$; if $T = 0$, output any $\omega'' \in \Omega$ satisfying $\omega''(0) = \omega(0)$); (b) assuming $T > 0$, continuously deform the time interval $[0, T)$ into $[0, T')$ for some $T' \in (0, \infty]$; let ω'' be the resulting function with the domain $[0, T')$; (c) if $T' = \infty$, output ω''; if $T' < \infty$ and $\lim_{t\to T'} \omega(t)$ exists in $\mathbb{R}$, extend ω'' to $[0, \infty)$ in any way making sure that the extension belongs to Ω and output the extension; otherwise, nothing is output. A set E is time-superinvariant if and only if application of these last three steps, (a)–(c), never leads outside E.

Remark 3.4 By the Dubins–Schwarz result [20] and Lemma 3.5 below, we can replace $\mathcal{W}_c$ in the statement of Theorem 3.1 by any probability measure P on $(\Omega, \mathcal{F})$

such that the process $X_t(\omega) := \omega(t)$ is a martingale with respect to P and the filtration $(\mathcal{F}_t)$, is unbounded P-a.s., is nowhere constant P-a.s., and satisfies $X_0 = c$ P-a.s.

Because of its generality, some aspects of Theorem 3.1 may appear counterintuitive. (For example, the conditions we impose on E imply that E contains all $\omega \in \Omega$ satisfying $\omega(0) = c$ whenever E contains the constant c.) In the rest of this section, we specialize Theorem 3.1 to the more intuitive case of divergent and nowhere constant price paths.

Formally, we say that $\omega \in \Omega$ is *nowhere constant* if there is no interval (t_1, t_2), where $0 \leq t_1 < t_2$, such that ω is constant on (t_1, t_2); we say that ω is *divergent* if there is no $c \in \mathbb{R}$ such that $\lim_{t\to\infty} \omega(t) = c$; and we let $\mathrm{DS} \subseteq \Omega$ stand for the set of all $\omega \in \Omega$ that are divergent and nowhere constant. Intuitively, the condition that the price path ω should be nowhere constant means that trading never stops completely, and the condition that ω should be divergent will be satisfied if ω's volatility does not eventually die away (cf. Remark 5.2 in Sect. 5 below). The conditions of being divergent and nowhere constant in the definition of DS are similar to, but weaker than, Dubins and Schwarz's [20] conditions of being unbounded and nowhere constant.

All unbounded and strictly increasing time transformations $f : [0, \infty) \to [0, \infty)$ form a group, which will be denoted $\mathcal{G}$. Let us say that an event E is *time-invariant* if it contains the whole orbit $\{\omega^f \mid f \in \mathcal{G}\}$ of each of its elements $\omega \in E$. It is clear that DS is time-invariant. Unlike $\mathcal{I}$, the time-invariant events form a σ-algebra: E^c is time-invariant whenever E is (cf. Remark 3.2).

The following two lemmas will be needed to specialize Theorem 3.1 to subsets of DS. First of all, it is not difficult to see that for subsets of DS, there is no difference between time-invariance and time-superinvariance (which makes the notion of time-superinvariance much more intuitive for subsets of DS).

Lemma 3.5 *An event $E \subseteq \mathrm{DS}$ is time-superinvariant if and only if it is time-invariant.*

Proof If E (not necessarily $E \subseteq \mathrm{DS}$) is time-superinvariant, $\omega \in E$ and $f \in \mathcal{G}$, we have $\psi := \omega^f \in E$ as $\psi^{f^{-1}} = \omega$. Therefore, time-superinvariance always implies time-invariance.

It is clear that for all $\psi \in \Omega$ and time transformations f, $\psi^f \notin \mathrm{DS}$ unless $f \in \mathcal{G}$. Let $E \subseteq \mathrm{DS}$ be time-invariant, $\omega \in E$, f a time transformation, and $\psi^f = \omega$. Since $\psi^f \in \mathrm{DS}$, we have $f \in \mathcal{G}$, and so $\psi = \omega^{f^{-1}} \in E$. Therefore, time-invariance implies time-superinvariance for subsets of DS. □

Lemma 3.6 *An event $E \subseteq \mathrm{DS}$ is time-superinvariant if and only if $\mathrm{DS} \setminus E$ is time-superinvariant.*

Proof This follows immediately from Lemma 3.5. □

For time-invariant events in DS, (3.2) can be strengthened to assert the coincidence of the outer and the inner content of E with $\mathcal{W}_c(E)$. However, the notions of outer and inner content have to be modified slightly.

For any $B \subseteq \Omega$, a restricted version of outer content can be defined by

$$\overline{\mathbb{P}}(E; B) := \inf\Big\{\mathfrak{S}_0 \;\Big|\; \forall\omega \in B : \liminf_{t\to\infty} \mathfrak{S}_t(\omega) \ge \mathbf{1}_E(\omega)\Big\} = \overline{\mathbb{P}}(E \cap B),$$

with $\mathfrak{S}$ again ranging over the positive capital processes. Intuitively, this is the definition obtained when Ω is replaced by B: we are told in advance that $\omega \in B$. The corresponding restricted version of inner content is

$$\underline{\mathbb{P}}(E; B) := 1 - \overline{\mathbb{P}}\big(E^c; B\big) = \underline{\mathbb{P}}\big(E \cup B^c\big).$$

We use these definitions only in the case where $\overline{\mathbb{P}}(B) = 1$. Lemma 7.3 below shows that in this case $\underline{\mathbb{P}}(E; B) \le \overline{\mathbb{P}}(E; B)$.

We say that $\overline{\mathbb{P}}(E; B)$ and $\underline{\mathbb{P}}(E; B)$ are *restricted to* B. It should be clear by now that these notions are not related to conditional probability $\mathbb{P}(E \mid B)$. Their analogs in measure-theoretic probability are the function $E \mapsto \mathbb{P}(E \cap B)$, in the case of outer content, and the function $E \mapsto \mathbb{P}(E \cup B^c)$, in the case of inner content (assuming B is measurable). Both functions coincide with $\mathbb{P}$ when $\mathbb{P}(B) = 1$.

We also use the restricted versions of the notions "null," "for typical," and "full." For example, E being B*-null* means $\overline{\mathbb{P}}(E; B) = 0$.

Theorem 3.1 immediately implies the following statement about the emergence of the Wiener measure in our trading protocol (another such statement, more general and constructive but also more complicated, will be given in Theorem 5.1(b)).

Corollary 3.7 *Let $c \in \mathbb{R}$. Each event $E \in \mathfrak{I}$ satisfies*

$$\overline{\mathbb{P}}\big(E; \omega(0) = c, \mathrm{DS}\big) = \underline{\mathbb{P}}\big(E; \omega(0) = c, \mathrm{DS}\big) = \mathcal{W}_c(E) \tag{3.3}$$

(*in this context*, $\omega(0) = c$ *stands for the event* $\{\omega \in \Omega \mid \omega(0) = c\}$ *and the comma stands for the intersection*).

Proof The events $E \cap \mathrm{DS} \cap \{\omega \mid \omega(0) = c\}$ and $E^c \cap \mathrm{DS} \cap \{\omega \mid \omega(0) = c\}$ belong to $\mathfrak{I}$; for the first of them, this immediately follows from $\mathrm{DS} \in \mathfrak{I}$ and $\mathfrak{I}$ being closed under intersections (cf. Remark 3.2), and for the second, it suffices to notice that $E^c \cap \mathrm{DS} = \mathrm{DS} \setminus (E \cap \mathrm{DS}) \in \mathfrak{I}$ (cf. Lemma 3.6). Applying (3.2) to these two events and making use of the inequality $\underline{\mathbb{P}} \le \overline{\mathbb{P}}$ (cf. Lemma 7.3 and Eq. (7.1) below), we obtain

$$\begin{aligned}\mathcal{W}_c(E) &= 1 - \mathcal{W}_c\big(E^c \cap \mathrm{DS} \cap \{\omega \mid \omega(0) = c\}\big) = 1 - \overline{\mathbb{P}}\big(E^c; \omega(0) = c, \mathrm{DS}\big)\\ &= \underline{\mathbb{P}}\big(E; \omega(0) = c, \mathrm{DS}\big) \le \overline{\mathbb{P}}\big(E; \omega(0) = c, \mathrm{DS}\big)\\ &= \mathcal{W}_c\big(E \cap \mathrm{DS} \cap \{\omega \mid \omega(0) = c\}\big) = \mathcal{W}_c(E). \qquad \square\end{aligned}$$

We can express the equality (3.3) by saying that the game-theoretic probability of E exists and is equal to $\mathcal{W}_c(E)$ when we restrict our attention to ω in DS satisfying $\omega(0) = c$.

4 Applications

The main goal of this section is to demonstrate the power of Theorem 3.1; in particular, we shall see that it implies the main results of [67] and [65]. One corollary (Corollary 4.5) of Theorem 3.1 solves an open problem posed in [65], and two other corollaries (Corollaries 4.6 and 4.7) give much more precise results. At the end of the section, we draw the reader's attention to several events such that Theorem 3.1 together with very simple game-theoretic arguments show that they are full, while the fact that they are full does not follow from Theorem 3.1 alone.

In this section, we deduce the main results of [67] and [65] and other results as corollaries of Theorem 3.1 and the corresponding results for measure-theoretic Brownian motion. It is, however, still important to have direct game-theoretic proofs such as those given in [65, 67]. This will be discussed in Remark 4.11.

The following obvious fact will be used constantly in this paper: restricted outer content is countably (in particular, finitely) subadditive. (Of course, this fact is obvious only because of our choice of definitions.)

Lemma 4.1 *For any $B \subseteq \Omega$ and any sequence of subsets $E_1, E_2, \ldots$ of Ω,*

$$\overline{\mathbb{P}}\left(\bigcup_{n=1}^{\infty} E_n; B\right) \leq \sum_{n=1}^{\infty} \overline{\mathbb{P}}(E_n; B).$$

In particular, a countable union of B-null sets is B-null.

4.1 Points of increase

Let us say that $t \in [0, \infty)$ is a *point of increase* for $\omega \in \Omega$ if there exists $\delta > 0$ such that $\omega(t_1) \leq \omega(t) \leq \omega(t_2)$ for all $t_1 \in ((t - \delta)^+, t]$ and $t_2 \in [t, t + \delta)$. Points of decrease are defined in the same way except that $\omega(t_1) \leq \omega(t) \leq \omega(t_2)$ is replaced by $\omega(t_1) \geq \omega(t) \geq \omega(t_2)$. We say that ω is *locally constant to the right of* $t \in [0, \infty)$ if there exists $\delta > 0$ such that ω is constant over the interval $[t, t + \delta]$.

A slightly weaker form of the following corollary was proved directly (by adapting Burdzy's [8] proof) in [67].

Corollary 4.2 *Typical ω have no points t of increase or decrease such that ω is not locally constant to the right of t.*

This result (without the clause about local constancy) was established by Dvoretzky, Erdős and Kakutani [24] for Brownian motion, and Dubins and Schwarz [20] noticed that their reduction of continuous martingales to Brownian motion shows that it continues to hold for all almost surely unbounded continuous martingales that are almost surely nowhere constant. We apply Dubins and Schwarz's observation in the game-theoretic framework.

Proof of Corollary 4.2 Let us first consider only the $\omega \in \Omega$ satisfying $\omega(0) = 0$. Consider the set E of all $\omega \in \Omega$ that have points t of increase or decrease such that

ω is not locally constant to the right of t and ω is not locally constant to the left of t (with the obvious definition of local constancy to the left of t; if $t = 0$, every ω is locally constant to the left of t). Since E is time-superinvariant (cf. Remark 3.3), Theorem 3.1 and the Dvoretzky–Erdős–Kakutani result show that the event E is null. And the following standard game-theoretic argument (as in [67], Theorem 1) shows that the event that ω is locally constant to the left but not locally constant to the right of a point of increase or decrease is null. For concreteness, we consider the case of a point of increase. It suffices (see Lemma 4.1) to show that for all rational numbers $b > a > 0$ and $D > 0$, the event that

$$\inf_{t\in[a,b]} \omega(t) = \omega(a) \le \omega(a) + D \le \sup_{t\in[a,b]} \omega(t) \tag{4.1}$$

is null. The simple capital process that starts from $\epsilon > 0$, bets $h_1 := 1/D$ at $\tau_1 = a$, and bets $h_2 := 0$ at time $\tau_2 := \min\{t \ge a \mid \omega(t) \in \{\omega(a) - D\epsilon, \omega(a) + D\}\}$ is positive and turns ϵ (an arbitrarily small amount) into 1 when (4.1) happens. (Notice that this argument works both when $t = 0$ and when $t > 0$.)

It remains to get rid of the restriction $\omega(0) = 0$. Fix a positive capital process $\mathfrak{S}$ satisfying $\mathfrak{S}_0 < \epsilon$ and reaching 1 on ω with $\omega(0) = 0$ that have at least one point t of increase or decrease such that ω is not locally constant to the right of t. Applying $\mathfrak{S}$ to $\omega - \omega(0)$ gives another positive capital process, which will achieve the same goal but without the restriction $\omega(0) = 0$. □

It is easy to see that the qualification about local constancy to the right of t in Corollary 4.2 is essential.

Proposition 4.3 *The outer content of the following event is one*: *There is a point t of increase such that ω is locally constant to the right of t.*

Proof This proof uses Lemma 7.2 stated in Sect. 7 below. Consider the continuous martingale which is Brownian motion that starts at 0 and is stopped as soon as it reaches 1. □

4.2 Variation index

For each interval $[u, v] \subseteq [0, \infty)$, each $p \in (0, \infty)$ and each $\omega \in \Omega$, the strong *p-variation* of ω over $[u, v]$ is defined as

$$\mathrm{v}_p^{[u,v]}(\omega) := \sup_\kappa \sum_{i=1}^{n_\kappa} \left|\omega(t_i) - \omega(t_{i-1})\right|^p, \tag{4.2}$$

where κ ranges over all partitions $u = t_0 \le t_1 \le \cdots \le t_{n_\kappa} = v$ of the interval $[u, v]$. It is obvious that there exists a unique number $\mathrm{vi}^{[u,v]}(\omega) \in [0, \infty]$, called the *variation index* of ω over $[u, v]$, such that $\mathrm{v}_p^{[u,v]}(\omega)$ is finite when $p > \mathrm{vi}^{[u,v]}(\omega)$ and infinite when $p < \mathrm{vi}^{[u,v]}(\omega)$; notice that $\mathrm{vi}^{[u,v]}(\omega) \notin (0, 1)$.

The following result was obtained in [65] (by adapting Bruneau's [7] proof); in measure-theoretic probability it was established by Lepingle ([40], Theorem 1 and Proposition 3) for continuous semimartingales and Lévy [41] for Brownian motion.

Corollary 4.4 *For typical $\omega \in \Omega$, the following is true: For any interval $[u, v] \subseteq [0, \infty)$ such that $u < v$, either $\mathrm{vi}^{[u,v]}(\omega) = 2$ or ω is constant over $[u, v]$.*

(The interval $[u, v]$ was assumed fixed in [65], but this assumption is easy to get rid of.)

Proof Without loss of generality, we restrict our attention to the ω satisfying $\omega(0) = 0$ (see the proof of Corollary 4.2). Consider the set of $\omega \in \Omega$ such that for some interval $[u, v] \subseteq [0, \infty)$, neither $\mathrm{vi}^{[u,v]}(\omega) = 2$ nor ω is constant over $[u, v]$. This set is time-superinvariant (cf. Remark 3.3), and so in conjunction with Theorem 3.1, Lévy's result implies that it is null. □

Corollary 4.4 says that, for typical ω,

$$\mathrm{v}_p^{[u,v]}(\omega) \begin{cases} < \infty & \text{if } p > 2 \\ = \infty & \text{if } p < 2 \text{ and } \omega \text{ is not constant.} \end{cases}$$

However, it does not say anything about the situation for $p = 2$. The following result completes the picture (solving the problem posed in [65], Sect. 5).

Corollary 4.5 *For typical $\omega \in \Omega$, the following is true: For any interval $[u, v] \subseteq [0, \infty)$ such that $u < v$, either $\mathrm{v}_2^{[u,v]}(\omega) = \infty$ or ω is constant over $[u, v]$.*

Proof Lévy [41] proves for Brownian motion that $\mathrm{v}_2^{[u,v]}(\omega) = \infty$ almost surely (for fixed $[u, v]$, which implies the statement for all $[u, v]$). Consider the set of $\omega \in \Omega$ such that for some interval $[u, v] \subseteq [0, \infty)$, neither $\mathrm{v}_2^{[u,v]}(\omega) = \infty$ nor ω is constant over $[u, v]$. This set is time-superinvariant, and so in conjunction with Theorem 3.1, Lévy's result implies that it is null. □

4.3 More precise results

Theorem 3.1 allows us to deduce much stronger results than Corollaries 4.4 and 4.5 from known results about Brownian motion.

Define $\ln^* u := 1 \vee |\ln u|$, $u > 0$, and let $\psi : [0, \infty) \to [0, \infty)$ be Taylor's [62] function

$$\psi(u) := \frac{u^2}{2 \ln^* \ln^* u}$$

(with $\psi(0) := 0$). For $\omega \in \Omega$, $T \in [0, \infty)$ and $\phi : [0, \infty) \to [0, \infty)$, set

$$\mathrm{v}_{\phi,T}(\omega) := \sup_\kappa \sum_{i=1}^{n_\kappa} \phi\bigl(\bigl|\omega(t_i) - \omega(t_{i-1})\bigr|\bigr),$$

where κ ranges over all partitions $0 = t_0 \leq t_1 \leq \cdots \leq t_{n_\kappa} = T$ of $[0, T]$. In the previous subsection we considered the case $\phi(u) := u^p$; another interesting case is $\phi := \psi$. See [6] for a much more explicit expression for $\mathrm{v}_{\psi,T}(\omega)$.

Corollary 4.6 *For typical ω,*

$$\forall T \in [0, \infty): \quad \mathrm{v}_{\psi,T}(\omega) < \infty.$$

Suppose $\phi : [0, \infty) \to [0, \infty)$ is such that $\psi(u) = o(\phi(u))$ as $u \to 0$. For typical ω,

$$\forall T \in [0, \infty): \quad \omega \textit{ is constant on } [0, T] \textit{ or } \mathrm{v}_{\phi,T}(\omega) = \infty.$$

Corollary 4.6 refines Corollaries 4.4 and 4.5; it will be further strengthened by Corollary 4.7.

The quantity $\mathrm{v}_{\psi,T}(\omega)$ is not nearly as fundamental as the following quantity introduced by Taylor [62]: for $\omega \in \Omega$ and $T \in [0, \infty)$, set

$$\mathrm{w}_T(\omega) := \lim_{\delta \to 0} \sup_{\kappa \in K_\delta[0,T]} \sum_{i=1}^{n_\kappa} \psi\left(\left|\omega(t_i) - \omega(t_{i-1})\right|\right), \tag{4.3}$$

where $K_\delta[0, T]$ is the set of all partitions $0 = t_0 \le \cdots \le t_{n_\kappa} = T$ of $[0, T]$ whose mesh is less than δ, i.e., $\max_i(t_i - t_{i-1}) < \delta$. Notice that the expression after $\lim_{\delta\to 0}$ in (4.3) is increasing in δ; therefore $\mathrm{w}_T(\omega) \le \mathrm{v}_{\psi,T}(\omega)$.

The following corollary contains Corollaries 4.4–4.6 as special cases. It is similar to Corollary 4.6 but stated in terms of the process w.

Corollary 4.7 *For typical ω,*

$$\forall T \in [0, \infty): \quad \omega \textit{ is constant on } [0, T] \textit{ or } \mathrm{w}_T(\omega) \in (0, \infty). \tag{4.4}$$

Proof First let us check that under the Wiener measure (4.4) holds for almost all ω. It is sufficient to prove that $\mathrm{w}_T = T$ for all $T \in [0, \infty)$ a.s. Furthermore, it is sufficient to consider only rational $T \in [0, \infty)$. Therefore, it is sufficient to consider a fixed rational $T \in [0, \infty)$. And for a fixed T, $\mathrm{w}_T = T$ a.s. follows from Taylor's result ([62], Theorem 1).

