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Quadratic BSDEs driven by a continuous martingale and applications to the utility maximization problem

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Abstract In this paper, we study a class of quadratic backward stochastic differential equations (BSDEs), which arises naturally in the utility maximization problem with portfolio constraints. We first establish the existence and uniqueness of solutions for such BSDEs and then give applications to the utility maximization problem. Three cases of utility functions, the exponential, power, and logarithmic ones, are discussed.

Keywords Backward stochastic differential equations (BSDEs) · Continuous filtration · Quadratic growth · Utility maximization · Portfolio constraints

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JEL Classification C60 · G11

1 Introduction

In this paper, the problem under consideration consists of maximizing the expected utility of the terminal value of a portfolio under constraints. The main objective is to give an expression for the value process of the utility maximization problem with utility function U and liability B, whose expression at time t is

$$V_t^B(x) = \operatorname*{ess\,sup}_{\nu \in \mathcal{A}_t} \mathbb{E}^{\mathcal{F}_t}\left(U\left(X_T^{\nu,t,x} - B\right)\right). \tag{1.1}$$

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In our model, $X_T^{\nu,t,x}$ is the terminal value of the wealth process associated with the strategy ν and equal to x at time t, and the essential supremum is taken over all trading strategies ν defined on $[t,T]$ and lying in an admissibility set denoted by $\mathcal{A}_t$. Since not any $\mathcal{F}_T$-measurable random variable B is replicable by a strategy in $\mathcal{A}_t$, the financial market is incomplete. This problem has further interest due to its connection with utility indifference valuation; in fact, the utility indifference price relates the two value processes V^B and V^0. Introduced by Hodges and Neuberger [12], the utility indifference selling price stands for the amount of money which makes the agent indifferent between selling or not selling the claim B.

Among previous studies of our problem, we refer to [2, 9, 17]. Becherer [2] studies both the utility maximization problem and the notion of utility indifference valuation in a discontinuous setting, whereas Mania and Schweizer [17] consider the same problem in a continuous framework. As in these papers, to solve the problem (1.1) in the case of nonconvex trading constraints, we rely on the dynamic programming methodology and on nonlinear BSDE theory (a major reference illustrating the connection between BSDE theory and finance is [10]). In the existing literature (see, e.g., [3, 18], or [20]), the convex duality method is widely used to study the unconditional case of the problem, but in the aforementioned papers, the authors either suppose that there are no constraints or assume the convexity of the constraint set, which is an assumption we relax here. We rather use the first method to handle dynamically the problem and, for this approach, some major references are [13] and [17]. Our contribution consists in extending the dynamic method to a general continuous setting in the presence of constraints. This requires to establish existence and uniqueness results for solutions to specific quadratic BSDEs and then use these results to characterize both the value process expressed at time t in (1.1) and the strategies attaining the supremum in this last expression.

The paper is structured as follows. Section 2 lays out the financial background and gives some preliminary tools and results about BSDEs. Then, the dynamic programming method is applied to derive an explicit BSDE. Section 3 investigates the existence and uniqueness results for solutions to the introduced BSDEs. In Sect. 4, applications to finance are developed, and the expression of the value process is provided for three types of utility functions. Lengthy proofs are relegated to the Appendix.

2 Statement of the problem and main results

2.1 The model and preliminaries

As usual, we consider a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ equipped with a right-continuous and complete filtration $\mathcal{F} = (\mathcal{F}_t)$ and with a continuous d-dimensional local martingale M. Throughout this paper, all processes are considered on $[0,T]$, T being a deterministic time, and we denote by $Z \cdot M$ the stochastic integral of Z with respect to M. We also assume that $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is a *continuous filtration*; this means that any $\mathbb{R}$-valued (square-integrable) $\mathcal{F}$-martingale K is continuous and can be written as

$$K = Z \cdot M + L$$

with a predictable $\mathbb{R}^d$-valued process Z and a (square-integrable) $\mathbb{R}$-valued martingale L strongly orthogonal to M (i.e., for each i, $\langle M^i, L\rangle = 0$). For a given square-integrable martingale M, the notation $\langle M\rangle$ stands for the quadratic variation process, and the notation $|\cdot|_\infty$ stands for the norm in $L^\infty(\mathcal{F}_T)$ of any bounded $\mathcal{F}_T$-measurable random variable.

From the Kunita–Watanabe inequality it follows that each component $\langle M^i, M^j\rangle$ ($i, j \in \{1, \dots, d\}$) is absolutely continuous with respect to $\tilde{C} = \sum_i \langle M^i\rangle$. Hence, there exists an increasing and bounded process C, for instance, $C_t = \arctan(\tilde{C}_t)$, such that $\langle M\rangle$ can be written

$$d\langle M\rangle_s = m_s m_s' \, dC_s,$$

where m is a predictable process taking values in $\mathbb{R}^{d\times d}$ (this expression has been used in [8] in an analogous continuous framework). The notation m' stands for the transposed matrix, and we also assume that, for any s, the matrix $m_s m_s'$ is invertible $\mathbb{P}$-a.s.

The financial background To bring further motivation, we explain the financial context and for this, we provide here all the definitions and common assumptions. We consider a financial market consisting of $d+1$ assets: one risk-free asset with zero interest rate and d risky assets. We model the price process S of the d risky assets as a process satisfying[1]

$$\frac{dS_s}{S_s} = dM_s + dA_s \quad \text{with } dA_s^j = \sum_{i=1}^d \lambda_s^i \, d\big\langle M^j, M^i\big\rangle_s, \quad j \in \{1, \dots, d\}, \tag{2.1}$$

and with an $\mathbb{R}^d$-valued process λ satisfying

$$(\mathrm{H}_\lambda)\ \exists a_\lambda > 0 \text{ such that } \int_0^T \lambda_s' \, d\langle M\rangle_s \, \lambda_s = \int_0^T |m_s\lambda_s|^2 \, dC_s \le a_\lambda, \ \ \mathbb{P}\text{-a.s.} \tag{2.2}$$

This definition is stronger than the usual structure condition, which only states that $\int_0^T \lambda_s' \, d\langle M\rangle_s \lambda_s < \infty$ $\mathbb{P}$-a.s. (we refer to [1] or [11] for this condition). Condition (2.2) stipulates that S has a bounded mean-variance tradeoff, and in particular, it implies that $\mathcal{E}(-\lambda\cdot M)$ is a strict martingale density for the price process S. In the financial application, we rely on (H_λ) to use the precise a priori estimates given in Lemma 3.1 in Sect. 3. We now state the definition of a wealth process X^ν and of the associated self-financing and constrained trading strategy ν.

Definition 2.1 A predictable $\mathbb{R}^d$-valued process $\nu = (\nu_s)_{s\in[t,T]}$ is called a *self-financing trading strategy* if it satisfies

1. $\nu_s \in \mathcal{C}$ $\mathbb{P}$-a.s. for all s, $\mathcal{C}$ being the constraint set (a closed and not necessarily convex set in $\mathbb{R}^d$).

[1] Both this decomposition already introduced in [7] and the assumption of almost sure invertibility of m_s for all s ensure that the no-arbitrage property holds.

2. The wealth process $X^\nu = X^{\nu,t,x}$ of an agent with strategy ν and wealth x at time t is defined as

$$\forall s \in [t,T], \quad X_s^\nu = x + \int_t^s \sum_{i=1}^d \frac{\nu_r^i}{S_r^i}\, dS_r^i, \tag{2.3}$$

and it is, by assumption on ν, in the space $\mathcal{H}^2$ of semimartingales.

In this definition, each component ν^i of the trading strategy corresponds to the amount of money invested in the ith asset. Due to the presence of portfolio constraints, there does not necessarily exist a strategy ν (such that, for all s, $\nu_s \in \mathcal{C}$) satisfying $X_T^\nu = B$ for a given $\mathcal{F}_T$-contingent claim B. Hence, we are facing an incomplete market. The utility maximization problem aims at giving an expression for the value process defined at any time t by (1.1) and at characterizing the set of optimal strategies, i.e., those achieving the esssup for the problem. In this study, we first consider the exponential utility maximization problem associated with the utility function $U_\alpha(x) = -\exp(-\alpha x)$ with $\alpha > 0$. Usually, the set of admissible trading strategies consists of all strategies such that the wealth process is bounded from below. To solve the problem analogously to [13], we need to enlarge the set of admissible strategies to a new set denoted by $\mathcal{A}_t$.

Definition 2.2 Let $\mathcal{C}$ be the constraint set, which is such that $0 \in \mathcal{C}$. The set $\mathcal{A}_t$ of *admissible strategies* consists of all d-dimensional predictable processes $\nu = (\nu_s)_{s\in[t,T]}$ satisfying $\nu_s \in \mathcal{C}$ $\mathbb{P}$-a.s. for all s, $\mathbb{E}(\int_t^T |m_s \nu_s|^2\, dC_s) < \infty$, and the uniform integrability of the family

$$\{\exp(-\alpha X_\tau^\nu) : \tau \text{ is an } \mathcal{F}\text{-stopping time taking values in } [t,T]\}.$$

This appears to be a restrictive condition on strategies, and it implies that we have to justify the existence of one optimal strategy admissible in this sense.

Preliminaries on quadratic BSDEs In the sequel, we consider one-dimensional BSDEs of the form

$$Y_t = B + \int_t^T F(s, Y_s, Z_s)\, dC_s + \frac{\beta}{2}\big(\langle L\rangle_T - \langle L\rangle_t\big) - \int_t^T Z_s\, dM_s - \int_t^T dL_s. \tag{2.4}$$

To refer to this BSDE, we use the notation BSDE(F, β, B). Usually, a BSDE is characterized by two parameters: its terminal condition B, assumed here to be bounded, and its generator $F = F(s, y, z)$, a $\mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^d)$-measurable function, continuous with respect to (y, z) ($\mathcal{P}$ denotes the σ-field of all predictable sets of $[0,T] \times \Omega$, and $\mathcal{B}(\mathbb{R})$ the Borel σ-field of $\mathbb{R}$). In our setting, we introduce another parameter β, which is assumed to be constant, and a financial meaning for β is given in the next paragraph. We also impose precise growth conditions on the generator; in particular, we study existence under the assumption of quadratic growth with respect to z.

