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Risk-neutral compatibility with option prices

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Abstract A common problem is to choose a "risk-neutral" measure in an incomplete market in asset pricing models. We show in this paper that in some circumstances it is possible to choose a unique "equivalent local martingale measure" by completing the market with option prices. We do this by modeling the behavior of the stock price X, together with the behavior of the option prices for a relevant family of options which are (or can theoretically be) effectively traded. In doing so, we need to ensure a kind of "compatibility" between X and the prices of our options, and this poses some significant mathematical difficulties.

Keywords Option prices · Risk neutral measures · Equity pricing · Equivalent martingale measures

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1 Introduction

A common practice for analyzing stock prices and the associated option pricing and hedging strategies has as a first step the choice of a model for the stock price. The

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Black–Scholes model is usually considered inaccurate, both due to volatility "smile" effects, and the widespread belief that price processes have "heavy tails" (cf., e.g., [4, 14]). This leads researchers to consider more sophisticated models such as diffusions, models with stochastic volatility, and/or models with jumps. In many cases this destroys the completeness of the model, thus rendering option pricing a perilous task and hedging imperfect. Moreover, the choice of the model and in particular the values allocated to the various parameters entering the model are also sometimes suspect.

On the other hand, many options are effectively traded, just as the stock itself is. The recorded prices of these options are used for determining the parameters of the model, through the "implied volatility" paradigm for example; and the options are also used for hedging.

A classical approach is to specify a parameterized form of a price process for a risky asset, then to estimate the parameters statistically (calibration); one then uses the calibrated model to determine risk-neutral prices of contingent claims in the case of a complete market; in the case of an incomplete market, one chooses a risk-neutral measure based on some criterion determined by the modeler.

Concerning option prices, ideally, one should derive them from the underlying model, and then enter the recorded prices to provide a check on one's estimates of the parameters. This proves impossible for two reasons at least. One is that option prices are *not* in general determined by the model when completeness fails, and another one is that even in the rare cases where option prices are determined by the model, they are almost never given in closed form, and actual computations are difficult or impossible.

However, there is also good news: If sufficiently many assets are traded, then the model becomes complete. This is a heuristic way of stating the following very general mathematical result. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, be a filtered probability space with an adapted process X (the stock price process). X represents the nondiscounted price process. If we take the time value of money into account, then we should replace X with $S_t = e^{-\int_0^t r_s\,ds} X_t$. However in this paper *we assume that the interest rate r is deterministic, and moreover $r \equiv 0$*. This can be a serious restriction, but it simplifies the already complicated mathematics of the paper.

Usually there are infinitely many measures $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which X is a martingale or local martingale (so completeness fails). But if we introduce a family $\mathcal{Y}$ of additional processes, the set of all measures $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which X and all processes Y in $\mathcal{Y}$ are martingales, shrinks. It may be empty, but it also may be a singleton, in which case the "model" (including all $Y \in \mathcal{Y}$) becomes complete. And it is always possible to find a big enough family $\mathcal{Y}$ such that the above set is a singleton (but typically, of course, $\mathcal{Y}$ is infinite, although countable as soon as the σ-field $\mathcal{F}$ is separable).

These considerations lead us to propose a scheme in which one models the behavior of the stock price X, together with the behavior of the option prices for a relevant family of options which are (or can theoretically be) effectively traded. In doing so, one should of course ensure a kind of "compatibility" between X and the prices of our options. This is similar to the modeling of bond prices, where one models the behavior of the interest rate together with the prices of bonds with various maturities. Our scheme is closely related to the term structure model of Heath, Jarrow, and

Morton [17], extended to the discontinuous case by Björk et al. in [1] and [2]. Unfortunately, however, an HJM-style model for option prices seems intrinsically much more complicated than it is for bonds.

The scheme we propose encompasses a lot of different "concrete" models, which all have in common that completeness holds. For simplicity, we suppose that there is a single stock price, but an extension to several stocks is straightforward. We also suppose that we have a single type of option which is traded, namely a European option with payoff function g (a nonnegative convex function, typically $g(x)=(x-K)^+$) and arbitrary expiration dates. Again, these ideas would work as well for other types of options. In fact, if we smooth the standard function $g(x)=(x-K)^+$ just a bit to make it C^2, then we shall see that one obtains a much more satisfying theory.

Indeed, this paper was originally conceived as an attempt to see if a term structure type model for option prices (broadly interpreted) could be seen to make sense in a general way. The problem is hard, due to the requirement in what we call the *full model* that the choice of the risk-neutral measure determine the value of the underlying martingale at the maturity time. What we have found is that if we insist that the standard function $g(x)=(x-K)^+$ be used, then the answer is negative. However if we are willing to tolerate a bit of smoothing of g, then we can make some sense of the problem, and it becomes tractable.

The simplicity (or lack thereof) of the problem is tied to when the options mature. If one is interested in a time interval $[0,T_\star]$ where the option in question matures *after* the time $T_\star$, then the problem is much simpler, since it is not necessary to force the ending value of the martingale representing the option price to agree with the value of the option, written as a function of the price process X at time $T_\star$, precisely because that time has not yet arrived. We consider this case first, where the ideas work well, before considering the more general case where the options mature on or before the time $T_\star$.

The idea of completing a market with option prices is not new. For example, there was a flurry of research appearing in 1997: Dengler and Jarrow [11] consider a simple model where the price process is pure jump, but with only two distinct jump sizes. They use two traded call options to complete the market, and the collection of martingale measures becomes a parameterized family, with only one of them being a martingale measure for all three securities. Dupire [13] considers a simple stochastic volatility model and lets the strike price vary but keeps time fixed. Differentiating twice with respect to the strike, he gets a partial differential equation, which, when solved, uniquely determines a martingale measure. Finally, Derman and Kani [12] also treat the stochastic volatility case and consider calls with varying strike price K. As does Dupire, they assume smoothness in K, differentiate twice, and get a density implied by the call option, which depends on K. The risk-neutral restriction leads to a PDE, and a numerical procedure for solving it is proposed. Schönbucher [28] in 1999 considers a Brownian paradigm, with the presence of stochastic volatility. He then finds conditions under which one has absence of arbitrage, which amounts to a condition that the over-specified model must satisfy (it is over-specified in the sense that there are more securities than sources of randomness). His approach revolves around implied volatilities, with the spot volatility taken as a fundamental variable.

In work essentially simultaneous to this paper, Schweizer and Wissel [29] have improved on these ideas using different maturities (rather than different strikes as in the 1997 literature), and in that sense their paper is closer in spirit to this one; see also the preprints by Davis and Obłoj [8] and by Schweizer and Wissel [30]. Finally, we also wish to mention the recent and very interesting related work of Carmona and Nadtochiy [3], which is similar in spirit to the work of Derman, Dupire, and others (in the continuous path case), but takes the ideas much further.

At this point it is prudent to make clear what we claim and also what we do not claim. In an incomplete market setting, in the absence of arbitrage opportunities, there are of course an infinite number of risk-neutral measures. It is theoretically possible to obtain enough information from contingent claim prices, given by the market, to match one or more of these risk-neutral measures to these extra prices. Such a matching implies that consistency conditions must be met. In some cases there may be only one such measure, or perhaps none at all. Having a unique measure implies a complete market, of course, but it need not be a stable complete market; that is, since the risk-neutral measure is determined by matching the option prices, it is possible there could be a paradigm shift. That is, the market has "chosen" (at least for now) a unique measure among all of the possible risk-neutral measures corresponding to the asset price process. The market could choose a different one at some future time. See [20] for the development and use of this idea. We wish to emphasize that we are not giving conditions for the absence of arbitrage opportunities; rather, we are giving conditions for when the price structure of derivatives is such that one can find an equivalent local martingale measure which is compatible with the price structure or alternatively is such that there cannot exist such an equivalent local martingale measure corresponding to the prices. When we can in fact find such a risk-neutral measure, then the theory of Harrison, Pliska, and Kreps, or more generally of Delbaen and Schachermayer, ensures the absence of arbitrage, properly interpreted. This is in contrast to the usual approach, where one risk neutral measure is chosen a priori, and then derivative prices are dictated by the model and this choice of risk-neutral measure.

A last warning is in order: We make no attempt to the calibration of the models we propose. For this, we should need to specify the model within a class of parametric models, and this would be the object of a totally different work; see however the end of Sect. 4.

A plan of the paper is as follows. In Sect. 2 we present the general framework we consider. The price process of a given stock will be assumed to follow what is known as an Itô process framework; it is an adapted càdlàg process consisting of a stochastic integral with respect to a Wiener process, and a jump component driven by a compensated Poisson random measure for the "small" jumps, the "large" jumps being arbitrary. To this we include models for the options, where we consider two different frameworks: the first with a finite trading time horizon $T_\star$ with options maturing after $T_\star$ (called *partial models*), and the second with $T_\star = \infty$ (called *full models*). We do not specify the prices via the model of the stock price, but rather consider them as given by the market.

In Sect. 3 we let $P(T) = (P(T)_t)_{t\geq 0}$ denote European call option prices through time, with a fixed strike price K and maturity time T. We keep K fixed, and we let the maturity time T vary. We then consider the collection of equivalent local martingale measures that are compatible with the given prices of the stock and options, and we characterize them, in terms of assumed structure.

In Sect. 4 we consider the entire collection of equivalent local martingale measures for partial models and determine when it has cardinality one. This obviously requires a compatibility condition among the option prices, since a priori we do not know the collection is not empty. Examples are given.

In Sect. 5 we consider the analogous situation as in Sect. 4 but for full models. We give an example of Markov-type stochastic volatility models.

2 A general model

We suppose that all sources of randomness are a family of Brownian motions and a possibly infinite Poisson random measure; this should cover most, if not all, applications. When all processes (stock prices, option prices) are continuous in time, we can dispense with the Poisson measure. When the stock prices are continuous but the volatility has jumps, we do need the Poisson measure, and even more so of course when the stock prices are discontinuous.

More specifically we have a filtered space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{P}$ is the historical measure, $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, and the space is endowed with a finite or countable family $(W^i)_{i\in I}$ of standard Brownian motions, and a Poisson random measure $\mu = \mu(dt, dx)$ on $\mathbb{R}_+ \times \mathbb{R}$ with intensity measure $\nu(dt, dx) = dt \otimes F(dx)$. We suppose that $\mathbb{F}$ is the smallest filtration to which the W^i and μ are adapted, and $\mathcal{F} = \mathcal{F}_\infty$.

We need to consider below the optional and predictable σ-fields $\mathcal{O}$ and $\mathcal{P}$ on the product $\Omega \times \mathbb{R}_+$. As a matter of fact, we also need to consider the products $\Omega \times \mathbb{R}_+ \times \mathbb{R}$ and $\Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$: all these products will be endowed with the product σ-fields of $\mathcal{F}$ with the Borel σ-fields of the other factors, and the measure which is the product of $\mathbb{P}$ with the relevant Lebesgue measure. We generically write $\widehat{\Omega}$ and $\widehat{\mathbb{P}}$ for these products and the associated measures. We also write $\widetilde{\mathcal{P}}$ for the product $\mathcal{P} \otimes \mathcal{R}$ on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$, where of course $\mathcal{R}$ denotes the Borel sets.

(1) *The stock model* We assume the following model for the stock price X. First, in the *continuous case* we suppose that

$$X_t = X_0 + \int_0^t a_s\, ds + \sum_{i\in I} \int_0^t \sigma_s^i\, dW_s^i. \tag{2.1}$$

In the general case, when there are jumps, we suppose that

$$X_t = X_0 + \int_0^t a_s\, ds + \sum_{i\in I} \int_0^t \sigma_s^i\, dW_s^i + \big(\psi 1_{\{|\psi|\leq 1\}}\big) * (\mu - \nu)_t + \big(\psi 1_{\{|\psi|>1\}}\big) * \mu_t. \tag{2.2}$$

Here and below we use the standard notation for stochastic integrals with respect to Wiener processes W^i and a random measure μ or compensated random measure $\mu - \nu$, see, e.g., [18]. In (2.2) $X_0 > 0$ is nonrandom, and the coefficients a, σ^i, and ψ are such that the integrals and sums above make sense; that is, a and σ^i are predictable, and ψ is $\widetilde{\mathcal{P}}$-measurable, and

$$\int_0^t \left(|a_s| + \sum_{i\in I} |\sigma_s^i|^2 + \int \big(\psi(s,x)^2 \wedge 1\big) F(dx) \right) ds < \infty \quad \text{a.s.}$$

for all t. Of course these coefficients should also be such that $X_t > 0$; this amounts to saying that they factorize as $a_t = X_{t-}\bar{a}_t$ and $\sigma_t^i = X_{t-}\bar{\sigma}_t^i$ and $\psi(t,x) = X_{t-}\bar{\psi}(t,x)$ with $\bar{\psi} > -1$ identically, but it is more convenient to use the form (2.2). Note that this is the most general semimartingale driven by μ and the W^i.

