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Option pricing with quadratic volatility: a revisit

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Abstract This paper considers the pricing of European options on assets that follow a stochastic differential equation with a quadratic volatility term. We correct several errors in the existing literature, extend the pricing formulas to arbitrary root configurations, and list alternative representations of option pricing formulas to improve computational performance. Our exposition is based entirely on probabilistic arguments, adding a fresh perspective and new intuition to the existing PDE-dominated literature on the subject. Our main tools are martingale methods and shifts of probability measures; the fact that the underlying process is typically a strict local martingale is carefully considered throughout the paper.

Keywords Quadratic volatility · Strict local martingale · Put and call option pricing · Hitting time densities · Fourier series · Method of images

Mathematics Subject Classification (2000) 91G20 · 91G80 · 60G40 · 60G46

JEL Classification G12 · G13

1 Introduction

Many authors (e.g., Blacher [5], Ingersoll [13], Lipton [19], and Zühlsdorff [29], to name a few) have suggested derivative pricing models where financial variables (e.g., foreign exchange rates, equity prices, or forward interest rates) follow diffusion processes with quadratic volatility. Consider therefore the fundamental problem of

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pricing European put and call options on an asset X that satisfies a stochastic differential equation (SDE) of the type

$$dX(t) = \left(\alpha + \beta X(t) + \gamma X(t)^2\right) dW(t), \quad X(0) = x_0, \tag{1.1}$$

where $\gamma \neq 0$. Rady [22] and Ingersoll [13], among several others, consider the bounded case where the function $A(x) = \alpha + \beta x + \gamma x^2$ has two real roots that straddle the initial value x_0. Albanese et al. [1] outline a general strategy that allows a transformation of the pricing PDE for (1.1) into the heat equation; this strategy is used in Lipton [19] to compute a call option pricing formula for the case where $A(x)$ has two negative roots and an absorbing barrier[1] has been inserted at $X = 0$. Other root configurations have been considered in Zühlsdorff [28], but several of the given option pricing results contain errors.

In this paper, we carefully analyze the characteristics of the process (1.1), paying particular attention to the circumstances under which X fails to be a martingale. This analysis, in turn, serves as the starting point for a probabilistic derivation of European put and call option pricing expressions for all non-trivial[2] root configurations of $A(x)$. In doing so, we take care to appropriately incorporate into the pricing expressions the strict local martingale property of (1.1), thereby avoiding the issues that plague existing results in the literature. We also discuss how to modify results if a range truncation through absorbing boundaries is desired. Our analysis contributes new intuition to SDEs with quadratic volatility and lists many new formulas for option pricing.

2 Two real roots left of x_0

We start out our analysis with the case where $A(x) = \alpha + \beta x + \gamma x^2$ in (1.1) has two real roots, both of which are located to the *left* of x_0. Subsequent sections extend the analysis to other root configurations and to the insertion of absorbing barriers.

2.1 Basic setup and results

We consider a semimartingale asset process X adapted to a filtration generated by a scalar Brownian motion. For simplicity, we also assume that interest rates are zero, an assumption that can easily be relaxed by the usual numeraire-deflation of the asset. From results in Delbaen and Schachermayer [8], the absence of a *free lunch with vanishing risk* (FLVR) is equivalent to the existence of a "risk-neutral" probability measure Q in which X is a local martingale. Without going into detail, we recall that

[1]Reference [19] does not explicitly state that the origin is absorbing, but the form of the given option pricing solution indicates that this must be the case.

[2]The bounded case where the roots straddle the initial value x_0 is well understood (see Rady [22] and Ingersoll [13]) so we skip a detailed analysis of this case. For completeness, Appendix A lists the known option pricing formulas.

FLVR is a slight strengthening of the usual definition of arbitrage from admissible[3] trading strategies; see [8] for the complete account.

Let us proceed to fix a scalar Brownian motion W on the probability space $(\Omega, \mathcal{F}, Q)$, with the standard filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by W. We assume that X satisfies the Q-SDE (1.1); clearly X is a local martingale. As mentioned above, we first consider the case where the quadratic polynomial in (1.1) has two real roots ℓ and u, $\ell < u$, lying to the left of x_0. Without loss of generality, we may then consider the normalized process

$$dX(t) = \frac{(X(t)-u)(X(t)-\ell)}{u-\ell}\, dW(t), \quad x_0 > u > \ell, \tag{2.1}$$

where W is a Brownian motion in the risk-neutral probability measure Q.

Let $p(t)$ and $c(t)$ denote the time t fair market prices of European put and call options, respectively. Assuming that the option strike is $K > u$ and the option maturity is $T > 0$, by definition of puts and calls we have the terminal payout conditions

$$p(T) = \big(K - X(T)\big)^+,$$
$$c(T) = \big(X(T) - K\big)^+.$$

A problem of fundamental interest in this paper is to establish closed-form expressions for the time 0 prices $p(0)$ and $c(0)$.

Remark 2.1 Instead of (2.1), in applications we often need to consider

$$dX(t) = \sigma(t)\frac{(X(t)-u)(X(t)-\ell)}{u-\ell}\, dW(t), \quad x_0 > u > \ell,$$

where $\sigma(t)$ is a bounded deterministic function of time. By the usual rules for time-change of Brownian motion, computation of $p(0)$ and $c(0)$ for this process proceeds by replacing, in the pricing formulas for the case (1.1), the maturity T with the integral

$$\int_0^T \sigma(s)^2\, ds.$$

We start by listing a few straightforward lemmas.

Lemma 2.2 *The range for the process* (2.1) *is* $X(t) \in (u, \infty)$. *In particular, the process for* $X(t)$ *does not explode in measure* Q.

Proof That $X(t)$ cannot go below u is obvious; further, Feller's boundary criteria (e.g., Karlin and Taylor [16], Chap. 15.6) establish that u is not accessible when

[3]While our treatment here is generally informal, the notion of "arbitrage" used in this paper is one that precludes the use of doubling strategies. More precisely, we require that all trading strategies be *admissible*, in the sense that trading gains are not allowed to go below some large (but finite) negative floor.

$x_0 > u$. As X is described by a time-homogeneous, one-dimensional SDE with zero drift, it cannot explode (Karatzas and Shreve [15], p. 332). □

Lemma 2.3 *The process X in (2.1) is a* strict *supermartingale in measure Q, i.e., $E_Q(X(T)|\mathcal{F}_t) < X(t)$, $t < T$, where $E_Q(\cdot)$ denotes expectation in measure Q.*

Proof In Appendix B.1. □

Lemma 2.4 *Suppose that X is the local martingale (2.1) in measure Q. Then the no-arbitrage put price at time $t \le T$ is*

$$p(t) = E_Q\big(p(T)\big|\mathcal{F}_t\big).$$

Proof A slight adaptation of the proof of Proposition 6.I in [9] shows that the put price $p(t)$ can be replicated by an admissible trading strategy, the value process of which is a local martingale in Q. In the absence of arbitrage, p must therefore be a local martingale in measure Q. As the put price is here bounded between 0 and $K-u$, it follows elementarily that, in fact, p must be a true Q-martingale. The result follows. □

2.2 The call option value

While the put pricing expression in Lemma 2.4 is noncontroversial, the unbounded payout of the call option introduces a number of complications in properly computing its value $c(t)$. Several differing opinions can be found in the literature and, as observed in [21], there is still some controversy about what constitutes the "best" pricing approach. While a complete account of the matter is beyond the scope of this paper, let us briefly present the main issues.

First, we observe that there are generally two main candidates for the call price solution,

$$c_{\min}(t) := E_Q\big(c(T)\big|\mathcal{F}_t\big), \tag{2.2}$$

$$c_{\max}(t) := p(t) + X(t) - K. \tag{2.3}$$

The first of these emerges from the usual risk-neutral valuation machinery, and the second from enforcing put-call parity. From Lemmas 2.3 and 2.4, we observe that

$$\begin{aligned} c_{\min}(t) &= E_Q\big(p(T) + X(T) - K\big|\mathcal{F}_t\big) \\ &= p(t) - K + E_Q\big(X(T)\big|\mathcal{F}_t\big) < c_{\max}(t). \end{aligned} \tag{2.4}$$

Both $c_{\min}(t) = c_{\min}(t, X(t))$ and $c_{\max}(t) = c_{\max}(t, X(t))$ satisfy the classical terminal value PDE

$$\frac{\partial c}{\partial t} + \frac{1}{2}\left(\frac{(x-u)(x-\ell)}{u-\ell}\right)^2 \frac{\partial^2 c}{\partial x^2} = 0,$$

subject to $c(T,x) = (x-K)^+$. Indeed, any convex combination of $c_{\min}$ and $c_{\max}$ would therefore solve this PDE, and could be considered a candidate for the call price solution.

Cox and Hobson [6] provide strong theoretical justification for $c_{\min}(t)$ by demonstrating that it equals the minimal cost of dynamically replicating the call payout with admissible trading strategies. As a consequence, they characterize $c_{\min}(t)$ as the *fair option price*.[4] On the other hand, in [6], it is also acknowledged that real-life trading constraints may make actual prices deviate from fair prices. In particular, the authors demonstrate (in their Corollary 5.3) that if an exchange requires posting of collateral in the amount of $\xi(X(s)-K)^+$, $\xi \in (0,1]$, at all times $s < T$, then the theoretical price of the option is $(1-\xi)c_{\min}(t) + \xi c_{\max}(t)$.