One essential motivation of this study is that such quadratic BSDEs[2] appear naturally when using the same dynamic method as in [13] to solve the problem (1.1). A solution of the BSDE(F,β,B) is a triple (Y,Z,L) in $S^\infty \times L^2(d\langle M\rangle \otimes d\mathbb{P}) \times \mathcal{M}^2([0,T])$ satisfying (2.4) and such that $\langle L,M\rangle = 0$ and $\int_0^T |F(s,Y_s,Z_s)|\,dC_s < \infty$ $\mathbb{P}$-a.s. The space S^∞ consists of all bounded continuous processes, $L^2(d\langle M\rangle \otimes d\mathbb{P})$ consists of all predictable processes Z such that $\mathbb{E}(\int_0^T |m_s Z_s|^2\,dC_s) < \infty$, and $\mathcal{M}^2([0,T])$ consists of all real square-integrable martingales of the filtration $\mathcal{F}$.

The stochastic exponential of a continuous semimartingale K denoted by $\mathcal{E}(K)$ is the unique process satisfying

$$\mathcal{E}_t(K) = 1 + \int_0^t \mathcal{E}_s(K)\,dK_s.$$

A process L is a BMO martingale if L is an $\mathcal{F}$-martingale and if there exists a constant $c>0$ such that, for any $\mathcal{F}$-stopping time τ,

$$\mathbb{E}^{\mathcal{F}_\tau}\big(\langle L\rangle_T - \langle L\rangle_\tau\big) \le c.$$

The dynamic method In this part, we use the same dynamic method as in [13] to characterize the value process of the optimization problem in terms of the solution of a BSDE with parameters (F^α,β,B). The expressions of F^α and β are obtained below in (2.6) by formal computations (these computations are justified in the last section of this paper).

To this end, we construct, for any strategy ν and fixed t, a process $R^\nu = (R^\nu_s)_{s\ge t}$ such that, for all s, $R^\nu_s = U_\alpha(X^\nu_s - Y_s)$ and such that the process Y solves a BSDE(F^α,β,B) of type (2.4). The terminal condition is the contingent claim B, and the parameters F^α and β have to be determined. Besides, this family (R^ν) is such that

(i) $R^\nu_T = U_\alpha(X^\nu_T - B)$ for any strategy ν.
(ii) $R^\nu_t = U_\alpha(x - Y_t)$ (x is assumed to be a constant[3]).
(iii) R^ν is a supermartingale for any strategy $\nu \in \mathcal{A}_t$ and a martingale for a particular strategy $\nu^* \in \mathcal{A}_t$.

We rely on (2.3) defining X^ν and on Itô's formula to get

$$X^\nu_s - Y_s = (x - Y_t) + \int_t^s (\nu_u - Z_u)\,dM_u - (L_s - L_t) \\ + \int_t^s F^\alpha(u,Z_u)\,dC_u + \frac{\beta}{2}\big(\langle L\rangle_s - \langle L\rangle_t\big) + \int_t^s (m_u\nu_u)'(m_u\lambda_u)\,dC_u.$$

[2]Such BSDEs have been considered in [17], where the authors already deal with the utility maximization problem but, contrary to the present paper, they do not assume the presence of trading constraints.

[3]This dynamic method can be extended to any attainable wealth x, i.e., any $\mathcal{F}_t$-measurable random variable such that $X^\nu_t = x$ for at least one admissible strategy ν defined on $[0,t]$.

Since $R_s^\nu = -\exp(-\alpha(X_s^\nu - Y_s))$ for all s, using the notation $\mathcal{E}_{t,T}(K) = \frac{\mathcal{E}_T(K)}{\mathcal{E}_t(K)}$ for a given local martingale K, we first write

$$\begin{aligned}\exp\Big(-\alpha\int_t^T(\nu_s - Z_s)\,dM_s\Big)\\ = \mathcal{E}_{t,T}\big(-\alpha(\nu - Z)\cdot M\big)\exp\Big(\frac{\alpha^2}{2}\int_t^T\big|m_s(\nu_s - Z_s)\big|^2\,dC_s\Big)\end{aligned}$$

and

$$\exp\big(\alpha(L_T - L_t)\big) = \mathcal{E}_{t,T}(\alpha L)\exp\Big(\frac{\alpha^2}{2}\big(\langle L\rangle_T - \langle L\rangle_t\big)\Big),$$

which leads to the multiplicative decomposition

$$R_s^\nu = -\exp\big(-\alpha(x - Y_t)\big)\mathcal{E}_{t,s}\big(-\alpha(\nu - Z)\cdot M\big)\mathcal{E}_{t,s}(\alpha L)\exp\big(A_s^\nu - A_t^\nu\big). \qquad (2.5)$$

Here, A^ν is such that

$$\begin{aligned}dA_s^\nu = \Big(-\alpha F^\alpha(s, Z_s) - \alpha(m_s\nu_s)'(m_s\lambda_s) + \frac{\alpha^2}{2}\big|m_s(\nu_s - Z_s)\big|^2\Big)dC_s\\ + \Big(\frac{\alpha^2 - \alpha\beta}{2}\Big)d\langle L\rangle_s.\end{aligned}$$

Since M and L are strongly orthogonal, we get

$$\mathcal{E}\big(-\alpha(\nu - Z)\cdot M\big)\mathcal{E}(\alpha L) = \mathcal{E}\big(-\alpha(\nu - Z)\cdot M + \alpha L\big).$$

By (2.5), R^ν is the product of a positive local martingale (as a stochastic exponential of a continuous local martingale) and a finite variation process. The process R^ν being negative, the multiplicative decomposition (2.5) and the increasing property of A^ν for all ν yield the supermartingale property of R^ν, and the process R^{ν^*} is a martingale for ν^* satisfying $dA^{\nu^*} \equiv 0$. These two last conditions on the family (A^ν) holding true for all $\nu \in \mathcal{A}_t$, we get

$$\begin{cases}-\alpha\frac{\beta}{2}\,d\langle L\rangle_s + \frac{\alpha^2}{2}\,d\langle L\rangle_s = 0, \quad \text{hence } \beta = \alpha,\\ -\alpha(F^\alpha(s, Z_s) + (m_s\nu_s)'(m_s\lambda_s)) + \frac{\alpha^2}{2}|m_s(\nu_s - Z_s)|^2 \geq 0.\end{cases}$$

This leads to

$$F^\alpha(s, z) = \inf_{\nu\in\mathcal{C}}\Big(\frac{\alpha}{2}\Big|m_s\Big(\nu - \Big(z + \frac{\lambda_s}{\alpha}\Big)\Big)\Big|^2\Big) - (m_s z)'(m_s\lambda_s) - \frac{1}{2\alpha}|m_s\lambda_s|^2. \qquad (2.6)$$

This method, explained for a fixed time t, relies on the dynamic programming principle and could therefore be extended without any additional difficulty to any $\mathcal{F}$-stopping time τ.

2.2 Statement of the assumptions and main results

Assumptions To study the existence of solutions for the BSDEs(F, β, B) of type (2.4), we assume in the sequel the boundedness of the terminal condition B. Moreover, we suppose that there exists a nonnegative predictable process $\bar{\alpha}$ such that $\int_0^T \bar{\alpha}_s \, dC_s \le a$ for a strictly positive constant a and such that with three strictly positive constants b, γ, and C_1, one of the three following conditions hold:

$$(\mathrm{H}_1) \quad \left|F(s,y,z)\right| \le \bar{\alpha}_s + b\bar{\alpha}_s|y| + \frac{\gamma}{2}|m_s z|^2 \quad \text{with } \gamma \ge |\beta| \text{ and } \gamma \ge b,$$

$$(\mathrm{H}_1') \quad \left|F(s,y,z)\right| \le \bar{\alpha}_s + \frac{\gamma}{2}|m_s z|^2,$$

$$(\mathrm{H}_1'') \quad -C_1\big(\bar{\alpha}_s + |m_s z|\big) \le F(s,y,z) \le \bar{\alpha}_s + \frac{\gamma}{2}|m_s z|^2.$$

Remark 2.3 We give here some comments:

- Assumption (H_1) is more general than the other two, but we only require these two last assumptions to establish the existence result. We first reduce assumption (H_1) to (H_1') by a classical truncation procedure, and we note that the additional assumption in (H_1'') is that the lower bound has at most linear growth in z. This condition has already been used by [4] in the Brownian setting to justify the existence of a minimal solution. We rely on the same construction to prove our existence result.
- The quadratic BSDE of the form (2.4) introduced in Sect. 2.1 has parameters $F = F^{\alpha}$, $\beta = \alpha$, and B (B standing for the liability in the optimization problem (1.1)). In particular, the generator F^{α} given by (2.6) satisfies (H_1). In fact, we have

  $$F^{\alpha}(s,z) \ge -(m_s z)'(m_s\lambda_s) - \frac{1}{2\alpha}|m_s\lambda_s|^2 \ge -|m_s z||m_s\lambda_s| - \frac{1}{2\alpha}|m_s\lambda_s|^2,$$

  which leads to

  $$F^{\alpha}(s,z) \ge -\left(\frac{\alpha}{2}|m_s z|^2 + \frac{1}{\alpha}|m_s\lambda_s|^2\right).$$

  Defining $\bar{\alpha}$, for all s, by $\bar{\alpha}_s = \frac{1}{\alpha}|m_s\lambda_s|^2$, we obtain that $\int_0^T \bar{\alpha}_s \, dC_s \le a$ $\mathbb{P}$-a.s. with the parameter a depending on α and a_λ (this last constant is defined in (2.2)). Since 0 is in $\mathcal{C}$, we get

  $$F^{\alpha}(s,z) \le \frac{\alpha}{2}|m_s z|^2.$$

- Even if we suppose that F is Lipschitz with respect to y and z, we cannot obtain directly an existence and uniqueness result for a BSDE of type (2.4) because of the presence of the additional term involving the quadratic variation process $\langle L \rangle$. This explains the introduction of another type of BSDEs, namely

$$\begin{cases} dU_s = -g(s,U_s,V_s)\,dC_s + V_s\,dM_s + dN_s, \\ U_T = e^{\beta B}. \end{cases} \tag{2.7}$$