The process X is fixed throughout; this determines the deterministic starting point X_0, and the coefficients a, σ^i and ψ up to a $\widehat{\mathbb{P}}$-nullset.

(2) *The associated option model* For the options, we consider a fixed payoff function g on $(0,\infty)$ which is *nonnegative and convex*, and we denote by $P(T)_t$ the price at time $t \in [0,T]$ of the option with payoff $g(X_T)$ at the expiration date T. We also assume that g is not affine, otherwise $P(T)_t = g(X_t)$ is of course the price of the option at time t, and there is nothing more to do.

We denote by $\mathcal{T}$ the set of expiration dates T corresponding to tradable options (always with the same given payoff function g), and by $T_\star$ the time horizon up to when trading may take place. Even when $T_\star < \infty$, there might be options with expiration date $T > T_\star$, so we need to specify the model up to infinity.

In practice $\mathcal{T}$ is a finite set, although perhaps quite large. For the mathematical analysis, it is much more convenient to take for $\mathcal{T}$ an interval, or perhaps a countable set which is dense in an interval. In this paper we mainly concentrate on two cases, although other obvious situations are also of interest:

- *The full models*, for which $T_\star = \infty$ and $\mathcal{T} = (0,\infty)$.
- *The partial models*, for which $T_\star < \infty$ and $\mathcal{T} = [T_0,\infty)$, with $T_0 > T_\star$.

Apart from the fact that $P(T)_T = g(X_T)$, the prices $P(T)_t$ are so far unspecified, and the idea is to model them on the basis of the same W^i and μ rather than X. However, since these are option prices, they should have some structural properties.

Indeed, if the option prices were derived in the customary way, we should have a measure $\mathbb{Q}$ which is equivalent to $\mathbb{P}$, and under which X is a martingale, $g(X_T)$ is $\mathbb{Q}$-integrable, and $P(T)_t = \mathbb{E}_{\mathbb{Q}}(g(X_T) \mid \mathcal{F}_t)$ for $t \le T$. Then of course $P(T)$ is a $\mathbb{Q}$-martingale indexed by $[0,T]$. But we can also look at how $P(T)_t$ varies as a function of the expiration date T, on the interval $[t,\infty)$. That is, we are taking the uncustomary step of fixing t and considering $P(T)_t$ *as a process where* T *varies*. Since X is a quasi-left-continuous martingale and g is convex, $T \mapsto g(X_T)$ is a quasi-left-continuous submartingale relative to $\mathbb{Q}$, and this implies that $T \mapsto P(T)_t$ is *nondecreasing and continuous* for $T \ge t$. Observe that this property holds $\mathbb{Q}$-almost surely, hence $\mathbb{P}$-almost surely as well because $\mathbb{P}$ and $\mathbb{Q}$ are equivalent.

We wish to emphasize that, for example in the case of European call options, the usual theory calls for $P(T)_t = \mathbb{E}_{\mathbb{Q}^\star}((X_T - K)^+ \mid \mathcal{F}_t)$ for some risk-neutral measure $\mathbb{Q}^\star$. We do not make this assumption here. Indeed, the previous paragraph is simply motivation for us to *assume a priori* that $T \mapsto P(T)_t$ is nondecreasing and continuous. This seems completely reasonable from the viewpoint of practice, where (in the absence of dividends or interest rate changes and anomalies) it is always observed that $T \mapsto P(T)_t$ is nondecreasing. In the language of practitioners, if it were not, it would imply a "negative pricing of the calendar", which makes no economic sense [22]. Nevertheless we warn the reader that there are pathological examples where this assumption does not hold. For example, if X is the reciprocal of a three-dimensional Bessel process starting at $X_0 = 1$, then X is a local martingale for its natural filtration, but $T \mapsto P(T)_0$ is not increasing, since $P(0)_0 = 0$, $P(T)_0 > 0$ for $T \in (0, \infty)$, but $\lim_{T\to\infty} P(T)_0 = 0$, hence $T \mapsto P(T)_0$ cannot be increasing for $T \geq 0$. Thus our assumption $T \mapsto P(T)_t$ is increasing in T rules out the possibility of the market being governed by such prices processes. (Note that the assumption that $T \mapsto P(T)_t$ is increasing in T is not essential and could be replaced simply with $T \mapsto P(T)_t$ is absolutely continuous as a function of T, provided that one makes the appropriate mathematics changes in the proofs.)

So one can consider the latter property as embedded into the fact that the $P(T)_t$ *are* option prices. We slightly strengthen this property by assuming that $T \mapsto P(T)_t$ is absolutely continuous on (t, ∞); we shall check that this assumption is satisfied by the Black–Scholes and many other usual models. An equivalent way of expressing these properties is to write

$$P(T)_t = g(X_t) + \int_t^T f(t,s)\,ds, \quad t \leq T, \tag{2.3}$$

where the $f(t,s)$ for $t < s$ are random nonnegative and $\mathcal{F}_t$-measurable variables, whose dynamics we now need to specify. (We remark that this is reminiscent of one of the "Greeks", namely theta, which measures the rate of change in the option's value with respect to time to expiration, namely $T - t$. Usually, however, it is implicit that T is fixed and that t is changing; here of course we are taking t fixed with T changing.) A priori $P(T)_t$ is not defined for $t > T$, but it is convenient to set $P(T)_t = g(X_T)$ for $t > T$ (recall that $P(T)_T = g(X_T)$). So in fact each process $P(T)$ is well defined over the whole positive line, and our extension of course does not alter the putative martingale property of $P(T)$; moreover since $T \mapsto P(T)_t$ is constant on $[0, t)$, we should have $f(t,s) = 0$ for $s < t$.

Now we have to specify the dynamics of the processes $(f(t,s), 0 \leq t < s)$. Note that in (2.3), for any (ω, t), the function $s \mapsto f(\omega, t, s)$ (for $s > t$) is a priori defined up to a Lebesgue-nullset only, since it is a density, and the nullset may depend on (ω, t). However it is convenient to assume that $f(t,s)$ is well defined for all $t < s$. In the *continuous case* the dynamics is given by

$$f(t,s) = f(0,s) + \int_0^t \alpha(u,s)\,du + \sum_{i\in I} \int_0^t \gamma^i(u,s)\,dW_u^i, \tag{2.4}$$

and in the general case it is

$$\begin{aligned} f(t,s) = f(0,s) &+ \int_0^t \alpha(u,s)\,du + \sum_{i\in I} \int_0^t \gamma^i(u,s)\,dW_u^i \\ &+ \big(\phi(.,s)1_{\{|\phi(.,s)|\leq 1\}}\big) * (\mu - \nu)_t + \big(\phi(.,s)1_{\{|\phi(.,s)|>1\}}\big) * \mu_t. \end{aligned} \tag{2.5}$$

The coefficients $\alpha(\omega,u,s)$, $\gamma^i(\omega,u,s)$, and $\phi(\omega,u,x,s)$ and the starting points $f(0,s)$ determine our option price model. The precise assumptions, which amount to nonnegativity and finiteness conditions, are given below.

Definition 2.1 A *full option model associated with* (X, g) is a family of processes $(P(T): T > 0)$ given by (2.3), where for each s, the process $(f(t,s): 0 \le t < s)$ has the form (2.5), and where

(1) $f(0,s)$ is a nonrandom nonnegative initial condition, measurable and locally integrable in s.
(2) $\alpha(\omega,u,s)$ and $\gamma^i(\omega,u,s)$, respectively $\phi(\omega,u,x,s)$, are measurable with respect to $\mathcal{P} \otimes \mathcal{R}_+$, respectively $\mathcal{P} \otimes \mathcal{R} \otimes \mathcal{R}_+$.
(3) α, γ^i, and ϕ are such that the integrals in (2.5) make sense for all $t < s$, and $(\omega,t,s) \mapsto f(t,s)(\omega)1_{\{t<s\}}$ is $\mathcal{O} \otimes \mathcal{R}_+$-measurable.
(4) We have $f(t,s) \ge 0$ for all $t < s$.
(5) We have $\int_t^T f(t,s)\,ds < \infty$ a.s. for all $t \le T$.

An important role will be played by the processes

$$\chi(s)_t = \int_0^t \left(\big|\alpha(u,s)\big| + \sum_{i \in I} \big|\gamma^i(u,s)\big|^2 + \int \big(\phi(u,x,s)^2 \wedge 1\big) F(dx) \right) du. \tag{2.6}$$

Under (2) above, (3) is equivalent to saying that $\chi(s)_t < \infty$ a.s. for all $t < s$.

Analogously to the stock model, one can modify α, γ^i, or ϕ on a $\widehat{\mathbb{P}}$-nullset without changing the option prices $P(T)_t$.

The above specification is necessary and sufficient for the $P(T)_t$ to be well defined, but it is not quite enough if we want to exchange the stochastic integrals in (2.5) and the ordinary integral in (2.3) in order to exhibit the dynamics of $P(T)$ itself. Note also that while we assume $\chi(T)_t < \infty$ for $t < T$, we do not suppose that $\chi(T)_T < \infty$. Therefore $f(T,T)$ might be not well defined, and this is indeed the case for the Black–Scholes model with payoff function $g(x) = (x-K)^+$. Nevertheless when $f(T,T)$ exists (as it does for some models), the models are much simpler to deal with. This leads us to consider more special models.

Definition 2.2 A full option model associated with (X, g) is said to be

(1) *Regular* if for almost all s, we have $\chi(s)_s < \infty$ a.s. (this implies that $f(s,s)$ is well defined);
(2) *Fair* if for all T, we have $\int_t^T \chi(s)_t\,ds < \infty$ a.s.
(3) *Strongly regular* if for all $T \in \mathbb{R}_+$, we have $\int_0^T \chi(s)_s\,ds < \infty$ a.s.

Note that strongly regular implies regular and fair, because $t \mapsto \chi(s)_t$ is increasing (but $s \mapsto \chi(s)_t$ is usually not increasing).

Now we turn to partial models, where trading takes place up to time $T_\star$ and expiration dates are all $T \ge T_0$, with $T_0 > T_\star$. In this case we have only to model $P(T)$ for $T \ge T_0$ and $t < T_0$, and of course (2.3) gives us

$$P(T)_t = P(T_0)_t + \int_{T_0}^T f(t,s)\,ds \tag{2.7}$$

Then if we know the dynamics of $P(T_0)$, the model will be fully specified by the dynamics of $t \mapsto f(t,s)$ for $s > T_0$, whereas we need only to characterize the dynamics of all processes up to time $T_\star$. This leads us to the following definition.

Definition 2.3 A $(T_\star, T_0)$ *partial option model associated with* (X, g) is a family of processes $(P(T) : T \geq T_0)$ with time interval $[0, T_\star]$ given by (2.7), where

(a) For each $s > T_0$, the process $(f(t,s) : t \leq T_\star)$ has the form (2.5), and all requirements of Definition 2.1 are satisfied for $t \leq T_\star$ and $s > T_0$.
(b) The process $P(T_0)$ is given for $t \leq T_\star$ by

$$P(T_0)_t = P(T_0)_0 + \int_0^t \overline{\alpha}_s \, ds + \sum_{i \in I} \int_0^t \overline{\gamma}_s^i \, dW_s^i \tag{2.8}$$

in the continuous case, and in the general case by

$$P(T_0)_t = P(T_0)_0 + \int_0^t \overline{\alpha}_s \, ds + \sum_{i \in I} \int_0^t \overline{\gamma}_s^i \, dW_s^i + \big(\overline{\phi} 1_{\{|\overline{\phi}| \leq 1\}}\big) * (\mu - \nu)_t \\ + \big(\overline{\phi} 1_{\{|\overline{\phi}| > 1\}}\big) * \mu_t, \tag{2.9}$$

where the above coefficients are predictable and satisfy

$$\int_0^{T_\star} \left(|\overline{\alpha}_t| + \sum_{i \in I} |\overline{\gamma}_t^i|^2 + \int (\overline{\phi}(t,x)^2 \wedge 1) F(dx) \right) dt < \infty$$

a.s., and further the (nonrandom) initial condition $P(T_0)_0$ and these coefficients are such that we have identically

$$t \in [0, T_\star] \quad \Longrightarrow \quad P(T_0)_t \geq g(X_t). \tag{2.10}$$

Finally the model is *fair* if we have $\int_{T_0}^T \chi(s)_{T_\star} \, ds < \infty$ a.s. for all $T > T_0$.