As usual, let us restrict our attention to the case $\omega(0) = 0$. In view of Theorem 3.1, it suffices to check that the complement of the event (4.4) is time-superinvariant, i.e., to check (3.1), where E is the complement of (4.4). In other words, it suffices to check that $\omega^f = \omega \circ f$ satisfies (4.4) whenever ω satisfies (4.4). This follows from Lemma 4.8 below, which says that $\mathrm{w}_T(\omega \circ f) = \mathrm{w}_{f(T)}(\omega)$. □

Lemma 4.8 *Let $T \in [0, \infty)$, $\omega \in \Omega$ and f be a time transformation. Then we have $\mathrm{w}_T(\omega \circ f) = \mathrm{w}_{f(T)}(\omega)$.*

Proof Fix $T \in [0, \infty)$, $\omega \in \Omega$, a time transformation f and $c \in [0, \infty]$. Our goal is to prove

$$\lim_{\delta \to 0} \sup_{\kappa \in K_\delta[0,f(T)]} \sum_{i=1}^{n_\kappa} \psi\left(\left|\omega(t_i) - \omega(t_{i-1})\right|\right) = c$$

$$\Longrightarrow \lim_{\delta \to 0} \sup_{\kappa \in K_\delta[0,T]} \sum_{i=1}^{n_\kappa} \psi\left(\left|\omega\left(f(t_i)\right) - \omega\left(f(t_{i-1})\right)\right|\right) = c, \tag{4.5}$$

in the notation of (4.3). Suppose the antecedent in (4.5) holds. Notice that the two $\lim_{\delta\to 0}$ in (4.5) can be replaced by $\inf_{\delta>0}$.

To prove that the limit on the right-hand side of (4.5) is $\le c$, take any $\epsilon > 0$. We assume $c < \infty$ (the case $c = \infty$ is trivial). Let $\delta > 0$ be so small that

$$\sup_{\kappa\in K_\delta[0,f(T)]} \sum_{i=1}^{n_\kappa} \psi\bigl(\bigl|\omega(t_i)-\omega(t_{i-1})\bigr|\bigr) < c+\epsilon.$$

Let $\delta' > 0$ be so small that $|t-t'| < \delta'$ implies that $|f(t)-f(t')| < \delta$. Since $f(\kappa) \in K_\delta[0, f(T)]$ whenever $\kappa \in K_{\delta'}[0,T]$,

$$\sup_{\kappa\in K_{\delta'}[0,T]} \sum_{i=1}^{n_\kappa} \psi\bigl(\bigl|\omega\bigl(f(t_i)\bigr)-\omega\bigl(f(t_{i-1})\bigr)\bigr|\bigr) < c+\epsilon.$$

To prove that the limit on the right-hand side of (4.5) is $\ge c$, take any $\epsilon > 0$ and $\delta' > 0$. We assume $c < \infty$ (the case $c = \infty$ can be considered analogously). Place a finite number N of points including 0 and T onto the interval $[0,T]$ so that the distance between any pair of adjacent points is less than δ'; this set of points will be denoted κ_0. Let $\delta > 0$ be so small that $\psi(|\omega(t'')-\omega(t')|) < \epsilon/N$ whenever $|t''-t'| < \delta$. Choose a partition $\kappa = \{t_0,\dots,t_n\} \in K_\delta[0, f(T)]$ satisfying

$$\sum_{i=1}^{n} \psi\bigl(\bigl|\omega(t_i)-\omega(t_{i-1})\bigr|\bigr) > c-\epsilon.$$

Let $\kappa' = \{t'_0,\dots,t'_n\}$ be a partition of the interval $[0,T]$ satisfying $f(\kappa') = \kappa$. This partition will satisfy

$$\sum_{i=1}^{n} \psi\bigl(\bigl|\omega\bigl(f(t'_i)\bigr)-\omega\bigl(f(t'_{i-1})\bigr)\bigr|\bigr) > c-\epsilon,$$

and the union $\kappa'' = \{t''_0,\dots,t''_{N+n}\}$ (with its elements listed in the increasing order) of κ_0 and κ' will satisfy

$$\sum_{i=1}^{N+n} \psi\bigl(\bigl|\omega\bigl(f(t''_i)\bigr)-\omega\bigl(f(t''_{i-1})\bigr)\bigr|\bigr) > c-2\epsilon.$$

Since $\kappa'' \in K_{\delta'}[0,T]$ and ϵ and δ' can be taken arbitrarily small, this completes the proof. □

The value $\mathrm{w}_T(\omega)$ defined by (4.3) can be interpreted as the quadratic variation of the price path ω over the time interval $[0,T]$. Another non-stochastic definition of quadratic variation (see (5.2)) will serve us in Sect. 5 as the basis for the proof of Theorem 3.1. For the equivalence of the two definitions, see Remark 5.6.

4.4 Limitations of Theorem 3.1

We said earlier that Theorem 3.1 implies the main result of [67] (see Corollary 4.2). This is true in the sense that the extra game-theoretic argument used in the proof of Corollary 4.2 was very simple. But this simple argument was essential: in this subsection, we shall see that Theorem 3.1 per se does not imply the full statement of Corollary 4.2.

Let $c \in \mathbb{R}$ and $E \subseteq \Omega$ be such that $\omega(0) = c$ for all $\omega \in E$. Suppose the set E is null. We can say that the equality $\overline{\mathbb{P}}(E) = 0$ can be deduced from Theorem 3.1 and the properties of Brownian motion if (and only if) $\mathcal{W}_c(\overline{E}) = 0$, where $\overline{E}$ is the smallest time-superinvariant set containing E (it is clear that such a set exists and is unique). It would be nice if all equalities $\overline{\mathbb{P}}(E) = 0$, for all null sets E satisfying $\forall \omega \in E : \omega(0) = c$, could be deduced from Theorem 3.1 and the properties of Brownian motion. We shall see later (Proposition 4.9) that this is not true even for some fundamental null events E; an example of such an event will now be given.

Let us say that a closed interval $[t_1, t_2] \subseteq [0, \infty)$ is an *interval of local maximum* for $\omega \in \Omega$ if (a) ω is constant on $[t_1, t_2]$ but not constant on any larger interval containing $[t_1, t_2]$, and (b) there exists $\delta > 0$ such that $\omega(s) \le \omega(t)$ for all $s \in ((t_1 - \delta)^+, t_1) \cup (t_2, t_2 + \delta)$ and all $t \in [t_1, t_2]$. In the case where $t_1 = t_2$, we say "point" instead of "interval." It is shown in [67] (Corollary 3) that for typical ω, all intervals of local maximum are points; this also follows from Corollary 4.2, and is very easy to check directly (using the same argument as in the proof of Corollary 4.2). Let E be the null event that $\omega(0) = c$ and not all intervals of local maximum of ω are points. Proposition 4.9 says that $\overline{\mathbb{P}}(E) = 0$ cannot be deduced from Theorem 3.1 and the properties of Brownian motion. This implies that Corollary 4.2 also cannot be deduced from Theorem 3.1 and the properties of Brownian motion, despite the fact that the deduction is possible with the help of a very easy game-theoretic argument.

Before stating and proving Proposition 4.9, we formally introduce the operator $E \mapsto \overline{E}$ and show that it is a bona fide closure operator. For each $E \subseteq \Omega$, $\overline{E}$ is defined to be the union of the trails of all points in E. It can be checked that $E \mapsto \overline{E}$ satisfies the standard properties of closure operators: $\overline{\emptyset} = \emptyset$ and $\overline{E_1 \cup E_2} = \overline{E_1} \cup \overline{E_2}$ are obvious, and $\overline{\overline{E}} = \overline{E}$ and $E \subseteq \overline{E}$ follow from the fact that the time transformations constitute a monoid. Therefore ([25], Theorem 1.1.3 and Proposition 1.2.7), $E \mapsto \overline{E}$ is the operator of closure in some topology on Ω, which may be called the *time-superinvariant topology*. A set $E \subseteq \Omega$ is closed in this topology if and only if it contains the trail of any of its elements.

Proposition 4.9 *Let $c \in \mathbb{R}$ and E be the set of all $\omega \in \Omega$ such that $\omega(0) = c$ and ω has an interval of local maximum that is not a point. Then E and $\overline{E}$ are events and*

$$0 = \mathcal{W}_c(E) = \overline{\mathbb{P}}(E) < \overline{\mathbb{P}}(\overline{E}) = \mathcal{W}_c(\overline{E}) = 1. \tag{4.6}$$

Proof For the equality $\overline{\mathbb{P}}(E) = 0$, see above. The equality $\mathcal{W}_c(E) = 0$ is a well-known fact (and follows from $\overline{\mathbb{P}}(E) = 0$ and Lemma 6.3 below). It suffices to prove that $\overline{E} \in \mathcal{F}$ and $\mathcal{W}_c(\overline{E}) = 1$; Theorem 3.1 will then imply $\overline{\mathbb{P}}(\overline{E}) = 1$. The inclusion $\overline{E} \in \mathcal{F}$ and the equality $\mathcal{W}_c(\overline{E}) = 1$ follow from the following explicit description of $\overline{E}$: this set consists of all $\omega \in \Omega$ with $\omega(0) = c$ that are not increasing functions. This can be seen from Remark 3.3 or from the following argument. If ω is increasing, ω^f will also be increasing for any time transformation f. Combining this with (3.1) we can see that the set of all ω that are not increasing is time-superinvariant; since this set contains E, it also contains $\overline{E}$. In the opposite direction, we are required to show that any $\omega \in \Omega$ that is not increasing is an element of $\overline{E}$, i.e., there exists a time transformation f such that $\omega^f \in E$. Fix such ω and find $0 \le a < b$ such that $\omega(a) > \omega(b)$. Let $m \in [0, b]$ be the smallest element of $\arg\max_{t\in[0,b]} \omega(t)$. Applying the time transformation

$$f(t) := \begin{cases} t & \text{if } t < m \\ m & \text{if } m \le t < m+1 \\ t-1 & \text{if } t \ge m+1 \end{cases}$$

to ω, we obtain an element of E. □

Remark 4.10 Another event E that satisfies (4.6) is the set of all $\omega \in \Omega$ such that $\omega(0) = c$ and ω has an interval of local maximum that is not a point, or has an interval of local minimum that is not a point (with the obvious definition of intervals and points of local minimum). Then $\overline{E}$ is the event that consists of all non-constant ω with $\omega(0) = c$. This is the largest possible $\overline{E}$ for E satisfying $\overline{\mathbb{P}}(E) = 0$ (provided we consider only ω with $\omega(0) = c$): indeed, if the constant c is in $\overline{E}$, c will also be in E, and so $\overline{\mathbb{P}}(E) = 1$.

Proposition 4.9 shows that Theorem 3.1 does not make all other game-theoretic arguments redundant. What is interesting is that already very simple arguments suffice to deduce all results in [65, 67].

Remark 4.11 Theorem 3.1 does not make the game-theoretic arguments in [65, 67] redundant also in another, perhaps even more important, respect. For example, Corollary 4.2 is an existence result: it asserts the existence of a trading strategy whose capital process is positive and increases from 1 to ∞ when ω has a point t of increase or decrease such that ω is not locally constant to the right of t. In principle, such a strategy could be extracted from the proof of Theorem 3.1, but it would be extremely complicated and non-intuitive; the result would remain essentially an existence result. The proof of Theorem 2 in [67], on the contrary, constructs an explicit trading strategy exploiting the existence of points of increase or decrease. Similarly, the proof of Theorem 1 in [65] constructs an explicit trading strategy whose existence is asserted in Corollary 4.4. The recent paper [63] partially extends Corollary 4.4 to discontinuous price paths, showing that $\mathrm{vi}^{[0,T]}(\omega) \le 2$ for all $T < \infty$ for typical ω. The trading strategy constructed in [63] for profiting from $\mathrm{vi}^{[0,T]}(\omega) > 2$ is especially intuitive: it just combines (following Stricker's [58] idea) the strategies for profiting from $\liminf_t \omega(t) < a < b < \limsup_t \omega(t)$ implicit in the standard proof of Doob's martingale convergence theorem.

Remark 4.12 All results discussed in this section are about sets of outer content zero or inner content one, and one might suspect that the class $\mathcal{J}$ is so small that

$\mathcal{W}_c(E) \in \{0, 1\}$ for all $c \in \mathbb{R}$ and all $E \in \mathcal{I}$ such that $\omega(0) = c$ when $\omega \in E$; this would have been another limitation of Theorem 3.1. However, it is easy to check that for each $p \in [0, 1]$ and each $c \in \mathbb{R}$, there exists $E \in \mathcal{I}$ satisfying $\omega(0) = c$ for all $\omega \in E$ and $\mathcal{W}_c(E) = p$. Indeed, without loss of generality we can take $c := p$, and we can then define E to be the event that $\omega(0) = p$, ω reaches levels 0 and 1, and ω reaches level 1 before reaching level 0.

5 Main result: constructive version

For each $n \in \{0, 1, \dots\}$, let $\mathbb{D}_n := \{k2^{-n} \mid k \in \mathbb{Z}\}$ and define a sequence of stopping times T_k^n, $k = -1, 0, 1, 2, \dots$, inductively by $T_{-1}^n := 0$,

$$T_0^n(\omega) := \inf\bigl\{t \ge 0 \bigm| \omega(t) \in \mathbb{D}_n\bigr\},$$
$$T_k^n(\omega) := \inf\bigl\{t \ge T_{k-1}^n \bigm| \omega(t) \in \mathbb{D}_n \text{ and } \omega(t) \ne \omega\bigl(T_{k-1}^n\bigr)\bigr\}, \quad k = 1, 2, \dots$$

(as usual, $\inf \emptyset := \infty$). For each $t \in [0, \infty)$ and $\omega \in \Omega$, define

$$A_t^n(\omega) := \sum_{k=0}^{\infty} \bigl(\omega\bigl(T_k^n \wedge t\bigr) - \omega\bigl(T_{k-1}^n \wedge t\bigr)\bigr)^2, \quad n = 0, 1, 2, \dots, \tag{5.1}$$

(cf. (4.2) with $p = 2$) and set

$$\overline{A}_t(\omega) := \limsup_{n\to\infty} A_t^n(\omega), \qquad \underline{A}_t(\omega) := \liminf_{n\to\infty} A_t^n(\omega). \tag{5.2}$$

We shall see later (Theorem 5.1(a)) that the event $\{\forall t \in [0, \infty) : \overline{A}_t = \underline{A}_t\}$ is full and that for typical ω the functions $\overline{A}(\omega) : t \in [0, \infty) \mapsto \overline{A}_t(\omega)$ and $\underline{A}(\omega) : t \in [0, \infty) \mapsto \underline{A}_t(\omega)$ are elements of Ω (in particular, they are finite). But in general we can only say that $\overline{A}(\omega)$ and $\underline{A}(\omega)$ are positive increasing functions (not necessarily strictly increasing) that can even take the value ∞. For each $s \in [0, \infty)$, define the stopping time

$$\tau_s := \inf\Bigl\{t \ge 0 \Bigm| \overline{A}|_{[0,t)} = \underline{A}|_{[0,t)} \in C[0, t) \text{ and } \sup_{u<t} \overline{A}_u = \sup_{u<t} \underline{A}_u \ge s\Bigr\}. \tag{5.3}$$

(We shall see in Lemma 8.3 that this is indeed a stopping time.) It will be convenient to use the following convention: an event stated in terms of A_∞, such as $A_\infty = \infty$, happens if and only if $\overline{A} = \underline{A} \in \Omega$ and $A_\infty := \overline{A}_\infty = \underline{A}_\infty$ satisfies the given condition.

Let P be a function defined on the power set of Ω and taking values in $[0, 1]$ (such as $\overline{\mathbb{P}}$ or $\underline{\mathbb{P}}$), and let $f : \Omega \to \Psi$ be a mapping from Ω to another set Ψ. The *pushforward* Pf^{-1} of P by f is the function on the power set of Ψ defined by

$$Pf^{-1}(E) := P\bigl(f^{-1}(E)\bigr), \quad E \subseteq \Psi.$$

An especially important mapping for this paper is the *normalizing time transformation* ntt $: \Omega \to \mathbb{R}^{[0,\infty)}$ defined as follows: for each $\omega \in \Omega$, ntt(ω) is the time-changed price path $s \mapsto \omega(\tau_s)$, $s \in [0, \infty)$, with $\omega(\infty)$ set to e.g. 0. (We call it "normalizing" since our goal is to ensure $\overline{A}_t(\text{ntt}(\omega)) = \underline{A}_t(\text{ntt}(\omega)) = t$ for all $t \ge 0$ for typical ω.) For each $c \in \mathbb{R}$, let

$$\overline{Q}_c := \overline{\mathbb{P}}\big(\cdot\,; \omega(0) = c, A_\infty = \infty\big)\,\mathrm{ntt}^{-1}, \tag{5.4}$$

$$\underline{Q}_c := \underline{\mathbb{P}}\big(\cdot\,; \omega(0) = c, A_\infty = \infty\big)\,\mathrm{ntt}^{-1} \tag{5.5}$$

(as before, the commas stand for conjunction in this context) be the pushforwards of the restricted outer and inner content

$$E \subseteq \Omega \mapsto \overline{\mathbb{P}}\big(E; \omega(0) = c, A_\infty = \infty\big),$$
$$E \subseteq \Omega \mapsto \underline{\mathbb{P}}\big(E; \omega(0) = c, A_\infty = \infty\big),$$

respectively, by the normalizing time transformation ntt.

As mentioned earlier, we use restricted outer and inner content $\overline{\mathbb{P}}(E; B)$ and $\underline{\mathbb{P}}(E; B)$ only when $\overline{\mathbb{P}}(B) = 1$. In Sect. 7, (7.2), we shall see that indeed $\overline{\mathbb{P}}(\omega(0) = c, A_\infty = \infty) = 1$.

The next theorem shows that the pushforwards of $\overline{\mathbb{P}}$ and $\underline{\mathbb{P}}$ we have just defined are closely connected with the Wiener measure. Remember that for each $c \in \mathbb{R}$, $\mathcal{W}_c$ is the probability measure on $(\Omega, \mathcal{F})$ which is the pushforward of the Wiener measure $\mathcal{W}_0$ by the mapping $\omega \in \Omega \mapsto \omega + c$ (i.e., $\mathcal{W}_c$ is the distribution of Brownian motion over the time period $[0, \infty)$ started from c).

Theorem 5.1 (a) *For typical ω, the function*

$$A(\omega) : t \in [0, \infty) \mapsto A_t(\omega) := \overline{A}_t(\omega) = \underline{A}_t(\omega)$$

exists, is an increasing element of Ω with $A_0(\omega) = 0$, and has the same intervals of constancy as ω.

(b) *For all $c \in \mathbb{R}$, the restriction of both $\overline{Q}_c$ and $\underline{Q}_c$ to $\mathcal{F}$ coincides with the measure $\mathcal{W}_c$ on Ω (in particular, $\underline{Q}_c(\Omega) = 1$).*

Remark 5.2 The value $A_t(\omega)$ can be interpreted as the total volatility of the price path ω over the time period $[0, t]$. Theorem 5.1(b) implies that typical ω satisfying $A_\infty(\omega) = \infty$ are unbounded (in particular, divergent). If $A_\infty(\omega) < \infty$, the total volatility $A_{t+1}(\omega) - A_t(\omega)$ of ω over $(t, t+1]$ tends to 0 as $t \to \infty$, and so the volatility of ω can be said to die away.