It has been observed by a number of authors (e.g., Pal and Protter [21] and Heston et al. [11]) that the fair value concept in [6] implies certain "pathological" results, such as the violation of put-call parity, infinite lookback option prices, and counterintuitive term structures of European option prices. Rejecting any violation of put-call parity, Lewis [18] defines $c_{\max}(t)$ to be the correct price whenever the underlying asset is a strict local martingale, but his choice must be characterized as ad hoc. A similar choice is implicit in the pricing formulas of Emmanuel and MacBeth [10], although it is doubtful that the authors were fully aware of the effects of strict local martingales. Madan and Yor [20] argue that the most appropriate definition of the call option price is $c_{\max}(t)$, since

$$c_{\max}(t) = \lim_{n\to\infty} E_Q\Big(\big(X(T\wedge T_n) - K\big)^+ \big| \mathcal{F}_t\Big)$$

where $T_n = \inf\{u \ge t : X(u) = n\}$ is a sequence of stopping times. According to [20], the right-hand side of this expression is the most meaningful definition of option value, as it implies that the option seller will only suffer bounded losses if he closes his position early.[5]

While $c_{\max}(t)$ satisfies put-call parity, this value candidate is not without its own oddities. For instance, the earlier proven inequality $c_{\max}(t) > c_{\min}(t) \ge 0$ holds for arbitrarily large strikes, so (and rather counterintuitively)

$$\lim_{K\to\infty} c_{\max}(t) = X(t) - E_Q\big(X(T)\big|\mathcal{F}_t\big) > 0.$$

Equivalently, from (2.3),

$$E_Q\big(X(T)\big|\mathcal{F}_t\big) = \lim_{K\to\infty}\big(K - p(t)\big), \tag{2.5}$$

which can be combined with (2.4) to express $c_{\min}(t)$ solely as a function of put prices.

It is not the goal of this paper to settle the highly complex debate outlined above, so we remain uncommitted and allow the reader to select which of the possible call

[4] Note that the time t fair value of a forward contract (i.e., a contract paying $X(T)$ at time T) will be less than $X(t)$, as an admissible dynamic strategy exists that will replicate the payout in a cheaper way than buy-and-hold.

[5] As pointed out by a referee, it is perhaps not entirely clear why only the option *seller*'s preferences should be considered when determining the option value.

option candidates he or she might prefer. For our purposes, it will suffice to provide explicit put option pricing expressions; these expressions can be used with the formulas (2.3) and (2.4), (2.5) to establish closed-form results for any of the candidates for the call option price discussed above. As a purely practical remark, we do note that if the option formulas in this paper are used to approximate call option prices from SDEs with bounded volatility (see, for instance, (6.1)), it is typically most reasonable to use $c_{\max}(t)$, as the model being approximated will certainly satisfy put-call parity. Also, domain truncation and other regularizations implicit in standard numerical methods (e.g., finite difference grids) will tend to result in artificial enforcement of put-call parity, so if the intent of the call pricing formulas is to replicate numerical results—something that is particularly useful if the pricing formulas are used for model calibration to vanilla options—then $c_{\max}(t)$ is again probably the best choice.

2.3 Option pricing formulas

Given the discussion above, we focus our attention on establishing the put price $p(0)$. For this purpose, let us note the useful equality

$$x - K = \frac{(x-u)(K-\ell) - (K-u)(x-\ell)}{u-\ell}$$

which allows us to write[6]

$$\begin{aligned} p(T) &= \frac{1}{u-\ell}\big((K-u)\big(X(T)-\ell\big) - \big(X(T)-u\big)(K-\ell)\big)^+ \\ &= \frac{(K-u)(X(T)-\ell)}{u-\ell}\mathbb{1}_{(K-u)(X(T)-\ell)-(X(T)-u)(K-\ell)>0} \\ &\quad - \frac{(X(T)-u)(K-\ell)}{u-\ell}\mathbb{1}_{(K-u)(X(T)-\ell)-(X(T)-u)(K-\ell)>0} \\ &=: p_1(T) - p_2(T). \end{aligned} \tag{2.6}$$

The payouts p_1 and p_2 have identical structure, so it suffices to focus our attention on pricing one of them, e.g., p_1.

From Lemma 2.4, we have $p_1(0) = E_Q(p_1(T))$, which we rewrite as

$$p_1(0) = \frac{K-u}{u-\ell} E_Q\big(\big(X(T)-\ell\big)\mathbb{1}_{(X(T)-u)/(X(T)-\ell)<(K-u)/(K-\ell)}\big). \tag{2.7}$$

At this point, our first instinct would be to perform a measure shift that eliminates the factor $X(T)-\ell$ in the expectation, i.e., we should like to introduce a new measure P such that $P(B) = (x_0-\ell)^{-1}E_Q((X(T)-\ell)\mathbb{1}_B)$, for any $\mathcal{F}_T$-measurable event B. We recall, however, that X (and, therefore, $X-\ell$) is not a martingale in Q, so such a measure shift cannot be performed outright. To get around this, we follow the

[6] $\mathbb{1}_B$ is the indicator function for the set B.

localization argument in Sin [25] and stop the process X at a finite level. Specifically, let us define a process $X^{(n)}$ as

$$X^{(n)}(t) = X(t \wedge \tau_n),$$

where τ_n is the stopping time

$$\tau_n = \inf\{t : X(t) - u = n\}.$$

The process for $L^{(n)}(t) := X^{(n)}(t) - \ell$ satisfies (up to τ_n)

$$dL^{(n)}(t) = L^{(n)}(t)\frac{X^{(n)}(t) - u}{u - \ell}\, dW(t), \quad L^{(n)}(0) = x_0 - \ell.$$

As $X^{(n)}(t) - u \le n$ is bounded from above for all t, it follows that $L^{(n)}$ is a true Q-martingale, so we can define a measure P^n by

$$P^n(B) = (x_0 - \ell)^{-1} E_Q\big(\big(X^{(n)}(T) - \ell\big)\mathbb{1}_B\big)$$

for any $\mathcal{F}_T$-measurable event B. Let $E^{(n)}$ denote expectation in measure P^n.

Lemma 2.5 *Set*

$$Y^{(n)}(T) = \frac{X^{(n)}(T) - u}{X^{(n)}(T) - \ell} = \frac{X^{(n)}(T) - u}{L^{(n)}(T)}.$$

Then

$$\begin{aligned} &E_Q\big(L^{(n)}(T)\mathbb{1}_{Y^{(n)}(T)<(K-u)/(K-\ell)}\mathbb{1}_{\tau_n>T}\big) \\ &\quad = (x_0 - \ell)E^{(n)}\big(\mathbb{1}_{Y^{(n)}(T)<(K-u)/(K-\ell)}\mathbb{1}_{\tau_n>T}\big) \end{aligned}$$

where $Y^{(n)}(T)$ satisfies, up to time τ_n,

$$dY^{(n)}(t) = Y^{(n)}(t)\, dW^{(n)}(t), \quad Y^{(n)}(0) = \frac{x_0 - u}{x_0 - \ell} < 1,$$

with $W^{(n)}$ being a P^n-Brownian motion.

Proof By Girsanov's theorem applied to the change of measure from Q to P^n. □

This lemma leads to the following proposition.

Proposition 2.6 *Let*

$$dY(t) = Y(t)\, dW(t), \quad Y(0) = \frac{x_0 - u}{x_0 - \ell} < 1,$$

be geometric Brownian motion in Q. Define $\tau = \inf\{t : Y(t) = 1\}$, and let $K > u$. Then $p_1(0)$ in (2.7) is given by

$$p_1(0) = \frac{(K - u)(x_0 - \ell)}{u - \ell} E_Q\big(\mathbb{1}_{Y(T)<(K-u)/(K-\ell)}\mathbb{1}_{\tau>T}\big). \tag{2.8}$$

Stated explicitly,

$$p_1(0) = K_1 \Phi\left(\frac{-\ln(x_1/K_1) + \frac{1}{2}T}{\sqrt{T}}\right) - x_2 \Phi\left(\frac{\ln(x_2/K_2) + \frac{1}{2}T}{\sqrt{T}}\right), \tag{2.9}$$

with Φ being the cumulative Gaussian distribution function, and

$$K_1 = \frac{(K-u)(x_0-\ell)}{u-\ell}, \qquad x_1 = \frac{(x_0-u)(K-\ell)}{u-\ell},$$
$$K_2 = \frac{(K-\ell)(x_0-\ell)}{u-\ell}, \qquad x_2 = \frac{(x_0-u)(K-u)}{u-\ell}.$$

Proof In Appendix B.2. □

Following similar steps leads to an expression for $p_2(0)$, which in turn leads to the following result for $p(0) = p_1(0) - p_2(0)$.

Proposition 2.7 *Let $K_i, x_i, i = 1, 2$, be given as in Proposition 2.6. Assuming $K > u$, the put price $p(0)$ for the model (2.1) has the explicit representation*

$$p(0) = K_1 \Phi\left(-d_-^{(1)}\right) - x_2 \Phi\left(d_+^{(2)}\right) - x_1 \Phi\left(-d_+^{(1)}\right) + K_2 \Phi\left(d_-^{(2)}\right),$$
$$d_\pm^{(i)} = \frac{\ln(x_i/K_i) \pm \frac{1}{2}T}{\sqrt{T}}, \quad i = 1, 2.$$

Remark 2.8 Proposition 2.7 corrects an erroneous result[7] in Zühlsdorff [28].

Turning briefly to call option pricing formulas, we note that $c_{\max}(0)$ in (2.3) can be established directly by put-call parity. To find $c_{\min}(0)$ in (2.2), we can use Proposition 2.7 to establish that

$$E_Q\left(X(T)\right) = \lim_{K\to\infty}\left(K - p(0)\right) = x_0 - (x_0-\ell)\Phi(d_-) - (x_0-u)\Phi(d_+) < x_0,$$

where

$$d_\pm = \frac{\ln\frac{x_0-u}{x_0-\ell} \pm \frac{1}{2}T}{\sqrt{T}}.$$

Application of (2.4), (2.5) then returns $c_{\min}(0)$.

[7] The formulas in [28] for option pricing with absorption at zero are also incorrect; see Sect. 3.2 for the correct expressions.