In the previous equation, $V \cdot M + N$ stands for the martingale part, and N is an $\mathbb{R}$-valued martingale orthogonal to M (the presence of such a martingale N is required, since M does not enjoy the predictable representation property). In the sequel, we denote it by BSDE$(g, e^{\beta B})$. This second type of BSDE is linked with the BSDE(F, β, B) of type (2.4) by using an exponential change of variable. Indeed, setting $U = e^{\beta Y}$, this leads to

$$g(s, u, v) = \left(\beta u F\left(s, \frac{\ln(u)}{\beta}, \frac{v}{\beta u}\right) - \frac{1}{2u}|m_s v|^2\right)\mathbf{1}_{u>0}.$$

This type of BSDE is simpler, since there is no term involving the quadratic variation process $\langle N \rangle$ in (2.7). Furthermore, these BSDEs having a generator g such that $(s, u, v) \longmapsto g(s, u, v)$ is uniformly Lipschitz with respect to u and v have been studied in [8] in a general continuous setting. Our aim is to establish a one-to-one correspondence between the solutions of the BSDE(F, β, B) of type (2.4) and those of the BSDE$(g, e^{\beta B})$ of type (2.7).

To prove a uniqueness result for solutions of the BSDE(F, β, B) of type (2.4), we impose that there exist two numbers μ and C_2, a nonnegative predictable process θ, and a constant c_θ such that

$$(\mathrm{H}_2) \quad \begin{cases} \forall z \in \mathbb{R}^d, \qquad \forall y^1, y^2 \in \mathbb{R}, \\ (y^1 - y^2)(F(s, y^1, z) - F(s, y^2, z)) \le \mu |y^1 - y^2|^2, \\ \exists \theta \quad \text{such that} \quad \int_0^T |m_s \theta_s|^2 \, dC_s \le c_\theta, \quad \forall y \in \mathbb{R},\ \forall z^1, z^2 \in \mathbb{R}^d, \\ |F(s, y, z^1) - F(s, y, z^2)| \le C_2(m_s\theta_s + |m_s z^1| + |m_s z^2|)|m_s(z^1 - z^2)|. \end{cases}$$

Remark 2.4 The first inequality in assumption (H_2) corresponds to a monotonicity assumption (this assumption is given in [19]). The second assumption on the increments in the variable z is a kind of local Lipschitz condition with respect to z, which is similar to the one in [13]. We check that (H_2) is satisfied by the generator F^α with $C_2 = \frac{\alpha}{2}$, $\theta := 4\frac{|m\lambda|}{\alpha}$, and $\mu = 0$, since F^α is independent of y. Indeed, for any z^1, z^2 in $\mathbb{R}^d$, we argue that the increments of F^α with respect to z satisfy

$$\begin{aligned} &\left|F^\alpha(s, z^1) - F^\alpha(s, z^2)\right| \\ &\quad \le \left|\frac{\alpha}{2}\left(\mathrm{dist}^2\left(m_s\left(z^1 + \frac{\lambda}{\alpha}\right), m_s\mathcal{C}\right) - \mathrm{dist}^2\left(m_s\left(z^2 + \frac{\lambda}{\alpha}\right), m_s\mathcal{C}\right)\right)\right| \\ &\qquad + \left|-(m_s z^1)'(m_s\lambda) + (m_s z^2)'(m_s\lambda)\right| \\ &\quad \le \frac{\alpha}{2}\left|m_s(z^1 - z^2)\right|\left(\left|m_s z^1\right| + \left|m_s z^2\right| + 2\frac{|m_s\lambda|}{\alpha}\right) + \left|m_s(z^1 - z^2)\right||m_s\lambda|. \end{aligned}$$

Main results To obtain existence and uniqueness results for solutions of BSDEs of type (2.4), we establish the same results for BSDEs of type (2.7). We now state the results which are justified in Sect. 3.

Theorem 2.5 (Existence) *Consider the* BSDE(F, β, B) *and assume both that the generator F satisfies* (H_1) *and that the terminal condition B is bounded. Then there exists a solution (Y, Z, L) in $S^\infty \times L^2(d\langle M\rangle \otimes d\mathbb{P}) \times \mathcal{M}^2([0,T])$ of the* BSDE.

Theorem 2.6 (Uniqueness) *For all* BSDEs(F, β, B) *of type* (2.4) *such that the generator F satisfies both* (H_1) *and* (H_2) *and such that the terminal condition is bounded, there exists a unique solution (Y, Z, L) in $S^\infty \times L^2(d\langle M\rangle \otimes d\mathbb{P}) \times \mathcal{M}^2([0,T])$.*

Theorem 2.7 (Comparison) *Consider two* BSDEs *of the form* (2.4) *given by (F^1, β, ξ^1) and (F^2, β, ξ^2), where F^1 and F^2 satisfy* (H_1) *and* (H_2), *and assume furthermore that (Y^1, Z^1, L^1) and (Y^2, Z^2, L^2) are respective solutions of each* BSDE *such that*

$$\big(\xi^1 \le \xi^2 \text{ and } F^1\big(s, Y^1_s, Z^1_s\big) \le F^2\big(s, Y^1_s, Z^1_s\big)\big) \quad \mathbb{P}\text{-a.s. for all } s.$$

Then we have $Y^1_s \le Y^2_s$ $\mathbb{P}$-a.s. for all s.

We only provide proofs for the two first theorems, since, without additional difficulty, we check that the comparison result given in Theorem 2.7 holds. To prove this, we proceed with a linearization of the generator similar to the one in Sect. 3.2; this consists of applying the Itô–Tanaka formula to the adapted and bounded process $\tilde{Y}^{1,2}_{\cdot} = \exp(2\mu C_{\cdot})|(Y^1_{\cdot} - Y^2_{\cdot})^+|^2$ and rewriting the same proof.

3 Results about quadratic BSDEs

3.1 A priori estimates

In this part, we obtain precise a priori estimates for solutions of BSDEs of type (2.4). Referring to previous studies on quadratic BSDEs (such as in [4] or [16]), these estimates are the starting point of the proof of the main existence result.

To prove these estimates, we assume the existence of a solution (Y, Z, L) of the BSDE(F, β, B) such that F satisfies (H_1) and we proceed analogously to [4]. However, since those authors work with a Brownian filtration, we have to generalize their method to our setting.

Lemma 3.1 *Consider a* BSDE *of type* (2.4) *given by (F, β, B) and assume both the boundedness of B and condition* (H_1) *for F. Then we can give explicitly in terms of the parameters γ, a, b given in* (H_1) *and $|B|_\infty$ three constants c, C, C' such that, for any solution (Y, Z, L) in $S^\infty \times L^2(d\langle M\rangle \otimes d\mathbb{P}) \times \mathcal{M}^2([0,T])$,*

(i) *$\mathbb{P}$-a.s. and for all t, $c \le Y_t \le C$,*
(ii) *for any $\mathcal{F}$-stopping time τ, $\mathbb{E}^{\mathcal{F}_\tau}\big(\int_\tau^T |m_s Z_s|^2\, dC_s + \langle L\rangle_T - \langle L\rangle_\tau\big) \le C'$.*

Proof By definition, the solution (Y, Z, L) is in $S^\infty \times L^2(d\langle M\rangle \otimes d\mathbb{P}) \times \mathcal{M}^2([0,T])$, and our first objective is to exhibit explicitly both a lower and an upper bound for Y.

Setting first $\tilde{a} = \frac{e^{ba}-1}{b}$, we aim at proving

$$\forall t, \quad \exp\big(\gamma |Y_t|\big) \le \exp\big(\gamma\big(\tilde{a} + |B|_\infty e^{ba}\big)\big). \tag{3.1}$$

To this end and for fixed t, we introduce $H = (U(s, |Y_s|))$ such $U(t, |Y_t|) = e^{\gamma |Y_t|}$ and, for all $s \ge t$,

$$H_s = U\big(s, |Y_s|\big) = \exp\left(\gamma\left(\frac{\exp(\int_t^s b\bar{\alpha}_u\, dC_u) - 1}{b}\right) + \gamma |Y_s| \exp\left(\int_t^s b\bar{\alpha}_u\, dC_u\right)\right).$$

Applying Itô's formula to the process H, we claim that it is a local submartingale; to this end, we prove that the predictable bounded variation process A in the canonical decomposition of the semimartingale H is increasing. For clarity, we first apply the Itô–Tanaka formula to $|Y|$ to get

$$\begin{aligned} d|Y_s| &= -\operatorname{sign}(Y_s) F(s, Y_s, Z_s)\, dC_s - \operatorname{sign}(Y_s)\frac{\beta}{2}\, d\langle L\rangle_s + d\ell_s \\ &\quad + \operatorname{sign}(Y_s)\big(Z_s\, dM_s + dL_s\big), \end{aligned}$$

ℓ being the local time of Y. Now, Itô's formula yields for A the expression

$$\begin{aligned} &\exp\left(-\int_t^s b\bar{\alpha}_u\, dC_u\right) dA_s \\ &\quad = H_s\left(\gamma\bar{\alpha}_s - \gamma \operatorname{sign}(Y_s) F(s, Y_s, Z_s) + \gamma b\bar{\alpha}_s |Y_s| + \frac{\gamma^2}{2} e^{\int_t^s b\bar{\alpha}_u\, dC_u} |m_s Z_s|^2\right) dC_s \\ &\qquad + H_s \gamma\, d\ell_s + H_s \gamma\left(\left(\frac{\gamma}{2} \exp\left(\int_t^s b\bar{\alpha}_u\, dC_u\right) - \operatorname{sign}(Y_s)\frac{\beta}{2}\right) d\langle L\rangle_s\right). \end{aligned}$$