Note that here the notions of regular or strongly regular models are irrelevant.

(3) *Equivalent (local) martingale measures* The definition of equivalent martingale measures is standard, but we need first to specify a number of points.

Here we diverge a bit from the standard theory of Delbaen and Schachermayer (see [9, 10]), where it is proved that the condition "No Free Lunch with Vanishing Risk" (known as NFLVR) is equivalent to the existence of an equivalent probability measure making the price process a local martingale in the continuous case. Instead, we consider locally equivalent probability measures; two probability laws Q and P are said to be *locally equivalent* if they are equivalent on each σ-field $\mathcal{F}_t$, $t < \infty$. A canonical example of two processes which give rise to locally equivalent probabilities, but not equivalent ones, are $W = (W_t)_{0 \leq t < \infty}$ and Z, where $Z_t = W_t + \alpha t$, $0 \leq t < \infty$ and $\alpha \neq 0$, for a standard Wiener process W.

For a full model, we then call $\mathcal{M}ar$, respectively $\mathcal{M}ar_{\mathrm{loc}}$, the set of all probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ which are *locally equivalent* to $\mathbb{P}$, and under which X and $P(T)$ for all $T \in \mathcal{T}$ are martingales, respectively local martingales.

Similarly, for a $(T_\star, T_0)$-partial model, we call $\mathcal{M}ar(T_\star, T_0)$ and $\mathcal{M}ar_{\mathrm{loc}}(T_\star, T_0)$ the sets of all probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F}_{T_\star})$ which are equivalent to $\mathbb{P}$ on $\mathcal{F}_{T_\star}$ and under which X and $P(T)$ for all $T \geq T_0$ are martingales, respectively local martingales, over the time interval $[0, T_\star]$.

According to the folklore of the subject, no-arbitrage means that the set $\mathcal{M}ar$ is not empty, and completeness means that it is a singleton. This of course comes from (local) martingale representation theory. From a mathematical point of view, $\mathcal{M}ar_{\mathrm{loc}}$ is much easier to deal with, and justified from the standpoint of the theory of Delbaen and Schachermayer ([9, 10]). We note that Delbaen and Schachermayer show that one can in general have only an equivalent sigma-martingale measure in the general case (i.e., not necessarily continuous) for the price process. However a sigma-martingale which is bounded below is a local martingale (see, e.g., [18] or [24]), and we are assuming nonnegativity of our risky asset price process, which of course is appropriate when modeling stock prices. So we do not need to consider sigma-martingales here.

We note in passing that the really meaningful sets would be the above sets of measures with the original filtration $(\mathcal{F}_t)$ replaced by the filtration $(\mathcal{G}_t)$ generated by the processes X and the $P(T)$ for $T \in \mathcal{T}$. But this is much more difficult to analyze when the inclusion $\mathcal{G}_t \subset \mathcal{F}_t$ is strict. It is often the case that the two filtrations agree, although it is of course easy to exhibit "counterexamples." We do not pursue this issue in this paper.

Throughout X is fixed, and of course the question of whether $\mathcal{M}ar_{\mathrm{loc}}$, resp. $\mathcal{M}ar$, is empty or not is irrelevant if we do not have (M), resp. (M′), below:

Hypothesis (M) *There exists at least one locally equivalent probability measure under which X is a local martingale.*

Hypothesis (M′) *There exists at least one locally equivalent probability measure under which X is a martingale.*

(4) *The Black–Scholes model* Here we check that option models as defined above encompass at least the Black–Scholes model, for which the price of options is completely determined by the stock price model. Other examples are provided later. We have a single Brownian motion $W^1 = W$ and no Poisson measure ($\mu = 0$). We suppose that $\mathbb{P}$ is the risk-neutral measure, so that

$$X_t = X_0 + \int_0^t \sigma X_s \, dW_s. \tag{2.11}$$

Under the traditional setting, the price of a European option with payoff function g having polynomial growth and maturity T, at time $t \leq T$, is $P(T,t) = C(X_t, T-t)$, where

$$C(x,t) = \mathbb{E}\big(g\big(x e^{\sigma U \sqrt{t} - \sigma^2 t/2}\big)\big), \tag{2.12}$$

and U is $\mathcal{N}(0,1)$. One knows that C is C^∞ on $(0,\infty)\times(0,\infty)$ and that (with obvious notation for the partial derivatives)

$$C'_t(x,t) = \frac{1}{2t}\int h(u)g\big(xe^{\sigma u\sqrt{t}-\sigma^2 t/2}\big)\big(u^2-1-u\sigma\sqrt{t}\big)\,du,$$

where h denotes the density function of the standard normal distribution. If g is C^2, then C is C^1 on $(0,\infty)\times[0,\infty)$ and $C'_t(x,0)=\frac{\sigma^2}{2}\,x^2g''(x)$, which is the value $\mathcal{L}g(x)$ with $\mathcal{L}$ the generator of X, at it should be. When g is convex but not C^2, one deduces from (2.12) that $C(x,t)-g(x)=\mathrm{O}(\sqrt{t})$ as $t\to 0$, and for the particular case $g(x)=(x-K)^+$, a simple computation shows that

$$\sqrt{t}C'_t(x,t)\to \frac{\sigma}{2\sqrt{2\pi}}1_{\{x=K\}}.$$

More generally, $C'_t(x,0)$ exists if g is twice continuously differentiable at x, and not if g is not twice differentiable at x.

From this discussion, it follows that (2.3) holds with

$$f(t,T)=C'_t(X_t,T-t),$$

and by Itô's formula we have (2.5) for $t<T$, with

$$\left.\begin{aligned} &f(0,T)=C'_t(X_0,T),\\ &\alpha(s,T)=-C''_{tt}(X_s,T-s)+\tfrac12\,C'''_{txx}(X_s,T-s)\sigma^2X_s^2,\\ &\gamma(s,T)=C''_{tx}(X_s,T-s)\sigma X_s.\end{aligned}\right\} \tag{2.13}$$

Therefore the prices $P(T)$ follow a full option model associated with (X,g), which is always fair. It is strongly regular if g is C^2, but it is not even regular if g is not twice differentiable.

3 A characterization of equivalent local martingale measures

In this section we aim to give a sort of "technical" characterization of equivalent local martingale measures. We start by showing that all processes $P(T)$ are of the type (2.7) and more specifically of the type (2.9) for suitable coefficients, and under the right assumptions. Let us first introduce the notation

$$\left.\begin{aligned} &\widetilde{\sigma}^i_t=\sigma^i_t g'(X_{t-}),\\ &\widetilde{\psi}(t,x)=g(X_{t-}+\psi(t,x))-g(X_{t-}),\\ &\overline{\psi}(t,x)=\widetilde{\psi}(t,x)-g'(X_{t-})\psi(t,x),\\ &\widetilde{a}_t=a_tg'(X_{t-})+\tfrac12\,g''(X_{t-})\textstyle\sum_{i\in I}(\sigma^i_t)^2+\int F(dx)\overline{\psi}(t,x)1_{\{|\psi(t,x)|\le 1\}}.\end{aligned}\right\} \tag{3.1}$$

Lemma 3.1 *For a strongly regular full model and if g is C^2, we have, for $t \le T$,*

$$P(T)_t = P(T)_0 + \int_0^t A(u,T)\,du + \sum_{i\in I} \int_0^t \Gamma^i(u,T)\,dW_u^i + \Phi(.,T) * (\mu - \nu)_t + \Phi'(.,T) * \mu_t, \tag{3.2}$$

where the coefficients above are, for $u \le T$,

$$\left.\begin{aligned} &A(u,T) = -f(u,u) + \widetilde{a}_u + \textstyle\int_u^T \alpha(u,s)\,ds, \\ &\Gamma^i(u,T) = \widetilde{\sigma}_u^i + \textstyle\int_u^T \gamma^i(u,s)\,ds, \\ &\Phi(u,x,T) = \widetilde{\psi}(u,x) 1_{\{|\psi(u,x)|\le 1\}} + \textstyle\int_u^T (\phi(u,x,s) 1_{\{|\phi(u,x,s)|\le 1\}})\,ds, \\ &\Phi'(u,x,T) = \widetilde{\psi}(u,x) 1_{\{|\psi(u,x)|>1\}} + \textstyle\int_u^T (\phi(u,x,s) 1_{\{|\phi(u,x,s)|>1\}})\,ds. \end{aligned}\right\} \tag{3.3}$$

Although it is not immediately apparent from the sight of (3.2), taking $t = T$ in this formula yields $P(T)_T = g(X_T)$; this happens because of the presence of the term $f(u,u)$ in the drift $A(u,T)$, plus the presence of $\widetilde{a}$, $\widetilde{\sigma}^i$, and $\widetilde{\psi}$ in the four terms of (3.3). Writing (3.2) implicitly means that all terms in it are meaningful, which amounts to saying that a.s.,

$$\int_0^T \left(|A(u,T)| + \sum_{i\in I} |\Gamma^i(u,T)|^2 + \int \left(\Phi(u,x,T)^2 + 1_{\{\Phi'(u,x,T)\neq 0\}}\right) F(dx) \right) du < \infty. \tag{3.4}$$

Proof By hypothesis $\int_0^T f(0,s)\,ds < \infty$, and also $\int_0^T \chi(s)_s\,ds < \infty$ a.s., hence for all s outside a Lebesgue-nullset N, the process $f(t \wedge s, s)$ is a semimartingale over $\mathbb{R}_+$. So a simple transformation of (2.3) gives

$$P(T)_t = P(T)_0 + g(X_t) - g(X_0) - \int_0^t f(s,s)\,ds + \int_0^T \big(f(t\wedge s, s) - f(0,s)\big)\,ds \tag{3.5}$$

if $t \le T$, provided that the last integral above makes sense, in which case the first one will also be finite (recall $f \ge 0$). Now each semimartingale $f(t \wedge s, s) - f(0,s)$ is "controlled" by the process $\chi(s)$, so $\int_0^T \chi(s)_s\,ds < \infty$ is exactly what we need to apply Fubini's theorem for ordinary (for the drift) and stochastic integrals. This means that the last integral in (3.5) is well defined and equal to

$$\int_0^t \overline{A}(u,T)\,du + \sum_{i\in I} \int_0^t \overline{\Gamma}^i(u,T)\,dW_s^i + \overline{\Phi}(.,T) * (\mu - \nu)_t + \overline{\Phi}'(.,T) * \mu_t,$$

where

$$\overline{A}(u,T) = \int_u^T \alpha(u,s)\,ds, \qquad \overline{\Phi}(u,x,T) = \int_u^T \big(\phi(u,x,s) 1_{\{|\phi(u,x,s)|\le 1\}}\big)\,ds,$$

$$\overline{\Gamma}^i(u,T) = \int_u^T \gamma^i(u,s)\,ds, \qquad \overline{\Phi}'(u,x,T) = \int_u^T \big(\phi(u,x,s) 1_{\{|\phi(u,x,s)|>1\}}\big)\,ds.$$

On the other hand, Itô's formula yields

$$g(X_t) = g(X_0) + \int_0^t \widetilde{a}_u \, du + \sum_{i\in I} \int_0^t \widetilde{\sigma}^i_s \, dW^i_s$$
$$+ \big(\widetilde{\psi} 1_{\{|\psi|\le 1\}}\big) * (\mu - \nu)_t + \big(\widetilde{\psi} 1_{\{|\psi|>1\}}\big) * \mu_t. \qquad (3.6)$$

Putting all these facts together readily gives us the result. □

Now for partial models, $P(T_0)$ is given by (2.9), so we really only need to write in a form like (3.2) the differences $Q(T) = P(T) - P(T_0)$ for $T \ge T_0$, which is

$$Q(T)_t = Q(T)_0 + \int_{T_0}^T \big(f(t,s) - f(0,s)\big)\, ds.$$

Then as soon as $\int_0^{T_\star} \chi(s)_{T_\star} \, ds$ is a.s. finite (recall that here $t \le T_\star$), we can apply Fubini's theorem as in the previous proof, and we readily get the following lemma; note that unlike in the previous lemma, we make no regularity assumption on g.