3 Extensions and other real root configurations

3.1 Roots to the right of x_0

Now, let the roots ℓ, u, $\ell < u$, both be to the *right* of x_0, and

$$dX(t) = \frac{(u - X(t))(\ell - X(t))}{u - \ell} dW(t), \quad x_0 < \ell < u. \tag{3.1}$$

As $X(t) < \ell$ for all t, the call option payout

$$c(T) = \big(X(T) - K\big)^+, \quad K < \ell,$$

is now bounded[8] so we concentrate on finding $c(0)$. Define the process $H(t) = \ell + u - X(t)$ such that $dH(t) = -dX(t)$, or

$$dH(t) = \frac{(H(t) - u)(H(t) - \ell)}{u - \ell} dW(t), \quad H(0) = \ell + u - x_0 > u.$$

Written in terms of $H(T)$ the call option payout is

$$c(T) = \big(K_H - H(T)\big)^+, \quad K_H := \ell + u - K, \tag{3.2}$$

where $K_H > u$. We recognize (3.2) as being of the form (2.1), with the call payout (3.2) being equivalent to a put payout on $H(T)$. Staying within the assumptions of Lemma 2.4, Proposition 2.7 then immediately gives us a pricing result for the call option $c(0)$.

Lemma 3.1 *Assume $K < \ell$, and define*

$$K_1 = \frac{(\ell - K)(u - x_0)}{u - \ell}, \qquad x_1 = \frac{(\ell - x_0)(u - K)}{u - \ell},$$
$$K_2 = \frac{(u - K)(u - x_0)}{u - \ell}, \qquad x_2 = \frac{(\ell - x_0)(\ell - K)}{u - \ell}.$$

For the process (3.1), *the call option price is*

$$c(0) = K_1 \Phi\big(-d_-^{(1)}\big) - x_2 \Phi\big(d_+^{(2)}\big) - x_1 \Phi\big(-d_+^{(1)}\big) + K_2 \Phi\big(d_-^{(2)}\big),$$
$$d_\pm^{(i)} = \frac{\ln(x_i/K_i) \pm \frac{1}{2}T}{\sqrt{T}}, \quad i = 1, 2.$$

The put option price is here equivalent to a call option in the root configuration of Sect. 2.1, and (candidates for) its price can be computed accordingly.

Remark 3.2 For the process (3.1), it is clear from Lemma 2.3 that X is a strict submartingale.

[8] The issues we discussed in Sect. 2.2 will now instead apply to the *put* option.

3.2 Absorption at zero

If X is supposed to model a nonnegative asset price, in some cases it may be desirable to insert an absorbing boundary at $X=0$. In a probabilistic framework, this is generally straightforward. To demonstrate, let the process for X be as in (2.1), with both roots to the left of x_0. Also, assume that $\ell < u < 0$, such that the unrestricted process X may go below zero. Finally, set

$$dY_\ell(t) = Y_\ell(t)\,dW(t), \qquad Y_\ell(0) = \frac{x_0-u}{x_0-\ell} < 1,$$
$$dY_u(t) = Y_u(t)\,dW(t), \qquad Y_u(0) = \frac{x_0-\ell}{x_0-u} > 1.$$

Lemma 3.3 *Let $p_1(0)$ and $p_2(0)$ be as defined in* (2.6), *and assume that X satisfies* (2.1), *with $\ell < u < 0$ and an absorbing barrier at zero. Define*

$$\tau_\ell = \inf\bigl\{t : Y_\ell(t) = 1 \text{ or } Y_\ell(t) = u/\ell\bigr\},$$
$$\tau_u = \inf\bigl\{t : Y_u(t) = 1 \text{ or } Y_u(t) = \ell/u\bigr\}.$$

Then

$$p_1(0) = \frac{(K-u)(x_0-\ell)}{u-\ell} E_Q\bigl(\mathbb{1}_{Y_\ell(T\wedge\tau_\ell)<(K-u)/(K-\ell)}\bigr),$$
$$p_2(0) = \frac{(K-\ell)(x_0-u)}{u-\ell} E_Q\bigl(\mathbb{1}_{Y_u(T\wedge\tau_u)>(K-\ell)/(K-u)}\bigr).$$

Proof An obvious extension of the argument in Appendix B.2, to insert an absorbing barrier in $X=0$. □

Computation of $p_1(0)$ and $p_2(0)$ can be done by classical means, using known expressions for the density of Brownian motion in the presence of two absorbing boundaries; for the relevant results, see, e.g., Cox and Miller [7], Chap. 5, and Bhattacharya and Waymire [4], Chap. 7.2. We notice that two different representations of the density are possible, either a Fourier sine-series or a series obtained by the method of images. Propositions 3.4 and 3.5 explore both possibilities.

Proposition 3.4 (Method of images) *For the process* (1.1), *assume $\ell < u < 0 < x_0$ and insert an absorbing boundary at $X=0$. With $K>0$, define*

$$F^{\pm}(x,z) = \Phi\left(\frac{x-z\pm\frac{1}{2}T}{\sqrt{T}}\right), \qquad \kappa = \ln\frac{(K-u)(x_0-\ell)}{(K-\ell)(x_0-u)},$$
$$z_u = \ln\frac{x_0-\ell}{x_0-u}, \qquad z_\ell = z_u - \ln(\ell/u).$$

The put price is

$$p(0) = \frac{(K-u)(x_0-\ell)}{u-\ell}\bigl\{e_1^+(\kappa)+e_2^+\bigr\} - \frac{(K-\ell)(x_0-u)}{u-\ell}\bigl\{e_1^-(\kappa)+e_2^-\bigr\},$$

where, with $z_n' = 2n(z_u - z_\ell)$ *and* $z_n'' = 2z_u - z_n'$,

$$e_1^\pm(\kappa) = \sum_{n=-\infty}^{\infty} \Big(e^{\mp\frac{1}{2}z_n'} \big(F^\pm(\kappa, z_n') - F^\pm(z_\ell, z_n') \big) - e^{\mp\frac{1}{2}z_n''} \big(F^\pm(\kappa, z_n'') - F^\pm(z_\ell, z_n'') \big) \Big),$$

$$e_2^\pm = \psi^\pm \mp \frac{e_1^\pm(z_u)}{D^\pm} \pm \frac{(\frac{x_0-u}{x_0-\ell})^{\pm 1} e_1^\mp(z_u)}{D^\pm}$$

and

$$\psi^+ = \frac{\ell}{\ell - x_0}, \qquad \psi^- = \frac{u}{u - x_0}, \qquad D^+ = 1 - u/\ell, \qquad D^- = \ell/u - 1.$$

Proof In Appendix B.3. □

Proposition 3.5 (Fourier series) *For the process* (1.1), *assume* $\ell < u < 0 < x_0$ *and insert an absorbing boundary at* $X = 0$. *Define*

$$\lambda_n = \frac{1}{2}\left(\frac{1}{4} + \frac{n^2\pi^2}{(z_u - z_\ell)^2}\right), \qquad a_n = \frac{n\pi z_\ell}{z_u - z_\ell}, \qquad k_n = \frac{n\pi(\kappa - z_\ell)}{z_u - z_\ell},$$

and let κ, z_u, z_ℓ *and* $\psi^\pm$ *be as in Proposition* 3.4. *Then the put price is*

$$p(0) = \frac{(K-u)(x_0-\ell)}{u-\ell}\{e_1^+ + e_2^+\} - \frac{(K-\ell)(x_0-u)}{u-\ell}\{e_1^- + e_2^-\},$$

where

$$e_1^\pm = \frac{1}{z_u - z_\ell} \sum_{n=1}^{\infty} \sin(a_n) \frac{e^{-\lambda_n T}}{\lambda_n} \times \left[\frac{a_n}{z_\ell} \big(e^{\mp\frac{1}{2}\kappa} \cos(k_n) - e^{\mp\frac{1}{2}z_\ell} \big) \pm \frac{1}{2} e^{\mp\frac{1}{2}\kappa} \sin(k_n) \right],$$

$$e_2^\pm = \psi^\pm + \frac{e^{\mp\frac{1}{2}z_\ell}}{(z_u - z_\ell)^2} \sum_{n=1}^{\infty} \frac{e^{-\lambda_n T}}{\lambda_n} n\pi \sin(a_n).$$

Proof In Appendix B.4. □

Remark 3.6 Lipton [19] uses classical PDE methods to list an alternative (but equivalent) form for the Fourier series in Proposition 3.5. Note that the series in [19] has a small typo: the constant ξ should be θ.

For the case where u/ℓ is large relative to the variance of $\log Y_\ell(T)$, the representation in Proposition 3.4 will typically require substantially fewer terms to converge than will the Fourier series representation in Proposition 3.5. On the other hand, the latter will be more convenient for the case where u/ℓ is small, i.e., when the roots are close together or the option maturity is large. An intelligent implementation will branch between the two solutions, in the manner discussed in, say, Andersen [2].

Remark 3.7 Insertion of an absorbing boundary at a non-zero level above u is a trivial extension of the results above, as is the insertion of an additional absorbing boundary *above* x_0 (see Sect. 4). If such an absorbing boundary is inserted, the process X is bounded both from above and below and will be a true martingale, hence put-call parity can be used to uniquely determine the call price.

We leave to the reader the case where zero is an absorbing boundary and the two real roots are to the right of x_0.

3.3 A single real root

Consider now the case where there is only a single root, i.e., $\ell = u$. Let us assume that $x_0 > u$; the case $x_0 < u$ can be solved by the symmetry arguments in Sect. 3.1. We write

$$dX(t) = \big(X(t) - u\big)^2 dW(t), \quad x_0 > u, \tag{3.3}$$

and note that the range of $X(t)$ is (u, ∞). It follows from the proof of Lemma 2.3 that X remains a strict supermartingale.