Using assumption (H_1) and the inequalities $|\beta| \le \alpha$ and $\bar{\alpha} \ge 0$, we get that the process A is increasing. Hence, H is a local submartingale, and there exists an increasing sequence (τ_k) of stopping times converging to T, taking values in $[t, T]$, and such that $(U(s \wedge \tau_k, |Y_{s\wedge\tau_k}|))$ is a submartingale. This entails

$$e^{\gamma |Y_t|} = U\big(t, |Y_t|\big) \le \mathbb{E}\big(U\big(T \wedge \tau_k, |Y_{T\wedge\tau_k}|\big)\big|\mathcal{F}_t\big).$$

Applying the bounded convergence theorem to $(\mathbb{E}(U(T \wedge \tau_k, |Y_{T\wedge\tau_k}|)|\mathcal{F}_t))_k$ and letting k tend to infinity, we obtain

$$e^{\gamma |Y_t|} \le \mathbb{E}\big(U\big(T, |Y_T|\big)\big|\mathcal{F}_t\big),$$

which gives (3.1). Hence, assertion (i) of Lemma 3.1 is satisfied with

$$C = \big(\tilde{a} + |B|_\infty e^{ba}\big) \quad \text{and} \quad c = -\big(\tilde{a} + |B|_\infty e^{ba}\big).$$

To prove assertion (ii), we apply Itô's formula to the bounded process $\tilde{\psi}(Y) = \psi_\gamma(Y + |c|)$ with ψ_γ given by

$$\psi_\gamma(x) = \frac{e^{\gamma x} - 1 - \gamma x}{\gamma^2}.$$

This function satisfies

$$\psi_\gamma'(x) \geq 0 \quad \text{for } x \geq 0 \quad \text{and} \quad -\gamma\psi_\gamma' + \psi_\gamma'' = 1, \tag{3.2}$$

and since c is the lower bound of Y, we have $Y + |c| \geq 0$ $\mathbb{P}$-a.s. We now consider an arbitrary stopping time τ of $(\mathcal{F}_t)_{t\in[0,T]}$. Taking the conditional expectation with respect to $\mathcal{F}_\tau$ in Itô's formula between τ and T, we get

$$\begin{aligned}\tilde{\psi}(Y_\tau) - \mathbb{E}^{\mathcal{F}_\tau}\big(\tilde{\psi}(Y_T)\big) = &-\mathbb{E}^{\mathcal{F}_\tau}\left(\int_\tau^T \tilde{\psi}'(Y_s)\Big(-F(s,Y_s,Z_s)\,dC_s - \frac{\beta}{2}\,d\langle L\rangle_s\Big)\right)\\ &- \mathbb{E}^{\mathcal{F}_\tau}\left(\int_\tau^T \tilde{\psi}'(Y_s)(Z_s\,dM_s + dL_s)\right)\\ &- \mathbb{E}^{\mathcal{F}_\tau}\left(\int_\tau^T \frac{\tilde{\psi}''(Y_s)}{2}\big(|m_sZ_s|^2\,dC_s + d\langle L\rangle_s\big)\right).\end{aligned}$$

Since $Z\cdot M$ and L are square-integrable martingales and $\tilde{\psi}'(Y)$ is a bounded process, the second conditional expectation on the right-hand side vanishes. Using both the upper bound on F in (H_1) and simple computations, we obtain

$$\begin{aligned}\tilde{\psi}(Y_\tau) - \mathbb{E}^{\mathcal{F}_\tau}\big(\tilde{\psi}(Y_T)\big) \leq &\ \mathbb{E}^{\mathcal{F}_\tau}\left(\int_\tau^T \tilde{\psi}'(Y_s)|\bar{\alpha}_s|\big(1 + b|Y|_{\mathcal{S}^\infty}\big)\,dC_s\right)\\ &+ \mathbb{E}^{\mathcal{F}_\tau}\left(\int_\tau^T \Big(\frac{\beta}{2}\tilde{\psi}' - \frac{1}{2}\tilde{\psi}''\Big)(Y_s)\,d\langle L\rangle_s\right)\\ &+ \mathbb{E}^{\mathcal{F}_\tau}\left(\int_\tau^T \Big(\frac{\gamma}{2}\tilde{\psi}' - \frac{1}{2}\tilde{\psi}''\Big)(Y_s)|m_sZ_s|^2\,dC_s\right).\end{aligned}$$

Using the properties of ψ_γ given by (3.2) and the fact that $\gamma \geq |\beta|$, we get

$$\Big(\frac{1}{2}\tilde{\psi}'' - \frac{\beta}{2}\tilde{\psi}'\Big)(Y_s) \geq \Big(\frac{1}{2}\tilde{\psi}'' - \frac{\gamma}{2}\tilde{\psi}'\Big)(Y_s) = \frac{1}{2} \quad \mathbb{P}\text{-a.s. for all } s.$$

Putting the two last terms into the left-hand side of Itô's formula applied to $\tilde{\psi}(Y)$, it follows from the two last inequalities that

$$\begin{aligned}&\frac{1}{2}\mathbb{E}^{\mathcal{F}_\tau}\left(\int_\tau^T |m_sZ_s|^2\,dC_s + \big(\langle L\rangle_T - \langle L\rangle_\tau\big)\right)\\ &\quad\leq \mathbb{E}^{\mathcal{F}_\tau}\left(\int_\tau^T \Big(\frac{1}{2}\tilde{\psi}'' - \frac{\gamma}{2}\tilde{\psi}'\Big)(Y_s)|m_sZ_s|^2\,dC_s + \int_\tau^T \Big(\frac{1}{2}\tilde{\psi}'' - \frac{\beta}{2}\tilde{\psi}'\Big)(Y_s)\,d\langle L\rangle_s\right)\\ &\quad\leq 2\big|\tilde{\psi}(Y)\big|_{\mathcal{S}^\infty} + \big|\tilde{\psi}'(Y)\big|_{\mathcal{S}^\infty}\int_0^T |\bar{\alpha}_s|\big(1 + b|Y|_{\mathcal{S}^\infty}\big)\,dC_s,\end{aligned}$$

and using the integrability assumption on $\bar{\alpha}$ given in Sect. 2.2, we get the existence of a constant C' as in assertion (ii), Lemma 3.1. This constant is independent of the stopping time τ and depends only on the parameters a, b, γ and $|B|_\infty$. □

3.2 The uniqueness result

Proof The key idea of this proof is to proceed by linearization and to justify as in [13] the use of Girsanov's theorem. Let (Y^1, Z^1, L^1) and (Y^2, Z^2, L^2) be two solutions of the BSDE(F, β, B) with F satisfying both (H_1) and (H_2) and B bounded. We define $Y^{1,2}$ by $Y^{1,2} = Y^1 - Y^2$ ($Z^{1,2}$ and $L^{1,2}$ are defined similarly) and consider the nonnegative and bounded semimartingale $\tilde{Y}^{1,2}$ defined by $\tilde{Y}^{1,2}_t = e^{2\mu C_t}|Y^{1,2}_t|^2$. We then use Itô's formula to get

$$d\tilde{Y}^{1,2}_s = 2\mu\tilde{Y}^{1,2}_s\, dC_s + e^{2\mu C_s} 2Y^{1,2}_s\, dY^{1,2}_s + \frac{1}{2}e^{2\mu C_s} 2\, d\langle Y^{1,2}\rangle_s.$$

Since Y^1 and Y^2 are solutions of the BSDE(F, β, B), we have

$$dY^{1,2}_s = -\big(F\big(s, Y^1_s, Z^1_s\big) - F\big(s, Y^2_s, Z^2_s\big)\big)\, dC_s - \frac{\beta}{2}d\big(\langle L^1\rangle_s - \langle L^2\rangle_s\big) + dK_s,$$

where $K = Z^{1,2} \cdot M + L^{1,2}$ stands for the martingale part. Hence, considering Itô's formula between t and an arbitrary $\mathcal{F}$-stopping time $\tau \geq t$, we get

$$\begin{aligned}\tilde{Y}^{1,2}_t - \tilde{Y}^{1,2}_\tau = &-\int_t^\tau 2\mu\tilde{Y}^{1,2}_s\, dC_s \\ &+\int_t^\tau e^{2\mu C_s} 2Y^{1,2}_s\big(F\big(s, Y^1_s, Z^1_s\big) - F\big(s, Y^2_s, Z^2_s\big)\big)\, dC_s \\ &+\int_t^\tau e^{2\mu C_s} 2Y^{1,2}_s \frac{\beta}{2}\, d\langle L^{1,2}, L^1 + L^2\rangle_s \\ &-\int_t^\tau e^{2\mu C_s} 2Y^{1,2}_s\big(Z^{1,2}_s\, dM_s + dL^{1,2}_s\big) \underbrace{-\int_t^\tau e^{2\mu C_s}\frac{1}{2}2\, d\langle Y^{1,2}\rangle_s}_{\leq 0}.\end{aligned}$$

The generator F satisfying (H_2), it follows that

$$2Y^{1,2}_s\big(F\big(s, Y^1_s, Z^1_s\big) - F\big(s, Y^2_s, Z^2_s\big)\big) \leq 2\mu\big|Y^{1,2}_s\big|^2 + 2Y^{1,2}_s(m_s\kappa_s)'\big(m_s Z^{1,2}_s\big),$$

where the $\mathbb{R}^d$-valued process κ is defined by

$$\kappa_s = \begin{cases} \frac{(F(s,Y^2_s,Z^1_s) - F(s,Y^2_s,Z^2_s))(Z^{1,2}_s)}{|m_s Z^{1,2}_s|^2} & \text{if } |m_s Z^{1,2}_s| \neq 0, \\ 0 & \text{otherwise.}\end{cases}$$

We introduce a new process A by

$$A_s = 2Y^{1,2}_s\big(F^1\big(s, Y^1_s, Z^1_s\big) - F^2\big(s, Y^2_s, Z^2_s\big) - \big(2\mu\big|Y^{1,2}_s\big|^2 + 2Y^{1,2}_s(m_s\kappa_s)'\big(m_s Z^{1,2}_s\big)\big)\big).$$