Lemma 3.2 *For a fair* $(T_\star, T_0)$ *partial model, we have, for* $t \le T_\star$ *and* $T \ge T_0$,

$$Q(T)_t = Q(T)_0 + \int_0^t \overline{A}(u,T)\, du + \sum_{i\in I} \int_0^t \overline{\Gamma}^i(u,T)\, dW^i_u$$
$$+ \overline{\Phi}(.,T) * (\mu - \nu)_t + \overline{\Phi}'(.,T) * \mu_t,$$

where the coefficients above are, for $u \le T_\star$,

$$\left.\begin{array}{ll} \overline{A}(u,T) = \int_{T_0}^T \alpha(u,s)\, ds, & \overline{\Phi}(u,x,T) = \int_{T_0}^T (\phi(u,x,s) 1_{\{|\phi(u,x,s)|\le 1\}})\, ds, \\ \overline{\Gamma}^i(u,T) = \int_{T_0}^T \gamma^i(u,s)\, ds, & \overline{\Phi}'(u,x,T) = \int_{T_0}^T (\phi(u,x,s) 1_{\{|\phi(u,x,s)|> 1\}})\, ds. \end{array}\right\}$$

Next, we recall how locally equivalent measures can be described. We introduce the set Υ_0 of all families $\mathcal{Y} = ((b^i)_{i\in I}, Y)$, where each b^i is a predictable process, and Y is a positive $\widetilde{\mathcal{P}}$-measurable function satisfying

$$D(\mathcal{Y})_t := \sum_{i\in I} \int_0^t \big(b^i_s\big)^2 ds + \int_0^t ds \int \big(\big|Y(s,x) - 1\big|^2 \wedge \big|Y(s,x) - 1\big|\big) F(dx) < \infty \qquad (3.7)$$

for all t. We denote by $\Upsilon_{0,\mathrm{b}}$ (the "b" is for bounded) the subset of all $\mathcal{Y} \in \Upsilon_0$ for which the random variables $D(\mathcal{Y})_t$ are bounded for each t. Two families $\mathcal{Y} = ((b^i)_{i\in I}, Y)$ and $\mathcal{Y}' = ((b'^i)_{i\in I}, Y')$ in Υ_0 are said to be equivalent if

$$i \in I \quad \Longrightarrow \quad b^i = b'^i \quad \widehat{\mathbb{P}}\text{-a.e.}, \qquad Y = Y' \quad \widehat{\mathbb{P}}\text{-a.e.} \qquad (3.8)$$

Finally, Υ and Υ_{b} are the sets of all equivalence classes (for this equivalence relation) of Υ_0 and $\Upsilon_{0,\mathrm{b}}$ respectively. Then Girsanov's theorem applied to the family $(W^i)_{i\in I}$ and to μ, with respect to which a martingale representation theorem holds, yields the following (see, e.g., [18], Sects. III.3 and III.4; let us emphasize that the "martingale representation theorem" here is different from the martingale representation with respect to a given process or a family of processes, as needed for complete models; this is the representation of all martingales as sums of stochastic integrals with respect to the W^i and the random martingale measure $\mu - \nu$).

Lemma 3.3

(a) *There is a subset* Υ_{eq} *of* Υ *which is in one-to-one correspondence with the set of all probability measures* $\mathbb{Q}$ *on* $(\Omega, \mathcal{F})$ *which are locally equivalent to* $\mathbb{P}$.
(b) *If* $\mathcal{Y} = ((b^i), Y) \in \Upsilon_{\mathrm{eq}}$ *and if* $\mathcal{Y}' = ((b'^i), Y') \in \Upsilon_{\mathrm{b}}$ *is such that* $Y + Y' > 1$ *identically, then the family* $\mathcal{Y}'' = ((b''^i), Y'')$, *where* $b''^i = b^i + b'^i$ *and* $Y'' = Y + Y' - 1$, *also belongs to* Υ_{eq}. *In particular* Υ_{eq} *contains* Υ_{b}.
(c) *If* $\mathcal{Y} = (b^i, Y) \in \Upsilon_{\mathrm{eq}}$, *then under the associated measure* $\mathbb{Q}$ *the processes* $W'^i_t = W^i_t - \int_0^t b^i_s\,ds$ *are independent Brownian motions, and the compensator of the random measure* μ (*no longer a Poisson measure in general*) *is* $\nu'(\omega, dt, dx) = Y(\omega, t, x)\nu(\omega, dt, dx)$. *Moreover the martingale representation theorem holds for* $\mathbb{Q}$, *with respect to the* W'^i *and* $\mu - \nu'$.

When the time horizon $T_\star$ is finite, we are interested in probability measures equivalent to $\mathbb{P}$ on $\mathcal{F}_{T_\star}$ only. The restrictions of these measures to $\mathcal{F}_{T_\star}$ are in one-to-one correspondence with the subset $\Upsilon'_{\mathrm{eq}}(T_\star)$ of Υ_{eq} consisting of those $((b^i), Y)$ satisfying $b^i_t = 0$ and $Y(t, x) = 0$ identically for $t > T_\star$.

We denote by Υ_{loc} the set of all $\mathcal{Y} \in \Upsilon_{\mathrm{eq}}$ which correspond, via the previous lemma, to a probability measure in $\mathcal{M}ar_{\mathrm{loc}}$. In a similar way, $\Upsilon'_{\mathrm{loc}}(T_\star, T_0)$ is the set of all $\mathcal{Y} \in \Upsilon'_{\mathrm{eq}}(T_\star)$ which correspond to a probability measure in $\mathcal{M}ar_{\mathrm{loc}}(T_\star, T_0)$.

For the next theorem, we recall the notation $\overline{\psi}$ of (3.1). We also note that one can modify the coefficients of the model outside a $\widehat{\mathbb{P}}$-nullset without affecting the model itself; so when these coefficients satisfy a property for $\widehat{\mathbb{P}}$-almost all (ω, s, T), we can thus find "good" versions of these coefficients which satisfy the property identically. The same holds for the elements $\mathcal{Y}$ of Υ.

Therorem 3.4 *For a full strongly regular model and if* g *is* C^2, *the set* Υ_{loc} *is the set of all* $((b^i), Y) \in \Upsilon_{\mathrm{eq}}$ *which, for a good version of the coefficients of the model, satisfy, for all* $s > 0$,

$$a_s + \sum_{i \in I} \sigma^i_s b^i_s + \int \psi(s, x)\big(Y(s, x) - 1_{\{|\psi(s,x)| \le 1\}}\big) F(dx) = 0, \tag{3.9}$$

$$f(s, s) = \frac{1}{2} g''(X_s) \sum_{i \in I} \big(\sigma^i_s\big)^2 + \int Y(s, x)\overline{\psi}(s, x) F(dx), \tag{3.10}$$

and also, for all $T \ge s$,

$$\alpha(s, T) + \sum_{i \in I} \gamma^i(s, T) b^i_s + \int \phi(s, x, T)\big(Y(s, x) - 1_{\{|\phi(s,x,T)| \le 1\}}\big) F(dx) = 0. \tag{3.11}$$

Remark 3.5 For any given s, relation (3.11) should hold for *all* $T \geq s$, whereas b^i_s and $Y(s,x)$ do not depend on T; so (3.11) basically specifies the drift when γ^i and ϕ are given. This is typical of term structure models and already present in the Heath–Jarrow–Morton model. On the other hand, relation (3.10) is somewhat surprising, probably difficult to check and even to interpret, and quite specific to the option model.

Before proceeding to the proof of Theorem 3.4 we need a lemma.

Lemma 3.6 *For a strongly regular full model with g being C^2, the set Υ_{loc} is the set of all $((b^i), Y) \in \Upsilon_{\mathrm{eq}}$ which, for all (ω, s) outside a $\widehat{\mathbb{P}}$-nullset, satisfy* (3.9) *and, for all $T > 0$,*

$$A(s,T) + \sum_{i \in I} \Gamma^i(s,T) b^i_s + \int \big(\Phi(s,x,T)\big(Y(s,x)-1\big) + Y(s,x)\Phi'(s,x,T)\big)F(dx) = 0. \quad (3.12)$$

Proof Let $\mathbb{Q}$ be the locally equivalent measure associated with $((b^i), Y) \in \Upsilon_{\mathrm{eq}}$. Since the coefficients $A(.,T)$, $\Gamma^i(.,T)$, $\Phi(.,T)$, and $\Phi'(.,T)$ are continuous in T, condition (3.11) is equivalent to having, for any given T, the property (3.12) for $\widehat{\mathbb{P}}$-almost all (ω, s). Hence we just have to show that X, respectively $P(T)$, is a $\mathbb{Q}$-local martingale if and only if we have (3.9), respectively (3.12), for any given T. We prove only the second property; the first one is similar, and in fact simpler.

With the notation of Lemma 3.1, we have

$$\begin{aligned} P(T)_t = P(T)_0 &+ \int_0^t A(s,T)\,ds + \sum_{i \in I} \int_0^t \Gamma^i(s,T)\,dW'^i_s \\ &+ \Phi(.,T) * (\mu - \nu')_t + \Phi'(.,T) * \mu_t \\ &+ \sum_{i \in I} \int_0^t \Gamma^i(s,T) b^i_s\,ds + \int_0^t ds \int \big(Y(s,x)-1\big)\Phi(s,x,T)\,F(dx), \end{aligned}$$

where all terms make sense because of (3.4) and (3.7). The third and fourth terms on the right side above are $\mathbb{Q}$-local martingales, and the second and the last two terms are continuous with finite variation, with a sum denoted by A_t. Then for $P(T)$ to be a $\mathbb{Q}$-local martingale, it is necessary and sufficient that the process $A'_t = \Phi'(.,T) * \nu'_t$, which is also $\int_0^t ds \int Y(s,x)\Phi'(s,x,T)F(dx)$, is well defined and satisfies $A_t + A'_t = 0$ a.s. But this amounts to saying that (3.12) holds $\widehat{\mathbb{P}}$-almost everywhere, and we are finished. □

Proof of Theorem 3.4 We fix a family $((b^i), Y) \in \Upsilon_{\mathrm{eq}}$. Denote by G_s, $H(s,T)$, and $K(s,T)$ the left sides (for $s < T$) of (3.9), (3.11), and (3.12) respectively, and by R_s the right side of (3.10). Set

$$M_s = -f(s,s) + \widetilde{a}_s + \sum_{i \in I} \widetilde{\sigma}^i_s b^i_s + \int \widetilde{\psi}(s,x)\big(Y(s,x) - 1_{\{|\psi(s,x)| \leq 1\}}\big)F(dx).$$

Some tedious but straightforward calculations based on (2.5) and (3.3) yield

$$K(s,T)=M_s+\int_s^T H(s,r)\,dr, \tag{3.13}$$

$$M_s=R_s-f(s,s)+g'(X_{s-})G_s. \tag{3.14}$$

Now we let N_1 be the set of all (ω,s) such that either $G_s(\omega)\neq 0$ or $K(s,T)(\omega)\neq 0$ for some $T>s$, N_1' the set of all (ω,s) such that either $G_s(\omega)\neq 0$ or $f(s,s)\neq R_s(\omega)$, and N_3 the set of all (ω,s,T) such that $H(s,T)(\omega)\neq 0$ and $T\geq s$. In view of Lemma 3.6, it is enough to prove that $\widehat{\mathbb{P}}(N_1)=0$ is equivalent to $\widehat{\mathbb{P}}(N_1')=\widehat{\mathbb{P}}(N_3)=0$.