If we make the variable transformation $U(t) = X(t) - u$, then

$$dU(t) = U(t)^2 dW(t), \quad U(0) > 0, \tag{3.4}$$

with the put option payout being

$$p(T) = \big(K - u - U(T)\big)^+, \quad K > u.$$

U is a constant elasticity of variance (CEV) process with a power of 2 and, as such, $p(0)$ can, in principle, be computed from the general CEV option pricing formulas in Schroder [24] (see also Andersen and Andreasen [3]). However, these involve infinite series of chi-square distributions and are unnecessarily complicated for the special case of (3.4). Instead, the simple formula below should be used.

Proposition 3.8 *For the process* (3.3), *the put option price is*

$$p(0) = (x_0 - u)(K - u)\sqrt{T}\big\{d_+\Phi(d_+) + \phi(d_+) - d_-\Phi(d_-) - \phi(d_-)\big\},$$

where $\phi(x) = (2\pi)^{-1/2}e^{-x^2/2}$ *is the Gaussian density, and*

$$d_\pm = \frac{\pm\frac{1}{x_0-u} - \frac{1}{K-u}}{\sqrt{T}}.$$

Proof Let us write

$$\begin{aligned} p(0) &= E_Q\big((K-u)\mathbb{1}_{X(T)-u<K-u}\big) - E_Q\big(\big(X(T)-u\big)\mathbb{1}_{X(T)-u<K-u}\big) \\ &= E_Q\big((K-u)\mathbb{1}_{1/(X(T)-u)>1/(K-u)}\big) \\ &\quad - E_Q\big(\big(X(T)-u\big)\mathbb{1}_{1/(X(T)-u)>1/(K-u)}\big). \end{aligned}$$

We notice that the process $(X(t)-u)^{-1}$ satisfies

$$d\big((X(t)-u)^{-1}\big) = -dW(t) + \big(X(t)-u\big)\,dt.$$

Let us therefore define $dZ(t) = -dW(t)$, with $Z(0) = (x_0 - u)^{-1} > 0$, and set $\tau = \inf\{t : Z(t) = 0\}$. Following the route of arguments that lead to Proposition 2.6, we can show that

$$\begin{aligned} E_Q\big((K-u)\mathbb{1}_{1/\{X(T)-u\}>1/(K-u)}\big) &= Z(0)^{-1}(K-u) \\ &\quad \times E_Q\big(Z(T)\mathbb{1}_{Z(T)>1/(K-u)}\mathbb{1}_{\tau>T}\big), \\ E_Q\big(\big(X(T)-u\big)\mathbb{1}_{1/\{X(T)-u\}>1/(K-u)}\big) &= Z(0)^{-1}E_Q\big(\mathbb{1}_{Z(T)>1/(K-u)}\mathbb{1}_{\tau>T}\big). \end{aligned}$$

Standard results for Brownian motion with an absorbing barrier (Cox and Miller [7], Chap. 5) allow us to easily evaluate these expressions in closed form. □

Remark 3.9 An alternative proof of Proposition 3.8 observes that for the process

$$dX(t) = \big(X(t)-u\big)\big(X(t)-\ell\big)\,dW(t), \quad \ell < u < x_0,$$

the put price can be computed from the result in Proposition 2.7, after a time-change, from T to $T(u-\ell)^2$; see Remark 2.1. Taking the limit of the put price as $\ell \uparrow u$ then establishes the result. For a pure PDE proof of the result in Proposition 3.8, see Zühlsdorff [28].

Remark 3.10 The process U in (3.4) is the canonical example of a strict supermartingale (see Johnson and Helms [14]) and can be represented as the inverse of a Bessel process of dimension three. As the transition density of a Bessel process of dimension three is known explicitly (see, e.g., Revuz and Yor [23]), the result of Proposition 3.8 can also be established directly by integrating the (suitably transformed) payout against this density. See also Pal and Protter [21] and Yen and Yor [27].

We leave the case where the lone real root lies to the right of x_0 to the reader.

3.4 A single real root: absorption at zero

Assume that X satisfies (3.3) with $u < 0$ and assume now that an absorbing barrier has been inserted at the origin, ensuring that X never goes negative. We can easily show the following result.

Lemma 3.11 *Assume that X satisfies (2.1), with $u < 0$ and an absorbing barrier at zero. Let*

$$dZ(t) = -dW(t), \quad Z(0) = \frac{1}{x_0 - u} > 0,$$

and define

$$\tau = \inf\big\{t : Z(t) = 0 \text{ or } Z(t) = -1/u\big\}.$$

Then the put price is given by

$$p(0) = (x_0 - u)(K - u)E_Q\big(Z(T \wedge \tau)\mathbb{1}_{Z(T\wedge\tau)>1/(K-u)}\big)$$
$$- (x_0 - u)E_Q\big(\mathbb{1}_{Z(T\wedge\tau)>1/(K-u)}\big). \tag{3.5}$$

Proof A simple extension of the argument in the proof of Proposition 3.8, to insert an absorbing barrier at $X = 0$. □

Evaluation of the two expectations in (3.5) is straightforward, and can proceed along the lines of the proofs of Propositions 3.4 and 3.5. In the interest of brevity, we omit the results, since they can be found by simply taking the limits $u \uparrow \ell$ in Propositions 3.4 and 3.5; see Remark 3.9.

When u is close to zero—that is, when the range $[-1/u, 0]$ is large—a series solution based on the method of images will require fewer terms to converge than a sine-solution. The opposite holds when u is far away from zero.

Remark 3.12 In Zühlsdorff [28], the expression for the single-root case with absorption is incorrect.

4 No real roots

We now consider the case where the polynomial $A(x)$ in (1.1) has no real roots. After suitable normalization,[9] our Q-SDE has the form

$$\begin{aligned} dX(t) &= b\left(1 + \left(\frac{X(t)-a}{b}\right)^2\right)dW(t) \\ &= \frac{1}{b}\big(\big(X(t)-a\big)^2 + b^2\big)\,dW(t) \\ &= \frac{1}{b}\big(X(t)-c_+\big)\big(X(t)-c_-\big)\,dW(t), \end{aligned} \tag{4.1}$$

where $c_\pm$ are two complex-valued roots,

$$c_\pm = a \pm ib, \quad b > 0,$$

with i being the imaginary unit, $i^2 = -1$.

Without further restrictions, the range for X is now the entire real line. Following the argument in the proof for Lemma 2.3, it can be demonstrated that X is a strict local martingale, but in the absence of lower and upper bounds on X, we cannot characterize X as either a supermartingale or a submartingale. Absence of FLVR dictates that put and call prices be local martingales in measure Q; as neither the put

[9] Our normalization follows that of Zühlsdorff [28].

nor the call have bounded payouts for the case of (4.1), neither can be argued to be martingales in measure Q. (We can, however, argue that both are supermartingales, as they are local martingales bounded from below at zero.)

To get firmer ground under our feet, we proceed to introduce explicit bounds on the process X, through the insertion[10] of *absorbing boundaries* x_L and x_U, with $x_L < x_0 < x_U$. X is thus a bounded local martingale, and hence a martingale. The same argument applies to put and call prices, therefore, we have the following result.

Lemma 4.1 *Assume that the process X is equipped with finite-valued absorbing boundaries x_L and x_U, with $x_L < x_0 < x_U$. Define*

$$\tau = \inf\{t : X(t) = x_L \text{ or } X(t) = x_U\}.$$

Then, with $x_L < K < x_U$,

$$c(0) = E_Q\big(\big(X(T \wedge \tau) - K\big)^+\big),$$
$$p(0) = E_Q\big(\big(K - X(T \wedge \tau)\big)^+\big).$$

Looking at the form of the diffusion term in (4.1) suggests, as in previous sections, to focus on the (complex-valued) ratio

$$Y(t) = \frac{X(t) - c_+}{X(t) - c_-} \tag{4.2}$$

as well as its logarithm. To gain some intuition, the following result is useful.

Lemma 4.2 *Let $Y(t)$ be as given in (4.2), and define the processes*[11] $R(t) = \frac{1}{2}\ln(-Y(t))$, $Z(t) = \mathrm{Im}(R(t))$. *Define τ as in Lemma 4.1. Then, for $t < \tau$,*

$$dY(t) = i2Y(t)\left(dW(t) - \frac{1}{b}\big(X(t) - a - ib\big)\,dt\right),$$
$$dR(t) = i\left(dW(t) - \frac{1}{b}\big(X(t) - a\big)\,dt\right),$$
$$dZ(t) = dW(t) - \frac{1}{b}\big(X(t) - a\big)\,dt = dW(t) - \tan Z(t)\,dt.$$

Proof The dynamics for Y and R follows from Itô's lemma, as does the first equation for $dZ(t)$. To show the second equality for $dZ(t)$, we only need to notice that, by the

[10] As should be obvious from previous results, it actually suffices to insert a single absorbing boundary to make *either* the put or the call payout bounded. For generality, we use two boundaries, with the understanding that one of them may, in fact, be set at either ∞ or $-\infty$. Indeed, a natural configuration of our bounds is to have $x_U = \infty$ and $x_L = 0$.

[11] We use $\mathrm{Im}(\cdot)$ to denote the imaginary part of a complex number.

definition of $Z(t)$,

$$Z(t) = \arctan\left(\frac{X(t) - a}{b}\right),$$

which is evident from the basic relation $\arctan(x) = \frac{1}{2} i(\ln(1 - ix) - \ln(1 + ix))$. □

The quantity $Z(t)$ in Lemma 4.2 is of particular interest, as (i) it is monotonic in $X(t)$; and (ii) it can be reduced to a Brownian motion by a change of measure. Acting on (ii), we introduce a probability measure P in which $d\tilde{W}(t) = dW(t) - \tan Z(t)\,dt$ defines a Brownian motion. To characterize the measure P, let $E(\cdot)$ denote expectation in measure P and introduce the P-martingale

$$\eta(t) = E\left(\frac{dQ}{dP}\middle|\mathcal{F}_t\right).$$

For $t < \tau$, Girsanov's theorem shows that then, in measure P,

$$\begin{aligned} d\eta(t)/\eta(t) &= -\tan Z(t)\,d\tilde{W}(t), \quad \eta(0) = 1, \\ dZ(t) &= d\tilde{W}(t), \quad Z(0) = \arctan\left(\frac{x_0 - a}{b}\right). \end{aligned} \tag{4.3}$$

From this, we can derive the following result.