Remark 5.3 Theorem 5.1 continues to hold if the restriction "; $\omega(0) = c, A_\infty = \infty$)" in the definitions (5.4) and (5.5) is replaced by "; $\omega(0) = c$, ω is unbounded)" (in analogy with [20]).

Remark 5.4 Theorem 5.1 depends on the arbitrary choice $(\mathbb{D}_n)$ of the sequence of grids to define the quadratic variation process A. To make this less arbitrary, we could consider all grids whose mesh tends to zero fast enough and which are definable in the standard language of set theory (similarly to Wald's [69] suggested requirement for von Mises's collectives). Dudley's [21] result shows that the rate of convergence $o(1/\log n)$ of the mesh to zero is sufficient for Brownian motion and partitions of the horizontal (time) axis, and de la Vega's [17] result shows that this rate is slowest possible. It is an open question what the optimal rate of convergence is when quadratic variation is defined via partitions of the vertical axis, as in the present paper.

Remark 5.5 In this paper, we construct the quadratic variation A and define the stopping times τ_s in terms of A. Dubins and Schwarz [20] construct τ_s directly (in a very similar way to our construction of A). An advantage of our construction (the game-theoretic counterpart of that in [34]) is that the function $A(\omega)$ is continuous for typical ω, whereas the event that the function $s \mapsto \tau_s(\omega)$ is continuous has inner content zero. (Dubins and Schwarz's extra assumptions make this function continuous for almost all ω.)

Remark 5.6 Theorem 3.1 implies that the two notions of quadratic variation that we have discussed so far, $\mathrm{w}_t(\omega)$ defined by (4.3) and $A_t(\omega)$, coincide for all t for typical ω. Indeed, since $\mathrm{w}_t = A_t = t$, $\forall t \in [0,\infty)$, holds almost surely in the case of Brownian motion (see Lemma 8.4 for $A_t = t$), it suffices to check that the complement of the event $\{\forall t \in [0,\infty) : \mathrm{w}_t = A_t\}$ belongs to $\mathcal{I}$. This follows from Lemma 4.8 and the analogous statement for A: if $\mathrm{w}_t(\omega) = A_t(\omega)$ for all t, we also have

$$\mathrm{w}_t(\omega \circ f) = \mathrm{w}_{f(t)}(\omega) = A_{f(t)}(\omega) = A_t(\omega \circ f)$$

for all t.

6 Functional generalizations

Theorems 3.1 and 5.1(b) are about outer content for sets, but the former and part of the latter can be generalized to cover the following more general notion of outer content for *functionals*, i.e., real-valued functions on Ω. The *outer content* of a positive functional F *restricted to* a set $B \subseteq \Omega$ is defined by

$$\overline{\mathbb{E}}(F; B) := \inf\left\{\mathfrak{S}_0 \,\middle|\, \forall \omega \in B : \liminf_{t\to\infty} \mathfrak{S}_t(\omega) \ge F(\omega)\right\}, \tag{6.1}$$

where $\mathfrak{S}$ ranges over the positive capital processes. This is the price of the cheapest positive superhedge for F when Reality is restricted to choosing $\omega \in B$. Restricted outer content for functionals generalizes restricted outer content for sets: $\overline{\mathbb{P}}(E; B) = \overline{\mathbb{E}}(\mathbf{1}_E; B)$ for all $E \subseteq \Omega$. When $B = \Omega$, we abbreviate $\overline{\mathbb{E}}(F; B)$ to $\overline{\mathbb{E}}(F)$ and refer to $\overline{\mathbb{E}}(F)$ as the *outer content* of F. Notice that $\overline{\mathbb{E}}(F; B) = \overline{\mathbb{E}}(F\,\mathbf{1}_B)$.

Let us say that a positive functional $F : \Omega \to [0,\infty)$ is $\mathcal{I}$*-measurable* if for each constant $c \in [0,\infty)$, the set $\{\omega \mid F(\omega) \ge c\}$ is in $\mathcal{I}$. (We need to spell out this definition since $\mathcal{I}$ is not a σ-algebra; cf. Remark 3.2.) Notice that the $\mathcal{I}$-measurability of F means that F is $\mathcal{F}$-measurable and, for each $\omega \in \Omega$ and each time transformation f,

$$F\left(\omega^f\right) \le F(\omega) \tag{6.2}$$

(cf. (3.1)).

Remark 6.1 The presence of $\le$ in (6.2) is natural as, intuitively, transforming ω into ω^f may involve cutting off part of ω (step (a) at the beginning of Remark 3.3). It is clear that $F(\omega^f) = F(\omega)$ when $f \in \mathcal{G}$.

In this paper, we shall in fact prove the following generalization of Theorem 3.1.

Theorem 6.2 *Let $c \in \mathbb{R}$. Each bounded positive $\mathcal{I}$-measurable functional $F : \Omega \to [0, \infty)$ satisfies*

$$\overline{\mathbb{E}}\big(F; \omega(0) = c\big) = \int F \, d\mathcal{W}_c. \tag{6.3}$$

The proof of the inequality $\geq$ in (6.3) is easy and is accomplished by the following lemma; it suffices to apply it to $\mathcal{W}_c$ in place of P and to $F \mathbf{1}_{\{\omega(0)=c\}}$ in place of F.

Lemma 6.3 *Let P be a probability measure on $(\Omega, \mathcal{F})$ such that the process $X_t(\omega) := \omega(t)$ is a martingale with respect to P and the filtration $(\mathcal{F}_t)$. Then $\int F \, dP \leq \overline{\mathbb{E}}(F)$ for any positive $\mathcal{F}$-measurable functional F.*

Proof Fix a positive $\mathcal{F}$-measurable functional F and let $\epsilon > 0$. Find a positive capital process $\mathfrak{S}$ of the form (2.2) such that $\mathfrak{S}_0 < \overline{\mathbb{E}}(F) + \epsilon$ and $\liminf_{t\to\infty} \mathfrak{S}_t(\omega) \geq F(\omega)$ for all $\omega \in \Omega$. It can be checked using the optional sampling theorem (it is here that the boundedness of Sceptic's bets is used) that each addend in (2.1) is a martingale, and so each partial sum in (2.1) is a martingale and (2.1) itself is a local martingale. Since each addend in (2.2) is a positive local martingale, it is a supermartingale. We can see that each addend in (2.2) is a positive continuous supermartingale. Using Fatou's lemma and the monotone convergence theorem, we now obtain

$$\begin{aligned}
\int F \, dP &\leq \int \liminf_{t\to\infty} \mathfrak{S}_t \, dP \leq \liminf_{t\to\infty} \int \mathfrak{S}_t \, dP \\
&= \liminf_{t\to\infty} \int \sum_{n=1}^{\infty} \mathcal{K}_t^{G_n, c_n} \, dP = \liminf_{t\to\infty} \sum_{n=1}^{\infty} \int \mathcal{K}_t^{G_n, c_n} \, dP \\
&\leq \liminf_{t\to\infty} \sum_{n=1}^{\infty} c_n = \mathfrak{S}_0 < \overline{\mathbb{E}}(F) + \epsilon,
\end{aligned} \tag{6.4}$$

where t can be assumed to take only integer values. Since ϵ can be arbitrarily small, this implies the statement of the lemma. □

We shall deduce the inequality $\leq$ in Theorem 6.2 from the following generalization of the part of Theorem 5.1(b) concerning $\overline{Q}_c$.

Theorem 6.4 *For any $c \in \mathbb{R}$ and any bounded positive $\mathcal{F}$-measurable functional $F : \Omega \to [0, \infty)$,*

$$\overline{\mathbb{E}}\big(F \circ \mathrm{ntt}; \omega(0) = c, A_\infty = \infty\big) = \int_\Omega F \, d\mathcal{W}_c \tag{6.5}$$

(*with $\circ$ standing for composition of two functions and with the convention that $(F \circ \mathrm{ntt})(\omega) := 0$ when $\omega \notin \mathrm{ntt}^{-1}(\Omega)$*).

We shall check that Theorem 6.4 (namely, the inequality $\le$ in (6.5)) indeed implies Theorem 5.1(b) in Sect. 10. In this section, we only prove the easy inequality $\ge$ in (6.5). In Lemma 8.4, we shall see that $\overline{A}_t(\omega) = \underline{A}_t(\omega) = t$ for all $t \in [0, \infty)$ for $\mathcal{W}_c$-almost all ω; therefore, $\mathrm{ntt}(\omega) = \omega$ for $\mathcal{W}_c$-almost all ω. In conjunction with Lemma 6.3, this implies the inequality $\ge$ in (6.5), i.e.,

$$\overline{\mathbb{E}}\big(F \circ \mathrm{ntt}; \omega(0) = c, A_\infty = \infty\big) = \overline{\mathbb{E}}\big((F \circ \mathrm{ntt})\, \mathbf{1}_{\{\omega(0)=c, A_\infty=\infty\}}\big)$$
$$\ge \int_\Omega (F \circ \mathrm{ntt})\, \mathbf{1}_{\{\omega(0)=c, A_\infty=\infty\}}\, d\mathcal{W}_c = \int_\Omega F\, d\mathcal{W}_c.$$

Remark 6.5 Theorem 6.2 gives the price of the cheapest superhedge for the contingent claim F, but it is not applicable to the usual contingent claims traded in financial markets, which are not $\mathcal{I}$-measurable. The theorem would be applicable to the imaginary contingent claim paying $f(\omega(\tau_S))$ at time τ_S (cf. (5.3); there is no payment if $\tau_S = \infty$), where $S > 0$ is a given constant and f is a given positive, bounded and measurable payoff function. (If the interest rate r is constant but different from 0, we can consider the contingent claim paying $e^{\tau_S r} f(\omega(\tau_S))$ at time τ_S.) The price of the cheapest superhedge will be $\int f(\psi(S))\mathcal{W}_c(d\psi)$, where $c := \omega(0)$, if there are no restrictions on $\omega \in \Omega$, but will become $\int f(\psi(S))\, \mathbf{1}_{\{\forall s \in [0,S]: \psi(s) \ge 0\}}\, \mathcal{W}_c(d\psi)$ if ω is restricted to be positive (as in many real financial markets).

Sections 7–11 are mainly devoted to the proof of the remaining statements in Theorems 5.1, 6.2 and 6.4, whereas Sect. 12 is devoted to the discussion of the financial meaning of our results and their connections with related probabilistic and financial literature.

7 Coherence

The following trivial result says that our trading game is *coherent*, in the sense that $\overline{\mathbb{P}}(\Omega) = 1$ (i.e., no positive capital process increases its value between time 0 and ∞ by more than a strictly positive constant for all $\omega \in \Omega$).

Lemma 7.1 $\overline{\mathbb{P}}(\Omega) = 1$. *Moreover, for each* $c \in \mathbb{R}$, $\overline{\mathbb{P}}(\omega(0) = c) = 1$.

Proof No positive capital process can strictly increase its value on a constant $\omega \in \Omega$. $\square$

Lemma 7.1, however, does not even guarantee that the set of non-constant elements of Ω has outer content one. The theory of measure-theoretic probability provides us with a plethora of non-trivial events of outer content one.

Lemma 7.2 *Let E be an event that almost surely contains the sample path of a continuous martingale with time interval* $[0, \infty)$. *Then* $\overline{\mathbb{P}}(E) = 1$.

Proof This is a special case of Lemma 6.3 applied to $F := \mathbf{1}_E$. □

In particular, applying Lemma 7.2 to Brownian motion started at $c \in \mathbb{R}$ gives

$$\overline{\mathbb{P}}\big(\omega(0) = c, \omega \in \mathrm{DS}\big) = 1 \tag{7.1}$$

and

$$\overline{\mathbb{P}}\big(\omega(0) = c, A_\infty = \infty\big) = 1 \tag{7.2}$$

(for the latter we also need Lemma 8.4 below). Both (7.1) and (7.2) have been used above.

Lemma 7.3 *Let $\overline{\mathbb{P}}(B) = 1$. For every set $E \subseteq \Omega$, $\underline{\mathbb{P}}(E; B) \le \overline{\mathbb{P}}(E; B)$.*

Proof Suppose $\underline{\mathbb{P}}(E; B) > \overline{\mathbb{P}}(E; B)$ for some E; by the definition of $\underline{\mathbb{P}}$, this would mean that $\overline{\mathbb{P}}(E; B) + \overline{\mathbb{P}}(E^c; B) < 1$. Since $\overline{\mathbb{P}}(\cdot; B)$ is finitely subadditive (see Lemma 4.1), this would imply $\overline{\mathbb{P}}(\Omega; B) < 1$, which is equivalent to $\overline{\mathbb{P}}(B) < 1$ and therefore contradicts our assumption. □

8 Existence of quadratic variation

In this paper, the set Ω is always equipped with the metric

$$\rho(\omega_1, \omega_2) := \sum_{d=1}^{\infty} 2^{-d} \sup_{t \in [0, 2^d]} \big(|\omega_1(t) - \omega_2(t)| \wedge 1\big) \tag{8.1}$$

(and the corresponding topology and Borel σ-algebra, the latter coinciding with $\mathcal{F}$). This makes it a complete and separable metric space. The main goal of this section is to prove that the sequence of continuous functions $t \in [0, \infty) \mapsto A_t^n(\omega)$ is convergent in Ω for typical ω; this is done in Lemma 8.2. This will establish the existence of $A(\omega) \in \Omega$ for typical ω, which is part of Theorem 5.1(a). It is obvious that when it exists, $A(\omega)$ is increasing and $A_0(\omega) = 0$. The last part of Theorem 5.1(a), asserting that the intervals of constancy of ω and $A(\omega)$ coincide for typical ω, will be proved in the next section (Lemma 9.4).

Lemma 8.1 *For each $T > 0$, for typical ω, $t \in [0, T] \mapsto A_t^n$ is a Cauchy sequence of functions in $C[0, T]$.*

Proof Fix a $T > 0$ and fix temporarily an $n \in \{1, 2, \ldots\}$. Let $\kappa \in \{0, 1\}$ be such that $T_0^{n-1} = T_\kappa^n$, and for each $k = 1, 2, \ldots$, let

$$\xi_k := \begin{cases} 1 & \text{if } \omega(T_{\kappa+2k}^n) = \omega(T_{\kappa+2k-2}^n) \\ -1 & \text{otherwise} \end{cases}$$

(this is only defined when $T_{\kappa+2k}^n < \infty$). If ω were generated by Brownian motion, ξ_k would be a random variable taking value j, $j \in \{1, -1\}$, with probability $1/2$; in particular, the expected value of ξ_k would be 0. As the standard backward induction procedure shows, this remains true in our current framework in the following game-theoretic sense: there exists a simple trading strategy that, when started with initial capital 0 at time $T^n_{\kappa+2k-2}$, ends with ξ_k at time $T^n_{\kappa+2k}$, provided both times are finite; moreover, the corresponding simple capital process is always between -1 and 1. (Namely, at time $T^n_{\kappa+2k-1}$ bet -2^n if $\omega(T^n_{\kappa+2k-1}) > \omega(T^n_{\kappa+2k-2})$ and bet 2^n otherwise.) Notice that the increment of the process $A^n_t - A^{n-1}_t$ over the time interval $[T^n_{\kappa+2k-2}, T^n_{\kappa+2k}]$ is

$$\eta_k := \begin{cases} 2(2^{-n})^2 = 2^{-2n+1} & \text{if } \xi_k = 1 \\ 2(2^{-n})^2 - (2^{-n+1})^2 = -2^{-2n+1} & \text{if } \xi_k = -1, \end{cases}$$

i.e., $\eta_k = 2^{-2n+1}\xi_k$.

The game-theoretic version of Hoeffding's inequality (see Theorem A.1 in the Appendix below) shows that for any constant $\lambda \in \mathbb{R}$, there exists a simple capital process $\mathfrak{S}^n$ with $\mathfrak{S}^n_0 = 1$ such that for all $K = 0, 1, 2, \ldots$,

$$\mathfrak{S}^n_{T^n_{\kappa+2K}} \geq \prod_{k=1}^{K} \exp\bigl(\lambda\eta_k - 2^{-4n+1}\lambda^2\bigr). \tag{8.2}$$

According to Eq. (A.1) in the Appendix (with x_n corresponding to η_k), such an $\mathfrak{S}^n$ can be defined as the capital process of the simple trading strategy betting the current capital times

$$\frac{e^{\lambda 2^{-2n+1}} - e^{-\lambda 2^{-2n+1}}}{2^{-2n+2}} \exp\left(-\frac{\lambda^2}{8}\bigl(2^{-2n+2}\bigr)^2\right)$$

on $A^n_t - A^{n-1}_t$ at each time $T^n_{\kappa+2k-2}$, $k \in \{1, 2, \ldots\}$. In terms of the original security, this simple trading strategy bets 0 on ω at each time $T^n_{\kappa+2k-2}$ and bets the current capital times

$$2\bigl(\omega\bigl(T^n_{\kappa+2k-2}\bigr) - \omega\bigl(T^n_{\kappa+2k-1}\bigr)\bigr)\frac{e^{\lambda 2^{-2n+1}} - e^{-\lambda 2^{-2n+1}}}{2^{-2n+2}} \exp\left(-\frac{\lambda^2}{8}\bigl(2^{-2n+2}\bigr)^2\right)$$

on ω at each time $T^n_{\kappa+2k-1}$, $k \in \{1, 2, \ldots\}$. It is clear that the process $\mathfrak{S}^n$ is positive: it is constant in each time interval $[T^n_{\kappa+2k-2}, T^n_{\kappa+2k-1}]$, and is linear in $\omega(t)$ in each time interval $[T^n_{\kappa+2k-1}, T^n_{\kappa+2k}]$; therefore, its overall positivity follows from its positivity (cf. (8.2)) at the points $T^n_{\kappa+2K}$, $K \in \{0, 1, 2, \ldots\}$.

Fix temporarily $\alpha > 0$. It is easy to see that since the sum of the positive capital processes $\mathfrak{S}^n$ over $n = 1, 2, \ldots$ with weights 2^{-n} will also be a positive capital process, none of these processes will ever exceed $2^n 2/\alpha$ except for a set of ω of outer content at most $\alpha/2$. The inequality

$$\prod_{k=1}^{K} \exp\bigl(\lambda\eta_k - 2^{-4n+1}\lambda^2\bigr) \leq 2^n \frac{2}{\alpha} \leq e^n \frac{2}{\alpha}$$

can be equivalently rewritten as

$$\lambda \sum_{k=1}^{K} \eta_k \le K\lambda^2 2^{-4n+1} + n + \ln\frac{2}{\alpha}. \tag{8.3}$$

Plugging in the identities

$$K = \frac{A^n_{T^n_{\kappa+2K}} - A^n_{T^n_\kappa}}{2^{-2n+1}},$$

$$\sum_{k=1}^{K} \eta_k = \left(A^n_{T^n_{\kappa+2K}} - A^n_{T^n_\kappa}\right) - \left(A^{n-1}_{T^n_{\kappa+2K}} - A^{n-1}_{T^n_\kappa}\right)$$

and taking $\lambda := 2^n$, we can transform (8.3) to

$$\left(A^n_{T^n_{\kappa+2K}} - A^n_{T^n_\kappa}\right) - \left(A^{n-1}_{T^n_{\kappa+2K}} - A^{n-1}_{T^n_\kappa}\right) \le 2^{-n}\left(A^n_{T^n_{\kappa+2K}} - A^n_{T^n_\kappa}\right) + \frac{n + \ln\frac{2}{\alpha}}{2^n}, \tag{8.4}$$

which implies

$$A^n_{T^n_{\kappa+2K}} - A^{n-1}_{T^n_{\kappa+2K}} \le 2^{-n} A^n_{T^n_{\kappa+2K}} + 2^{-2n+1} + \frac{n + \ln\frac{2}{\alpha}}{2^n}. \tag{8.5}$$

This is true for any $K = 0, 1, 2, \dots$; choosing the largest K such that $T^n_{\kappa+2K} \le t$, we obtain

$$A^n_t - A^{n-1}_t \le 2^{-n} A^n_t + 2^{-2n+2} + \frac{n + \ln\frac{2}{\alpha}}{2^n} \tag{8.6}$$

for any $t \in [0, \infty)$ (the simple case $t < T^n_\kappa$ has to be considered separately). Proceeding in the same way but taking $\lambda := -2^n$, we obtain

$$\left(A^n_{T^n_{\kappa+2K}} - A^n_{T^n_\kappa}\right) - \left(A^{n-1}_{T^n_{\kappa+2K}} - A^{n-1}_{T^n_\kappa}\right) \ge -2^{-n}\left(A^n_{T^n_{\kappa+2K}} - A^n_{T^n_\kappa}\right) - \frac{n + \ln\frac{2}{\alpha}}{2^n}$$

instead of (8.4) and

$$A^n_{T^n_{\kappa+2K}} - A^{n-1}_{T^n_{\kappa+2K}} \ge -2^{-n} A^n_{T^n_{\kappa+2K}} - 2^{-2n+1} - \frac{n + \ln\frac{2}{\alpha}}{2^n}$$

instead of (8.5), which gives

$$A^n_t - A^{n-1}_t \ge -2^{-n} A^n_t - 2^{-2n+2} - \frac{n + \ln\frac{2}{\alpha}}{2^n} \tag{8.7}$$

instead of (8.6). We know that (8.6) and (8.7) hold for all $t \in [0, \infty)$ and all $n = 1, 2, \dots$ except for a set of ω of outer content at most α.