Suppose first $\widehat{\mathbb{P}}(N_1')=0$ and $\widehat{\mathbb{P}}(N_3)=0$. By Fubini, the set N_1'' of all (ω,s) such that $\int_s^\infty 1_{N_3}(\omega,s,T)\,dT>0$ has $\widehat{\mathbb{P}}(N_1'')=0$, and (3.13) yields that $K(s,T)(\omega)=M_s(\omega)$ for all $T\geq s$ if $(\omega,s)\in N_1''$. Then $N_1\subset N_1'\cup N_1''$, and thus $\widehat{\mathbb{P}}(N_1)=0$.

Second, suppose $\widehat{\mathbb{P}}(N_1)=0$. If $(\omega,s)\notin N_1$, (3.13) yields that $H(s,r)(\omega)=0$ for dr-almost all $r\geq s$, and also $M_s(\omega)=0$. We deduce that $\widehat{\mathbb{P}}(N_3)=0$, and also obviously that $\widehat{\mathbb{P}}(N_1')=0$, and upon using our comments about "good versions" for the coefficients, the proof is finished. □

In exactly the same way, one proves the following theorem.

Therorem 3.7 *For a $(T_\star,T_0)$ partial fair model, the set $\Upsilon'_{\mathrm{loc}}(\mathcal{T},T_\star)$ is the set of all $((b^i),Y)$ in $\Upsilon'_{\mathrm{eq}}(T_\star)$ which, for a good version of the coefficients of the model, satisfy (3.9) for $s\leq T_\star$ a.s., (3.11) for $s\leq T_\star$ and $T\geq T_0$ a.s., and also, for $s\leq T_\star$ a.s.,*

$$\overline{\alpha}_s+\sum_{i\in I}\overline{\gamma}^i_s b^i_s+\int\overline{\phi}(s,x)\big(Y(s,x)-1_{\{|\overline{\phi}(s,x)|\leq 1\}}\big)F(dx)=0. \tag{3.15}$$

4 Completeness for partial fair models

In this section we consider $(T_\star,T_0)$ partial models. There is no problem for the existence of such models with no-arbitrage (a nonempty set $\mathcal{M}ar_{\mathrm{loc}}(T_\star,T_0)$), or at least not any more than for a one- or multidimensional stock price model for having no arbitrage; indeed the model basically amounts to specifying the processes X and $P(T)$ for $T\geq T_0$, with no restrictions except for $g(X_t)\leq P(T)_t\leq P(T')_t$ when $t<T_\star<T_0\leq T<T'$.

Therefore we concentrate on the completeness of such (fair) models. Since for us completeness means that the set $\mathcal{M}ar_{\mathrm{loc}}(T_\star,T_0)$ is a singleton, this is a relatively simple matter, at least from a theoretical point of view (see, e.g., [23]).

So we start with a partial fair model. For any (ω,s), we denote by $\Upsilon'(\omega,s)$ the set of all families $\zeta=((\beta_i)_{i\in I},y)$ where $\beta_i\in\mathbb{R}$, and y is a function on $\mathbb{R}$ satisfying the conditions

$$\left.\begin{array}{l}\sum_{i\in I}\beta_i^2<\infty,\qquad \int(y(x)^2\wedge|y(x)|)F(dx)<\infty,\\ x\mapsto\psi(\omega,s,x)y(x)\quad\text{and}\quad x\mapsto\overline{\phi}(\omega,s,x)y(x)\quad\text{are }F\text{-integrable},\\ T\geq T_0\quad\Longrightarrow\quad x\mapsto\phi(\omega,s,x,T)y(x)\quad\text{is }F\text{-integrable}.\end{array}\right\} \tag{4.1}$$

Therorem 4.1 *Consider a* $(T_\star, T_0)$ *partial fair model such that the set* $\mathcal{M}ar_{\mathrm{loc}}(T_\star, T_0)$ *is not empty. Then this set is a singleton if and only if, for a good version of the coefficients of the model, we have the following property: The system of linear equations*

$$\sum_{i\in I}\sigma_s^i(\omega)\beta_i + \int \psi(\omega,s,x)y(x)F(dx) = 0, \tag{4.2}$$

$$\sum_{i\in I}\overline{\gamma}_s^i(\omega)\beta_i + \int \overline{\phi}(\omega,s,x)y(x)F(dx) = 0, \tag{4.3}$$

$$T \ge T_0 \quad \Longrightarrow \quad \sum_{i\in I}\gamma^i(s,T)(\omega)\beta_i + \int \phi(\omega,s,x,T)y(x)\,F(dx) = 0, \tag{4.4}$$

all for $s > 0$ *a.s., where* $((\beta_i), y) \in \Upsilon'(\omega, s)$, *has the only solution* $\beta_i = 0$ *and* $y = 0$ *up to an* $F(dx)$*-nullset.*

Proof Let us first observe that, since

$$(x+y)^2 \wedge |x+y| \le 6\big(x^2 \wedge |x| + y^2 \wedge |y|\big),$$

each set $\Upsilon'(\omega, s)$ is clearly a linear space (for the obvious vector operations).

By hypothesis there exists a $\mathbb{Q} \in \mathcal{M}ar_{\mathrm{loc}}(T_\star, T_0)$, associated with the family $\mathcal{Y} = ((b^i), Y) \in \Upsilon'_{\mathrm{eq}}(T_\star, T_0)$ in such a way that, provided we choose a good version of the coefficients and of $\mathcal{Y}$, we have (3.9), (3.15), and (3.11) for all $s \le T_\star$ and $T \ge T_0$, and also $\sum_{i\in I}|b_s^i|^2 < \infty$ and $\int (Y(\omega,s,x)-1)^2 \wedge |Y(\omega,s,x)-1|)F(dx) < \infty$ for $s \le T_\star$. In particular the family $((b_s^i(\omega), Y(\omega,s,.)-1)$ belongs to $\Upsilon'(\omega,s)$.

Let us now prove the sufficient condition. If $\mathbb{Q}'$ is another measure in $\mathcal{M}ar_{\mathrm{loc}}(T_\star, T_0)$, associated with the family $\mathcal{Y}' = ((b'^i), Y') \in \Upsilon'_{\mathrm{eq}}(T_\star, T_0)$, then $\mathcal{Y}'$ has the same properties as above (we can of course choose a good version of the coefficients which works for both $\mathbb{Q}$ and $\mathbb{Q}'$). Then if, for some (ω, s) with $s \le T_\star$, we set $\beta_i = b_s^i(\omega) - b_s'^i(\omega)$ and $y(x) = Y(\omega,s,x) - Y'(\omega,s,x)$, we readily deduce from (3.9), (3.15), and (3.11), written for $\mathcal{Y}$ and for $\mathcal{Y}'$, that $((\beta_i), y)$ belongs to $\Upsilon'(\omega,s)$ and satisfies (4.2), (4.3), and (4.4). Then the condition of the theorem implies that $b^i = b'^i$ and $Y = Y'$, that is, $Q' = \mathbb{Q}$, and we have the sufficient condition.

For the converse, suppose that the condition is not satisfied. Using a section theorem (or, measurable selection theorem), one can construct predictable processes b'^i and a predictable function $Y' > 0$ on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$ such that

$$Y + Y' > 1, \qquad \sum_{i\in I}(b_s'^i)^2 + \int \big(\big(Y'(s,x)-1\big)^2 \wedge \big|Y'(s,x)-1\big|\big)F(dx) \le 1$$

and that the family $((\beta_s'^i(\omega)), Y'(\omega,s,.)-1)$ belongs to $\Upsilon'(\omega,s)$ and solves (4.2), (4.3), and (4.4), and also such that the set

$$A = \big\{(\omega,s) : s \le T_\star, b_s'^i(\omega) = 0 \forall i \in I, Y'(\omega,s,.) = 1\ F(dx)\text{-a.e.}\big\}$$

has a non-$\widehat{\mathbb{P}}$-negligible complement A^c in $\Omega \times [0, T_\star]$. Then $\mathcal{Y}' = ((b'^i), Y') \in \Upsilon_b$, and the family $\mathcal{Y}'' = ((b''^i), Y'')$, where $b'^i = b^i + b''^i$ and $Y'' = Y + Y' - 1$, satisfies (3.9), (3.15), and (3.11). Hence by Lemma 3.3(b) and Theorem 3.7 there is a measure $\mathbb{Q}''$ associated with $\mathcal{Y}''$, and this measure belongs to $\mathcal{M}ar_{\text{loc}}(T_\star, T_0)$. Moreover since $\widehat{\mathbb{P}}(A^c) > 0$, we deduce that $\mathcal{Y}$ and $\mathcal{Y}''$ are not equivalent in the sense of (3.8), that is, $\mathbb{Q}'' \neq \mathbb{Q}$; this proves the necessary condition. □

Remark 4.2 In practice the set $\mathcal{T}$ of the expiration dates of all tradable options is finite. In that case, and still in the "partial" situation studied here, there are evidently other, and perhaps simpler, ways to pose the problem. Namely, instead of describing the dynamics of $f(t, s)$, we can as well give directly the dynamics of $P(T)$ for each $T \in \mathcal{T}$ as we did for $P(T_0)$ in (2.9), provided we add the compatibility relationship $P(T)_t \leq P(T')_t$ if $T < T'$ (like we did in (2.10) for $P(T_0)$). In this case the family $(P(T) : T \in \mathcal{T})$ behaves like a family of stock prices, especially as far as completeness is concerned; for example, in the continuous case we have completeness if and only if there are not more driving Wiener processes than the number of options, plus 1 (the stock itself), as soon as some nondegeneracy holds. See, in this regard, for example, the work of Romano and Touzi [27] or Davis [7].

The continuous case As an example, let us consider the continuous case, that is, the model given by (2.1), (2.4), and (2.8).

In this case, the second terms on the left sides of (4.2), (4.3), and (4.4) vanish, and the following result is then a trivial consequence of Theorem 4.1. Below, when I is infinite, we denote by $\ell^2(I)$ the subspace of $\mathbb{R}^I$ of all vectors whose sum of squared components is finite (so at most a countable number of them are nonzero), and this space is endowed with the usual ℓ^2 topology. The same applies when I is finite, but then of course $\ell^2(I) = \mathbb{R}^I$. Note that one can consider $(\beta_i)_{i \in I}$ in (4.1) as an element of $\ell^2(I)$, and similarly for $(\sigma^i(\omega)_s)_{i \in I}$, $(\gamma^i(\omega, s, T))_{i \in I}$, and $(\overline{\gamma}^i(\omega)_s)_{i \in I}$ just above, at least for a good version.

Therorem 4.3 *Suppose that we have the* $(T_\star, T_0)$ *partial fair model described by* (2.1), (2.4), *and* (2.8), *and also that the set* $\mathcal{M}ar_{\text{loc}}(T_\star, T_0)$ *is not empty. Then there are two alternatives*:

(a) *Either, for a good version of the coefficients, for all* $s \leq T_\star$ *and* ω, *the closed linear subspace of* $\ell^2(I)$ *spanned by the vectors* $(\sigma^i(\omega)_s)_{i \in I}$, $(\overline{\gamma}^i(\omega)_s)_{i \in I}$, *and* $(\gamma^i(\omega, s, T))_{i \in I}$ *for* $T \geq T_0$, *is equal to* $\ell^2(I)$ *itself; in this case the set* $\mathcal{M}ar_{\text{loc}}(T_\star, T_0)$ *is a singleton.*
(b) *Or this property fails, and the set* $\mathcal{M}ar_{\text{loc}}(T_\star, T_0)$ *is infinite.*

So in this case it is very simple to obtain models that are complete. Checking that (a) above holds for a "good" version of the coefficients may be tricky. However in practice it is quite likely that the coefficients σ^i_s, $\overline{\gamma}^i_s$, and $\gamma^i(s, T)$ are taken continuous in s and T; then they are automatically such a good version, and (a) can be checked directly.