Proposition 4.3 *Define* $z_L = \arctan(\frac{x_L - a}{b})$, $z_U = \arctan(\frac{x_U - a}{b})$, *where we choose* $z_L, z_U \in (-\pi/2, \pi/2)$. *Set*

$$\tau = \inf\{t : Z(t) = z_L \text{ or } Z(t) = z_U\},$$

where Z is a Brownian motion in P, started at level $\arctan(\frac{x_0 - a}{b})$. *With $\tilde{T} := T \wedge \tau$ we have, for $K \in (x_L, x_U)$,*

$$p(0) = b\sqrt{1 + \left(\frac{x_0 - a}{b}\right)^2} E\left(e^{\frac{1}{2}\tilde{T}}\left(\tilde{K}\cos Z(\tilde{T}) - \sin Z(\tilde{T})\right)^+\right), \quad \tilde{K} = \frac{K - a}{b}.$$

Proof In Appendix B.5. □

Writing $1 = \mathbb{1}_{\{\tau > T\}} + \mathbb{1}_{\{\tau \le T, Z(\tau) = z_L\}} + \mathbb{1}_{\{\tau \le T, Z(\tau) = z_U\}}$ allows us to decompose the result in Proposition 4.3 as

$$\begin{aligned} p(0) &= \sqrt{bA(x_0)} e^{\frac{1}{2}T} E\left(\left(\tilde{K}\cos Z(T) - \sin Z(T)\right)^+ \mathbb{1}_{\{\tau > T\}}\right) \\ &\quad + \sqrt{bA(x_0)}(\tilde{K}\cos z_L - \sin z_L) E\left(e^{\frac{1}{2}\tau}\mathbb{1}_{\{\tau \le T, Z(\tau) = z_L\}}\right) \\ &= \sqrt{bA(x_0)} e^{\frac{1}{2}T} E\left(\left(\tilde{K}\cos Z(T) - \sin Z(T)\right)^+ \mathbb{1}_{\{\tau > T\}}\right) \\ &\quad + \sqrt{\frac{A(x_0)}{A(x_L)}}(K - x_L) E\left(e^{\frac{1}{2}\tau}\mathbb{1}_{\{\tau \le T, Z(\tau) = z_L\}}\right), \end{aligned} \tag{4.4}$$

where $A(x) = b[1 + ((x - a)/b)^2]$. We have used the fact that the payout of the put is zero whenever $Z(\tau) = z_U$ (since we assume that $x_L < K < x_U$). The expectations involved in the expression above can be computed analytically, using results similar to those used to prove Propositions 3.4 and 3.5. Again, we have at least two representations, either as a sine-series or as a series based on the method of images. The sine-series result is listed in Proposition 4.4 below.

Proposition 4.4 (Fourier series) *Consider the model* (4.1) *with absorbing barriers at x_L and x_U. Let z_L, z_U, and $\tilde{K}$ be as in Proposition* 4.3, *and define*

$$z_0 = \arctan\left(\frac{x_0 - a}{b}\right), \qquad \alpha_n = \frac{n^2\pi^2}{2(z_U - z_L)^2}, \qquad a_n = \frac{n\pi(z_L - z_0)}{z_U - z_L}.$$

Then $p(0)$ is given by (4.4), *with*

$$E\Big(\big(\tilde{K}\cos Z(T) - \sin Z(T)\big)^+ \mathbb{1}_{\{\tau > T\}}\Big)$$
$$= \frac{2}{z_U - z_L}\sum_{n=1}^{\infty} e^{-\alpha_n T}\sin(-a_n)\big(\tilde{K} I_n^{(c)} - I_n^{(s)}\big),$$

$$E\big(e^{\frac{1}{2}\tau}\mathbb{1}_{\{\tau \le T, Z(\tau) = z_L\}}\big) = \frac{\sin(z_U - z_0)}{\sin(z_U - z_L)} - \frac{\sum_{n=1}^{\infty} n\pi \sin(-a_n)\frac{e^{-(\alpha_n - \frac{1}{2})T}}{\alpha_n - \frac{1}{2}}}{(z_U - z_L)^2}.$$

Here, the terms $I_n^{(c)}$ and $I_n^{(s)}$ are given in closed form in (*B*.2) *and* (*B*.1) *in Appendix* *B*.6.

Proof In Appendix B.6. □

Zühlsdorff [28] lists an alternative representation of the Fourier sine-series in Proposition 4.4. As written, the series in [28] suffers from overflow issues[12] and, additionally, will typically require the computation of many 100's of terms (the author lists 200–300 terms as an average number). In contrast, the series representation above will, on average, converge with 5–10 terms or less.

Application of the method of images here does not lead to a closed form solution (or so we believe), but the put price can still be computed by one-dimensional numerical integration. In cases where the sine-series in Proposition 4.4 is slow to converge (x_L and x_U far apart, small value of T), the method of images result may still be worthwhile pursuing. We list it below.

Proposition 4.5 (Method of images) *Consider the model* (4.1) *with absorbing barriers at x_L and x_U. Let z_L, z_U, and $\tilde{K}$ be as in Proposition* 4.3, *and define*

$$z_0 = \arctan\left(\frac{x_0 - a}{b}\right), \qquad z_n' = 2n(z_U - z_L), \qquad z_n'' = 2(z_U - z_0) - z_n'.$$

[12]The series in [28] involves rapidly growing terms of the form $\exp((\text{const}\, n^2\pi^2 - 1)T/2)$ where const > 0. We should also note that there are several typos in the result in [28].

Also, set $k = \arctan(\tilde{K}) - z_0$ *and*

$$I_c(y) = \frac{1}{\sqrt{2\pi T}} \int_{z_L - z_0}^{k} \exp\left\{-\frac{(z-y)^2}{2T}\right\} \cos(z + z_0)\, dz, \tag{4.5}$$

$$I_s(y) = \frac{1}{\sqrt{2\pi T}} \int_{z_L - z_0}^{k} \exp\left\{-\frac{(z-y)^2}{2T}\right\} \sin(z + z_0)\, dz, \tag{4.6}$$

and

$$\begin{aligned} F(t) = {} & \sum_{n=-\infty}^{\infty} \frac{z_U - z_0 - z_n'}{z_U - z_L} \left[\Phi\left(\frac{z_U - z_0 - z_n'}{\sqrt{t}}\right) - \Phi\left(\frac{z_L - z_0 - z_n'}{\sqrt{t}}\right)\right] \\ & - \sum_{n=-\infty}^{\infty} \frac{z_U - z_0 - z_n''}{z_U - z_L} \left[\Phi\left(\frac{z_U - z_0 - z_n''}{\sqrt{t}}\right) - \Phi\left(\frac{z_L - z_0 - z_n''}{\sqrt{t}}\right)\right] \\ & + \frac{\sqrt{t}}{z_U - z_L} \sum_{n=-\infty}^{\infty} \left[\phi\left(\frac{z_U - z_0 - z_n'}{\sqrt{t}}\right) - \phi\left(\frac{z_U - z_0 - z_n''}{\sqrt{t}}\right) \right. \\ & \left. - \phi\left(\frac{z_L - z_0 - z_n'}{\sqrt{t}}\right) + \phi\left(\frac{z_L - z_0 - z_n''}{\sqrt{t}}\right)\right]. \end{aligned}$$

Then $p(0)$ *is given by* (4.4), *with*

$$\begin{aligned} & E\left(\left(\tilde{K}\cos Z(T) - \sin Z(T)\right)^{+} \mathbb{1}_{\{\tau > T\}}\right) \\ & \quad = \sum_{n=-\infty}^{n=\infty} \left(\tilde{K}\left(I_c(z_n') - I_c(z_n'')\right) - \left(I_s(z_n') - I_s(z_n'')\right)\right), \\ & E\left(e^{\frac{1}{2}\tau} \mathbb{1}_{\{\tau \le T, Z(\tau) = z_L\}}\right) = -\int_0^T \frac{\partial F(t)}{\partial t} e^{\frac{1}{2}t}\, dt. \end{aligned}$$

Proof In Appendix B.7. □

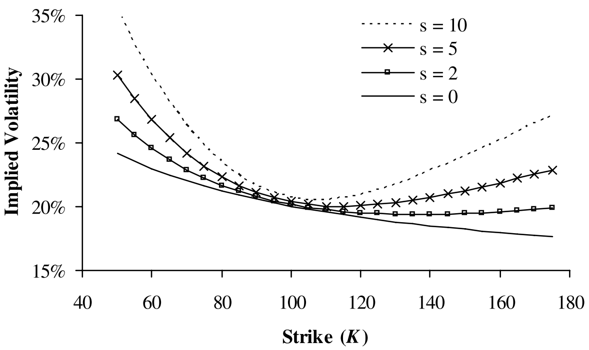
Remark 4.6 While the topic is somewhat outside the scope of this paper, we note that several computational tricks can be used to optimize the computation of the integrals in Proposition 4.5. For instance, using Euler's formulas for $\sin(\cdot)$ and $\cos(\cdot)$, the integrals I_c and I_s in (4.5), (4.6) can be rewritten in terms of the *complex error function*, allowing for quick computation using well-known techniques (e.g., Weideman [26]).