Now we have all ingredients to complete the proof. Suppose there exists $\alpha > 0$ such that (8.6) and (8.7) hold for all $n = 1, 2, \dots$ (this is true for typical ω). First let us show that the sequence A^n_T, $n=1,2,\dots$, is bounded. Define a new sequence B^n, $n=0,1,2,\dots$, as follows: $B^0 := A^0_T$ and B^n, $n=1,2,\dots$, are defined inductively by

$$B^n := \frac{1}{1-2^{-n}}\left(B^{n-1} + 2^{-2n+2} + \frac{n+\ln\frac{2}{\alpha}}{2^n}\right) \tag{8.8}$$

(notice that this is equivalent to (8.6) with B^n in place of A^n_t and $=$ in place of $\leq$). As $A^n_T \leq B^n$ for all n, it suffices to prove that B^n is bounded. If it is not, $B^N \geq 1$ for some N. By (8.8), $B^n \geq 1$ for all $n \geq N$. Therefore, again by (8.8),

$$B^n \leq B^{n-1}\frac{1}{1-2^{-n}}\left(1 + 2^{-2n+2} + \frac{n+\ln\frac{2}{\alpha}}{2^n}\right), \quad n > N,$$

and the boundedness of the sequence (B^n) follows from $B^N < \infty$ and

$$\prod_{n=N+1}^{\infty}\frac{1}{1-2^{-n}}\left(1 + 2^{-2n+2} + \frac{n+\ln\frac{2}{\alpha}}{2^n}\right) < \infty.$$

Now it is obvious that the sequence (A^n_t) is Cauchy in $C[0,T]$: (8.6) and (8.7) imply

$$\left|A^n_t - A^{n-1}_t\right| \leq 2^{-n}A^n_T + 2^{-2n+2} + \frac{n+\ln\frac{2}{\alpha}}{2^n} = O\left(n/2^n\right). \qquad \square$$

Lemma 8.1 implies that for typical ω, the sequence $t \in [0,\infty) \mapsto A^n_t$ is Cauchy in Ω. Therefore, we have the following implication.

Lemma 8.2 *The event that the sequence of functions $t \in [0,\infty) \mapsto A^n_t$ converges in Ω is full.*

We can see that the first term in the conjunction in (5.3) holds for typical ω; let us check that τ_s itself is a stopping time.

Lemma 8.3 *For each $s \geq 0$, the function τ_s defined by (5.3) is a stopping time.*

Proof It suffices to check that the condition $\tau_s \leq t$ can be written as

$$\forall (q_1,q_2) \subseteq (0,s)\ \exists q \in (0,t)\cap\mathbb{Q} : \overline{A}_q = \underline{A}_q \in (q_1,q_2), \tag{8.9}$$

where (q_1,q_2) ranges over the non-empty intervals with rational end-points. Let T be the largest number in $[0,\infty]$ such that the functions $\overline{A}|_{[0,T)}$ and $\underline{A}|_{[0,T)}$ coincide and are continuous; we use A' as the common notation for $\overline{A}|_{[0,T)} = \underline{A}|_{[0,T)}$. The condition $\tau_s \leq t$ means that for some $t' \in [0,t]$, the domain of A' includes $[0,t')$ and $\sup_{u<t'} A'_u = s$. Now it is clear that the condition (8.9) is satisfied if $\tau_s \leq t$. In the opposite direction, suppose (8.9) is satisfied. Then $\overline{A}_u = \underline{A}_u$ whenever $u \in (0,t)$ satisfies $\underline{A}_u < s$. Indeed, if we had $\underline{A}_u < \overline{A}_u$ for such u, we could choose $(q_1,q_2) \subseteq (0,s)$ satisfying $\underline{A}_u < q_1 < q_2 < \overline{A}_u$ and there would be no q satisfying the required properties in (8.9): if $q \leq u$, $\underline{A}_q \leq \underline{A}_u < q_1$, and if $q \geq u$, $\overline{A}_q \geq \overline{A}_u > q_2$. Combining this result with (8.9), we can see that there is a function A'' with a domain $[0, t'') \subseteq [0, t)$ such that $A''_u = \overline{A}_u = \underline{A}_u$ for all $u \in [0, t'')$ and $\sup A'' = s$. The function A'' is increasing and, by (8.9), continuous; this implies $\tau_s \le t$. □

Let us now consider the case of Brownian motion.

Lemma 8.4 *For any* $c \in \mathbb{R}$, $\mathcal{W}_c(\forall t \in [0, \infty) : \overline{A}_t = \underline{A}_t = t) = 1$.

Proof It suffices to consider only rational values of t and, therefore, a fixed value of t. The convergence $A^n_t \to t$ (see (5.1)) in $\mathcal{W}_c$-probability can be deduced from the law of large numbers applied to T^n_k:

- the law of large numbers implies that $A^n_t \to t$ in $\mathcal{W}_c$-probability because $\int (T^n_k - T^n_{k-1})\, d\mathcal{W}_c = 2^{-2n}$ (this is a combination of the second statement of Theorem 2.49 in [45], which is a corollary of Wald's second lemma, with the strong Markov property of Brownian motion);
- the law of large numbers is applicable because $\int (T^n_k - T^n_{k-1})^2\, d\mathcal{W}_c < \infty$ (see the proof of the second statement of Theorem 2.49 in [45]).

It remains to apply Lemma 8.2, which, in combination with Lemma 6.3 (applied to the indicator functions of events), implies that the sequence (A^n) converges in Ω $\mathcal{W}_c$-almost surely. □

Remark 8.5 This section is about the quadratic variation of the price path, but in finance the quadratic variation of the stochastic logarithm (see e.g. [33], p. 134) of a price process is usually even more important than the quadratic variation of the price process itself. A pathwise version of the stochastic logarithm has been studied by Norvaiša in [48, 49]. Consider an $\omega \in \Omega$ such that $A(\omega)$ exists, belongs to Ω, and has the same intervals of constancy as ω; Theorem 5.1(a) says that these conditions are satisfied for typical ω. Fix a time horizon $T > 0$ and suppose additionally that $\inf_{t \in [0,T]} \omega(t) > 0$. The limit

$$R_t(\omega) := \lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{\omega(T^n_k \wedge t) - \omega(T^n_{k-1} \wedge t)}{\omega(T^n_{k-1} \wedge t)}$$

(where we use the same notation as in (5.1)) exists for all $t \in [0, T]$, and the function $R(\omega) : t \in [0, T] \mapsto R_t(\omega)$ satisfies ([49], Proposition 56)

$$R_t(\omega) = \ln \frac{\omega(t)}{\omega(0)} + \frac{1}{2} \int_0^t \frac{dA_s(\omega)}{\omega^2(s)}, \quad t \in [0, T].$$

In financial terms, the value $R_t(\omega)$ is the cumulative return of the security ω over $[0, t]$ ([48], Sect. 2); in probabilistic terms, $R(\omega)$ is the pathwise stochastic logarithm of ω. The quadratic variation of $R(\omega)$ can be defined as

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} \big(R_{T^n_k \wedge t}(\omega) - R_{T^n_{k-1} \wedge t}(\omega)\big)^2 = \int_0^t \frac{dA_s(\omega)}{\omega^2(s)}$$

(the existence of the limit and the equality are also parts of Proposition 56 in [49]).

9 Tightness

In this section, we do some groundwork for the proof of Theorems 5.1(b) and 6.4, and also finish the proof of Theorem 5.1(a). We start from the results that show (see the next section) that $\underline{Q}_c$ is tight in the topology induced by the metric (8.1).

Lemma 9.1 *For each $\alpha > 0$ and $S \in \{1, 2, 4 \ldots\}$,*

$$\underline{\mathbb{P}}\big(\forall\delta \in (0, 1)\ \forall s_1, s_2 \in [0, S] : (0 \le s_2 - s_1 \le \delta \textit{ and } \tau_{s_2} < \infty) \\ \Longrightarrow \big|\omega(\tau_{s_2}) - \omega(\tau_{s_1})\big| \le 230\,\alpha^{-1/2} S^{1/4}\delta^{1/8}\big) \ge 1 - \alpha. \tag{9.1}$$

Proof Let $S = 2^d$, where $d \in \{0, 1, 2, \ldots\}$. For each $m = 1, 2, \ldots,$ divide the interval $[0, S]$ into 2^{d+m} equal subintervals of length 2^{-m}. Fix for a moment such an m, and set $\beta = \beta_m := (2^{1/4} - 1)2^{-m/4}\alpha$ (where $2^{1/4} - 1$ is the normalizing constant ensuring that the β_m sum to α) and

$$t_i := \tau_{i2^{-m}}, \qquad \omega_i := \omega(t_i), \quad i = 0, 1, \ldots, 2^{d+m} \tag{9.2}$$

(we shall be careful to use ω_i only when $t_i < \infty$).

We first replace the quadratic variation process A (in terms of which the stopping times τ_s are defined) by a version of A^ℓ for a large enough ℓ. If τ is any stopping time (we are interested in $\tau = t_i$ for various i), set, in the notation of (5.1),

$$A_t^{n,\tau}(\omega) := \sum_{k=0}^{\infty}\big(\omega\big(\tau \vee T_k^n \wedge t\big) - \omega\big(\tau \vee T_{k-1}^n \wedge t\big)\big)^2, \quad t \ge \tau,\ n = 1, 2, \ldots$$

(we omit parentheses in expressions of the form $x \vee y \wedge z$ since we have $(x \vee y) \wedge z = x \vee (y \wedge z)$, provided $x \le z$). The intuition is that $A_t^{n,\tau}(\omega)$ is the version of $A_t^n(\omega)$ that starts at time τ rather than 0.

For $i = 0, 1, \ldots, 2^{d+m} - 1$, let $\mathfrak{E}_i$ be the event that $t_i < \infty$ implies that (8.7), with α replaced by $\gamma > 0$ and A_t^n replaced by A_t^{n,t_i}, holds for all $n = 1, 2, \ldots$ and $t \in [t_i, \infty)$. Applying a trading strategy similar to that used in the proof of Lemma 8.1 but starting at time t_i rather than 0, we can see that the inner content of $\mathfrak{E}_i$ is at least $1 - \gamma$. The inequality

$$A_t^{n,t_i} - A_t^{n-1,t_i} \ge -2^{-n} A_t^{n,t_i} - 2^{-2n+2} - \frac{n + \ln\frac{2}{\gamma}}{2^n}$$

holds for all $t \in [t_i, t_{i+1}]$ and all n on the event $\{t_i < \infty\} \cap \mathfrak{E}_i$. For the value $t := t_{i+1}$, this inequality implies

$$A_{t_{i+1}}^{n,t_i} \ge \frac{1}{1 + 2^{-n}}\left(A_{t_{i+1}}^{n-1,t_i} - 2^{-2n+2} - \frac{n + \ln\frac{2}{\gamma}}{2^n}\right)$$

(including the case $t_{i+1} = \infty$). Applying the last inequality to $n = \ell + 1, \ell + 2, \ldots$ (where ℓ will be chosen later), we obtain that

$$A^{\infty,t_i}_{t_{i+1}} \geq \left(\prod_{n=\ell+1}^{\infty} \frac{1}{1+2^{-n}}\right) A^{\ell,t_i}_{t_{i+1}} - \sum_{n=\ell+1}^{\infty} \left(2^{-2n+2} + \frac{n + \ln\frac{2}{\gamma}}{2^n}\right) \tag{9.3}$$

holds on the whole of $\{t_i < \infty\} \cap \mathfrak{E}_i$ except perhaps a null set. The qualification "except a null set" allows us not only to assume that $A^{\infty,t_i}_{t_{i+1}}$ exists in (9.3), but also to assume that $A^{\infty,t_i}_{t_{i+1}} = A_{t_{i+1}} - A_{t_i} = 2^{-m}$. Let $\gamma := \frac{1}{3}2^{-d-m}\beta$ and choose $\ell = \ell(m)$ so large that (9.3) implies $A^{\ell,t_i}_{t_{i+1}} \leq 2^{-m+1/2}$ (this can be done as both the product and the sum in (9.3) are convergent, and so the product can be made arbitrarily close to 1 and the sum can be made arbitrarily close to 0). Doing this for all $i = 0, 1, \ldots, 2^{d+m} - 1$ will ensure that the inner content of

$$\left\{t_i < \infty \Longrightarrow A^{\ell,t_i}_{t_{i+1}} \leq 2^{-m+1/2},\ i = 0, 1, \ldots, 2^{d+m} - 1\right\} \tag{9.4}$$

is at least $1 - \beta/3$.

An important observation for what follows is that the process defined as 0 for $t < t_i$ and as $(\omega(t) - \omega(t_i))^2 - A^{\ell,t_i}_t$ for $t \geq t_i$ is a simple capital process (corresponding to betting $2(\omega(T^\ell_k) - \omega(t_i))$ at each time $T^\ell_k > t_i$). Now we can see that

$$\sum_{i=1,\ldots,2^{d+m}:t_i<\infty} (\omega_i - \omega_{i-1})^2 \leq 2^{1/2}\frac{3}{\beta}S \tag{9.5}$$

will hold on the event (9.4), except for a set of ω of outer content at most $\beta/3$: indeed, there is a positive simple capital process taking value at least $2^{1/2}S + \sum_{i=1}^{j}(\omega_i - \omega_{i-1})^2 - j2^{-m+1/2}$ at time t_j on the conjunction of the events (9.4) and $\{t_j < \infty\}$ for all $j = 0, 1, \ldots, 2^{d+m}$, and this simple capital process will make at least $2^{1/2}\frac{3}{\beta}S$ at time τ_S (in the sense of $\liminf$ if $\tau_S = \infty$) out of initial capital $2^{1/2}S$, if (9.4) happens but (9.5) fails to happen.

For each $\omega \in \Omega$, define

$$J(\omega) := \left\{i = 1, \ldots, 2^{d+m} : t_i < \infty \text{ and } |\omega_i - \omega_{i-1}| \geq \epsilon\right\},$$

where $\epsilon = \epsilon_m$ will be chosen later. It is clear that $|J(\omega)| \leq 2^{1/2}3S/\beta\epsilon^2$ on the set (9.5). Consider the simple trading strategy whose capital increases by $(\omega(t_i) - \omega(\tau))^2 - A^{\ell,\tau}_{t_i}$ between each time $\tau \in [t_{i-1}, t_i] \cap [0, \infty)$ when $|\omega(\tau) - \omega_{i-1}| = \epsilon$ for the first time during $[t_{i-1}, t_i] \cap [0, \infty)$ (this is guaranteed to happen when $i \in J(\omega)$) and the corresponding time t_i, $i = 1, \ldots, 2^{d+m}$, and which is not active (i.e., sets the bet to 0) otherwise. (Such a strategy exists, as explained in the previous paragraph.) This strategy will make at least ϵ^2 out of $(2^{1/2}3S/\beta\epsilon^2)2^{-m+1/2}$, provided all three of the events (9.4), (9.5) and

$$\left\{\exists i \in \left\{1, \ldots, 2^{d+m}\right\} : t_i < \infty \text{ and } |\omega_i - \omega_{i-1}| \geq 2\epsilon\right\}$$

happen. (And we can make the corresponding simple capital process positive by being active for at most $2^{1/2}3S/\beta\epsilon^2$ values of i and setting the bet to 0 as soon as (9.4) becomes violated.) This corresponds to making at least 1 out of $(2^{1/2}3S/\beta\epsilon^4)2^{-m+1/2}$. Solving the equation $(2^{1/2}3S/\beta\epsilon^4)2^{-m+1/2} = \beta/3$ for ϵ gives $\epsilon = (2 \times 3^2 S 2^{-m}/\beta^2)^{1/4}$. Therefore,

$$\max_{i=1,\dots,2^{d+m}:t_i<\infty} |\omega_i - \omega_{i-1}| \le 2\epsilon = 2\big(2 \times 3^2 S 2^{-m}/\beta^2\big)^{1/4}$$
$$= 2^{5/4}3^{1/2}\big(2^{1/4}-1\big)^{-1/2}\alpha^{-1/2}S^{1/4}2^{-m/8} \qquad (9.6)$$

except for a set of ω of outer content β. By the countable subadditivity of outer content (Lemma 4.1), (9.6) holds for all $m = 1, 2, \dots$ except for a set of ω of outer content at most $\sum_m \beta_m = \alpha$.

We have now allowed m to vary and so write t_i^m instead of t_i defined by (9.2). Fix an $\omega \in \Omega$ satisfying $A(\omega) \in \Omega$ and (9.6) for $m = 1, 2, \dots$. Intervals of the form $[t_{i-1}^m(\omega), t_i^m(\omega)] \subseteq [0, \infty)$, for $m \in \{1, 2, \dots\}$ and $i \in \{1, 2, 3, \dots, 2^{d+m}\}$, will be called *predyadic* (*of order* m). Given an interval $[s_1, s_2] \subseteq [0, S]$ of length at most $\delta \in (0, 1)$ and with $\tau_{s_2} < \infty$, we can cover $(\tau_{s_1}(\omega), \tau_{s_2}(\omega))$ (without covering any points in the complement of $[\tau_{s_1}(\omega), \tau_{s_2}(\omega)]$) by adjacent predyadic intervals with disjoint interiors such that, for some $m \in \{1, 2, \dots\}$; there are between one and two predyadic intervals of order m; for $i = m+1, m+2, \dots$, there are at most two predyadic intervals of order i (start from finding the point in $[s_1, s_2]$ of the form $j2^{-k}$ with integer j and k and the smallest possible k, and cover $(\tau_{s_1}(\omega), \tau_{j2^{-k}}]$ and $[\tau_{j2^{-k}}, \tau_{s_2}(\omega))$ by predyadic intervals in the greedy manner). Combining (9.6) and $2^{-m} \le \delta$, we obtain

$$\begin{aligned}
\big|\omega(\tau_{s_2}) - \omega(\tau_{s_1})\big| &\le 2^{9/4}3^{1/2}\big(2^{1/4}-1\big)^{-1/2}\alpha^{-1/2}S^{1/4} \\
&\quad \times \big(2^{-m/8} + 2^{-(m+1)/8} + 2^{-(m+2)/8} + \cdots\big) \\
&= 2^{9/4}3^{1/2}\big(2^{1/4}-1\big)^{-1/2}\big(1-2^{-1/8}\big)^{-1}\alpha^{-1/2}S^{1/4}2^{-m/8} \\
&\le 2^{9/4}3^{1/2}\big(2^{1/4}-1\big)^{-1/2}\big(1-2^{-1/8}\big)^{-1}\alpha^{-1/2}S^{1/4}\delta^{1/8},
\end{aligned}$$

which is stronger than (9.1) as $2^{9/4}3^{1/2}(2^{1/4}-1)^{-1/2}(1-2^{-1/8})^{-1} \approx 228.22$. □

Now we can prove the following elaboration of Lemma 9.1, which will be used in the next two sections.