On the other hand checking that, to begin with, the set $\mathcal{M}ar_{\text{loc}}(T_\star, T_0)$ is not empty is quite another matter. This depends primarily on the structure of the vectors

$\sigma_s, \overline{\gamma}_s$, and $\gamma(s,T)$, considered as elements of $\ell^2(I)$. When all coefficients (including the drifts a_s, $\overline{\alpha}_s$, and $\alpha(s,T)$) are continuous in s and T, then as soon as for all ω and $s < T_\star$ there is a countable dense subset $D(\omega, s)$ of (T_0, ∞) such that the vectors $\sigma_s(\omega)$, $\overline{\gamma}_s(\omega)$, and $\gamma(\omega, s, T)$ for $T \in D(\omega, s)$ are linearly independent, then $\mathcal{M}ar_{\mathrm{loc}}(T_\star, T_0)$ is not empty (note that this implies that I is infinite). Otherwise this property may fail; if, for example, $\gamma(s,T)^i = 0$ for all i and some $s < T_\star$ and $T > T_0$, then one should have $\alpha(s,T) = 0$ as well if $\mathcal{M}ar_{\mathrm{loc}}(T_\star, T_0)$ is to be not empty.

The Lévy driven case Here we assume that all our processes are driven by the same real-valued Lévy process Z, say with square-integrable jumps, and whose characteristics are denoted by (B, C, F). That is, we replace (2.2), (2.5), and (2.9) by the equations

$$\left.\begin{array}{l} X_t = X_0 + \int_0^t \delta_s \, dZ_s, \\ f(t,T) = f(0,T) + \int_0^t \rho(u,T) \, dZ_u, \\ P(T_0)_t = P(T_0)_0 + \int_0^t \eta_u \, dZ_u. \end{array}\right\} \tag{4.5}$$

This is a particular case of our general models; indeed, F is the restriction to $\mathbb{R}\backslash\{0\}$ of the image of Lebesgue measure on $\mathbb{R}$ by a Borel function θ, and either $I = \{1\}$ and $W = W^1 = Z^c/\sqrt{C}$ when the unit variance C of the Gaussian part Z^c of Z is positive, or $I = \emptyset$ if $C = 0$, so that

$$Z_t = Bt + \sqrt{C} W_t + \theta * (\mu - \nu)_t, \tag{4.6}$$

where $\mu = \mu^Z$, the jump measure of Z. Then (4.5) amounts to (2.2), (2.5), and (2.9), with the coefficients

$$\left.\begin{array}{ll} a_s = \delta_s(B - \int_{\{|\delta_s \theta(x)|>1\}} \theta(x) F(dx)), & \sigma_s^1 = \delta_s \sqrt{C}, \\ \alpha(s,T) = \rho(s,T)(B - \int_{\{|\rho(s,T)\theta(x)|>1\}} \theta(x) F(dx)), & \gamma^1(s,T) = \rho(s,T)\sqrt{C}, \\ \overline{\alpha}_s = \eta_s(B - \int_{\{|\eta_s \theta(x)|>1\}} \theta(x) F(dx)), & \overline{\gamma}_s^1 = \eta_s \sqrt{C}, \\ \psi(s,x) = \delta_s \theta(x), \phi(s,x,T) = \rho(s,T)\theta(x), & \overline{\phi}(s,x) = \eta_s \theta(x). \end{array}\right\} \tag{4.7}$$

Two popular examples in the literature are geometric Lévy processes and the exponential of a Lévy process. A geometric Lévy process is often expressed in the form

$$X_t = 1 + \int_0^t \sigma X_{s-} \, dZ_s + \int_0^t \mu X_{s-} \, ds \tag{4.8}$$

with the caveat that the jumps of the Lévy process Z are strictly larger than -1, so that the price process X remains positive. An exponential Lévy process has the particularly simple form

$$Y_t = \exp(\sigma Z_t + \mu t), \tag{4.9}$$

and in this case there is no a priori restriction on the jumps of Z. Since t is also a (degenerate) Lévy process, one can show (using Itô's formula) that X in (4.8) and Y in (4.9) can both be expressed in the form (4.5). Thus this situation is quite general, and analogous considerations show that almost all previously proposed models fall within this scope. Indeed, if one wants to consider even quite general semimartingale Markov processes as models of stock prices, using the work of Çinlar and Jacod [5], up to a change of time and change of space, such models can be represented as solutions of stochastic differential equations driven by Wiener processes and compensated Poisson random measures, and these too fit into this framework.

We do not examine here the question of whether $\mathcal{M}ar_{\mathrm{loc}}(T_\star, T_0)$ is empty or not (both cases are possible) and concentrate on the completeness.

Proposition 4.4 *In the Lévy case described above, the set* $\mathcal{M}ar_{\mathrm{loc}}(T_\star, T_0)$ *is not a singleton, unless either* $C = 0$ *and the support of* F *contains one point, or* $C > 0$ *and* $F = 0$.

Proof Equations (4.2)–(4.4) amount to the single equation

$$\sqrt{C}\beta_1 + \int \theta(x) y(x) F(dx) = 0. \tag{4.10}$$

The result is thus obvious. □

Hence, as far as completeness is concerned, we are in a situation similar to what happens for the stock price only, namely a one-factor driving Lévy process (for the asset price and option prices all together) does not provide completeness except when completeness holds already for the asset price by itself.

On the one hand this is natural for a model like (4.5), which is thus probably irrelevant in practice: If we take any specific equivalent martingale measure for the asset price, supposed to be given by (4.5), and then compute the option prices under that particular measure, we do *not* obtain option prices driven by (4.5), except in the two special cases above. In fact, the option prices are driven by the Wiener part *and* the jump measure of Z, but not by Z itself.

On the other hand this is a noticeable difference with what happens for generalizations of the Heath–Jarrow–Morton model driven by Lévy processes, see, e.g., [15] or [16], and despite the fact that the two kinds of models look very similar: For interest rate models driven by a one-factor Lévy process, we typically have completeness as soon as bonds with all maturities are tradable.

An example of a complete model in the discontinuous case The reason completeness fails in the previous example is the way the coefficient ϕ in (4.7) factors. In contrast, completeness does not fail for the analogous situation for bonds. Indeed, this is because, in the case for bond models, (2.3) is replaced by an equation where the (random) interest rate enters, implying that even when the processes $f(., s)$ are all driven by the same Lévy process Z, the prices $P(T)$ (prices of bonds) are *not* driven by Z, but rather by its jump measure which is a Poisson measure.

Having this in mind, it is simple enough to produce an example where completeness holds, in the discontinuous case, even though the stock price is driven by a Lévy process. So let Z be as in (4.6), with say $C = 0$ (we are in the "purely discontinuous case"). The stock price X and the option price $P(T_0)$ are still driven by Z, that is, by

(4.5), and the associated coefficients a, ψ, $\overline{\alpha}$, and $\overline{\phi}$ are still given by (4.7), whereas $\sigma = 0$ and $\overline{\gamma} = 0$ because no Wiener process is present. What differs is the dynamics for the $f(t,s)$: In (2.5) we take $\gamma = 0$ (again, there is no Wiener process) and ϕ as

$$\phi(\omega, s, x, T) = \Phi_s(\omega) g(x, T - s)$$

for some predictable process Φ which does not vanish and some function g on $\mathbb{R} \times (T_0 - T_\star, \infty)$. We have left $\alpha(s,T)$ unspecified, since it plays no role as far as completeness is concerned. But of course it plays a fundamental role as to whether the set $\mathcal{M}ar_{\mathrm{loc}}(T_\star, T_0)$ is empty or not, see Theorem 3.7 above.

Proposition 4.5 *For the model described above, and if the set $\mathcal{M}ar_{\mathrm{loc}}(T_\star, T_0)$ is not empty, then it is a singleton as soon as the family of functions $(x \mapsto g(x,t) : t > 0)$ is "measure-determining" in the sense that any two measures $G(dx) = y(x)F(dx)$ and $G'(dx) = y'(x)F(dx)$ on $\mathbb{R}$, with $y \geq 0$ and $y' \geq 0$ satisfying the second property in* (4.1), *and having $\int g(x,t)G(dx) = \int g(t,x)G'(dx)$ for all $t > 0$, coincide.*

Proof In our case, (4.2)–(4.4) amount to

$$\left.\begin{array}{l} \int \theta(x) y(x) F(dx) = 0, \\ \Phi_s \int g(x, T - s) y(x) F(dx) = 0 \quad \forall T > T_0. \end{array}\right\}$$

Then again the result is obvious, since $\Phi_s \neq 0$. □

An example of a function g satisfying the above requirement is $g(x,t) = e^{-th(x)}$, where h is a bijection from $\mathbb{R}$ into $(0,\infty)$ such that $\int e^{-th(x)} F(dx) < \infty$ for all $t > 0$. If we further assume that F belongs to a given parametric family of measures and that Φ belongs to a parametric family of processes, then one could attempt to do calibration on empirical data.

What this shows is the intuitively appealing observation that if we have enough information obtained from contingent claims, which in some sense spans the hidden noise leading to the original incompleteness, then we can in fact "complete" the market with this extra information.

5 Full regular models: existence and completeness

Now we consider full models. A full model describes in particular what happens when time increases and reaches the expiration times $T \in \mathcal{T}$, and thus *some compatibility between the stock price and the option model is required*, in addition to $P(T)_t \geq g(X_t)$. This property strongly restricts the class of possible option models associated with (X, g), if we want them to be arbitrage-free, and the question of their existence is naturally posed (from the Black–Scholes example seen above we know that there are some . . .).

The main results in that direction are the following two theorems. For their statement, we need to recall the Itô–Meyer–Tanaka formula

$$g(X_t) = g(X_0) + \int_0^t h_s\, ds + M_t + N_t + L_t, \tag{5.1}$$

where M is a local martingale, and in fact the sum of stochastic integrals with respect to the W^i and $\mu - \nu$, h is locally integrable, N is an integral with respect to μ having locally finitely many jumps, and L is an increasing continuous adapted process such that the measure dL_t is singular with respect to Lebesgue measure (a local time). When g is C^2, we have $L = 0$, but otherwise L typically does not vanish (the fact that L does not vanish depends in a rather complicated way on how the continuous martingale part of X behaves when X reaches the points where g is not twice differentiable).

Therorem 5.1

(a) *If the process L in* (5.1) *does not vanish a.s.* (*hence g is not twice differentiable*), *for any full strongly regular option model associated with* (X, g), *the set* $\mathcal{M}ar_{\rm loc}$ *is empty.*
(b) *If g is of class C^2 and if* (*M*) *holds*, *there exists at least a sequence of stopping times S_n increasing to ∞ and for each n a full regular option model associated with* (X^{S_n}, g) *such that* $\mathcal{M}ar_{\rm loc}$ *is not empty* (*here X^{S_n} is the stopped process* $X_t^{S_n} = X_{S_n \wedge t}$).
(c) *If g is of class C^2 with at most linear growth and if* (*M'*) *holds*, *there exists at least a full regular option model associated with* (X, g) *such that* $\mathcal{M}ar_{\rm loc}$ *is not empty.*

Remark 5.2 Theorem 5.1 shows that one cannot have models such as we propose for classic European options, since the function $g(x) = (x - K)^+$ is not C^2 on $(0, \infty)$. This essentially implies that if our models are correct, then classic financial derivatives such as European calls create arbitrage opportunities, since there cannot be a risk-neutral measure for which all prices are compatible. At first blush this is horrifying, but some reflection leads one to reconsider the concept of NFLVR, where the "vanishing risk" occurs as one passes to the limiting case. If one were to approximate g with C^2 functions g_n, then there could exist compatible pricing measures Q_n for each g_n, and these measures would vary continuously with g_n (see [26]). This would correspond to practice, where the difference in a practical economic sense between $g(x) = (x - K)^+$ and a close approximation of g is essentially negligible. In other words, the arbitrage opportunity only exists in the limit, and thus in reality cannot be used in practice to economic advantage.

Nevertheless, this is unsatisfying, since any reasonable model should be able to handle a function of the form $g(x) = (x - K)^+$ and not only C^2 approximations of it. This shows that our approach of constructing a type of term structure model is theoretically unsatisfying, although for practical purposes, it is essentially the same as if functions such as $g(x) = (x - K)^+$ were included exactly as they are. We note that the (less general) models of [3, 6, 29], and [31] do not have this shortcoming, although they of course have their own limitations. Basically, this shortcoming is due to the NFLVR condition (and the NFL condition before it) being justified primarily by its equivalence to the existence of an equivalent sigma-martingale probability measure, and not due to its inherent nature as a lack of arbitrage in any practical sense. In other words, the arbitrage which would be present in our model if we included functions such as $g(x) = (x - K)^+$ is an arbitrage obtainable only "in the limit," and to exploit it one would essentially need to be able to use a strategy which exploits a process of the nature of a local time, an essentially impossible task in practice (see in this regard [19], for example).