5 Parametrization and numerical example

In practical applications, the quadratic volatility model may be parametrized through the intuitive form

$$dX(t) = \sigma\left(qX(t) + (1-q)x_0 + \frac{1}{2}s\frac{(X(t)-x_0)^2}{x_0}\right) dW(t), \tag{5.1}$$

Fig. 1 Implied volatility smile for quadratic model. The model setup was as in (5.1), with $\sigma = 20\%$, $q = 0.5$, and $x_0 = 100$. The option maturity is $T = 1$ and the convexity parameter s varies as indicated in the graph. We assumed absorption in zero, the effect of which is minimal for the strike range in the figure



where $\sigma > 0$ is a proxy for the at-the-money volatility level,[13] q is a volatility slope or "skew" parameter, and s is a measure of the convexity of the quadratic volatility function. The three new parameters σ, q, s map to the parameters α, β, γ in (1.1) in an obvious fashion. The roots of the quadratic function in (5.1) are

$$\left(s - q \pm \sqrt{q^2 - 2s}\right)\frac{x_0}{s},$$

and there will be two real roots if $q^2 > 2s$, one real root if $q^2 = 2s$, and no real roots if $q^2 < 2s$.

Figure 1 shows an example of the types of implied Black–Scholes volatility[14] smile that can be produced by the model (5.1). In computing the smile, we used the option formulas in this paper with appropriate scaling on the maturity T, as outlined in Remark 2.1.

Let us pick the case in the figure with $s = 10$ for further numerical scrutiny. As here $q^2 < 2s$, the quadratic polynomial has no real roots for this case, so pricing must be done through one of the infinite series in Propositions 4.4 or 4.5. As mentioned earlier, the former proposition is typically the most useful when $\sigma^2 T$ is relatively large, whereas the latter is most useful when this quantity is small. To illustrate, we compute the two series solutions, assuming that we truncate sums running to ∞ and $-\infty$ to sums running to N and $-N$, with N being some positive integer. Table 1 below shows the convergence behavior of the two series solutions, as a function of N. While both series here perform reasonably well for both scenarios in the table, in the scenario with high volatility and long maturity, the Fourier series clearly does best and essentially converges in a single step ($N = 1$). When volatility and maturity are low, however, the situation is reversed and the series based on the method of images converges immediately. If intelligent branching between the two series solutions is used, numerical effort will always be low.

[13] Notice that the volatility function reduces to σx_0 whenever $X(t) = x_0$. As discussed in Remark 2.1, we can easily allow σ to be a function of time.

[14] See any finance textbook (e.g., [12]) for the definition of "implied" Black–Scholes volatility.

Table 1 Convergence of series solutions from Propositions 4.4 or 4.5. Table shows put option prices for a strike of $K = 100$ (at-the-money), as a function of truncation level N. The model setup was as in (5.1), with $s = 10$, $q = 0.5$, and $x_0 = 100$. Absorption bounds were $x_L = 0$ and $x_U = 10^{10}$. "Scenario I": $\sigma = 40\%$ and $T = 30$. "Scenario II": $\sigma = 20\%$ and $T = 0.5$

N	Scenario I		Scenario II	
	Fourier	Method of images	Fourier	Method of images
1	98.51573	103696.9	7.901845	5.725993
2	98.51573	8846.244	7.534609	5.725993
3	98.51573	294.7279	6.037472	5.725993
4	98.51573	99.75636	5.904599	5.725993
5	98.51573	98.51775	5.770191	5.725993
6	98.51573	98.51573	5.739344	5.725993
7	98.51573	98.51574	5.730691	5.725993
8	98.51573	98.51573	5.726600	5.725993
9	98.51573	98.51573	5.726314	5.725993
10	98.51573	98.51573	5.726009	5.725993
11	98.51573	98.51573	5.726006	5.725993
12	98.51573	98.51573	5.725993	5.725993
13	98.51573	98.51573	5.725993	5.725993

6 Conclusion

As should be obvious at this point, call and put option pricing in the quadratic volatility model is a rather delicate problem that scrapes against the limits of no-arbitrage theory. We have here provided a careful analysis which we hope clarifies some confusion in the existing literature and solves the problem once and for all. The numerous pricing formulas listed in the paper should be of use to practitioners who are interested in quick calibration of quadratic volatility models to quoted put and call prices. Due to their tractability, quadratic volatility models may serve as a convenient backbone to more complicated models, such as the "universal" local-stochastic volatility model in Lipton [19] and others.

As a final comment, we note that we should expect practitioners to find it convenient in numerical work to "regularize" the quadratic volatility model to something like

$$dX(t) = \max\Big(\sigma_{\min} X(t), \min\big(\sigma_{\max} X(t), A(X(t)t)\big)\Big)\, dW(t), \tag{6.1}$$

in effect stitching on linear tails to the quadratic form $A(x)$. See Andersen and Andreasen [3] for similar ideas in a CEV setting. Option computations for such a model would necessarily require numerical methods, but if $\sigma_{\min}$ and $\sigma_{\max}$ are small and large, respectively, the formulas in this paper will typically give an excellent approximation of European put and call prices for the model (6.1).

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Appendix A: Option pricing solution for bounded case

In this Appendix, we list known option pricing formulas for the case where the quadratic polynomial $A(x)$ has two real roots that straddle the initial condition x_0. Normalizing in the same fashion as in (2.1), we consider the Q-measure SDE

$$dX(t) = \frac{(u - X(t))(X(t) - \ell)}{u - \ell}\, dW(t), \quad u > x_0 > \ell.$$

Clearly this process is bounded to the interval (ℓ, u) for all t, so X is a here a true martingale. For a call option on X, it has been shown by Ingersoll [13] (by PDE methods) and Rady [22] (by probabilistic arguments) that, for $K \in (\ell, u)$,

$$c(0) = E_Q\big(\big(X(T) - K\big)^+\big) = x^* \Phi(d_+) - K^* \Phi(d_-),$$
$$d_\pm = \frac{\ln(x^*/K^*) \pm \frac{1}{2}T}{\sqrt{T}},$$

where $\Phi(\cdot)$ is the Gaussian CDF and where we have defined

$$K^* = \frac{(K - \ell)(u - x_0)}{u - \ell}, \qquad x^* = \frac{(x_0 - \ell)(u - K)}{u - \ell}.$$

The put option price may be computed by put-call parity.[15]

Appendix B: Proofs

B.1 Proof of Lemma 2.3

There are several ways of proving Lemma 2.3; we show two of them. For the first approach, set $U(t) = X(t) - u$, such that

$$dU(t) = \frac{U(t)(U(t) + u - \ell)}{u - \ell}\, dW(t) = U(t)V(t)\, dW(t), \quad V(t) := \frac{U(t) + u - \ell}{u - \ell}.$$

The process for V is therefore

$$dV(t) = \frac{1}{u - \ell} U(t)V(t)\, dW(t) = \big(V(t) - 1\big)V(t)\, dW(t).$$

According to the arguments of Sin [25], U—and, therefore, $X = U + u$—will be a strict local martingale provided that the "augmented" process

$$d\hat{V}(t) = \big(\hat{V}(t) - 1\big)\hat{V}(t)^2\, dt + \big(\hat{V}(t) - 1\big)\hat{V}(t)\, dW(t)$$

[15] As both the put and call payouts are here bounded, the complications surrounding put-call parity discussed in Sect. 2.2 do not appear.

explodes in finite time. Application of the standard Feller boundary criteria for SDEs (e.g., Karlin and Taylor [16], Chap. 15.6) shows that ∞ is accessible by $\hat{V}$, proving that X is a strict local martingale. As a strict local martingale bounded from below is a strict supermartingale, the lemma follows.

A second approach[16] to proving the strict local martingale property is to note that the speed measure for (2.1) is

$$m(dx) = \frac{2(u-\ell)^2}{(x-u)^2(x-\ell)^2}\, dx,$$

whereby

$$\int_{u+1}^{\infty} x\, m(dx) < \infty.$$

From Theorem 1 in Kotani [17], it follows that the process (2.1) is a strict local martingale. □

B.2 Proof of Proposition 2.6

Adopting the notation of Lemma 2.5, the fact that X does not explode in Q (see Lemma 2.2) shows that

$$p_1(T) = \lim_{n\to\infty} L^{(n)}(T)\mathbb{1}_{Y^{(n)}(T)<(K-u)/(K-\ell)}\mathbb{1}_{\tau_n>T}.$$

By dominated convergence, we then have from Lemma 2.5 that

$$\begin{aligned} p_1(0) = E_Q\big(p_1(T)\big) &= \lim_{n\to\infty} E_Q\big(L^{(n)}(T)\mathbb{1}_{Y^{(n)}(T)<(K-u)/(K-\ell)}\mathbb{1}_{\tau_n>T}\big) \\ &= (x_0-\ell)\lim_{n\to\infty} E^{(n)}\big(\mathbb{1}_{Y^{(n)}(T)<(K-u)/(K-\ell)}\mathbb{1}_{\tau_n>T}\big). \end{aligned}$$

The event $X^{(n)}(T) - u = n$ translates to $Y^{(n)}(T) = \frac{n}{n+u-\ell}$, the limit of which is 1 for $n \to \infty$. As $Y^{(n)}$ is a geometric Brownian motion in measure P^n up to the hitting time τ_n, the result (2.8) follows.