Lemma 9.2 *For each* $\alpha > 0$,

$$\underline{\mathbb{P}}\Big(\forall S \in \{1, 2, 4, \dots\}\ \forall \delta \in (0, 1)\ \forall s_1, s_2 \in [0, S] : (0 \le s_2 - s_1 \le \delta \text{ and } \tau_{s_2} < \infty)$$
$$\Longrightarrow \big|\omega(\tau_{s_2}) - \omega(\tau_{s_1})\big| \le 430\,\alpha^{-1/2}S^{1/2}\delta^{1/8}\Big) \ge 1 - \alpha. \qquad (9.7)$$

Proof Replacing α in (9.1) by $\alpha_S := (1 - 2^{-1/2})S^{-1/2}\alpha$ for $S = 1, 2, 4, \dots$ (where $1 - 2^{-1/2}$ is the normalizing constant ensuring that the α_S sum to α over S), we obtain

$$\underline{\mathbb{P}}\Big(\forall\delta\in(0,1)\ \forall s_1,s_2\in[0,S]:(0\le s_2-s_1\le\delta \text{ and } \tau_{s_2}<\infty)$$
$$\Longrightarrow \big|\omega(\tau_{s_2})-\omega(\tau_{s_1})\big|\le 230\,\big(1-2^{-1/2}\big)^{-1/2}\alpha^{-1/2}S^{1/2}\delta^{1/8}\Big)$$
$$\ge 1-\big(1-2^{-1/2}\big)S^{-1/2}\alpha.$$

The countable subadditivity of outer content now gives

$$\underline{\mathbb{P}}\Big(\forall S\in\{1,2,4,\dots\}\ \forall\delta\in(0,1)\ \forall s_1,s_2\in[0,S]:(0\le s_2-s_1\le\delta \text{ and } \tau_{s_2}<\infty)$$
$$\Longrightarrow \big|\omega(\tau_{s_2})-\omega(\tau_{s_1})\big|\le 230\,\big(1-2^{-1/2}\big)^{-1/2}\alpha^{-1/2}S^{1/2}\delta^{1/8}\Big)$$
$$\ge 1-\alpha,$$

which is stronger than (9.7) as $230\,(1-2^{-1/2})^{-1/2}\approx 424.98$. □

The following lemma develops inequality (9.5) and will be useful in the proof of Theorem 5.1.

Lemma 9.3 *For each* $\alpha>0$,

$$\underline{\mathbb{P}}\Bigg(\forall S\in\{1,2,4,\dots\}\ \forall m\in\{1,2,\dots\}:$$
$$\sum_{i=1,\dots,S2^m:t_i<\infty}\big(\omega(t_i)-\omega(t_{i-1})\big)^2\le 64\,\alpha^{-1}S^2 2^{m/16}\Bigg)\ge 1-\alpha, \tag{9.8}$$

in the notation of (9.2).

Proof Replacing $\beta/3$ in (9.5) with $2^{-1}(2^{1/16}-1)S^{-1}2^{-m/16}\alpha$, where S ranges over $\{1,2,4,\dots\}$ and m over $\{1,2,\dots\}$, we obtain

$$\underline{\mathbb{P}}\Bigg(\sum_{i=1,\dots,S2^m:t_i<\infty}\big(\omega(t_i)-\omega(t_{i-1})\big)^2\le 2^{3/2}\big(2^{1/16}-1\big)^{-1}\alpha^{-1}S^2 2^{m/16}\Bigg)$$
$$\ge 1-2^{-1}\big(2^{1/16}-1\big)S^{-1}2^{-m/16}\alpha. \tag{9.9}$$

By the countable subadditivity of outer content this implies

$$\underline{\mathbb{P}}\Bigg(\forall S\in\{1,2,4,\dots\}\ \forall m\in\{1,2,\dots\}:$$
$$\sum_{i=1,\dots,S2^m:t_i<\infty}\big(\omega(t_i)-\omega(t_{i-1})\big)^2\le 2^{3/2}\big(2^{1/16}-1\big)^{-1}\alpha^{-1}S^2 2^{m/16}\Bigg)\ge 1-\alpha,$$

which is stronger than (9.8) as $2^{3/2}(2^{1/16}-1)^{-1}\approx 63.88$. □

The following lemma completes the proof of Theorem 5.1(a).

Lemma 9.4 *For typical* ω, $A(\omega)$ *has the same intervals of constancy as* ω.

Proof The definition of A immediately implies that $A(\omega)$ is always constant on every interval of constancy of ω (provided $A(\omega)$ exists). Therefore, we are only required to prove that typical ω are constant on every interval of constancy of $A(\omega)$.

The proof can be extracted from the proof of Lemma 9.1. It suffices to prove that for any $\alpha > 0$, $S \in \{1, 2, 4, \dots\}$, rational $c > 0$, and interval $[a, b]$ with rational end-points a and b such that $a < b$, the outer content of the following event is at most α: ω changes by at least c over $[a, b]$, A is constant over $[a, b]$, and $[a, b] \subseteq [0, \tau_S]$. Fix such α, S, c and $[a, b]$, and let E stand for the event described in the previous sentence. Choose $m \in \{1, 2, \dots\}$ such that $2^{-m+1/2}/c^2 \le \alpha/2$ and choose the corresponding $\ell = \ell(m)$ as in the proof of Lemma 9.1 but with $1-\beta/3$ replaced by $1-\alpha/2$ (cf. (9.4)). The positive simple capital process $2^{-m+1/2} + (\omega(t) - \omega(a))^2 - A_t^{\ell,a}$, started at time a and stopped when t reaches $b \wedge \tau_S$, when $A_t^{\ell,a}$ reaches $2^{-m+1/2}$, or when $|\omega(t) - \omega(a)|$ reaches c, whichever happens first, makes c^2 out of $2^{-m+1/2}$ on the conjunction of (9.4) and the event E. Therefore, the outer content of the conjunction is at most $\alpha/2$, and the outer content of E is at most α. □

In view of Lemma 9.4, we can strengthen (9.7) to

$$\begin{aligned}&\underline{\mathbb{P}}\big(\forall S \in \{1, 2, 4, \dots\}\ \forall \delta \in (0, 1)\ \forall t_1, t_2 \in [0, \infty) :\\&\quad \big(|A_{t_2} - A_{t_1}| \le \delta \text{ and } A_{t_1} \in [0, S] \text{ and } A_{t_2} \in [0, S]\big)\\&\quad \Longrightarrow \big|\omega(t_2) - \omega(t_1)\big| \le 430\,\alpha^{-1/2} S^{1/2} \delta^{1/8}\big) \ge 1 - \alpha.\end{aligned}$$

10 Proof of the remaining parts of Theorems 5.1(b) and 6.4

Let $c \in \mathbb{R}$ be a fixed constant. The results of the previous section imply the tightness of $\underline{Q}_c$ (for details, see below).

Lemma 10.1 *For each $\alpha > 0$ there exists a compact set $\mathfrak{K} \subseteq \Omega$ such that $\underline{Q}_c(\mathfrak{K}) \ge 1 - \alpha$.*

In particular, Lemma 10.1 asserts that $\underline{Q}_c(\Omega) = 1$. This fact and the results of Sect. 7 allow us to check that Theorem 6.4 implies Theorem 5.1(b). First, the inequality $\le$ in (6.5) implies

$$\begin{aligned}\overline{Q}_c(E) &= \overline{\mathbb{P}}\big(\mathrm{ntt}^{-1}(E);\ \omega(0) = c,\ A_\infty = \infty\big)\\&= \overline{\mathbb{E}}\big(\mathbf{1}_E \circ \mathrm{ntt};\ \omega(0) = c,\ A_\infty = \infty\big) \le \int_\Omega \mathbf{1}_E \, d\mathcal{W}_c = \mathcal{W}_c(E)\end{aligned}$$

for all $E \in \mathcal{F}$. Therefore,

$$\begin{aligned}\underline{Q}_c(E) &= \underline{\mathbb{P}}\big(\mathrm{ntt}^{-1}(E);\ \omega(0) = c,\ A_\infty = \infty\big)\\&= 1 - \overline{\mathbb{P}}\big(\mathrm{ntt}^{-1}\big(E^c\big) \cup \big(\mathrm{ntt}^{-1}(\Omega)\big)^c;\ \omega(0) = c,\ A_\infty = \infty\big)\\&= 1 - \overline{\mathbb{P}}\big(\mathrm{ntt}^{-1}\big(E^c\big);\ \omega(0) = c,\ A_\infty = \infty\big)\\&\ge 1 - \mathcal{W}_c\big(E^c\big) = \mathcal{W}_c(E)\end{aligned} \tag{10.1}$$

and so, by Lemma 7.3 and (7.2),

$$\overline{Q}_c(E) = \underline{Q}_c(E) = \mathcal{W}_c(E)$$

for all $E \in \mathfrak{F}$. The equality in line (10.1) follows from

$$\underline{\mathbb{P}}\big(\mathrm{ntt}^{-1}(\Omega); \omega(0) = c, A_\infty = \infty\big) = 1,$$

which in turn follows from (and is in fact equivalent to) $\underline{Q}_c(\Omega) = 1$. Therefore, we only need to finish the proof of Theorem 6.4.

More precise results than Lemma 10.1 can be stated in terms of the *modulus of continuity* of a function $\psi \in \mathbb{R}^{[0,\infty)}$ on an interval $[0, S] \subseteq [0, \infty)$, defined by

$$\mathrm{m}_\delta^S(\psi) := \sup_{s_1,s_2 \in [0,S]:|s_1-s_2|\le\delta} \big|\psi(s_1) - \psi(s_2)\big|, \quad \delta > 0;$$

it is clear that $\lim_{\delta\to 0} \mathrm{m}_\delta^S(\psi) = 0$ if and only if ψ is continuous (equivalently, uniformly continuous) on $[0, S]$.

Lemma 10.2 *For each* $\alpha > 0$,

$$\underline{Q}_c\big(\forall S \in \{1, 2, 4, \dots\}\ \forall\delta \in (0, 1) : \mathrm{m}_\delta^S \le 430\,\alpha^{-1/2} S^{1/2}\delta^{1/8}\big) \ge 1 - \alpha.$$

Lemma 10.2 immediately follows from Lemma 9.2, and Lemma 10.1 immediately follows from Lemma 10.2 and the Arzelà–Ascoli theorem (as stated in [35], Theorem 2.4.9).

We start the proof of the remaining part of Theorem 6.4 from a series of reductions. To establish the inequality $\le$ in (6.5) we only need to establish

$$\overline{\mathbb{E}}\big(F \circ \mathrm{ntt}; \omega(0) = c, A_\infty = \infty\big) < \int F\, d\mathcal{W}_c + \epsilon$$

for each positive constant ϵ.

(a) We can assume that F in (6.5) is lower semicontinuous on Ω. Indeed, if it is not, the Vitali–Carathéodory theorem (see e.g. [52], Theorem 2.25) gives for any compact $\mathfrak{K} \subseteq \Omega$ (assumed non-empty) the existence of a lower semicontinuous function G on $\mathfrak{K}$ such that $G \ge F$ on $\mathfrak{K}$ and $\int_{\mathfrak{K}} G\, d\mathcal{W}_c \le \int_{\mathfrak{K}} F\, d\mathcal{W}_c + \epsilon$. Without loss of generality we assume $\sup G \le \sup F$, and we extend G to all of Ω by setting $G := \sup F$ outside $\mathfrak{K}$. Choosing $\mathfrak{K}$ with large enough $\mathcal{W}_c(\mathfrak{K})$ (which can be done since the probability measure $\mathcal{W}_c$ is tight: see e.g. [5], Theorem 1.4), we have $G \ge F$ and $\int G\, d\mathcal{W}_c \le \int F\, d\mathcal{W}_c + 2\epsilon$. Achieving $\mathfrak{S}_0 \le \int G\, d\mathcal{W}_c + \epsilon$ and $\liminf_{t\to\infty} \mathfrak{S}_t(\omega) \ge (G \circ \mathrm{ntt})(\omega)$, where $\mathfrak{S}$ is a positive capital process, will automatically achieve $\mathfrak{S}_0 \le \int F\, d\mathcal{W}_c + 3\epsilon$ and $\liminf_{t\to\infty} \mathfrak{S}_t(\omega) \ge (F \circ \mathrm{ntt})(\omega)$.

(b) We can further assume that F is continuous on Ω. Indeed, since each lower semicontinuous function on a metric space is the limit of an increasing sequence of continuous functions (see e.g. [25], Problem 1.7.15(c)), given a lower semicontinuous positive function F on Ω we can find a series of positive continuous functions G^n on Ω, $n = 1, 2, \dots$, such that $\sum_{n=1}^\infty G^n = F$. The sum $\mathfrak{S}$

of positive capital processes $\mathfrak{S}^1, \mathfrak{S}^2, \ldots$ achieving $\mathfrak{S}_0^n \le \int G^n \, d\mathcal{W}_c + 2^{-n}\epsilon$ and $\liminf_{t\to\infty} \mathfrak{S}_t^n(\omega) \ge (G^n \circ \mathrm{ntt})(\omega)$, $n = 1, 2, \ldots$, will achieve $\mathfrak{S}_0 \le \int F \, d\mathcal{W}_c + \epsilon$ and $\liminf_{t\to\infty} \mathfrak{S}_t(\omega) \ge (F \circ \mathrm{ntt})(\omega)$.

(c) We can further assume that F depends on $\psi \in \Omega$ only via $\psi|_{[0,S]}$ for some $S \in (0, \infty)$. Indeed, let us fix $\epsilon > 0$ and prove

$$\overline{\mathbb{E}}\big(F \circ \mathrm{ntt}; \omega(0) = c, A_\infty = \infty\big) \le \int F \, d\mathcal{W}_c + C\epsilon$$

for some positive constant C assuming

$$\overline{\mathbb{E}}\big(G \circ \mathrm{ntt}; \omega(0) = c, A_\infty = \infty\big) \le \int G \, d\mathcal{W}_c$$

for all continuous positive G that depend on $\psi \in \Omega$ only via $\psi|_{[0,S]}$ for some $S \in (0, \infty)$. Choose a compact set $\mathfrak{K} \subseteq \Omega$ with $\mathcal{W}_c(\mathfrak{K}) > 1 - \epsilon$ and $\underline{Q}_c(\mathfrak{K}) > 1 - \epsilon$ (cf. Lemma 10.1). Set $F^S(\psi) := F(\psi^S)$, where ψ^S is defined by $\psi^S(s) := \psi(s \wedge S)$ and S is sufficiently large in the following sense. Since F is uniformly continuous on $\mathfrak{K}$ and the metric is defined by (8.1), F and F^S can be made arbitrarily close in $C(\mathfrak{K})$; in particular, let $\|F - F^S\|_{C(\mathfrak{K})} < \epsilon$. Choose positive capital processes $\mathfrak{S}^0$ and $\mathfrak{S}^1$ such that

$$\mathfrak{S}_0^0 \le \int F^S \, d\mathcal{W}_c + \epsilon, \qquad \liminf_{t\to\infty} \mathfrak{S}_t^0(\omega) \ge \big(F^S \circ \mathrm{ntt}\big)(\omega),$$
$$\mathfrak{S}_0^1 \le \epsilon, \qquad \liminf_{t\to\infty} \mathfrak{S}_t^1(\omega) \ge (\mathbf{1}_{\mathfrak{K}^c} \circ \mathrm{ntt})(\omega),$$

for all $\omega \in \Omega$ with $\omega(0) = c$ and $A_\infty(\omega) = \infty$. The sum $\mathfrak{S} := \mathfrak{S}^0 + (\sup F)\mathfrak{S}^1 + \epsilon$ will satisfy

$$\begin{aligned}\mathfrak{S}_0 &\le \int F^S \, d\mathcal{W}_c + (\sup F + 2)\epsilon \le \int_{\mathfrak{K}} F^S \, d\mathcal{W}_c + (2\sup F + 2)\epsilon \\ &\le \int_{\mathfrak{K}} F \, d\mathcal{W}_c + (2\sup F + 3)\epsilon \le \int F \, d\mathcal{W}_c + (2\sup F + 3)\epsilon\end{aligned}$$

and

$$\liminf_{t\to\infty} \mathfrak{S}_t(\omega) \ge \big(F^S \circ \mathrm{ntt}\big)(\omega) + (\sup F)(\mathbf{1}_{\mathfrak{K}^c} \circ \mathrm{ntt})(\omega) + \epsilon \ge (F \circ \mathrm{ntt})(\omega),$$

provided $\omega(0) = c$ and $A_\infty(\omega) = \infty$. We assume $S \in \{1, 2, 4, \ldots\}$ without loss of generality.

(d) We can further assume that $F(\psi)$ depends on $\psi \in \Omega$ only via the values $\psi(iS/N)$, $i = 1, \ldots, N$ (remember that we are interested in the case $\psi(0) = c$), for some $N \in \{1, 2, \ldots\}$. Indeed, let us fix $\epsilon > 0$ and prove

$$\overline{\mathbb{E}}\big(F \circ \mathrm{ntt}; \omega(0) = c, A_\infty = \infty\big) \le \int F \, d\mathcal{W}_c + C\epsilon$$

for some positive constant C assuming

$$\overline{\mathbb{E}}\big(G \circ \mathrm{ntt}; \omega(0) = c, A_\infty = \infty\big) \le \int G \, d\mathcal{W}_c$$

for all continuous positive G that depend on $\psi \in \Omega$ only via $\psi(iS/N)$, $i = 1, \dots, N$, for some N. Let $\mathfrak{K} \subseteq \Omega$ be the compact set in Ω defined as

$$\mathfrak{K} := \big\{\psi \in \Omega \bigm| \psi(0) = c \text{ and } \forall \delta > 0 : m_\delta^S(\psi) \le f(\delta)\big\}$$

for some $f : (0, \infty) \to (0, \infty)$ satisfying $\lim_{\delta \to 0} f(\delta) = 0$ (cf. the Arzelà–Ascoli theorem) and chosen in such a way that $\mathcal{W}_c(\mathfrak{K}) > 1 - \epsilon$ and $\underline{Q}_c(\mathfrak{K}) > 1 - \epsilon$. Let $g(\delta) := \sup_{\psi_1, \psi_2 \in \mathfrak{K} : \rho(\psi_1, \psi_2) \le \delta} |F(\psi_1) - F(\psi_2)|$ be the modulus of continuity of F on $\mathfrak{K}$; we know that $\lim_{\delta \to 0} g(\delta) = 0$. Set $F_N(\psi) := F(\psi_N)$, where ψ_N is the piecewise linear function whose graph is obtained by joining the points $(iS/N, \psi(iS/N))$, $i = 0, 1, \dots, N$, and $(\infty, \psi(S))$, and N is so large that $g(f(S/N)) \le \epsilon$. Since

$$\psi \in \mathfrak{K} \Longrightarrow \|\psi - \psi_N\|_{C[0,S]} \le f(S/N) \Longrightarrow \rho(\psi, \psi_N) \le f(S/N)$$

(we assume, without loss of generality, that the graph of ψ is horizontal over $[S, \infty)$), we have $\|F - F_N\|_{C(\mathfrak{K})} \le \epsilon$. Choose positive capital processes $\mathfrak{S}^0$ and $\mathfrak{S}^1$ such that

$$\mathfrak{S}_0^0 \le \int F_N \, d\mathcal{W}_c + \epsilon, \qquad \liminf_{t \to \infty} \mathfrak{S}_t^0(\omega) \ge (F_N \circ \mathrm{ntt})(\omega),$$
$$\mathfrak{S}_0^1 \le \epsilon, \qquad \liminf_{t \to \infty} \mathfrak{S}_t^1(\omega) \ge (\mathbf{1}_{\mathfrak{K}^c} \circ \mathrm{ntt})(\omega),$$

provided $\omega(0) = c$ and $A_\infty(\omega) = \infty$. The sum $\mathfrak{S} := \mathfrak{S}^0 + (\sup F)\mathfrak{S}^1 + \epsilon$ will satisfy

$$\begin{aligned}\mathfrak{S}_0 &\le \int F_N \, d\mathcal{W}_c + (\sup F + 2)\epsilon \le \int_{\mathfrak{K}} F_N \, d\mathcal{W}_c + (2 \sup F + 2)\epsilon \\ &\le \int_{\mathfrak{K}} F \, d\mathcal{W}_c + (2 \sup F + 3)\epsilon \le \int F \, d\mathcal{W}_c + (2 \sup F + 3)\epsilon\end{aligned}$$

and

$$\liminf_{t \to \infty} \mathfrak{S}_t(\omega) \ge (F_N \circ \mathrm{ntt})(\omega) + (\sup F)(\mathbf{1}_{\mathfrak{K}^c} \circ \mathrm{ntt})(\omega) + \epsilon \ge (F \circ \mathrm{ntt})(\omega),$$

provided $\omega(0) = c$ and $A_\infty(\omega) = \infty$.