This process being almost surely nonpositive, we obtain

$$\begin{aligned}\tilde{Y}_t^{1,2} - \tilde{Y}_\tau^{1,2} = & \underbrace{\int_t^\tau A_s\, dC_s - \int_t^\tau e^{2\mu C_s} \frac{1}{2} 2\, d\langle Y^{1,2}\rangle_s}_{\leq 0} \\ & + \int_t^\tau 2Y_s^{1,2} e^{2\mu C_s} (m_s \kappa_s)' \big(m_s Z_s^{1,2}\big)\, dC_s \\ & + \int_t^\tau 2Y_s^{1,2} e^{2\mu C_s} \frac{\beta}{2}\, d\langle L^{1,2}, L^1 + L^2\rangle_s \\ & - \int_t^\tau 2e^{2\mu C_s} Y_s^{1,2} Z_s^{1,2}\, dM_s - \int_t^\tau 2e^{2\mu C_s} Y_s^{1,2}\, dL_s^{1,2}.\end{aligned}$$

We then consider the stochastic integrals

$$\tilde{N} = \big(2e^{2\mu C} Y^{1,2} Z^{1,2}\big) \cdot M \quad \text{and} \quad \bar{N} = \kappa \cdot M, \quad \text{on the one hand,}$$

$$\tilde{L} = \big(2Y^{1,2} e^{2\mu C}\big) \cdot L^{1,2} \quad \text{and} \quad \bar{L} = \frac{\beta}{2}(L^1 + L^2), \quad \text{on the other hand.}$$

From (H_2) we deduce

$$\exists C > 0, \quad |m_s \kappa_s| \leq C\big(|m_s \theta_s| + \big|m_s Z_s^1\big| + \big|m_s Z_s^2\big|\big).$$

Using both assertion (ii) in Lemma 3.1 and the assumption on θ given by (H_2), we get that $\kappa \cdot M + \frac{\beta}{2}(L^1 + L^2)$ is a BMO martingale. Hence according to [14], $\mathcal{E}(\kappa \cdot M + \frac{\beta}{2}(L^1 + L^2))$ is a true martingale. Defining $\mathbb{Q}$ by $\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(\kappa \cdot M + \frac{\beta}{2}(L^1 + L^2))$, Girsanov's theorem entails that $K = \tilde{N} + \tilde{L} - \langle \tilde{N} + \tilde{L}, \kappa \cdot M + \frac{\beta}{2}(L^1 + L^2)\rangle$ is a local martingale under $\mathbb{Q}$. This implies the existence of a sequence (τ^k) converging to T such that each τ^k may be assumed greater than t and such that $K_{\cdot \wedge \tau^k}$ is a martingale under $\mathbb{Q}$. Hence, between $t = t \wedge \tau^k$ and τ^k, the adapted process $\tilde{Y}$ satisfies

$$\tilde{Y}_t^{1,2} = \tilde{Y}_{\tau^k}^{1,2} + \int_t^{\tau^k} A_s\, dC_s + (K_{\tau^k} - K_t).$$

Taking the conditional expectation with respect to $\mathcal{F}_t$ under $\mathbb{Q}$ in that last equality and using the martingale property of $K_{\cdot \wedge \tau^k}$, we get

$$\tilde{Y}_t^{1,2} \leq \mathbb{E}^{\mathbb{Q}}\big(\tilde{Y}_{\tau^k}^{1,2} \big| \mathcal{F}_t\big). \tag{3.3}$$

As in the proof of Lemma 3.1, the use of the bounded convergence theorem on the right-hand side of (3.3) entails

$$\tilde{Y}_t^{1,2} \leq \lim_k \mathbb{E}^{\mathbb{Q}}\big(\tilde{Y}_{\tau^k}^{1,\,2} \big| \mathcal{F}_t\big) = 0.$$

Hence, it implies

$$\forall t, \quad \tilde{Y}_t^{1,2} \leq 0 \quad \mathbb{Q}\text{-a.s.} \quad (\text{and } \mathbb{P}\text{-a.s., because of the equivalence of } \mathbb{P} \text{ and } \mathbb{Q}),$$

which ends the proof, $\tilde{Y}^{1,2}$ being a nonnegative adapted process. □

3.3 Existence

3.3.1 Main steps of the proof of Theorem 2.5

In this part, to prove the existence result (Theorem 2.5), we proceed with three main steps.

In a first step, we prove that, to solve a BSDE of type (2.4) under assumption (H_1), it suffices to solve the same BSDE under simpler assumption (H'_1).

In a second step, we introduce an intermediate BSDE of the form (2.7) and establish a one-to-one correspondence between the existence of a solution of a BSDE of the form (2.4) and one of the form (2.7).

The third and last step consists in constructing a solution of the BSDE of the form (2.7) when its generator g satisfies (H'_1) and in establishing a "monotone stability" result analogous to the one given in [16].

Step 1: Truncation in y We rely on the a priori estimates given in Lemma 3.1 to strengthen the assumption on the generator and obtain precise estimates for an intermediate BSDE. More precisely, we show that it is sufficient to study existence under simpler assumption (H'_1) (instead of (H_1)), namely

$$(\mathrm{H}'_1)\ \ \exists\bar{\alpha} \geq 0 \ \text{ such that } \int_0^T \bar{\alpha}_s\, dC_s \leq a \ (a > 0) \ \text{ and } \ \big|F(s, y, z)\big| \leq \bar{\alpha}_s + \frac{\gamma}{2}|m_s z|^2.$$

Assuming that we have a solution of the BSDE(F, β, B) of type (2.4) under assumption (H'_1) on F, we deduce the existence of a solution of this BSDE under (H_1). For this, we define K by $K = |c| + |C|$ with constants c and C given in assertion (i) in Lemma 3.1, and we introduce

$$\begin{cases} dY_s^K = -F^K(s, Y_s^K, Z_s^K)\, dC_s - \frac{\beta}{2}\, d\langle L^K\rangle_s + Z_s^K\, dM_s + dL_s^K, \\ Y_T^K = B, \end{cases}$$

where the generator F^K and the truncation function ρ_K are respectively defined by $F^K(s, y, z) = F(s, \rho_K(y), z)$ and

$$\rho_K(y) = \begin{cases} -K & \text{if } y < -K, \\ y & \text{if } |y| \leq K, \\ K & \text{if } y > K. \end{cases}$$

Hence, we have

$$\forall y \in \mathbb{R},\ z \in \mathbb{R}^d, \quad \big|F^K(s, y, z)\big| \leq \bar{\alpha}_s\big(1 + b\big|\rho_K(y)\big|\big) + \frac{\gamma}{2}|m_s z|^2.$$

Since $|\rho_K(y)| \leq |y|$, F^K again satisfies (H_1) with the same parameters as F. Using Lemma 3.1, K is an upper bound of Y^K in S^∞ for any solution (Y^K, Z^K, L^K) of BSDE(F^K, β, B). Besides, if we replace $\bar{\alpha}$ by $\tilde{\alpha} = \bar{\alpha}(1 + bK)$, F^K satisfies (H'_1). Due to the initial assumption, there exists a solution denoted by (Y^K, Z^K, L^K) of

BSDE(F^K, β, B). Since $|Y^K| \le K$, F^K and F coincide along the trajectories of this solution, and hence, (Y^K, Z^K, L^K) is a solution of BSDE(F, β, B) with F satisfying (H_1).

Step 2: An intermediate BSDE To establish the one-to-one correspondence, we first assume the existence of a solution (Y, Z, L) of BSDE(F, β, B) with F satisfying (H_1') and we set $U = e^{\beta Y}$. Using Itô's formula, we check that U solves a BSDE of the type (2.7) and that the generator g is given by

$$g(s,u,v) = \left(\beta u F\left(s, \frac{\ln(u)}{\beta}, \frac{v}{\beta u}\right) - \frac{1}{2u}|m_s v|^2\right)\mathbf{1}_{u>0}. \tag{3.4}$$

A solution of the BSDE($g, e^{\beta B}$) of type (2.7) is given by the triple (U, V, N) such that $U_s = e^{\beta Y_s}$, $V_s = \beta U_s Z_s$, and $N = \beta U \cdot L$. Our aim is to prove that the converse is true, i.e., if we can solve the BSDE(g, $e^{\beta B}$) of type (2.7) under assumption (H_1') on g, then we obtain a solution of the BSDE(F, β, B) of type (2.4) by setting

$$Y = \frac{\ln(U)}{\beta}, \qquad Z = \frac{V}{\beta U}, \quad \text{and} \quad L = \frac{1}{\beta U} \cdot N. \tag{3.5}$$

To achieve this, we give precise estimates of U in S^∞ for any solution (U, V, N) of the BSDE(g, $e^{\beta B}$) of type (2.7). Due to the singularity of the expression (3.4) of g with respect to u, we first rely on a truncation argument and for this, we introduce a new generator G by

$$G(s,u,v) = \beta\rho_{c^2}(u) F\left(s, \frac{\ln(u \vee c^1)}{\beta}, \frac{v}{\beta(u \vee c^1)}\right) - \frac{1}{2(u \vee c^1)}|m_s v|^2.$$

The two positive constants c^1 and c^2 are defined later, and the function ρ_{c^2} is the same as in the first step. Since F satisfies (H_1') and since $\rho_{c^2}(u) \le c^2$, we obtain that G also satisfies (H_1'). Hence, for any positive constants c^1 and c^2, there exists a solution of the BSDE($G, e^{\beta B}$) of type (2.7). We denote it by $(U^{c^1,c^2}, V^{c^1,c^2}, N^{c^1,c^2})$. Thanks to the estimates

$$\begin{aligned} \left|G(s,u,v)\right| &\le \beta\rho_{c^2}(u)\left(\bar{\alpha}_s + \frac{\gamma|m_s v|^2}{2|\beta c^1|^2}\right) + \frac{|m_s v|^2}{2c^1} \\ &\le \beta\bar{\alpha}_s|u| + \frac{\hat{\gamma}}{2}|m_s v|^2, \quad \text{with } \hat{\gamma} = \frac{\gamma c^2}{|\beta||c^1|^2} + \frac{1}{c^1}, \end{aligned}$$

we obtain that G satisfies (H_1) with parameters a, b, and γ defined by

$$a = \int_0^T |\beta|\bar{\alpha}_s\, dC_s, \qquad b = 1, \qquad \gamma = \hat{\gamma}.$$