To prove (2.9), we write $Z(t) = \ln Y(t)$, such that

$$dZ(t) = -\frac{1}{2}dt + dW(t), \qquad Z(0) = \ln Y(0) < 0.$$

The absorbing barrier at $Y_\ell = 1$ becomes an absorbing barrier at the origin of Z. The expectation in the expression

$$p_1(0) = \frac{(K-u)(x_0-\ell)}{u-\ell} E_Q\big(\mathbb{1}_{Z(T)<\ln k}\mathbb{1}_{\tau>T}\big), \qquad k := (K-u)/(K-\ell),$$

[16] We thank one of the referees for pointing our attention to the convenient results for time-homogeneous SDEs in Kotani [17].

can be evaluated from standard methods for absorbed Brownian motion with drift; see Cox and Miller [7], Chap. 5. The result is

$$E_Q\big(\mathbb{1}_{Z(T)<\ln k}\mathbb{1}_{\tau>T}\big)=\Phi\left(\frac{\ln k-Z(0)+\frac{1}{2}T}{\sqrt{T}}\right)-e^{Z(0)}\Phi\left(\frac{\ln k+Z(0)+\frac{1}{2}T}{\sqrt{T}}\right).$$

A few simplifications lead to (2.9). □

B.3 Proof of Proposition 3.4

Let us focus on the evaluation of $p_1(0)$. For this, set $Z_\ell(t)=\ln Y_\ell(t)-\ln Y_\ell(0)$, such that

$$\begin{aligned}p_1(0)&=\frac{(K-u)(x_0-\ell)}{u-\ell}E_Q\big(\mathbb{1}_{Y_\ell(T\wedge\tau_\ell)<(K-u)/(K-\ell)}\big)\\&=\frac{(K-u)(x_0-\ell)}{u-\ell}E_Q\big(\mathbb{1}_{Z_\ell(T\wedge\tau_\ell)<\kappa}\big),\quad \kappa:=\ln\frac{K-u}{K-\ell}-\ln\frac{x_0-u}{x_0-\ell}.\end{aligned}$$

Clearly Z_ℓ is a drifting Brownian motion, started at zero, i.e.,

$$dZ_\ell(t)=-\frac{1}{2}dt+dW(t),\quad Z_\ell(0)=0.$$

If Y_ℓ hits 1, Z_ℓ hits $z_u=-\ln Y_\ell(0)$; if Y_ℓ hits u/ℓ, Z_ℓ hits $z_\ell=\ln(u/\ell)+z_u$. Note that $1=\mathbb{1}_{\tau_\ell>T}+\mathbb{1}_{\{\tau_\ell\le T,Z_\ell(\tau_\ell)=z_\ell\}}+\mathbb{1}_{\{\tau_\ell\le T,Z_\ell(\tau_\ell)=z_u\}}$, such that

$$\begin{aligned}p_1(0)&=\frac{(K-u)(x_0-\ell)}{u-\ell}\big\{E_Q\big(\mathbb{1}_{Z_\ell(T\wedge\tau_\ell)<\kappa}\mathbb{1}_{\tau_\ell>T}\big)+E_Q\big(\mathbb{1}_{\{\tau_\ell\le T,Z_\ell(\tau_\ell)=z_\ell\}}\big)\big\}\\&=:\frac{(K-u)(x_0-\ell)}{u-\ell}\big\{e_1^+(\kappa)+e_2^+\big\},\end{aligned}$$

where we have used the fact that $\mathbb{1}_{Z_\ell(T\wedge\tau_\ell)<\kappa}=0$ whenever $\mathbb{1}_{\{\tau_\ell\le T,Z_\ell(\tau_\ell)=z_u\}}=1$. The two expectations $e_1^+(\kappa)$ and e_2^+ can be found by standard means. First, from the interior density given in Cox and Miller [7], Example 5.1,

$$\begin{aligned}e_1^+(\kappa)&=E_Q\big(\mathbb{1}_{Z_\ell(T\wedge\tau_\ell)<\kappa}\mathbb{1}_{\tau_\ell>T}\big)\\&=\int_{z_\ell}^{\kappa}\frac{1}{\sqrt{2\pi T}}\sum_{n=-\infty}^{\infty}\left(\exp\left\{-\frac{z_n'}{2}-\frac{(z-z_n'+\frac{1}{2}T)^2}{2T}\right\}\right.\\&\qquad\left.-\exp\left\{-\frac{z_n''}{2}-\frac{(z-z_n''+\frac{1}{2}T)^2}{2T}\right\}\right)dz,\end{aligned}$$

where $z_n'=2n(z_u-z_\ell)$, $z_n''=2z_u-z_n''$. Evaluating this integral leads to the expression for $e_1^+(\kappa)$ given in Proposition 3.4.

The expectation e_2^+ is the probability of Z_ℓ hitting the barrier z_ℓ (i) before time T, and (ii) before z_u is hit. To compute this probability, we first compute $\psi(x)$, the outright probability that z_ℓ will be hit before z_u, conditional on $Z_\ell(0)=x$. From standard results (e.g., [16], Chap. 15.3) we have, for $x \in (z_\ell, z_u)$,

$$\psi(x) = \frac{\int_x^{z_u} \exp(z - z_\ell)\, dz}{\int_{z_\ell}^{z_u} \exp(z - z_\ell)\, dz} = \frac{1 - \exp(x - z_u)}{1 - \exp(z_\ell - z_u)}.$$

According to Bhattacharya and Waymire [4], pp. 406–407, we can now compute e_2^+ as

$$e_2^+ = \psi(0) - \int_{z_\ell}^{z_u} \frac{\psi(z)}{\sqrt{2\pi T}} \sum_{n=-\infty}^{\infty} \left(\exp\left\{ -\frac{z_n'}{2} - \frac{(z - z_n' + \frac{1}{2}T)^2}{2T} \right\} \right.$$
$$\left. - \exp\left\{ -\frac{z_n''}{2} - \frac{(z - z_n'' + \frac{1}{2}T)^2}{2T} \right\} \right) dz.$$

Computing the integral leads to the expression for e_2^+ given in the proposition.

As for $p_2(0)$, we have

$$p_2(0) = \frac{(K-\ell)(x_0 - u)}{u - \ell} E_Q\left(\mathbb{1}_{Y_u(T \wedge \tau_u) > (K-\ell)/(K-u)} \right)$$
$$= \frac{(K-\ell)(x_0 - u)}{u - \ell} E_Q\left(\mathbb{1}_{Z_u(T \wedge \tau_u) \le \kappa} \right),$$

where κ is as defined earlier and $Z_u(t) = -\ln Y_u(T \wedge \tau_u) + \ln Y_u(0)$; Z_u is a Brownian motion with drift $+\frac{1}{2}$ (rather than $-\frac{1}{2}$, as above) and starting point $Z_u(0) = 0$. The expectation $E_Q(\mathbb{1}_{Z_u(T \wedge \tau_u) \le \kappa})$ can consequently be computed easily from the expressions above (essentially by changing sign on all terms that involve $\frac{1}{2}T$). We omit the details. □

B.4 Proof of Proposition 3.5

The proof proceeds as for Proposition 3.4 (see Appendix B.3 above), but now we use Fourier series representations for survival and absorption probabilities. Adopting the notation of Appendix B.3 everywhere, we first notice, from Cox and Miller [7], Example 5.1,

$$e_1^+ = E_Q\left(\mathbb{1}_{Z_\ell(T \wedge \tau_\ell) < \kappa} \mathbb{1}_{\tau_\ell > T} \right)$$
$$= \frac{2}{z_u - z_\ell} \int_{z_\ell}^{\kappa} e^{-\frac{1}{2}z} \sum_{n=1}^{\infty} e^{-\lambda_n T} \sin\left(n\pi \frac{-z_\ell}{z_u - z_\ell} \right) \sin\left(n\pi \frac{z - z_\ell}{z_u - z_\ell} \right) dz$$
$$= \sum_{n=1}^{\infty} \frac{2 \sin(n\pi \frac{-z_\ell}{z_u - z_\ell}) e^{-\lambda_n T}}{z_u - z_\ell} \int_{z_\ell}^{\kappa} e^{-\frac{1}{2}z} \sin\left(n\pi \frac{z - z_\ell}{z_u - z_\ell} \right) dz,$$

where

$$\lambda_n = \frac{1}{2}\left(\frac{1}{4} + \frac{n^2\pi^2}{(z_u - z_\ell)^2} \right).$$

Setting $a_n = \frac{n\pi z_\ell}{z_u - z_\ell}$ and $k_n = \frac{n\pi(\kappa - z_\ell)}{z_u - z_\ell}$, it is easy to demonstrate that

$$\int_{z_\ell}^{\kappa} e^{-\frac{1}{2}z} \sin\left(n\pi \frac{z - z_\ell}{z_u - z_\ell}\right) dz = \frac{a_n}{2 z_\ell \lambda_n}\left(e^{-\frac{1}{2}z_\ell} - e^{-\frac{1}{2}\kappa}\cos(k_n)\right) - \frac{e^{-\frac{1}{2}\kappa}}{4\lambda_n}\sin(k_n).$$

So,

$$\begin{aligned} e_1^+ &= \frac{1}{z_u - z_\ell}\sum_{n=1}^{\infty}\sin(-a_n)e^{-\lambda_n T} \\ &\quad \times \left[\frac{a_n}{\lambda_n z_\ell}\left(e^{-\frac{1}{2}z_\ell} - e^{-\frac{1}{2}\kappa}\cos(k_n)\right) - \frac{e^{-\frac{1}{2}\kappa}}{2\lambda_n}\sin(k_n)\right]. \end{aligned}$$

As for the lower-boundary absorption probability e_2^+, using the same technique as in Appendix B.3,

$$\begin{aligned} e_2^+ &= E_Q\left(\mathbb{1}_{\{\tau_\ell \le T, Z_\ell(\tau_\ell) = z_\ell\}}\right) \\ &= \psi(0) - \int_{z_\ell}^{z_u} \psi(z) \sum_{n=1}^{\infty} \frac{2\sin(-a_n)e^{-\lambda_n T}}{z_u - z_\ell} \int_{z_\ell}^{z_u} e^{-\frac{1}{2}z}\sin\left(n\pi\frac{z - z_\ell}{z_u - z_\ell}\right) dz, \end{aligned}$$

where $\psi(x) = (1 - \exp(x - z_u))/(1 - \exp(z_\ell - z_u))$. Evaluating the integral gives

$$e_2^+ = \psi^+ + \frac{e^{-\frac{1}{2}z_\ell}}{(z_u - z_\ell)^2}\sum_{n=1}^{\infty}\frac{e^{-\lambda_n T}}{\lambda_n} n\pi \sin(a_n),$$

with ψ^+ as given in the proposition. Computation of e_1^- and e_2^- proceeds in the same way, after a shift of drift of Z_ℓ from $-\frac{1}{2}$ to $+\frac{1}{2}$. □