(e) We can further assume that

$$F(\psi) = U\big(\psi(S/N), \psi(2S/N), \dots, \psi(S)\big) \tag{10.2}$$

where the function $U : \mathbb{R}^N \to [0, \infty)$ is not only continuous but also has compact support. (We sometimes say that U is the *generator* of F.) Indeed, let us fix $\epsilon > 0$ and prove

$$\overline{\mathbb{E}}\big(F \circ \mathrm{ntt}; \omega(0) = c, A_\infty = \infty\big) \le \int F \, d\mathcal{W}_c + C\epsilon$$

for some positive constant C assuming

$$\overline{\mathbb{E}}\big(G \circ \mathrm{ntt}; \omega(0) = c, A_\infty = \infty\big) \le \int G \, d\mathcal{W}_c$$

for all G whose generator has compact support. Let B_R be the open ball of radius R and centered at the origin in the space $\mathbb{R}^N$ with the ℓ_∞ norm. We can rewrite (10.2) as $F(\psi) = U(\sigma(\psi))$, where $\sigma : \Omega \to \mathbb{R}^N$ reduces each $\psi \in \Omega$ to $\sigma(\psi) := (\psi(S/N), \psi(2S/N), \dots, \psi(S))$. Choose $R > 0$ so large that $\mathcal{W}_c(\sigma^{-1}(B_R)) > 1 - \epsilon$ and $\underline{Q}_c(\sigma^{-1}(B_R)) > 1 - \epsilon$ (the existence of such R follows from the Arzelà–Ascoli theorem and Lemma 10.1). Alongside F, whose generator is denoted U, we also consider F^* with generator

$$U^*(z) := \begin{cases} U(z) & \text{if } z \in \overline{B_R} \\ 0 & \text{if } z \in B_{2R}^c \end{cases}$$

(where $\overline{B_R}$ is the closure of B_R in $\mathbb{R}^N$); in the remaining region $B_{2R} \setminus \overline{B_R}$, U^* is defined arbitrarily (but making sure that U^* is continuous and takes values in $[0, \sup U]$; this can be done by the Tietze–Urysohn theorem, [25], Theorem 2.1.8). Choose positive capital processes $\mathfrak{S}^0$ and $\mathfrak{S}^1$ such that

$$\begin{aligned} \mathfrak{S}_0^0 &\le \int F^* \, d\mathcal{W}_c + \epsilon, & \liminf_{t\to\infty} \mathfrak{S}_t^0(\omega) &\ge \big(F^* \circ \mathrm{ntt}\big)(\omega), \\ \mathfrak{S}_0^1 &\le \epsilon, & \liminf_{t\to\infty} \mathfrak{S}_t^1(\omega) &\ge (\mathbf{1}_{(\sigma^{-1}(B_R))^c} \circ \mathrm{ntt})(\omega), \end{aligned}$$

provided $\omega(0) = c$ and $A_\infty(\omega) = \infty$. The sum $\mathfrak{S} := \mathfrak{S}^0 + (\sup F)\mathfrak{S}^1$ will satisfy

$$\begin{aligned} \mathfrak{S}_0 &\le \int F^* \, d\mathcal{W}_c + (\sup F + 1)\epsilon \le \int_{\sigma^{-1}(B_R)} F^* \, d\mathcal{W}_c + (2 \sup F + 1)\epsilon \\ &= \int_{\sigma^{-1}(B_R)} F \, d\mathcal{W}_c + (2 \sup F + 1)\epsilon \le \int F \, d\mathcal{W}_c + (2 \sup F + 1)\epsilon \end{aligned}$$

and

$$\liminf_{t\to\infty} \mathfrak{S}_t(\omega) \ge \big(F^* \circ \mathrm{ntt}\big)(\omega) + (\sup F)(\mathbf{1}_{(\sigma^{-1}(B_R))^c} \circ \mathrm{ntt})(\omega) \ge (F \circ \mathrm{ntt})(\omega),$$

provided $\omega(0) = c$ and $A_\infty(\omega) = \infty$.

(f) Since every continuous $U : \mathbb{R}^N \to [0, \infty)$ with compact support can be arbitrarily well approximated in $C(\mathbb{R}^N)$ by an infinitely differentiable (positive) function with compact support (see e.g. [1], Theorem 2.29), we can further assume that the generator U of F is an infinitely differentiable function with compact support.

(g) By Lemma 10.1, it suffices to prove that, given $\epsilon > 0$ and a compact set $\mathfrak{K}$ in Ω, some positive capital process $\mathfrak{S}$ with $\mathfrak{S}_0 \le \int F \, d\mathcal{W}_c + \epsilon$ achieves $\liminf_{t\to\infty} \mathfrak{S}_t(\omega) \ge (F \circ \mathrm{ntt})(\omega)$ for all $\omega \in \mathrm{ntt}^{-1}(\mathfrak{K})$ such that $\omega(0) = c$ and $A_\infty(\omega) = \infty$. Indeed, we can choose $\mathfrak{K}$ with $\underline{Q}_c(\mathfrak{K})$ so close to 1 that the sum of $\mathfrak{S}$ and a positive capital process eventually attaining $\sup F$ on $(\mathrm{ntt}^{-1}(\mathfrak{K}))^c$ will give a positive capital process starting from at most $\int F\,d\mathcal{W}_c + 2\epsilon$ and attaining $(F \circ \mathrm{ntt})(\omega)$ in the limit, provided $\omega(0) = c$ and $A_\infty(\omega) = \infty$.

From now on we fix a compact $\mathfrak{K} \subseteq \Omega$, assuming without loss of generality that the statements inside the outer parentheses in (9.7) and (9.8) are satisfied for some $\alpha > 0$ when $\mathrm{ntt}(\omega) \in \mathfrak{K}$.

In the rest of the proof, we use, often following [55], Sect. 6.2, the standard method going back to Lindeberg [42]. For $i = N - 1$, define a function $\overline{U}_i : \mathbb{R} \times [0, \infty) \times \mathbb{R}^i \to \mathbb{R}$ by

$$\overline{U}_i(x, D; x_1, \dots, x_i) := \int_{-\infty}^{\infty} U_{i+1}(x_1, \dots, x_i, x + z)\mathcal{N}_{0,D}(dz), \tag{10.3}$$

where U_N stands for U and $\mathcal{N}_{0,D}$ is the Gaussian probability measure on $\mathbb{R}$ with mean 0 and variance $D \geq 0$. Next define, for $i = N - 1$,

$$U_i(x_1, \dots, x_i) := \overline{U}_i(x_i, S/N; x_1, \dots, x_i). \tag{10.4}$$

Finally, we can alternately use (10.3) and (10.4) for $i = N - 2, \dots, 1, 0$ to define inductively other $\overline{U}_i$ and U_i (with (10.4) interpreted as $U_0 := \overline{U}_0(c, S/N)$ when $i = 0$). Notice that $U_0 = \int F\,d\mathcal{W}_c$.

Informally, the functions (10.3) and (10.4) constitute Sceptic's goal: assuming $\mathrm{ntt}(\omega) \in \mathfrak{K}$, $\omega(0) = c$, and $A_\infty(\omega) = \infty$, he will keep his capital at time $\tau_{iS/N}$, $i = 0, 1, \dots, N$, close to $U_i(\omega(\tau_{S/N}), \omega(\tau_{2S/N}), \dots, \omega(\tau_{iS/N}))$ and his capital at any other time $t \in [0, \tau_S]$ close to $\overline{U}_i(\omega(t), D; \omega(\tau_{S/N}), \omega(\tau_{2S/N}), \dots, \omega(\tau_{iS/N}))$, where $i := \lfloor NA_t/S \rfloor$ and $D := (i + 1)S/N - A_t$. This will ensure that his capital at time τ_S is close to or exceeds $(F \circ \mathrm{ntt})(\omega)$ when his initial capital is $U_0 = \int F\,d\mathcal{W}_c$, $\omega(0) = c$, and $A_\infty(\omega) = \infty$.

The proof is based on the fact that each function $\overline{U}_i(x, D; x_1, \dots, x_i)$ satisfies the heat equation in the variables x and D, i.e.,

$$\frac{\partial \overline{U}_i}{\partial D}(x, D; x_1, \dots, x_i) = \frac{1}{2}\frac{\partial^2 \overline{U}_i}{\partial x^2}(x, D; x_1, \dots, x_i) \tag{10.5}$$

for all $x \in \mathbb{R}$, all $D > 0$, and all $x_1, \dots, x_i \in \mathbb{R}$. This can be checked by direct differentiation.

Sceptic will only bet at the times of the form $\tau_{kS/LN}$, where $L \in \{1, 2, \dots\}$ is a constant that will later be chosen large and k is integer. For $i = 0, \dots, N$ and $j = 0, \dots, L$, let us set

$$t_{i,j} := \tau_{iS/N + jS/LN}, \qquad X_{i,j} := \omega(t_{i,j}), \qquad D_{i,j} := S/N - jS/LN.$$

For any array $Y_{i,j}$, we set $dY_{i,j} := Y_{i,j+1} - Y_{i,j}$.

Using Taylor's formula and omitting the arguments $\omega(\tau_{S/N}), \dots, \omega(\tau_{iS/N})$, we obtain, for $i = 0, \dots, N - 1$ and $j = 0, \dots, L - 1$,

$$
\begin{aligned}
d\overline{U}_i(X_{i,j}, D_{i,j}) &= \frac{\partial \overline{U}_i}{\partial x}(X_{i,j}, D_{i,j})\, dX_{i,j} + \frac{\partial \overline{U}_i}{\partial D}(X_{i,j}, D_{i,j}) dD_{i,j} \\
&\quad + \frac{1}{2}\frac{\partial^2 \overline{U}_i}{\partial x^2}\big(X'_{i,j}, D'_{i,j}\big)(dX_{i,j})^2 + \frac{\partial^2 \overline{U}_i}{\partial x \partial D}\big(X'_{i,j}, D'_{i,j}\big)\, dX_{i,j} dD_{i,j} \\
&\quad + \frac{1}{2}\frac{\partial^2 \overline{U}_i}{\partial D^2}\big(X'_{i,j}, D'_{i,j}\big)(dD_{i,j})^2, \qquad (10.6)
\end{aligned}
$$

where $(X'_{i,j}, D'_{i,j})$ is a point strictly between $(X_{i,j}, D_{i,j})$ and $(X_{i,j+1}, D_{i,j+1})$. Applying Taylor's formula to $\partial^2 \overline{U}_i / \partial x^2$, we find

$$
\begin{aligned}
\frac{\partial^2 \overline{U}_i}{\partial x^2}\big(X'_{i,j}, D'_{i,j}\big) &= \frac{\partial^2 \overline{U}_i}{\partial x^2}(X_{i,j}, D_{i,j}) \\
&\quad + \frac{\partial^3 \overline{U}_i}{\partial x^3}\big(X''_{i,j}, D''_{i,j}\big)\Delta X_{i,j} + \frac{\partial^3 \overline{U}_i}{\partial D \partial x^2}\big(X''_{i,j}, D''_{i,j}\big)\Delta D_{i,j},
\end{aligned}
$$

where $(X''_{i,j}, D''_{i,j})$ is a point strictly between $(X_{i,j}, D_{i,j})$ and $(X'_{i,j}, D'_{i,j})$, and $\Delta X_{i,j}$ and $\Delta D_{i,j}$ satisfy $|\Delta X_{i,j}| \le |dX_{i,j}|$, $|\Delta D_{i,j}| \le |dD_{i,j}|$. Plugging this equation and the heat equation (10.5) into (10.6), we obtain

$$
\begin{aligned}
d\overline{U}_i(X_{i,j}, D_{i,j}) &= \frac{\partial \overline{U}_i}{\partial x}(X_{i,j}, D_{i,j})\, dX_{i,j} + \frac{1}{2}\frac{\partial^2 \overline{U}_i}{\partial x^2}(X_{i,j}, D_{i,j})\big((dX_{i,j})^2 + dD_{i,j}\big) \\
&\quad + \frac{1}{2}\frac{\partial^3 \overline{U}_i}{\partial x^3}\big(X''_{i,j}, D''_{i,j}\big)\Delta X_{i,j}(dX_{i,j})^2 \\
&\quad + \frac{1}{2}\frac{\partial^3 \overline{U}_i}{\partial D \partial x^2}\big(X''_{i,j}, D''_{i,j}\big)\Delta D_{i,j}(dX_{i,j})^2 \\
&\quad + \frac{\partial^2 \overline{U}}{\partial x \partial D}\big(X'_{i,j}, D'_{i,j}\big)\, dX_{i,j} dD_{i,j} \\
&\quad + \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial D^2}\big(X'_{i,j}, D'_{i,j}\big)(dD_{i,j})^2. \qquad (10.7)
\end{aligned}
$$

To show that Sceptic can achieve his goal, we describe a simple trading strategy that results in an increase of his capital of approximately (10.7) during the time interval $[t_{i,j}, t_{i,j+1}]$ (we shall make sure that the cumulative error of our approximation is small with high probability, which will imply the statement of the theorem). We shall see that there is a trading strategy resulting in the capital increase equal to the first addend on the right-hand side of (10.7), that there is another trading strategy resulting in the capital increase approximately equal to the second addend, and that the last four addends are negligible. The sum of the two trading strategies will achieve our goal.

The trading strategy whose capital increase over $[t_{i,j}, t_{i,j+1}]$ is the first addend is obvious: it bets $\partial \overline{U}_i / \partial x$ at time $t_{i,j}$. The bet is bounded as average of $\partial U_{i+1} / \partial x_{i+1}$, the boundedness of which can be seen from the recursive formula

$$U_k(x_1,\dots,x_k)=\int_{-\infty}^{\infty}U_{k+1}(x_1,\dots,x_k,x_k+z)\mathcal{N}_{0,S/N}(dz),$$
$$k=i+1,\dots,N-1,$$

and $U_N=U$ being an infinitely differentiable function with compact support.

The second addend involves the expression

$$(dX_{i,j})^2+dD_{i,j}=(\omega_{i,j+1}-\omega_{i,j})^2-S/LN.$$

To analyze it, we need the following lemma.

Lemma 10.3 *For all $\delta>0$ and $\beta>0$, there exists a positive integer ℓ such that*

$$t_{i,j+1}<\infty\Longrightarrow\left|\frac{A^{\ell,t_{i,j}}_{t_{i,j+1}}}{S/LN}-1\right|<\delta$$

holds for all $i=0,\dots,N-1$ and $j=0,\dots,L-1$ except for a set of ω of outer content at most β.

Lemma 10.3 can be proved similarly to (9.4). (The inequality in (9.4) is one-sided, so it was sufficient to use only (8.7); for Lemma 10.3, both (8.7) and (8.6) should be used.)

We know that $(\omega(t)-\omega(t_{i,j}))^2-A^{\ell,t_{i,j}}_t$ is a simple capital process (see the proof of Lemma 9.1). Therefore, there is indeed a simple trading strategy resulting in a capital increase approximately equal to the second addend on the right-hand side of (10.7), with a cumulative approximation error that can be made arbitrarily small on a set of ω of inner content arbitrarily close to 1. (Analogously to the analysis of the first addend, $\partial^2\overline{U}_i/\partial x^2$ is bounded as average of $\partial^2U_{i+1}/\partial x^2_{i+1}$.)

Let us show that the last four terms on the right-hand side of (10.7) are negligible when L is sufficiently large (assuming S, N, and U fixed). All the partial derivatives involved in those terms are bounded: the heat equation implies

$$\frac{\partial^3\overline{U}_i}{\partial D\partial x^2}=\frac{\partial^3\overline{U}_i}{\partial x^2\partial D}=\frac{1}{2}\frac{\partial^4\overline{U}_i}{\partial x^4},$$
$$\frac{\partial^2\overline{U}_i}{\partial x\partial D}=\frac{1}{2}\frac{\partial^3\overline{U}_i}{\partial x^3},$$
$$\frac{\partial^2\overline{U}_i}{\partial D^2}=\frac{1}{2}\frac{\partial^3\overline{U}_i}{\partial D\partial x^2}=\frac{1}{4}\frac{\partial^4\overline{U}_i}{\partial x^4},$$

and $\partial^3\overline{U}_i/\partial x^3$ and $\partial^4\overline{U}_i/\partial x^4$, being averages of $\partial^3U_{i+1}/\partial x^3_{i+1}$ and $\partial^4U_{i+1}/\partial x^4_{i+1}$, respectively, are bounded. We can assume that

$$|dX_{i,j}|\le C_1L^{-1/8},\qquad\sum_{i=0}^{N-1}\sum_{j=0}^{L-1}(dX_{i,j})^2\le C_2L^{1/16}$$

(cf. (9.7) and (9.8), respectively) for $\operatorname{ntt}(\omega) \in \mathfrak{K}$ and some constants C_1 and C_2 (remember that S, N, U and, of course, α are fixed; without loss of generality we can assume that N and L are powers of 2). This makes the cumulative contribution of the four terms have at most the order of magnitude $O(L^{-1/16})$; therefore, Sceptic can achieve his goal for $\operatorname{ntt}(\omega) \in \mathfrak{K}$ by making L sufficiently large.

To ensure that his capital is always positive, Sceptic stops playing as soon as his capital hits 0. Increasing his initial capital by a small amount, we can make sure that this will never happen when $\operatorname{ntt}(\omega) \in \mathfrak{K}$ (for L sufficiently large).