Using (i) in Lemma 3.1, the solution $(U^{c^1,c^2}, V^{c^1,c^2}, N^{c^1,c^2})$ satisfies

$$U^{c^1,c^2} \le e^a - 1 + \left|e^{\beta B}\right|_\infty e^a \quad \mathbb{P}\text{-a.s.}$$

Defining c^2 by $c^2 = e^a - 1 + |e^{\beta B}|_\infty e^a$, this provides an upper bound independent of γ. To prove the existence of a strictly positive lower bound, we consider a solution (U, V, N) of the BSDE$(G, e^{\beta B})$ and we introduce the adapted process $\Psi(U)$ for all t by $\Psi(U_t) = e^{-\int_0^t \tilde{\beta}_s \, dC_s} U_t$ (we check that $\tilde{\beta} = |\beta|\bar{\alpha} \operatorname{sign}(U_s)$ satisfies $\int_0^T |\tilde{\beta}_s| \, dC_s \leq a$ $\mathbb{P}$-a.s.). Applying then Itô's formula to $\Psi(U)$ between t and T, we get

$$\begin{aligned}
&\Psi(U_t) - \Psi(U_T) \\
&\quad = \int_t^T \big(e^{-\int_0^s \tilde{\beta}_u \, dC_u} \big(G(s, U_s, V_s) + \tilde{\beta}_s U_s\big)\big) \, dC_s \\
&\qquad - \int_t^T e^{-\int_0^s \tilde{\beta}_u \, dC_u} (V_s \, dM_s + dN_s) \\
&\quad = \int_t^T e^{-\int_0^s \tilde{\beta}_u \, dC_u} A_s \, dC_s - \int_t^T \frac{\gamma}{2} e^{-\int_0^s \tilde{\beta}_u \, dC_u} |m_s V_s|^2 \, dC_s \\
&\qquad - \int_t^T \big(e^{-\int_0^s \tilde{\beta}_u \, dC_u} (V_s \, dM_s + dN_s)\big)
\end{aligned}$$

with the process A such that $A_s = G(s, U_s, V_s) + (\tilde{\beta}_s U_s + \frac{\gamma}{2} |m_s V_s|^2)$, which is almost surely positive. Since $-\frac{\gamma}{2}(V \cdot M)$ is a BMO martingale (thanks to (ii) in Lemma 3.1), we introduce a probability measure by defining $\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(-\frac{\gamma}{2} V \cdot M)$. The Girsanov transform $\tilde{M}$ of M, i.e., $\tilde{M} = M + \frac{\gamma}{2} \langle V \cdot M, M \rangle$, is a local martingale under $\mathbb{Q}$, and it follows that $\Psi(U)$ is the sum of a local martingale (under $\mathbb{Q}$) and an increasing process. Using the standard localization procedure and the boundedness assumption on $\Psi(U)$, we conclude that

$$\Psi(U_t) \geq \mathbb{E}^{\mathbb{Q}}\big(\Psi(U_T) \big| \mathcal{F}_t\big).$$

Hence, $U_t \geq \mathbb{E}^{\mathbb{Q}}\big((\inf U_T) e^{-\int_t^T \tilde{\beta}_s \, dC_s} \big| \mathcal{F}_t\big)$, and if c^1 is defined by $c^1 = e^{-|\beta|(|B|_\infty + a)}$, it is a lower bound of U. For these choices of c^1, c^2, the generator G satisfies (H_1) and, for any solution (U, V, N),

$$c^1 \leq U_s \leq c^2 \quad \mathbb{P}\text{-a.s. for all s.}$$

Since $G(s, U_s, V_s) = g(s, U_s, V_s)$ $\mathbb{P}$-a.s. for all s, (U, V, N) is a solution of the BSDE$(g, e^{\beta B})$. The process U being strictly positive and bounded, we can define (Y, Z, L) by (3.5) and applying Itô's formula to $\frac{\ln(U)}{\beta}$, we check that (Y, Z, L) is a solution of the BSDE(F, β, B).

Step 3: Approximation To prove the existence of a solution to the BSDE(F, β, B) of type (2.4) under (H_1), the above two steps show that it is sufficient to prove the existence to a solution of the BSDE$(g, e^{\beta B})$ of type (2.7). Assuming here that F satisfies (H_1'), Step 2 entails that we only need to prove existence for the second type of BSDE under assumption (H_1') on g. Analogously to [16], we construct an approximating sequence (U^n, V^n, N^n) satisfying

- These triples are solutions of the BSDEs$(g^n, e^{\beta B})$,

- The sequence (g^n) is increasing and converges, $\mathbb{P}$-a.s. for all s, to the function $(y,z) \longmapsto g(s,y,z))$.

From now and for the remainder of Sect. 3.3.1, we impose

$$\text{Assumption 1:} \quad \text{The generator } g \text{ satisfies } (\mathrm{H}_1''). \tag{3.6}$$

We then proceed by defining g^n by inf-convolution, i.e.,

$$g^n(s,u,v) = \operatorname*{ess\,inf}_{u',v' \in \mathbb{Q}^d} \big(g(s,u',v') + n\big|m_s(v-v')\big| + n|u-u'|\big).$$

Each g^n is well defined and globally Lipschitz-continuous, which means that

$$\begin{aligned}&\forall u^1,u^2,v^1,v^2,\\ &\big|g^n\big(s,u^1,v^1\big) - g^n\big(s,u^2,v^2\big)\big| \le n\big(\big|m_s\big(v^1-v^2\big)\big| + \big|u^1-u^2\big|\big).\end{aligned} \tag{3.7}$$

Since (g^n) is increasing and converges, $\mathbb{P}$-a.s. for all s, to $g:(u,v) \longmapsto g(s,u,v)$, which is continuous with respect to (u,v), Dini's theorem implies that the convergence is uniform over compact sets. Besides, using that $g^n \le g$, we obtain

$$\sup_n \big|g^n(s,0,0)\big| \le \bar{\alpha}_s. \tag{3.8}$$

The existence of a unique solution (U^n, V^n, N^n) of the BSDEs given by $(g^n, e^{\beta B})$ in $S^2 \times L^2(d\langle M\rangle \otimes d\mathbb{P}) \times \mathcal{M}^2([0,T])$[4] follows from (3.7) and (3.8) (a detailed proof of this existence result can be found in [8], where it is obtained in a general continuous setting). Furthermore, applying Theorem 2.7 for these BSDEs of type (2.7) and using that $(g^n)_n$ is increasing, we get $U^n \le U^{n+1}$. The following result entails that, for all n, U^n is in S^∞.

Proposition 3.2 *Let (U^n, V^n, N^n) be a solution in $S^2 \times L^2(d\langle M\rangle \otimes d\mathbb{P}) \times \mathcal{M}^2([0,T])$ of a* BSDE *of type* (2.7) *given by the parameters $(g^n, \bar{B})$, a generator g^n, which is L_n-Lipschitz, and a bounded terminal condition $\bar{B}$. Then we have*

$$\begin{aligned}&\exists K(L_n,T) > 0, \forall t,\\ &\big|U_t^n\big|^2 \le K(L_n,T)\mathbb{E}\bigg(|\bar{B}|_\infty^2 + \bigg(\int_t^T \big|g^n(s,0,0)\big|\,dC_s\bigg)^2 \bigg| \mathcal{F}_t\bigg).\end{aligned} \tag{3.9}$$

The proof, relegated to the Appendix, is adapted from the results given in Proposition 2.1 in [5]. Relying on (3.8) and on the assumption on $\bar{\alpha}$, Proposition 3.2 implies that U^n is in S^∞. Furthermore, since each generator g^n satisfies assumption (H_1'') (and hence (H_1) with the same parameters), assertion (i) in Lemma 3.1 ensures that (U^n) is uniformly bounded in S^∞.

[4]The space S^2 consists of all continuous processes U such that $\mathbb{E}\big(\sup_{t\in[0,T]}|U_t|^2\big) < \infty$.

Step 4: Convergence of the approximations To prove the convergence of the solutions of the BSDEs$(g^n, e^{\beta B})$ under Assumption 1 (see (3.6)), we introduce the triple $(\tilde{U}, \tilde{V}, \tilde{N})$ as the limit (in a specific sense) of (U^n, V^n, N^n). (U^n) being increasing, we set $\tilde{U}_s = \lim_n \nearrow (U^n_s)$ $\mathbb{P}$-a.s. for all s. Any generator g^n satisfying (H_1'') and hence (H_1) with the same parameters, estimate (ii) in Lemma 3.1 holds true for each term of $(V^n)_n$ and $(N^n)_n$ (uniformly in n). As bounded sequences in Hilbert spaces, there exist subsequences of (V^n) and (N^n_T) such that $V^n \xrightarrow{w} \tilde{V}$ (in $L^2(d\langle M\rangle \otimes d\mathbb{P})$) and $N^n_T \xrightarrow{w} \tilde{N}_T$ in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$. This implies the weak convergence in $L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ of N^n_t to $\tilde{N}_t$ if we define $\tilde{N}_t$ by $\tilde{N}_t = \mathbb{E}^{\mathcal{F}_t}(\tilde{N}_T)$. However, to justify the passage to the limit in the BSDEs given by $(g^n, e^{\beta B})$, we need the strong convergence of (V^n), possibly along a subsequence, to $\tilde{V}$ in $L^2(d\langle M\rangle \otimes d\mathbb{P})$ (resp. (N^n) to $\tilde{N}$ in $\mathcal{M}^2([0,T])$). We give one essential result (similar to the stability result in [16]) which is the key ingredient in the last step of the proof of Theorem 2.5.