Remark 5.3 In the continuous case for the stock price, that is, (2.1), the process L associated by (5.1) is always nonvanishing when g is not twice differentiable, and thus there is no arbitrage-free option model associated with (X, g) which is strongly regular. This is due to the fact that L has paths which are singular with respect to Lebesgue measure and thus creates an arbitrage opportunity, which has been observed long ago (see, for example, the book of Karatzas and Shreve [21, p. 329], where it is a consequence of Theorem B.2, or alternatively the recent detailed treatment in [25]). One may perhaps find nonregular arbitrage-free option models; see, for example, [29] for results in this direction in a different but related (continuous) context.

Therorem 5.4 *Assume that g is of class C^2. A locally equivalent measure $\mathbb{Q}$ cannot be in the sets $\mathcal{M}ar_{\mathrm{loc}}$ corresponding to two different full strongly regular option models associated with (X, g).*

Remark 5.5 This result is in deep contrast with what happens for bond models à la Heath–Jarrow–Morton and also for the usual stock prices models. In the latter case, for instance, if we have two stock prices X and Y under, say, some risk-neutral measure, the volatility of X obviously does not determine the volatility of Y; here, however, if $\mathbb{P}$ is a risk-neutral measure and in the continuous case, say (that is $\mu = \nu = 0$), knowing the volatility of X (that is, the σ^i) determines the volatility of the $P(T)$ (that is, the $\gamma^i(., T)$).

Remark 5.6 This result does not mean that there is a single arbitrage-free strongly regular option model associated with (X, g). There might be several, and even infinitely many, each one corresponding to a different equivalent local martingale measure, as we shall see in the stochastic volatility example provided at the end of the paper. Rather it shows that these ideas are well defined.

Remark 5.7 However, for the Black–Scholes model (2.11) with W being the only source of randomness, $\mathcal{M}ar_{\mathrm{loc}}$ is either empty or is the singleton $\{\mathbb{P}\}$ because $\mathbb{P}$ is the only measure under which X is a (local) martingale. Then Theorem 5.4 yields that there is at most a full strongly regular which is arbitrage-free, and of course there is one, exhibited in (2.13). In view of the PDE satisfied by the function C, it is clear from (2.13) that the conditions of Theorem 3.4 are satisfied with $b = b^i = 0$. More generally, if we start with a stock model which by itself is already complete, then the associated full strongly regular model which is arbitrage-free is unique; in other words, the theory developed in this paper is useless in that case, as we know from the start since the stock model is enough for hedging any claim!

Proof of Theorems 5.1 and 5.4 (a) Suppose that we have a full strongly regular model and that g is such that the process L in (5.1) is not almost surely zero. Then (3.5) holds. If we try to derive (3.2), we get the same formula, for possibly different coefficients A, Γ^i, and Ψ, plus the extra summand L_t.

Now, by Girsanov's theorem, under any locally equivalent measure $\mathbb{Q}$, the process $P(T)$ is the sum of stochastic integrals with respect to W'^i and $\mu - \nu'$ (notation of Lemma 3.3), plus an absolutely continuous drift which may possibly be set to 0 by the right change of measure, plus the continuous increasing process L which, being singular with respect to Lebesgue measure, is always here. Then $P(T)$ cannot be a local martingale under any locally equivalent measure.

(b) We assume that g is C^2 and that (M) holds. Consider $((b^i), Y) \in \Upsilon_{\mathrm{eq}}$ associated with a measure $\mathbb{Q}$ under which X is a local martingale. This amounts to assuming (3.9) for a good version of a, σ^i, and ψ, and under $\mathbb{Q}$, the local martingale X can be written as

$$X_t = X_0 + \sum_{i\in I} \int_0^t \sigma_s^i \, dW_s'^i + \psi \star (\mu - \nu')_t.$$

Then Itô's formula yields that, under $\mathbb{Q}$,

$$g(X_t) = g(X_0) + \int_0^t R_s \, ds + \sum_{i\in I} \int_0^t \widetilde{\sigma}_s^i \, dW_s'^i + \widetilde{\psi} \star (\mu - \nu')_t, \tag{5.2}$$

where

$$R_s = \frac{1}{2} g''(X_s) \sum_{i\in I} \big(\sigma_s^i\big)^2 + \int F(dx) Y(s,x) \overline{\psi}(s,x),$$

which has $R_s \ge 0$ because g is convex. In particular $\int_0^t R_s \, ds < \infty$ a.s. for all t. Then we choose the increasing sequence of stopping times S_n such that, for example, $\int_0^{S_n} R_s \, ds \le n$. We shall prove the existence of a full regular option model associated with (X^{S_n}, g), which is arbitrage-free for each fixed n.

Replace a_s and $\psi(s,x)$ by $a_s 1_{\{s \le S_n\}}$ and $\psi(s,x) 1_{\{s \le S_n\}}$; this does not affect (3.9), and it implies that X is replaced by X^{S_n} and R_s by $R_s 1_{\{s \le S_n\}}$. Therefore it is in fact enough to prove the existence of an arbitrage-free option model associated with (X, g), under the additional assumption that

$$\int_0^t \mathbb{E}_{\mathbb{Q}}(R_s) \, ds < \infty \quad \forall t. \tag{5.3}$$

Under (5.3) it is also possible to set the coefficients a and ψ to be 0 at any time s such that $\mathbb{E}_{\mathbb{Q}}(|R_s| 1_{\{s \le S_n\}}) = \infty$ without altering the model. Therefore we can also suppose that R_s is $\mathbb{Q}$-integrable for all s.

The process $U(s)_t = \mathbb{E}_{\mathbb{Q}}(R_s | \mathcal{F}_t)$ is a $\mathbb{Q}$-martingale. Using the notation of Lemma 3.3(b) and the representation theorem for $\mathbb{Q}$-martingales, we obtain

$$U(s)_t = u(s) + \sum_{i\in I} \int_0^t \gamma^i(u,s) \, dW_u^i + \phi(.,s) * (\mu - \nu')_t, \tag{5.4}$$

with $u(s) = \mathbb{E}_{\mathbb{Q}}(R_s)$ and some predictable processes $\gamma^i(.,s)$ and $\widetilde{\mathcal{P}}$-measurable

$\phi(.,s)$, which are determined, uniquely up to $\widehat{\mathbb{P}}$-nullsets, by the properties

$$\big[U(s), W'^i\big]_t = \int_0^t \gamma^i(u,s)\,du, \qquad \phi(.,s) = \mathbb{M}\big(\Delta U(s) \mid \widetilde{\mathcal{P}}\big), \tag{5.5}$$

where $\Delta U(s)_t(\omega,x) = U(s)_t(\omega) - U(s)_{t-}(\omega)$, and $\mathbb{M}$ denotes the positive $\widetilde{\mathcal{P}}$-σ-finite measure on $(\Omega \times \mathbb{R}_+ \times \mathbb{R}, \mathcal{F} \otimes \mathcal{R}_+ \otimes \mathcal{R})$ given by $\mathbb{M}(d\omega, dt, dx) = \mathbb{Q}(d\omega)\mu(\omega, dt, dx)$. Since $(\omega, s) \mapsto R_s(\omega)$ is $\mathcal{F}\otimes\mathcal{R}_+$-measurable, there is a version of $U(s)_t(\omega)$ which is $\mathcal{F}\otimes\mathcal{R}_+\otimes\mathcal{R}_+$-measurable in (ω, t, s). Hence by (5.5) we can find a version for γ^i and ϕ which has the properties (2) of Definition 2.1, and such that $\gamma^i(u,s) = \phi(u,x,s) = 0$ if $u > s$ because $U(s)_t = U(s)_s$ if $t \geq s$. Moreover, since the right side of (5.4) makes sense, for all t and s, we have $\mathbb{P}$-almost surely

$$C(s)_t := \int_0^t \Bigg(\sum_{i\in I} \big|\gamma^i(u,s)\big|^2 + \int \Big(Y(u,s)\big(\phi(u,x,s)^2 \wedge \big|\phi(u,x,s)\big|\big)\Big) F(dx)\Bigg) du \\ < \infty.$$

Then we use the elementary estimates

$$\begin{aligned} &\phi^2 \wedge 1 \leq 2Y\big(\phi^2 \wedge |\phi|\big) + 4\big((Y-1)^2 \wedge |Y-1|\big), \\ &\big|\phi(Y - 1_{\{|\phi|\leq 1\}})\big| \leq Y\big(\phi^2 \wedge |\phi|\big) + \big((Y-1)^2 \wedge |Y-1|\big) \\ &\qquad + \sqrt{\big(\phi^2 \wedge 1\big)\big((Y-1)^2 \wedge |Y-1|\big)} \end{aligned}$$

plus Cauchy–Schwarz to get, with D_t denoting the (finite) left side of (3.7),

$$\int_0^t \Bigg(\sum_{i\in I} \big|\gamma^i(u,T)\big|^2 + \int \big(\phi(u,x,T)^2 \wedge 1\big) F(dx)\Bigg) du \leq 2C(T)_t + 4D_t, \tag{5.6}$$

$$\int_0^t \Bigg(\sum_{i\in I} \big|\gamma^i(u,T) b^i_u\big| + \int \big|\phi(u,x,T)\big(Y(u,x) - 1_{\{|\phi(u,x,T)|\leq 1\}}\big)\big| F(dx)\Bigg) du \\ \leq \sqrt{C(T)_t\big(2C(T)_t + 4D_t\big)} + C(T)_t + D_t. \tag{5.7}$$

Equation (5.7) allows to define the coefficient α in such a way that (3.11) holds $\widehat{\mathbb{P}}$-a.e., and this coefficient satisfies the same measurability property as the γ^i, and by using also (5.6) we get that the processes $\chi(T)$ of (2.6) are finite-valued.

At this stage we set $f(t,s) = U(s)_t$, and we define $P(T)_t$ by (2.3), so in particular $f(0,s) = u(s)$. Since $R_s \geq 0$ and (5.3) holds, we clearly have the properties (1), (4), (5) of Definition 2.1, and (2) and (3) were proved above. It remains to show that f is given by (2.5) and that $P(T)$ is a $\mathbb{Q}$-local martingale for all T; we then have an option model associated with (X, g) which will necessarily be regular because $\chi(T)_T < \infty$, as seen above, and for which $\mathbb{Q} \in \mathcal{M}ar_{\mathrm{loc}}$.

That (2.5) holds follows from a simple calculation based on (5.4) and $f(0,s)=u(s)$ and on the form of α and the fact that $W_t^{\prime i}=W_t^i-\int_0^t b_s^i\,ds$ and $\nu'=Y\cdot\nu$. For the local martingale property, one could think that, since (3.10) holds by construction of f (we have $f(s,s)=R_s$), it follows from Theorem 3.4. This is not correct, because we do not know that the model is strongly regular (although we suspect it is!). However, we can use (3.6) and perform a calculation similar to the one for getting (3.14), plus use the fact that (3.9) holds, to obtain (5.2). Therefore $P(T)_t$ is the sum of a $\mathbb{Q}$-local martingale, plus the process

$$\int_0^t \big(R_s-f(s,s)\big)\,ds+\int_0^T \big(f(t,s)-f(0,s)\big)\,ds=\mathbb{E}_{\mathbb{Q}}\bigg(\int_0^T R_s\,ds\,\bigg|\,\mathcal{F}_t\bigg)-\int_0^T \mathbb{E}_{\mathbb{Q}}(R_s)\,ds,$$

because $f(t,s)=f(s,s)=R_s$ if $t\geq s$. Due to (5.3), this is a $\mathbb{Q}$-martingale, and the proof of (b) of Theorem 5.1 is finished.

(c) For (c) of Theorem 5.1, we can reproduce the proof of (b), choosing for $\mathbb{Q}$ a measure under which X is a martingale, hence $g(X)$ is a submartingale. Recall that (3.6) under $\mathbb{Q}$ gives

$$g(X_t)=g(X_0)+\int_0^t R_s\,ds+M_t$$

for a $\mathbb{Q}$-local martingale M, and $R_s\geq 0$. This gives the Doob–Meyer decomposition of the $\mathbb{Q}$-submartingale $g(X)$. Since X is a $\mathbb{Q}$-martingale, it is locally of class (D) (for $\mathbb{Q}$) and, since g is of linear growth here, the process $g(X)$ is also locally of class (D). It follows that the increasing process in the Doob–Meyer decomposition is integrable for each time t, that is, we have (5.3) without any stopping at the times S_n as before. The result follows from the end of the proof of (b).