11 Proof of the inequality ≤ in Theorem 6.2

Fix a bounded positive $\mathcal{I}$-measurable functional F. Let $a := \int F \, d\mathcal{W}_c$; our goal is to show that $\overline{\mathbb{E}}(F; \omega(0) = c) \leq a$. Define Ω' to be the set of all $\omega \in \Omega$ such that $\omega(0) = c$ and $\forall t \in [0, \infty) : \overline{A}_t(\omega) = \underline{A}_t(\omega) = t$. We know (Lemma 8.4) that $\mathcal{W}_c(\Omega') = 1$. It is clear that $\tau_s(\omega) = s$ for all $\omega \in \Omega'$, and so $\operatorname{ntt}(\omega) = \omega$ for all $\omega \in \Omega'$. By Theorem 6.4,

$$\overline{\mathbb{E}}(F \mathbf{1}_{\Omega'}) = \overline{\mathbb{E}}\big(F; \Omega'\big) = \overline{\mathbb{E}}\big(F \circ \operatorname{ntt}; \Omega'\big) \leq \overline{\mathbb{E}}\big(F \circ \operatorname{ntt}; \omega(0) = c, A_\infty = \infty\big) = a$$

(we do not need the opposite inequality in that theorem). Thus, for any $\epsilon > 0$ there is a positive capital process $\mathfrak{S}$ such that $\mathfrak{S}_0 \leq a + \epsilon$ and $\liminf_{t\to\infty} \mathfrak{S}_t \geq F \mathbf{1}_{\Omega'}$. We assume without loss of generality that $\mathfrak{S}$ is bounded. Moreover, the proof of Theorem 6.4 shows that $\mathfrak{S}$ can be chosen *time-invariant*, in the sense that $\mathfrak{S}_{f(t)}(\omega) = \mathfrak{S}_t(\omega \circ f)$ for all time transformations f and all $t \in [0, \infty)$. This property will also be assumed to be satisfied until the end of this section. In conjunction with the time-superinvariance of F (which is equivalent to (6.2)) and the last statement of Theorem 5.1(a), it implies, for typical $\omega \in \Omega$ satisfying $\omega(0) = c$ and $A_\infty(\omega) = \infty$,

$$\liminf_{t\to\infty} \mathfrak{S}_t(\omega) = \liminf_{t\to\infty} \mathfrak{S}_t\big(\psi^f\big) = \liminf_{t\to\infty} \mathfrak{S}_{f(t)}(\psi) \tag{11.1}$$

$$\geq (F \mathbf{1}_{\Omega'})(\psi) = F(\psi) \geq F(\omega), \tag{11.2}$$

where ψ is any element of Ω' that satisfies $\psi^f = \omega$ for some time transformation f, necessarily satisfying $\lim_{t\to\infty} f(t) = \infty$ (we can always take $\psi := \operatorname{ntt}(\omega)$ and $f := A(\omega)$; $\omega = \operatorname{ntt}(\omega) \circ A(\omega)$ follows from $\omega(t) = \omega(\tau_{A_t(\omega)})$). It is easy to modify $\mathfrak{S}$ so that $\mathfrak{S}_0$ is increased by at most ϵ and the inequality between the two extreme terms in (11.1) becomes true for all, rather than for typical, $\omega \in \Omega$ satisfying $\omega(0) = c$ and $A_\infty(\omega) = \infty$.

Let us now consider $\omega \in \Omega$ such that $\omega(0) = c$ but $A_\infty(\omega) = \infty$ is not satisfied. Without loss of generality, we assume that $A(\omega)$ exists and is an element of Ω with the same intervals of constancy as ω and that the statement in the outermost parentheses in (9.7) holds for some $\alpha > 0$. Set $b := A_\infty(\omega) < \infty$. Suppose $\liminf_{t\to\infty} \mathfrak{S}_t(\omega) \leq F(\omega) - \delta$ for some $\delta > 0$; to complete the proof, it suffices to arrive at a contradiction. By the statement in the outermost parentheses in (9.7), the function $\operatorname{ntt}(\omega)|_{[0,b)}$ can be continued to the closed interval $[0, b]$ so that it becomes an element g of $C[0,b]$. Let $\Gamma(g)$ be the set of all extensions of g that are elements of Ω. By the time-superinvariance of F, all $\psi \in \Gamma(g)$ satisfy $F(\psi) \geq F(\omega)$. Since $\liminf_{t\to b-} \mathfrak{S}_t(\psi) \leq F(\omega) - \delta$ (remember that $\mathfrak{S}$ is time-invariant) and the function $t \mapsto \mathfrak{S}_t$ is lower semicontinuous (see (2.2)), $\mathfrak{S}_b(\psi) \leq F(\omega) - \delta \leq F(\psi) - \delta$, for each $\psi \in \Gamma(g)$. Continue g, which is now fixed, by measure-theoretic Brownian motion starting from $g(b)$, so that the extension is an element of Ω' with probability one. Let us represent $\mathfrak{S}$ in the form (2.2) and use the argument in the proof of Lemma 6.3. We can see that $\mathfrak{S}_t(\xi)$, $t \geq b$, where ξ is g extended by the trajectory of Brownian motion starting from $g(b)$, is a measure-theoretic stochastic process which is the sum of a sequence of positive continuous supermartingales on the time interval $[b,\infty)$. Now we have the following analog of (6.4) that

$$\begin{aligned}\int_{\Gamma(g)} \liminf_{t\to\infty} \mathfrak{S}_t \, dP &\leq \liminf_{t\to\infty} \int_{\Gamma(g)} \mathfrak{S}_t \, dP = \liminf_{t\to\infty} \int_{\Gamma(g)} \sum_{n=1}^{\infty} \mathfrak{K}_t^{G_n,c_n}(\psi) P(d\psi) \\ &= \liminf_{t\to\infty} \sum_{n=1}^{\infty} \int_{\Gamma(g)} \mathfrak{K}_t^{G_n,c_n}(\psi) P(d\psi) \\ &\leq \liminf_{t\to\infty} \sum_{n=1}^{\infty} \int_{\Gamma(g)} \mathfrak{K}_b^{G_n,c_n}(\psi) P(d\psi) \\ &= \int_{\Gamma(g)} \sum_{n=1}^{\infty} \mathfrak{K}_b^{G_n,c_n}(\psi) P(d\psi) \\ &= \int_{\Gamma(g)} \mathfrak{S}_b(\psi) P(d\psi) \leq \int_{\Gamma(g)} F(\psi) P(d\psi) - \delta,\end{aligned}$$

P referring to the underlying probability measure of the Brownian motion (concentrated on $\Gamma(g)$). However, $\int_{\Gamma(g)} \liminf_{t\to\infty} \mathfrak{S}_t \, dP < \int_{\Gamma(g)} F \, dP$ contradicts the choice of $\mathfrak{S}$: cf. (11.1) and Lemma 8.4.

12 Other connections with the literature

This section discusses several areas of stochastics (in Sect. 12.1) and mathematical finance (in Sects. 12.2 and 12.3) which are especially closely connected with the present paper's approach.

12.1 Stochastic integration

The natural financial interpretation of the stochastic integral is that $\int_0^t \pi_s \, dX_s$ is the trader's profit at time t from holding π_s units at time s of a financial security with price path X (see e.g. [57], Remark III.5a.2). It is widely believed that $\int_0^t \pi_s \, dX_s$ cannot in general be defined pathwise; since our picture does not involve a probability measure on Ω, we restricted ourselves to countable combinations (see (2.2)) of integrals of simple integrands (see (2.1)). This definition served our purposes well, but in this subsection, we discuss other possible definitions, always assuming that X_s is a continuous function of s.

The pathwise definition of $\int_0^t \pi_s \, dX_s$ is straightforward when the total variation (i.e., strong 1-variation in the terminology of Sect. 4.2) of X over $[0, t]$ is finite; it can be defined as e.g. the Lebesgue–Stieltjes integral. It has been known for a long time that the Riemann–Stieltjes definition also works in the case $1/\operatorname{vi}(\pi) + 1/\operatorname{vi}(X) > 1$ (Youngs' theory; see e.g. [22], Sect. 2.2). Unfortunately, in the most interesting case $\operatorname{vi}(\pi) = \operatorname{vi}(X) = 2$, this condition is not satisfied.

Another pathwise definition of stochastic integrals is due to Föllmer [26]. Föllmer considers a sequence of partitions of the interval $[0, \infty)$ and assumes that the quadratic variation of X exists, in a suitable sense, along this sequence. Our definition of quadratic variation given in Sect. 5 resembles Föllmer's definition; in particular, our Theorem 5.1(a) implies that Föllmer's quadratic variation exists for typical ω along the sequence of partitions T^n (as defined at the beginning of Sect. 5). In the statement of his theorem ([26], p. 144), Föllmer defines the pathwise integral $\int_0^t f(X_s) \, dX_s$ for a C^1 function f assuming that the quadratic variation of X exists and proves Itô's formula for his integral. In particular, Föllmer's pathwise integral $\int_0^t f(\omega(s)) \, d\omega(s)$ along T^n exists for typical ω and satisfies Itô's formula. There are two obstacles to using Föllmer's definition in this paper: in order to prove the existence of the quadratic variation, we already need our simple notion of integration (which defines the notion of "typical" in Theorem 5.1(a)); and the class of integrals $\int_0^t f(\omega(s)) \, d\omega(s)$ with $f \in C^1$ is too restrictive for our purposes, and using it would complicate the proofs.

An interesting development of Youngs' theory is Lyons's [44] theory of rough paths. In Lyons's theory, we can deal directly only with the rough paths X satisfying $\operatorname{vi}(X) < 2$ (by means of Youngs' theory). In order to treat rough paths satisfying $\operatorname{vi}(X) \in [n, n+1)$, where $n = 2, 3, \dots$, we need to postulate the values of the iterated integrals $X^i_{s,t} := \int_{s<u_1<\cdots<u_i<t} dX_{u_1} \cdots dX_{u_i}$ for $i = 2, \dots, n$ (satisfying the so-called Chen's consistency condition). According to Corollary 4.4, only the case $n = 2$ is relevant for our idealized market, and in this case Lyons's theory is much simpler than in general (but to establish Corollary 4.4 we already used our simple integral). Even in the case $n = 2$ there are different natural choices of $X^2_{s,t}$ (e.g., those leading to Itô-type and to Stratonovich-type integrals); and in the case $n > 2$ the choice would inevitably become even more ad hoc.

Another obstacle to using Lyons's theory in this paper is that the smoothness restrictions that it imposes are too strong for our purposes. In principle, we could use the integral $\int_0^t G \, d\omega$ to define the capital brought by a strategy G for trading in ω by time t. However, similarly to Föllmer's, Lyons's theory requires that G should take a position of the form $f(\omega(t))$ at time t, where f is a differentiable function whose derivative f' is a Lipschitz function ([12], Theorems 3.2 and 3.6). This restriction would again complicate the proofs.

12.2 Fundamental theorems of asset pricing

The first and second fundamental theorems of asset pricing (FTAPs, for brevity) are families of mathematical statements; e.g., we have different statements for one-period, multi-period, discrete-time, and continuous-time markets. A very special case of the second FTAP, the one covering binomial models, was already discussed briefly in Sect. 1. In the informal comparisons of our results and the FTAPs in this subsection, we only consider the case of one security whose price path X is assumed to be continuous. (In the background, there is also an implicit security, such as cash or bond, serving as our numéraire.)

The first FTAP says that a stochastic model for the security price path X admits no arbitrage (or satisfies a suitable modification of this condition, such as no free lunch with vanishing risk) if and only if there is an equivalent martingale measure (or a suitable modification thereof, such as an equivalent sigma-martingale measure). The second FTAP says that the market is complete if and only if there is only one equivalent martingale measure (as e.g. in the case of the classical Black–Scholes model). The completeness of the market means that each contingent claim has a unique fair price defined in terms of hedging.

Theorems 3.1 and 6.2 are connected (admittedly, somewhat loosely) with the second FTAP, namely its part saying that each contingent claim has a unique fair price provided there is a unique equivalent martingale measure. For example, Theorem 3.1 and Corollary 3.7 essentially say that each contingent claim of the form $\mathbf{1}_E$, where $E \in \mathcal{I}$ and $\omega(0) = c$ for all $\omega \in E$, has a fair price and its fair price is equal to the Wiener measure $\mathcal{W}_c(E)$ of E. The scarcity of contingent claims that we can show to have a fair price is not surprising: it is intuitively clear that our market is heavily incomplete. According to Remark 3.4, we can replace the Wiener measure by many other measures. The proofs of both the second FTAP and our Theorems 3.1 and 6.2 construct fair prices of contingent claims using hedging arguments. Extending this paper's results to a wider class of contingent claims is an interesting direction of further research.

Theorems 3.1 and 6.2 are much more closely connected with a generalized version of the second FTAP (see [27], Theorem 5.32, for a discrete-time version) which says, in the first approximation, that the range of arbitrage-free prices of a contingent claim coincides with the range of the expectations of its payoff function with respect to the equivalent martingale measures. We can even say (completely disregarding mathematical rigour for a moment) that Theorem 6.2 is a special case of the generalized second FTAP: by the Dubins–Schwarz result, ω is a time-changed Brownian motion under the martingale measures, and so the $\mathcal{I}$-measurability of F implies that the unique fair price of the contingent claim with the payoff function F is $\int F \, d\mathcal{W}_{\omega(0)}$.

The conditions of the first, second and generalized second FTAP include a given probability measure on the sample space (our stochastic model of the market). In the case of continuous time, it is this postulated probability measure that allows one to use Itô's notion of stochastic integral for defining basic financial notions such as the resulting capital of a trading strategy. No such condition is needed in the case of our results.

The notion of arbitrage is pivotal in mathematical finance; in particular, it enters both the first FTAP and the generalized second FTAP. This paper's results and discussions were not couched in terms of arbitrage, although there were two places where arbitrage-type notions did enter the picture.

First, we used the notion of coherence in Sect. 7. The most standard notion of arbitrage is that no trading strategy can start from zero capital and end up with positive capital that is strictly positive with a strictly positive probability. Our condition of coherence is similar but much weaker; and of course, it does not involve probabilities. We show that this condition is satisfied automatically in our framework.

The second place where we need arbitrage-type notions is in the interpretation of results such as Corollaries 4.2 and 4.4–4.7. For example, Corollary 4.4 implies that $\mathrm{vi}^{[0,1]}(\omega) \in \{0, 2\}$ for typical ω. Remembering our definitions, this means that either $\mathrm{vi}^{[0,1]}(\omega) \in \{0, 2\}$ or a predefined trading strategy makes infinite capital (at time 1) starting from one monetary unit and never risking going into debt. If we do not believe that making infinite capital risking only one monetary unit is possible for a predefined trading strategy (i.e., that the market is "efficient," in a very weak sense), we should expect $\mathrm{vi}^{[0,1]}(\omega) \in \{0, 2\}$. This looks like an arbitrage-type argument, but there are two important differences:

- Our condition of market efficiency is only needed for the interpretation of our results; their mathematical statements do not depend on it. The standard no-arbitrage conditions are used directly in mathematical theorems (such as the first FTAP and the generalized second FTAP).
- The usual no-arbitrage conditions are conditions on the currently observed prices or our stochastic model of the market (or both). On the contrary, our condition of market efficiency describes what we expect to happen, or not to happen, on the actual price path.

It should be noted that our condition of market efficiency (a predefined trading strategy is not expected to make infinite capital risking only one monetary unit) is much closer to Delbaen and Schachermayer's [18] version of the no-arbitrage condition, which is known as NFLVR (no free lunch with vanishing risk), than to the classical no-arbitrage condition. The classical no-arbitrage condition only considers trading strategies that start from 0 and never go into debt, whereas the NFLVR condition allows trading strategies that start from 0 and are permitted to go into slight debt. Our condition of market efficiency allows risking one monetary unit, but this can be rescaled so that the trading strategies considered start from zero and are only allowed to go into debt limited by an arbitrarily small $\epsilon > 0$.

Remark 12.1 Mathematical statements of the first FTAP sometimes involve the condition that X should be a semimartingale: see e.g. Delbaen and Schachermayer's version in [18], Theorem 1.1. However, this condition is not a big restriction: in the same paper, Delbaen and Schachermayer show that the NFLVR condition already implies that X is a semimartingale under some additional conditions, such as X being locally bounded; see [18], Theorem 7.2. A direct proof of the last result, using financial arguments and not depending on the Bichteler–Dellacherie theorem, is given in the recent paper [3].

We could have used the notion of arbitrage to restate part of Theorem 6.2: if the contingent claim with a bounded and $\mathcal{I}$-measurable payoff function $F : \Omega \to [0, \infty)$ is worth strictly more than $\int F \, d\mathcal{W}_{\omega(0)}$ at time 0, we can turn capital 0 at time 0 into capital 1 at time ∞. Indeed, we can short such a contingent claim and divide the proceeds $\int F \, d\mathcal{W}_{\omega(0)} + \epsilon$, where $\epsilon > 0$, into two parts: investing $\int F \, d\mathcal{W}_{\omega(0)} + \epsilon/2$ into a trading strategy bringing capital $F(\omega)$ at time ∞ allows us to meet our obligation; we keep the remaining $\epsilon/2$ (and we can scale up our portfolio to replace $\epsilon/2$ by 1). We did not introduce the corresponding notion of arbitrage formally since this restatement does not seem to add much to the theorem.

12.3 Model uncertainty and robust results

In this subsection, we discuss some known approaches to mathematical finance that do not assume from the outset a given probability model.

One natural relaxation of the standard framework replaces the probability model with a family, more or less extensive, of probability models (there is a "model uncertainty"). Results proved under model uncertainty may be called robust. We get some robustness for free already in the standard Black–Scholes framework: option prices do not depend on the drift parameter μ in the probability model $dX_t/X_t = \mu\, dt + \sigma\, dW_t$, W being Brownian motion. "Volatility uncertainty," i.e., uncertainty about the value of σ, is much more serious. A natural assumption, sometimes called the "uncertain volatility model," is that σ can change dynamically between known limits $\underline{\sigma}$ and $\overline{\sigma}$, $\underline{\sigma} < \overline{\sigma}$. The study of volatility uncertainty under this assumption was originated by Avellaneda et al. [2] and Lyons [43] and has been the object of intensive study recently; whereas older paper concentrated on robust pricing of contingent claims whose payoff depends on the underlying security's value at one maturity date, recent work treats the much more difficult case of general path-dependent contingent claims. This research has given rise to two important developments: Denis and Martini's [19] "almost pathwise" theory of stochastic calculus and Peng's [50, 51] G-stochastic calculus (in our current context, G refers to the function $G(y) := \sup_{\sigma\in[\underline{\sigma},\overline{\sigma}]} \sigma^2 y$).

Definitions similar to our (2.3) and (6.1) are standard in the literature on model uncertainty: see e.g. Mykland [46], (3.3), Denis and Martini [19] (the definition of $\Lambda(f)$ on p. 834), or Cassese [9], (4.4). Different terms corresponding to our "outer content" have been used, such as "conservative ask price" (Mykland) and "cheapest riskless superreplication price" (Denis and Martini); we use Mykland's "conservative ask price" as a generic notion. A major difficulty for such definitions lies in defining the class of capital processes; it is here that pre-specifying a family of probability models proves to be particularly useful.

Finally, we discuss approaches that are completely model-free. Bick and Willinger [4] use Föllmer's construction of stochastic integrals discussed in Sect. 12.1 to define capital processes of trading strategies. Even though their framework is not stochastic, the conditions that they impose on the price paths in order for dynamic hedging to be successful are not so different from the standard conditions. The assumption used in their Proposition 1 is, in their notation, $[Y, Y]_t = Y^0 + \sigma^2 t$, where $S(t) = \exp(Y(t))$ is the price path and $[Y, Y]_t$ is the pathwise quadratic variation of its logarithm; this is similar to the Black–Scholes model. They also consider (in Proposition 3) a more general case $d[S, S]_t = \beta^2(S(t), t)$, but β has to be a continuous function that is known in advance.

Section 4 of Dawid et al.'s [14] can be recast as a study of the conservative ask price of the American option paying $f(X_t^*)$, where f is a fixed positive and increasing function, t is the exercise time (chosen by the option's owner), $X_t^* := \max_{s \le t} X_s$ (time is discrete in [14]), and $X_s \ge 0$ is the price of the underlying security at time s. Corollary 2 in [14] implies that the conservative ask price of this option is $X_0 \int_{X_0}^{\infty} \frac{f(x)}{x^2}\,dx$. This is compatible with Theorem 6.2 since X_0/x^2, $x \in [X_0, \infty)$, is the density of the maximum of Brownian motion started at X_0 and stopped when it hits 0 (cf. the first statement of Theorem 2.49 in [45]).