Lemma 3.3 *Let (g^n) and $(\tilde{B}^n)$ be two sequences associated with the* BSDEs$(g^n, \tilde{B}^n)$ *of type* (2.7) *and satisfying*

- *$\mathbb{P}$-a.s. and for all s, $(g^n : (u,v) \longmapsto g^n(s,u,v))$ converges increasingly with respect to n and uniformly on the compact sets of $\mathbb{R} \times \mathbb{R}^d$ to $g : (u,v) \longmapsto g(s,u,v)$ (g is continuous with respect to (u,v)).*
- *For all n, each g^n satisfies (H_1''), with the same parameters as g (independent of n),*
- *$(\tilde{B}^n)$ is a uniformly bounded sequence of $\mathcal{F}_T$-measurable random variables, which converges almost surely to $\tilde{B}$ and increasingly with respect to n.*

If there exist solutions (U^n, V^n, N^n) of the BSDEs *given by $(g^n, \tilde{B}^n)$ such that the sequence $(U^n)_n$ is increasing, then the sequence (U^n, V^n, N^n) converges to $(\tilde{U}, \tilde{V}, \tilde{N})$ in the sense that*

$$\mathbb{E}\Big(\sup_{t\in[0,T]} \big|U^n_t - \tilde{U}_t\big|\Big) \to 0 \quad \text{as } n \to \infty$$

and

$$\mathbb{E}\bigg(\int_0^T \big|m_s(\tilde{V}_s - V^n_s)\big|^2 dC_s + \big|\tilde{N}_T - N^n_T\big|^2\bigg) \to 0 \quad \text{as } n \to \infty,$$

and the triple $(\tilde{U}, \tilde{V}, \tilde{N})$ solves the BSDE$(g, \tilde{B})$ *of type* (2.7).

Remark 3.4 The "stability" result stated in Lemma 3.3 holds also for the solution of the BSDE(F, β, B) of type (2.4) (this results from the correspondence established in the second step).

We relegate to Sect. 3.3.2 the technical point in the proof of Lemma 3.3, i.e., the strong convergence in their respective Hilbert spaces of the sequences (V^n) and (N^n). Assuming this, we prove the existence of a solution for BSDE$(g, \tilde{B})$ by justifying the passage to the limit in the BSDEs$(g^n, \tilde{B}^n)$, i.e.,

$$U^n_t = \tilde{B}^n + \int_t^T g^n\big(s, U^n_s, V^n_s\big)\, dC_s - \int_t^T V^n_s\, dM_s - \big(N^n_T - N^n_t\big).$$

To this end, we check that, $\mathbb{P}$-a.s. for all t,

$$\text{(i)}\quad V^n \to \tilde{V} \quad \big(\text{in } L^2\big(d\langle M\rangle \otimes d\mathbb{P}\big)\big) \quad \text{as } n \to \infty,$$

$$\text{(ii)}\quad N^n \to \tilde{N} \quad \big(\text{in } \mathcal{M}^2\big([0,T]\big)\big) \quad \text{as } n \to \infty,$$

$$\text{(iii)}\quad \mathbb{E}\bigg(\int_0^t \big|g^n\big(s, U_s^n, V_s^n\big) - g(s, \tilde{U}_s, \tilde{V}_s)\big|\, dC_s\bigg) \to 0 \quad \text{as } n \to \infty.$$

Assertions (i) and (ii) are consequences of the strong convergence of the sequences (V^n) (resp. (N^n)) in $L^2(d\langle M\rangle \times d\mathbb{P})$ (resp. in $\mathcal{M}^2([0,T])$). To prove (iii), we justify the convergence in $L^1(ds \otimes d\mathbb{P})$ using the two following results:

- The convergence in $dC_s \otimes d\mathbb{P}$-measure of $(m_s V_s^n)$ and (U_s^n) (at least along suitable subsequences) and the properties of (g^n) which ensure the convergence of $(g^n(s, U_s^n, V_s^n))$ to $g(s, \tilde{U}_s, \tilde{V}_s)$ in $dC_s \otimes d\mathbb{P}$-measure.
- The uniform integrability of the family $(g^n(s, U_s^n, V_s^n))$ resulting from the estimates of g^n given by (H_1') and from the fact that $(|mV^n|^2)$ is a uniformly integrable sequence, since it is strongly convergent in $L^1(dC \times d\mathbb{P})$.

Passing to the limit as n goes to ∞, we get that the triple $(\tilde{U}, \tilde{V}, \tilde{N})$ is a solution of the BSDE$(g, e^{\beta B})$.

To obtain a solution of the BSDE(F, β, B), we rely on the results of the two first steps and we define $(\tilde{Y}, \tilde{Z}, \tilde{L})$ using formula (3.5). □

Now, we relax Assumption 1 given by (3.6), i.e., we proceed with the case where g only satisfies (H_1'). In this case, the lower bound is no more Lipschitz and, for the procedure, we refer once again to [4]. The idea consists in using two successive approximations. For this, we define $(g^{n,p})$ by

$$\begin{aligned} g^{n,p}(s,u,v) = \operatorname*{ess\,inf}_{u',v'}\big(g^+(s,u',v') + n\big|m_s(v-v')\big| + n|u-u'|\big) \\ - \operatorname*{ess\,inf}_{u',v'}\big(g^-(s,u',v') + p\big|m_s(v-v')\big| + p|u-u'|\big), \end{aligned}$$

which is increasing with respect to n and decreasing with respect to p. The entire proof can be rewritten by passing to the limit as n goes to ∞ (p being fixed) and then as p goes to ∞.

3.3.2 Proof of the "stability" result in Lemma 3.3

Following the same method as in [16], we establish the strong convergence of the sequences $(V^n)_n$ and $(N^n)_n$ to $\tilde{V}$ and $\tilde{N}$ (this requires the a priori estimates established in Lemma 3.1 for the solutions of the BSDEs given by $(g^n, \tilde{B}^n)$). We first introduce the nonnegative semimartingale $\Phi_L(U^n - U^p) = (\Phi_L(U^{n,p}))_{n\geq p}$ with Φ_L given by

$$\Phi_L(x) = \frac{e^{Lx} - Lx - 1}{L^2}. \tag{3.10}$$

This function Φ_L satisfies $\Phi_L \geq 0$, $\Phi_L(0) = 0$, $\Phi_L'' - L\Phi_L' = 1$, $\Phi_L'(x) \geq 0$, and $\Phi_L''(x) \geq 1$ if $x \geq 0$. Since $V^{n,p} \cdot M$ and $N^{n,p}$ are square-integrable martingales, their expectations are constant. Applying Itô's formula to $\Phi_L(U^{n,p})$ between 0 and T, we get

$$\begin{aligned}
&\mathbb{E}\Phi_L\big(U_0^{n,p}\big) - \mathbb{E}\Phi_L\big(U_T^{n,p}\big)\\
&\quad = \mathbb{E}\bigg(\int_0^T \big(\Phi'_L\big(U_s^{n,p}\big)\big(g^n\big(s, U_s^n, V_s^n\big) - \big(g^p\big(s, U_s^p, V_s^p\big)\big)\big)\, dC_s\big)\\
&\qquad - \mathbb{E}\bigg(\int_0^T \frac{\Phi''_L}{2}\big(U_s^{n,p}\big)\big|m_s\big(V_s^{n,p}\big)\big|^2\, dC_s\bigg) - \mathbb{E}\bigg(\int_0^T \frac{\Phi''_L}{2}\big(U_s^{n,p}\big)\, d\big\langle N^{n,p}\big\rangle_s\bigg).
\end{aligned}$$

Then, since both g^n and g^p satisfy (H'_1) with the same parameters,

$$\begin{aligned}
&\big|g^n\big(s, U_s^n, V_s^n\big) - g^p\big(s, U_s^p, V_s^p\big)\big|\\
&\quad \le 2\bar{\alpha}_s + \frac{\gamma}{2}\big|m_s\big(V_s^n\big)\big|^2 + \frac{\gamma}{2}\big|m_s\big(V_s^p\big)\big|^2\\
&\quad \le 2\bar{\alpha}_s + \frac{3\gamma}{2}\big(\big|m_s\big(V_s^{n,p}\big)\big|^2 + \big|m_s\big(V_s^p - \tilde{V}_s\big)\big|^2 + \big|m_s\tilde{V}_s\big|^2\big)\\
&\qquad + \gamma\big(\big|m_s\big(V_s^p - \tilde{V}_s\big)\big|^2 + |m_s\tilde{V}_s|^2\big)\\
&\quad \le 2\bar{\alpha}_s + \frac{3\gamma}{2}\big(\big|m_s\big(V_s^{n,p}\big)\big|^2\big) + \frac{5\gamma}{2}\big(\big|m_s\big(V_s^p - \tilde{V}_s\big)\big|^2 + |m_s\tilde{V}_s|^2\big).
\end{aligned}$$

The two last inequalities result from the convexity of $z \mapsto |z|^2$. Using these estimates and transferring both

$$\mathbb{E}\bigg(\int_0^T \frac{\Phi''_L}{2}\big(U_s^{n,p}\big)\big|m_s\big(V_s^{n,p}\big)\big|^2\, dC_s\bigg) \quad \text{and}$$
$$\mathbb{E}\bigg(\int_0^T \Phi'_L\big(U_s^{n,p}\big)\frac{3\gamma}{2}\big|m_s\big(V_s^{n,p}\big)\big|^2\, dC_s\bigg)$$

to the left-hand side of Itô's formula applied to $\Phi_L(U^{n,p})$, we obtain

$$\begin{aligned}
&\mathbb{E}\big(\Phi_L\big(U_0^{n,p}\big)\big) + \frac{1}{2}\,\mathbb{E}\big(\big|N_T^{n,p}\big|^2\big)\\
&\qquad + \mathbb{E}\bigg(\int_0^T \bigg(\frac{\Phi''_L}{2} - \frac{3\gamma}{2}\Phi'_L\bigg)\big(U_s^{n,p}\big)\big|m_s\big(V_s^{n,p}\big)\big|^2\, dC_s\bigg)\\
&\quad \le \mathbb{E}\big(\Phi_L\big(\tilde{B}^n - \tilde{B}^p\big)\big)\\
&\qquad + \mathbb{E}\bigg(\int_0^T \Phi'_L\big(U_s^{n,p}\big)\bigg(2\bar{\alpha}_s + \frac{5\gamma}{2}\big(\big|m_s\big(V_s^p - \tilde{V}_s\big)\big|^2 + |m_s\tilde{V}_s|^2\big)\bigg)\, dC_s\bigg).
\end{aligned} \tag{3.11}$$