B.5 Proof of Proposition 4.3

The form of (4.3) suggests that, for $t \le \tau$,

$$\eta(t) = q(t)\cos Z(t)$$

for some deterministic function q. Applying Itô's lemma, we get

$$\begin{aligned} d\eta(t) &= -q(t)\sin Z(t)\, d\tilde{W}(t) - \frac{1}{2}q(t)\cos Z(t)\, dt + \frac{dq(t)}{dt}\cos Z(t)\, dt \\ &= -\eta(t)\tan Z(t)\, d\tilde{W}(t) + \cos Z(t)\left(\frac{dq(t)}{dt} - \frac{1}{2}q(t)\right) dt. \end{aligned}$$

From (4.3), it follows that the function $q(t)$ must satisfy

$$\frac{dq(t)}{dt} = \frac{1}{2}q(t), \quad q(0)\cos Z(0) = 1,$$

i.e.,

$$q(t) = q(0)e^{\frac{1}{2}t}, \quad q(0) = \frac{1}{\cos(\arctan(\frac{x_0-a}{b}))} = \sqrt{1 + \left(\frac{x_0 - a}{b}\right)^2}.$$

By Lemma 4.1 and the usual rules of measure change,

$$\begin{aligned} p(0) &= E_Q\Big(\big(K - X(\tilde{T})\big)^+\Big) = E\Big(\eta(\tilde{T})\big(K - X(\tilde{T})\big)^+\Big) \\ &= q(0E\Big(e^{\frac{1}{2}\tilde{T}} \cos Z(\tilde{T})\big(K - a - b \tan Z(\tilde{T})\big)^+\Big) \\ &= q(0)bE\Big(e^{\frac{1}{2}\tilde{T}}\big(\tilde{K} \cos Z(\tilde{T}) - \sin Z(\tilde{T})\big)^+\Big), \quad \tilde{K} = \frac{K - a}{b}. \end{aligned}$$ □

B.6 Proof of Proposition 4.4

Let us first turn to the computation of $e_1 = E((\tilde{K} \cos Z(T) - \sin Z(T))^+ 1_{\tau > T})$, where we recall that τ is the first time that the Brownian motion Z hits either z_U or z_L. We define $z_0 = \arctan(\frac{x_0-a}{b})$, such that $M = Z - z_0$ is a regular Brownian motion started at zero. We first notice that $\tilde{K} \cos Z(T) - \sin Z(T) > 0$ if and only if $M(T) < \arctan(\tilde{K}) - z_0 =: k$, such that, from results similar to those in Appendix B.4,

$$e_1 = \int_{m_L}^{k} n(z; m_L, m_U, T)\big(\tilde{K} \cos(z + z_0) - \sin(z + z_0)\big)\, dz,$$

where $m_L = z_L - z_0$, $m_U = z_U - z_0$, and

$$n(z; m_L, m_U, T) = \frac{2}{m_U - m_L} \sum_{n=1}^{\infty} e^{-\alpha_n T} \sin\left(n\pi \frac{-m_L}{m_U - m_L}\right) \sin\left(n\pi \frac{z - m_L}{m_U - m_L}\right),$$

with

$$\alpha_n = \frac{n^2 \pi^2}{2(m_U - m_L)^2}.$$

It follows that

$$e_1 = \frac{2}{m_U - m_L} \sum_{n=1}^{\infty} e^{-\alpha_n T} \sin(-a_n)\big(\tilde{K} I_n^{(c)} - I_n^{(s)}\big),$$

where with $a_n = n\pi \frac{m_L}{m_U - m_L}$,

$$I_n^{(c)} = \int_{m_L}^{k} \sin\left(a_n \frac{z}{m_L} - a_n\right) \cos(z + z_0)\, dz,$$

$$I_n^{(s)} = \int_{m_L}^{k} \sin\left(a_n \frac{z}{m_L} - a_n\right) \sin(z + z_0)\, dz.$$

Defining $b_n^{\pm} = 1 \pm a_n/m_L$ and $c_n^{\pm} = b_n^{\pm} k \mp a_n + z_0$, it is easily shown that

$$I_n^{(c)} = \frac{m_L}{2}\left[\frac{\cos(c_n^-)}{m_L - a_n} - \frac{\cos(c_n^+)}{m_L + a_n} + \frac{2\cos(m_L)a_n}{a_n^2 - m_L^2}\right], \tag{B.1}$$

where we have used that $b_n^{\pm} m_L = m_L \pm a_n = z_L - z_0 \pm a_n$. Similarly, we get

$$I_n^{(s)} = \frac{m_L}{2}\left[\frac{\sin(c_n^-)}{m_L - a_n} - \frac{\sin(c_n^+)}{m_L + a_n} + \frac{2\sin(m_L)a_n}{a_n^2 - m_L^2}\right]. \tag{B.2}$$

As an aside, notice that

$$\cos(z_L) = \cos\left(\arctan\left(\frac{x_L - a}{b}\right)\right) = \frac{1}{\sqrt{A(x_L)/b}},$$
$$\sin(z_L) = \sin\left(\arctan\left(\frac{x_L - a}{b}\right)\right) = \frac{x_L - a}{\sqrt{bA(x_L)}}.$$

Having now computed an explicit expression for e_1, we turn to the computation of $e_2 = E(e^{\frac{1}{2}\tau}\mathbb{1}_{\{\tau \le T, Z(\tau)=z_L\}})$. First, from a standard result, the likelihood that the Brownian motion M will hit m_L before m_U, given $M(0) = x$, is

$$\psi(x) = \frac{m_U - x}{m_U - m_L}.$$

Proceeding as in Appendix B.3, it then follows that

$$\begin{aligned} E\big(\mathbb{1}_{\{\tau \le T, M(\tau)=m_L\}}\big) &= E\big(\mathbb{1}_{\{\tau \le T, Z(\tau)=z_L\}}\big) \\ &= \psi(0) - \int_{m_L}^{m_U} \psi(z) n(z; m_L, m_U, T)\, dz. \end{aligned}$$

The (defective) density of the random time τ_L at which M gets absorbed at m_L is therefore (employing somewhat loose notation)

$$\begin{aligned} \varphi(t) &:= P\big(\tau_L \in [t, t+dt]\big)/dt = -\int_{m_L}^{m_U} \psi(z) \frac{\partial}{\partial t} n(z; m_L, m_U, t)\, dz \\ &= \frac{1}{(m_U - m_L)^2} \sum_{n=1}^{\infty} e^{-\alpha_n t} n\pi \sin(-a_n). \end{aligned}$$

Consequently,

$$\begin{aligned} e_2 &= E\big(e^{\frac{1}{2}\tau}\mathbb{1}_{\{\tau \le T, Z(\tau)=z_L\}}\big) = \int_0^T \varphi(t) e^{\frac{1}{2}t}\, dt \\ &= \frac{1}{(m_U - m_L)^2} \sum_{n=1}^{\infty} n\pi \sin(-a_n) \int_0^T e^{-(\alpha_n - \frac{1}{2})t}\, dt \\ &= \frac{1}{(m_U - m_L)^2} \sum_{n=1}^{\infty} n\pi \sin(-a_n) \frac{1 - e^{-(\alpha_n - \frac{1}{2})T}}{\alpha_n - \frac{1}{2}} \\ &= \frac{\sin(m_U)}{\sin(m_U - m_L)} - \frac{1}{(m_U - m_L)^2} \sum_{n=1}^{\infty} n\pi \sin(-a_n) \frac{e^{-(\alpha_n - \frac{1}{2})T}}{\alpha_n - \frac{1}{2}}. \end{aligned}$$

□

B.7 Proof of Proposition 4.5

We borrow all notation from Appendix B.6 above. As before, we have

$$e_1 = \int_{m_L}^{k} n(z; m_L, m_U, T)\big(\tilde{K}\cos(z+z_0) - \sin(z+z_0)\big)\,dz,$$

where we now use a method of images representation for $n(z; m_L, m_U, T)$, namely

$$n(z; m_L, m_U, T) = \frac{1}{\sqrt{2\pi T}} \sum_{n=-\infty}^{n=\infty} \left[\exp\left\{-\frac{(z-z_n')^2}{2T}\right\} - \exp\left\{-\frac{(z-z_n'')^2}{2T}\right\}\right],$$

with $z_n' = 2n(m_U - m_L)$, $z_n'' = 2m_U - z_n'$. Defining $I_c(\cdot)$ and $I_s(\cdot)$ as in (4.5), (4.6) we get

$$e_1 = \sum_{n=-\infty}^{n=\infty} \Big(\tilde{K}\big(I_c(z_n') - I_c(z_n'')\big) - \big(I_s(z_n') - I_s(z_n'')\big)\Big).$$

As for the computation of $e_2 = E(e^{\frac{1}{2}\tau} 1_{\{\tau \le T, Z(\tau)=z_L\}})$, we first define τ_L to be the time at which the Brownian motion M gets absorbed at m_L. Proceeding as in Appendix B.6, the (defective) density of τ_L can be computed as

$$\varphi(t) := P\big(\tau_L \in [t, t+dt]\big)/dt = -\frac{\partial}{\partial t} \int_{m_L}^{m_U} \psi(z) n(z; m_L, m_U, t)\,dz,$$

where $\psi(z) = (m_U - z)/(m_U - m_L)$. Evaluating the integral shows that

$$\int_{m_L}^{m_U} \psi(z) n(z; m_L, m_U, t)\,dz = F(t)$$

where $F(t)$ is as given in Proposition 4.5. Finally, from the definition of e_2,

$$e_2 = \int_0^T \varphi(t) e^{\frac{1}{2}t}\,dt. \qquad \square$$

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