Let us assume for simplicity that $X_0 = 1$ (as in [28]). The simplest American option with payoff $f(X_t^*)$ is the one corresponding to the identity function $f(x) = x$; it is a kind of a perpetual lookback option (as discussed in e.g. [23], Sect. 5). The conservative ask price of this option is, of course, infinite: $\int_1^{\infty}(1/x)\,dx = \infty$. To get a finite price, we can fix a finite maturity date T and consider a European option with payoff $X_T^* := \sup_{t \le T} X_t$ (we no longer assume that time is discrete). To find a non-trivial conservative ask price of this *European lookback option*, Hobson [28] considers trading strategies that trade not only in the underlying security X but also in call options on X with maturity date T and all possible strike prices (making some regularity assumptions about the call prices); he also finds the conservative ask prices for some modifications of European lookback options. In order to avoid the use of the stochastic integral, the dynamic part of the trading strategies that he considers is very simple; there is only finite trading activity in each security. Hobson's paper has been developed in various directions: see e.g. the recent review [29] and references therein. One important issue that arises when we specify the prices of vanilla options at the outset is whether these prices lead to arbitrage opportunities; it has been investigated, for various notions of arbitrage, in [13] and [10].

An advantage of the present paper's main results is that the prices they provide are "almost" two-sided (serve as both ask and bid prices): cf. Corollary 3.7. Their disadvantage is that they allow us only to price such a narrow class of contingent claims: their payoff functions are required to be $\mathcal{I}$-measurable. In principle, Hobson's idea of using vanilla options for pricing exotic options may lead to interesting developments of the present paper's approach. One could consider a whole spectrum of trading frameworks, even in the case of one underlying security X. One extreme is the framework of the present paper and, in the case of a discontinuous price path, ref. [63]. The security is not supported by any derivatives, which leads to the paucity of contingent claims that can be priced. The other extreme is where, alongside X, we are allowed to trade in call or put options of all possible strikes and maturity dates. Perhaps the most interesting research questions arise in between the two extremes, where only some call or put options are available for use in hedging.

Acknowledgements The final statement of Theorem 3.1 is due to Peter McCullagh's insight and Tamas Szabados's penetrating questions. The game-theoretic version of Hoeffding's inequality is inspired by a question asked by Yoav Freund. Rimas Norvaiša's useful comments and explanations are gratefully appreciated. This paper very much benefitted from the feedback from several anonymous reviewers, the Associate Editor, and Professor Martin Schweizer. Their contributions ranged from very specific, such as noticing a mistake in the statement of Corollary 3.7 in an early version, to general comments that have led e.g. to new applications of Theorem 3.1, to Theorems 6.2 and 6.4, to the inclusion of Sect. 12, and to improved presentation. At the latest stages of the work on this paper I have benefitted from comments by Wouter Koolen, Roman Chychyla, and John Shawe-Taylor.

Supported in part by EPSRC (grant EP/F002998/1).

Appendix: Hoeffding's process

In this appendix, we check that Hoeffding's original proof of his inequality ([30], Theorem 2) remains valid in the game-theoretic framework. This observation is fairly obvious, but all details will be spelled out for convenience of reference. This appendix is concerned with the case of discrete time, and it will be convenient to redefine some notions (such as "process").

Perhaps the most useful product of Hoeffding's method is a positive supermartingale starting from 1 and attaining large values when the sum of bounded martingale differences is large. Hoeffding's inequality can be obtained by applying the maximal inequality to this supermartingale. However, we do not need Hoeffding's inequality in this paper, and instead of Hoeffding's positive supermartingale, we have a positive "supercapital process," to be defined below.

Here is a version of the basic forecasting protocol from [55]:

Game of forecasting bounded variables

Players: Sceptic, Forecaster, Reality

Protocol:

Sceptic announces $\mathcal{K}_0 \in \mathbb{R}$.
FOR $n = 1, 2, \dots$:
Forecaster announces interval $[a_n, b_n] \subseteq \mathbb{R}$ and number $\mu_n \in (a_n, b_n)$.
Sceptic announces $M_n \in \mathbb{R}$.
Reality announces $x_n \in [a_n, b_n]$.
Sceptic announces $\mathcal{K}_n \leq \mathcal{K}_{n-1} + M_n(x_n - \mu_n)$.

On each round n of the game Forecaster outputs an interval $[a_n, b_n]$ which, in his opinion, will cover the actual observation x_n to be chosen by Reality, and also outputs his expectation μ_n for x_n. The forecasts are being tested by Sceptic, who is allowed to gamble against them. The expectation μ_n is interpreted as the price of a ticket which pays x_n after Reality's move becomes known; Sceptic is allowed to buy any number M_n, positive or negative (perhaps zero), of such tickets. When x_n falls outside $[a_n, b_n]$, Sceptic becomes infinitely rich; without loss of generality we include the requirement $x_n \in [a_n, b_n]$ in the protocol; furthermore, we always assume that $\mu_n \in (a_n, b_n)$. Sceptic is allowed to choose his initial capital $\mathcal{K}_0$ and is allowed to throw away part of his money at the end of each round.

It is important that the game of forecasting bounded variables is a perfect-information game: each player can see the other players' moves before making his or her (Forecaster and Sceptic are male and Reality is female) own move; there is no randomness in the protocol.

A *process* is a real-valued function defined on all finite sequences

$$(a_1, b_1, \mu_1, x_1, \dots, a_N, b_N, \mu_N, x_N), \qquad N = 0, 1, \dots,$$

of Forecaster's and Reality's moves in the game of forecasting bounded variables. If we fix a strategy for Sceptic, Sceptic's capital $\mathcal{K}_N$, $N = 0, 1, \dots$, becomes a function of Forecaster's and Reality's previous moves; in other words, Sceptic's capital becomes a process. The processes that can be obtained this way are called *supercapital processes*.

The following theorem is essentially inequality (4.16) in [30].

Theorem A.1 *For any $h \in \mathbb{R}$, the process*

$$\prod_{n=1}^{N} \exp\left(h(x_n - \mu_n) - \frac{h^2}{8}(b_n - a_n)^2\right)$$

is a supercapital process.

Proof Assume without loss of generality that Forecaster is additionally required to always set $\mu_n := 0$. (Adding the same number to a_n, b_n and μ_n on each round will not change anything for Sceptic.) Now we have $a_n < 0 < b_n$.

It suffices to prove that on round n Sceptic can turn a capital of $\mathcal{K}$ into a capital of at least

$$\mathcal{K} \exp\left(hx_n - \frac{h^2}{8}(b_n - a_n)^2\right);$$

in other words, that he can obtain a payoff of at least

$$\exp\left(hx_n - \frac{h^2}{8}(b_n - a_n)^2\right) - 1$$

using the available tickets (paying x_n and costing 0). This will follow from the inequality

$$\exp\left(hx_n - \frac{h^2}{8}(b_n - a_n)^2\right) - 1 \le x_n \frac{e^{hb_n} - e^{ha_n}}{b_n - a_n} \exp\left(-\frac{h^2}{8}(b_n - a_n)^2\right), \quad \text{(A.1)}$$

which can be rewritten as

$$\exp(hx_n) \le \exp\left(\frac{h^2}{8}(b_n - a_n)^2\right) + x_n \frac{e^{hb_n} - e^{ha_n}}{b_n - a_n}. \quad \text{(A.2)}$$

Our goal is to prove (A.2). By the convexity of the function exp, it suffices to prove

$$\frac{x_n - a_n}{b_n - a_n} e^{hb_n} + \frac{b_n - x_n}{b_n - a_n} e^{ha_n} \le \exp\left(\frac{h^2}{8}(b_n - a_n)^2\right) + x_n \frac{e^{hb_n} - e^{ha_n}}{b_n - a_n}, \quad \text{(A.3)}$$

i.e.,

$$\frac{b_n e^{ha_n} - a_n e^{hb_n}}{b_n - a_n} \le \exp\left(\frac{h^2}{8}(b_n - a_n)^2\right), \quad \text{(A.4)}$$

i.e.,

$$\ln\left(b_n e^{ha_n} - a_n e^{hb_n}\right) \le \frac{h^2}{8}(b_n - a_n)^2 + \ln(b_n - a_n). \quad \text{(A.5)}$$

(The logarithm on the left-hand side of (A.5) is well defined since the numerator of the left-hand side of (A.4) is strictly positive, which follows from the left-hand side of (A.4) being the value at $x_n = 0$ of the left-hand side of (A.3), linear in x_n and strictly positive for both $x_n = a_n$ and $x_n = b_n$.) The derivative of the left-hand side of (A.5) in h is

$$\frac{a_n b_n e^{ha_n} - a_n b_n e^{hb_n}}{b_n e^{ha_n} - a_n e^{hb_n}}$$

and the second derivative, after cancellations and regrouping, is

$$(b_n - a_n)^2 \frac{(b_n e^{ha_n})(-a_n e^{hb_n})}{(b_n e^{ha_n} - a_n e^{hb_n})^2}.$$

The last ratio is of the form $u(1-u)$ where $0 < u < 1$. Hence it does not exceed $1/4$, and the second derivative itself does not exceed $(b_n - a_n)^2/4$. Inequality (A.5) now follows from the second-order Taylor expansion of the left-hand side around $h = 0$. □

References

1. Adams, R.A., Fournier, J.J.F.: Sobolev Spaces, 2nd edn. Academic Press, Amsterdam (2003)
2. Avellaneda, M., Levy, A., Parás, A.: Pricing and hedging derivative securities in markets with uncertain volatilities. Appl. Math. Finance **2**, 73–88 (1995)
3. Beiglböck, M., Schachermayer, W., Veliyev, B.: A direct proof of the Bichteler–Dellacherie theorem and connections to arbitrage. Ann. Probab. **39**, 2424–2440 (2011)
4. Bick, A., Willinger, W.: Dynamic spanning without probabilities. Stoch. Process. Appl. **50**, 349–374 (1994)
5. Billingsley, P.: Convergence of Probability Measures. Wiley, New York (1968)
6. Bruneau, M.: Mouvement brownien et F-variation. J. Math. Pures Appl. **54**, 11–25 (1975)
7. Bruneau, M.: Sur la p-variation des surmartingales. In: Dellacherie, C., Meyer, P.-A., Weil, M. (eds.) Séminaire de Probabilités de Strasbourg, XIII. LNM, vol. 721, pp. 227–232. Springer, Berlin (1979)
8. Burdzy, K.: On nonincrease of Brownian motion. Ann. Probab. **18**, 978–980 (1990)
9. Cassese, G.: Asset pricing with no exogenous probability measure. Math. Finance **18**, 23–54 (2008)
10. Cox, A.M.G., Obłój, J.: Robust pricing and hedging of double no-touch options. Finance Stoch. **15**, 573–605 (2011)
11. Dambis, K.E.: On the decomposition of continuous submartingales. Theory Probab. Appl. **10**, 401–410 (1965)
12. Davie, A.M.: Differential equations driven by rough paths: an approach via discrete approximation. Appl. Math. Res. Express (2008). doi:10.1093/amrx/abm009
13. Davis, M.H.A., Hobson, D.G.: The range of traded option prices. Math. Finance **17**, 1–14 (2007)
14. Dawid, A.P., de Rooij, S., Shafer, G., Shen, A., Vereshchagin, N., Vovk, V.: Insuring against loss of evidence in game-theoretic probability. Stat. Probab. Lett. **81**, 157–162 (2011)
15. Dawid, A.P.: Statistical theory: the prequential approach (with discussion). J. R. Stat. Soc. A **147**, 278–292 (1984)
16. Dawid, A.P., Vovk, V.: Prequential probability: principles and properties. Bernoulli **5**, 125–162 (1999)
17. de la Vega, F.W.: On almost sure convergence of quadratic Brownian variation. Ann. Probab. **2**, 551–552 (1973)
18. Delbaen, F., Schachermayer, W.: A general version of the fundamental theorem of asset pricing. Math. Ann. **300**, 463–520 (1994)
19. Denis, L., Martini, C.: A theoretical framework for the pricing of contingent claims in the presence of model uncertainty. Ann. Appl. Probab. **16**, 827–852 (2006)
20. Dubins, L.E., Schwarz, G.: On continuous martingales. Proc. Natl. Acad. Sci. **53**, 913–916 (1965)
21. Dudley, R.M.: Sample functions of the Gaussian process. Ann. Probab. **1**, 66–103 (1973)

22. Dudley, R.M., Norvaiša, R.: Concrete Functional Calculus. Springer, Berlin (2010)
23. Duffie, J.D., Harrison, J.M.: Arbitrage pricing of Russian options and perpetual lookback options. Ann. Appl. Probab. **3**, 641–651 (1993)
24. Dvoretzky, A., Erdős, P., Kakutani, S.: Nonincrease everywhere of the Brownian motion process. In: Neyman, J. (ed.) Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, vol. II. Contributions to Probability Theory, pp. 103–116. University of California Press, Berkeley (1961)
25. Engelking, R.: General Topology, 2nd edn. Heldermann, Berlin (1989)
26. Föllmer, H.: Calcul d'Itô sans probabilités. In: Azéma, J., Yor, M. (eds.) Séminaire de Probabilités de Strasbourg, XV. Lecture Notes in Mathematics, vol. 850, pp. 143–150. Springer, Berlin (1981)
27. Föllmer, H., Schied, A.: Stochastic Finance: Introduction in Discrete Time, 3rd edn. Walter de Gruyter, Berlin (2011)
28. Hobson, D.G.: Robust hedging of the lookback option. Finance Stoch. **2**, 329–347 (1998)
29. Hobson, D.G.: The Skorokhod embedding problem and model-independent bounds for option prices. In: Carmona, R.A., Çinlar, E., Ekeland, I., Jouini, E., Scheinkman, J.A., Touzi, N. (eds.) Paris-Princeton Lectures on Mathematical Finance 2010. Lecture Notes in Mathematics, vol. 2003, pp. 267–318. Springer, Berlin (2011)
30. Hoeffding, W.: Probability inequalities for sums of bounded random variables. J. Am. Stat. Assoc. **58**, 13–30 (1963)
31. Hoffmann-Jørgensen, J.: The general marginal problem. In: Kurepa, S., Kraljević, H., Butković, D. (eds.) Functional Analysis II. Lecture Notes in Mathematics, vol. 1242, pp. 77–367. Springer, Berlin (1987)
32. Horikoshi, Y., Takemura, A.: Implications of contrarian and one-sided strategies for the fair-coin game. Stoch. Process. Appl. **118**, 2125–2142 (2008)
33. Jacod, J., Shiryaev, A.N.: Limit Theorems for Stochastic Processes, 2nd edn. Springer, Berlin (2003)
34. Karandikar, R.L.: On the quadratic variation process of a continuous martingale. Ill. J. Math. **27**, 178–181 (1983)
35. Karatzas, I., Shreve, S.E.: Brownian Motion and Stochastic Calculus, 2nd edn. Springer, New York (1991)
36. Kumon, M., Takemura, A.: On a simple strategy weakly forcing the strong law of large numbers in the bounded forecasting game. Ann. Inst. Stat. Math. **60**, 801–812 (2008)
37. Kumon, M., Takemura, A., Takeuchi, K.: Game-theoretic versions of strong law of large numbers for unbounded variables. Stochastics **79**, 449–468 (2007)
38. Kumon, M., Takemura, A., Takeuchi, K.: Capital process and optimality properties of a Bayesian skeptic in coin-tossing games. Stoch. Anal. Appl. **26**, 1161–1180 (2008)
39. Kumon, M., Takemura, A., Takeuchi, K.: Sequential optimizing strategy in multi-dimensional bounded forecasting games. Stoch. Process. Appl. **121**, 155–183 (2011)
40. Lepingle, D.: La variation d'ordre p des semi-martingales. Z. Wahrscheinlichkeitstheor. Verw. Geb. **36**, 295–316 (1976)
41. Lévy, P.: Le mouvement brownien plan. Am. J. Math. **62**, 487–550 (1940)
42. Lindeberg, J.W.: Eine neue Herleitung des Exponentialgesetzes in der Wahrscheinlichkeitsrechnung. Math. Z. **15**, 211–225 (1922)
43. Lyons, T.J.: Uncertain volatility and the risk-free synthesis of derivatives. Appl. Math. Finance **2**, 117–133 (1995)
44. Lyons, T.J.: Differential equations driven by rough signals. Rev. Mat. Iberoam. **14**, 215–310 (1998)
45. Mörters, P., Peres, Y.: Brownian Motion. Cambridge University Press, Cambridge (2010)
46. Mykland, P.A.: Conservative delta hedging. Ann. Appl. Probab. **10**, 664–683 (2000)
47. Neumann, P.M., Stoy, G.A., Thompson, E.C.: Groups and Geometry. Oxford University Press, Oxford (1994). Reprinted in 2002
48. Norvaiša, R.: Modelling of stock price changes: a real analysis approach. Finance Stoch. **4**, 343–369 (2000)
49. Norvaiša, R.: Quadratic variation, p-variation and integration with applications to stock price modelling. Technical report. arXiv:math.ca/0108090 (August 2001)
50. Peng, S.: G-expectation, G-Brownian motion and related stochastic calculus of Itô's type. In: Benth, F.E., Nunno, G., Lindstrom, T., Øksendal, B., Zhang, T. (eds.) Stochastic Analysis and Applications. Abel Symposia, vol. 2, pp. 541–567 (2007)
51. Peng, S.: Nonlinear expectations and stochastic calculus under uncertainty: with robust central limit theorem and G-Brownian motion. Technical report. arXiv:1002.4546 [math.pr] (February 2010)

52. Rudin, W.: Real and Complex Analysis, 3rd edn. McGraw-Hill, New York (1987)
53. Schwarz, G.: Time-free continuous processes. Proc. Natl. Acad. Sci. **60**, 1183–1188 (1968)
54. Schwarz, G.: On time-free functions. Trans. Am. Math. Soc. **167**, 471–478 (1972)
55. Shafer, G., Vovk, V.: Probability and Finance: It's Only a Game! Wiley, New York (2001)
56. Shafer, G., Vovk, V., Takemura, A.: Lévy's zero-one law in game-theoretic probability. Ann. Inst. Stat. Math. **63**, 873–885 (2011)
57. Shiryaev, A.N.: Essentials of Stochastics in Finance: Facts, Models, Theory. World Scientific, Singapore (1999)
58. Stricker, C.: Sur la p-variation des surmartingales. Sémin. Probab. Strasbg. **13**, 233–237 (1979)
59. Takeuchi, K.: Mathematics of Betting and Financial Engineering. Saiensusha, Tokyo (2004) (in Japanese)
60. Takeuchi, K., Kumon, M., Takemura, A.: A new formulation of asset trading games in continuous time with essential forcing of variation exponent. Bernoulli **15**, 1243–1258 (2009)
61. Takeuchi, K., Kumon, M., Takemura, A.: Multistep Bayesian strategy in coin-tossing games and its application to asset trading games in continuous time. Stoch. Anal. Appl. **28**, 842–861 (2010)
62. Taylor, S.J.: Exact asymptotic estimates of Brownian path variation. Duke Math. J. **39**, 219–241 (1972)
63. Vovk, V.: Rough paths in idealized financial markets. Lith. Math. J. **51**, 274–285 (2011). Expanded version: http://probabilityandfinance.com, Working Paper 35
64. Vovk, V.: Forecasting point and continuous processes: prequential analysis. Test **2**, 189–217 (1993)
65. Vovk, V.: Continuous-time trading and the emergence of volatility. Electron. Commun. Probab. **13**, 319–324 (2008)
66. Vovk, V.: Game-theoretic Brownian motion. Technical report. arXiv:0801.1309 [math.pr] (January 2008)
67. Vovk, V.: Continuous-time trading and the emergence of randomness. Stochastics **81**, 455–466 (2009)
68. Vovk, V., Shafer, G.: A game-theoretic explanation of the $\sqrt{dt}$ effect. The Game-Theoretic Probability and Finance Project. http://probabilityandfinance.com, Working Paper 5 (January 2003)
69. Wald, A.: Die Widerspruchfreiheit des Kollektivbegriffes der Wahrscheinlichkeitsrechnung. Ergeb. Math. Kolloqu. **8**, 38–72 (1937)