(d) It remains to prove Theorem 5.4. We fix the stock model (2.2). Suppose that we have a locally equivalent measure $\mathbb{Q}$ which is a local martingale measure for our stock model and for two distinct full strongly regular option models associated with (X,g), with g of class C^2; one is the model (2.3)–(2.5), the other one is similar with initial data $f'(0,T)$ and coefficients α', γ'^i, and ϕ'.

The measure $\mathbb{Q}$ is associated with $((b^i),Y)\in\Upsilon_{\mathrm{loc}}$. The left side of (3.10) can be written (with the notation of Lemma 3.3 again) as

$$f(0,s)+\sum_{i\in I}\int_0^t \gamma^i(u,s)\,dW_u^{\prime i}+\phi(.,s)*(\mu-\nu')_s,$$

and we have a similar expression if we use the second option model, whereas the right side of (3.10) depends on X and Y (hence on $\mathbb{Q}$), but not on the option model. Therefore if $f''=f-f'$ and $\gamma''^i=\gamma^i-\gamma'^i$ and $\phi''=\phi-\phi'$, we deduce that for all (ω,s), $s\geq 0$, outside a $\widehat{\mathbb{P}}$-nullset,

$$f''(0,s)+\sum_{i\in I}\int_0^t \gamma''^i(u,s)\,dW_s^{\prime i}+\phi''(.,s)*(\mu-\nu')_s=0. \tag{5.8}$$

Now, as said before, any $\mathbb{Q}$-integrable martingale can be written as the sum of a constant plus stochastic integrals with respect to the W'^i and $\mu - \nu'$, this decomposition is unique, and the integrands are unique as well up to nullsets. Applying this to the "variable" 0, we deduce from (5.8) that $f(0,s) = f'(0,s)$ for almost all s, $\gamma^i = \gamma'^i$ for $\widehat{\mathbb{P}}$-almost all (ω, u, s), and $\phi = \phi'$ for $\widehat{\mathbb{P}}$-almost all (ω, u, x, s). If we finally apply (3.11) for the two option models and use the previous result, we obtain also that $\alpha = \alpha'$ for $\widehat{\mathbb{P}}$-almost all (ω, u, s). Hence the two option models coincide. □

Now let us turn to the completeness of our model. We start with a full strongly regular model with coefficients as above. Similarly to (4.1), for any (ω, s), we denote by $\Upsilon(\omega, s)$ the set of all families $\zeta = ((\beta_i)_{i \in I}, y)$, where $\beta_i \in \mathbb{R}$, and y is a function on $\mathbb{R}$ satisfying the first condition in (4.1), and also (recall (3.1) for $\overline{\psi}$)

$$\left.\begin{array}{l} x \mapsto \psi(\omega, s, x) y(x) \quad \text{and} \quad x \mapsto \overline{\psi}(\omega, s, x) y(x) \quad \text{are } F\text{-integrable,} \\ T \geq s \quad \Rightarrow \quad x \mapsto \phi(\omega, s, x, T) y(x) \quad \text{is } F\text{-integrable.} \end{array}\right\}$$

Then the proof of the following characterization of completeness is the same as for Theorem 4.1.

Therorem 5.8 *Assume that our model is full and strongly regular, that g is C^2, and that the set $\mathcal{M}ar_{\mathrm{loc}}$ is not empty. Then this set is a singleton if and only if, for a good version of the coefficients of the model, we have the following property: The system of linear equations consisting of* (4.2) *and*

$$\int \overline{\psi}(\omega, s, x) y(x)\, F(dx) = 0, \tag{5.9}$$

$$T \geq s \quad \Longrightarrow \quad \sum_{i \in I} \gamma^i(s, T)(\omega) \beta_i + \int \phi(\omega, s, x, T) y(x)\, F(dx) = 0, \tag{5.10}$$

where $((\beta_i), y) \in \Upsilon(\omega, s)$, has the only solutions $\beta_i = 0$ and $y = 0$ up to an $F(dx)$-nullset.

In particular when the model is "continuous," we have, similarly to Theorem 4.3:

Therorem 5.9 *Suppose that we have a full strongly regular model described by* (2.1) *and* (2.4), *with further g being C^2 and $\mathcal{M}ar_{\mathrm{loc}}$ being nonempty. Then there are two alternatives*:

(a) *Either, for a good version of the coefficients, for all $s \geq 0$ and ω, the closed linear subspace of $\ell^2(I)$ spanned by the vectors $(\sigma^i(\omega)_s)_{i \in I}$ and $(\gamma^i(\omega, s, T)_{i \in I}$ for $T \geq s$ is equal to $\ell^2(I)$ itself; in this case the set $\mathcal{M}ar_{\mathrm{loc}}$ is a singleton.*
(b) *Or this property fails, and the set $\mathcal{M}ar_{\mathrm{loc}}$ is infinite.*

In the Lévy driven case, we also have a result similar to Proposition 4.4. Consider again the situation where our full model is driven by the Lévy process Z, that is, the first two equations in (4.5) hold. Then the coefficients a, σ^i, and ψ on the one side, α, γ^i, and ϕ on the other side, are given by (4.7). Then we have:

Proposition 5.10 *In the Lévy case described above, the set $\mathcal{M}ar_{\mathrm{loc}}$ is not a singleton, unless either $C = 0$ and the support of F has at most two points, or $C > 0$ and the support of F has at most one point.*

Proof Equations (4.2), (5.9), and (5.10) amount to the two equations (4.10) and

$$\int \big(g\big(X_{s-} + \delta_s\theta(x)\big) - g(X_{s-}) - g'(X_{s-})\delta_s\theta(x)\big)y(x)F(dx) = 0.$$

The result is thus obvious. □

Hence, as far as completeness is concerned, we are in the same situation as we are for a pure (that is, there are no options being considered) model for two stocks driven by a Lévy process. However, although we cannot prove it, we suspect that in this case the set $\mathcal{M}ar_{\mathrm{loc}}$ usually is empty. To support this, one can look at the proof of Theorem 5.1(b); the key to the proof is to establish the analogue of (3.6), but to achieve a model like (4.5), we need that ϕ factorizes as $\phi(u,x,s) = \rho(u,s)\theta(x)$, and this is of course usually not the case.

On the other hand, Proposition 4.5 holds for full models, with exactly the same statement.

Remark 5.11 A different approach to the completion of Lévy driven markets is that of introducing new securities, based on the sums of the powers of the jumps, such that these sums are convergent (which implies the imposition of some light hypotheses on the existence of moments). For this, see Corcuera et al. [6].

A family of Markov-type stochastic volatility models This example is a typical, and quite general, "Markov-type" stochastic volatility model. Namely, within our general setting (a filtered space endowed with the Wiener processes W^i and the Poisson random measure μ), the dynamics of the stock price is

$$X_t = X_0 + \int_0^t a(X_s, Y_s)\,ds + \int_0^t \sigma(X_s, Y_s)\,dW_s^1. \tag{5.11}$$

Now Y is an auxiliary process, which we take for simplicity to be d-dimensional and locally integrable (more general versions would do as well), and the basic assumption is that the pair (X, Y) is a semimartingale driven by $(W^i)_{i\in I}$ and μ and also a Markov process. This amounts to saying that, in addition to (5.11), we have

$$\begin{aligned} Y_t = Y_0 &+ \int_0^t h(X_s, Y_s)\,ds + \sum_{i\in I}\int_0^t k_i(X_s, Y_s)\,dW_s^i \\ &+ \int_0^t \int_{\mathbb{R}} H(X_s, Y_{s-}, x)(\mu - \nu)(ds, dx) \end{aligned} \tag{5.12}$$

for Borel functions h and k_i on $\mathbb{R} \times \mathbb{R}^d$ and H on $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ (the fact that the pair (X, Y) is driven by the two equations (5.11) and (5.12) is in fact necessary if it is a Markov process *and* a semimartingale with absolutely continuous characteristics, with in addition X continuous; see [5]). We assume that a, σ, h, k_i, and H are nice enough for these equations to have a unique (strong) solution for every (non-random) starting point (X_0, Y_0), and this solution (X, Y) is consequently a Markov process. This covers in particular the usual globally Markov stochastic volatility models, where $\sigma(x, y) = y$, $H = 0$, and $I = \{1, 2\}$, and also cases when the volatility has jumps.

In the classical paradigm, we choose a locally equivalent martingale measure $\mathbb{Q}$ (meaning that X is a martingale, or a local martingale, under $\mathbb{Q}$; no reason of course for Y to be a martingale), and we evaluate the prices $P(T)_t$ of options under $\mathbb{Q}$. These prices may be discontinuous. Among the (many) locally equivalent martingale measures, some preserve the Markov property, namely the "extremal" ones in the convex set of all locally equivalent martingale measures. Let us suppose that it is the case for $\mathbb{Q}$ and denote by (Q_t) the semigroup of the Markov process (X, Y) under $\mathbb{Q}$, and $(\mathcal{L}, \mathcal{D}_{\mathcal{L}})$ its infinitesimal generator. We also assume that $\mathcal{D}_{\mathcal{L}}$ contains all C^2 functions. Recall that, under $\mathbb{Q}$, the pair (X, Y) satisfy (5.11) and (5.12) with the same σ, k_i, and H, but with modified coefficients a and h, other Wiener processes W'^i, and another compensator ν' for μ.

Now, the price $P(T)_t$ (under $\mathbb{Q}$) takes the form

$$P(T)_t = \mathbb{E}_{\mathbb{Q}}\big(g(X_T)\,\big|\,\mathcal{F}_t\big) = Q_{T-t}G(X_t, Y_t)$$

by the Markov property, where $G(x, y) = g(x)$. If g is C^2, hence G as well, we have $G \in \mathcal{D}_{\mathcal{L}}$, and thus we may write

$$P(T)_t = g(X_t) + \int_t^T Q_{s-t}\mathcal{L}G(X_t, Y_t)\,ds.$$

Hence (2.3) holds, with $f(t, s) = Q_{s-t}\mathcal{L}G(X_t, Y_t)$. Note that the process $(f(t, s) : 0 \le t \le s)$ is a $\mathbb{Q}$-martingale and also that it is discontinuous when Y is discontinuous (that is, the process H in (5.11) is not identically 0). If further g and the coefficients driving (X, Y) are smooth enough, the function $Q_{s-t}\mathcal{L}G(x, y)$ is C^2 in (x, y) and C^1 in t over $[0, s]$, so we can apply Itô's formula to obtain (2.5) (relative to W'^i and $\mu - \nu'$) with coefficients of the form

$$\left.\begin{aligned} \alpha(s, T) &= A(T - s, X_s, Y_{s-}),\\ \gamma^i(s, T) &= \Gamma^i(T - s, X_s, Y_{s-}),\\ \phi(s, x, T) &= \Phi(T - s, X_s, Y_{s-}, x) \end{aligned}\right\}$$

for functions A, Γ^i, and Φ which are easily computable in terms of the function $Q_t\mathcal{L}G(x, y)$ and its derivatives in t, x, and y. That is, we have obtained a full model for our option prices, which is necessarily regular and with the corresponding $\mathcal{M}ar_{\mathrm{loc}}$ not empty because it contains at least $\mathbb{Q}$ itself.

In other words, under some regularity conditions, we have at least as many full regular option models associated with (X, g), with no-arbitrage, as there are locally equivalent martingale measures under which (X, Y) is a Markov process with a nice enough semigroup and X is a (local) martingale (and actually many more because there is a full option price model with no-arbitrage for any convex combination of measures $\mathbb{Q}$ as above).

When g is convex but not C^2, then G does not usually belong to the domain of $\mathcal{L}$. However under some ellipticity condition, $Q_\varepsilon G$ does for all $\varepsilon > 0$. Then the same argument shows that $T \mapsto P(t,T)$ is absolutely continuous on (t,∞). Since it is also continuous at $T = t$ (because g is continuous), we still have (2.3) and a full option model, no longer regular.

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