Setting $L = 8\gamma$ and using the definition (3.10), we check that

$$\Phi''_L - 8\gamma\Phi'_L = 1, \tag{3.12}$$

which entails, in particular, the positiveness of the last term on the left-hand side of (3.11). Then, thanks to the weak convergence of (V^n) to $\tilde{V}$ (and of (N^n) to $\tilde{N}$) and the convexity of $z \mapsto |z|^2$, we have

$$\liminf_{n\to\infty} \mathbb{E}\biggl(\int_0^T \biggl(\frac{\Phi_L''}{2} - \frac{3\gamma}{2}\Phi_L'\biggr)\bigl(U_s^{n,p}\bigr)\bigl|m_s\bigl(V_s^{n,p}\bigr)\bigr|^2\, dC_s\biggr)$$
$$\geq \mathbb{E}\biggl(\int_0^T \biggl(\frac{\Phi_L''}{2} - \frac{3\gamma}{2}\Phi_L'\biggr)\bigl(\tilde{U}_s - U_s^p\bigr)\bigl(\bigl|m_s\bigl(\tilde{V}_s - V_s^p\bigr)\bigr|^2\bigr)\, dC_s\biggr). \tag{3.13}$$

Similarly, we get

$$\liminf_{n\to\infty} \mathbb{E}\bigl(\bigl|N_T^{n,p}\bigr|^2\bigr) \geq \mathbb{E}\bigl(\bigl|\tilde{N}_T - N_T^p\bigr|^2\bigr). \tag{3.14}$$

Using the almost sure convergence of the increasing sequence (U^n) to $\tilde{U}$, the dominated convergence theorem yields

$$\Phi_L'\bigl(U_s^{n,p}\bigr)\biggl(\frac{5\gamma}{2}\bigl(\bigl|m_s\bigl(\tilde{V}_s - V_s^p\bigr)\bigr|^2 + |m_s \tilde{V}_s|^2\bigr) + 2\bar{\alpha}_s\biggr)$$
$$\leq \Phi_L'\bigl(\tilde{U}_s - U_s^p\bigr)\biggl(\frac{5\gamma}{2}\bigl(\bigl|m_s\bigl(\tilde{V}_s - V_s^p\bigr)\bigr|^2 + |m_s \tilde{V}_s|^2\bigr) + 2\bar{\alpha}_s\biggr),$$

which holds uniformly in n. Besides, the process on the right-hand side of (3.15) is integrable with respect to dC, as a product of a bounded process and a sum of integrable processes. To obtain a lower bound for the left-hand side of inequality (3.11), we use both (3.13) and (3.14). Then, for the right-hand side of (3.11), we rely on (3.15) and on the almost sure and increasing convergence of $(\tilde{B}^n)$ to $\tilde{B}$ to get

$$\mathbb{E}\bigl(\Phi_L\bigl(\tilde{U}_0 - U_0^p\bigr)\bigr) + \frac{1}{2}\mathbb{E}\bigl(\bigl|\tilde{N}_T - N_T^p\bigr|^2\bigr)$$
$$+ \mathbb{E}\biggl(\int_0^T \biggl(\frac{\Phi_L''}{2} - \frac{3\gamma}{2}\Phi_L'\biggr)\bigl(\tilde{U}_s - U_s^p\bigr)\bigl|m_s\bigl(\tilde{V}_s - V_s^p\bigr)\bigr|^2\, dC_s\biggr)$$
$$\leq \mathbb{E}\bigl(\Phi_L(\tilde{B} - \tilde{B}^p)\bigr)$$
$$+ \mathbb{E}\biggl(\int_0^T \Phi_L'\bigl(\tilde{U}_s - U_s^p\bigr)\biggl(\frac{5\gamma}{2}\bigl|m_s\bigl(\tilde{V}_s - V_s^p\bigr)\bigr|^2 + 2\bar{\alpha}_s + \frac{5\gamma}{2}|m_s \tilde{V}_s|^2\biggr)\, dC_s\biggr).$$

Transferring now $\mathbb{E}\bigl(\int_0^T \Phi_L'(\tilde{U}_s - U_s^p)(\frac{5\gamma}{2}|m_s(\tilde{V}_s - V_s^p)|^2)\, dC_s\bigr)$ to the left-hand side of this inequality and using the properties of Φ_L and in particular (3.12), we obtain

$$\mathbb{E}\bigl(\Phi_L\bigl(\tilde{U}_0 - U_0^p\bigr)\bigr) + \frac{1}{2}\mathbb{E}\biggl(\int_0^T \bigl|m_s\bigl(\tilde{V}_s - V_s^p\bigr)\bigr|^2\, dC_s + \bigl|\tilde{N}_T - N_T^p\bigr|^2\biggr)$$
$$\leq \mathbb{E}\biggl(\Phi_L\bigl(\tilde{B} - \tilde{B}^p\bigr) + \int_0^T \Phi_L'\bigl(\tilde{U}_s - U_s^p\bigr)\biggl(2\bar{\alpha}_s + \frac{5\gamma}{2}|m_s \tilde{V}_s|^2\biggr)\, dC_s\biggr).$$

Thanks to the convergence of $(\tilde{U}_s - U_s^p)$ to 0 (holding true $\mathbb{P}$-a.s. for all s) and since $|m\tilde{V}|^2$ and $\bar{\alpha}$ are in $L^1(dC \otimes d\mathbb{P})$, the dominated convergence theorem entails the convergence of the right-hand side to 0. Taking the lim sup over p on the left-hand side yields

$$\limsup_{p\to\infty} \frac{1}{2}\mathbb{E}\left(\int_0^T \left|m_s\left(\tilde{V}_s - V_s^p\right)\right|^2 dC_s + \left|\tilde{N}_T - N_T^p\right|^2\right) \leq 0,$$

which ends the proof. □

4 Applications to finance

In this section, we study the problem (1.1) stated in the introduction for three types of utility functions.

4.1 The case of the exponential utility

Theorem 4.1 (1) *For any fixed t, the value process given at time t by V_t^B can be expressed in terms of the unique solution (Y, Z, L) of a BSDE of type (2.4) given by (F^α, β, B) as*

$$V_t^B(x) = U_\alpha(x - Y_t). \tag{4.1}$$

The constant β corresponds to the risk aversion parameter α, B is the contingent claim, and F^α is the generator given by

$$F^\alpha(s,z) = \inf_{\nu\in\mathcal{C}}\left(\frac{\alpha}{2}\left|m_s\left(\nu - \left(z + \frac{\lambda_s}{\alpha}\right)\right)\right|^2\right) - (m_s z)'(m_s\lambda_s) - \frac{1}{2\alpha}|m_s\lambda_s|^2.$$

(2) *There exists an optimal strategy $\nu^* = (\nu_s^*)_{s\in[t,T]}$ such that $\nu^* \in \mathcal{A}_t$ and satisfying, $\mathbb{P}$-a.s. for all s,*

$$\nu_s^* \in \arg\min_{\nu\in\mathcal{C}}\left|m_s\left(\nu - \left(Z_s + \frac{\lambda_s}{\alpha}\right)\right)\right|^2. \tag{4.2}$$

(3) *Extending the definition of $V_t^B(x)$ to an arbitrary stopping time τ, we set*

$$V_\tau^B(x) = \operatorname*{ess\,sup}_{\nu} \mathbb{E}^{\mathcal{F}_\tau}\left(U_\alpha\left(x + \int_\tau^T \sum_i \nu_u^i \frac{dS_u^i}{S_u^i} - B\right)\right),$$

where, in this expression, all trading strategies ν are defined on $[\tau, T]$. Then, for any τ,

$$V_\tau^B(x) = U_\alpha(x - Y_\tau) = R_\tau^{\nu^*},$$

and we recover the formulation of the dynamic programming principle as

$$\forall \tau, \sigma, \tau \leq \sigma, \mathcal{F}\text{-stopping times}, \quad V_\tau^B(x) = \mathbb{E}^{\mathcal{F}_\tau}\left(V_\sigma^B\left(X_\sigma^{\nu^*,\tau,x}\right)\right). \tag{4.3}$$

Remark 4.2 To give sense to the expression $V_\sigma^B(X_\sigma^{\nu^*,\tau,x})$, we refer to the footnote 3. Indeed, by definition, $X_\sigma^{\nu^*,\tau,x} = x + \int_\tau^\sigma \nu_u^* \frac{dS_u}{S_u}$ is an attainable wealth at time σ when starting from x at time τ.

Proof of Theorem 4.1 To prove (4.1), we rely on the results obtained in Sect. 3 to get the existence of a unique solution (Y, Z, U) to the BSDE(F^α, α, B). Then, using the expression of $R^\nu = U_\alpha(X^\nu - Y)$ obtained in the last paragraph of Sect. 2.1, we write

$$\forall s \in [t, T], \quad R_s^\nu = R_t^\nu \tilde{M}_{t,s}^\nu \exp\bigl(A_s^\nu - A_t^\nu\bigr),$$

with $\tilde{M}_{t,s}^\nu = \mathcal{E}_{t,s}(-\alpha(\nu - Z)\cdot M + \alpha L)$. Since the continuous stochastic exponential is a positive local martingale and since $A^\nu \geq 0$, there exists a sequence of stopping times (τ_n) such that $(R_{\cdot\wedge\tau_n}^\nu)$ are supermartingales (for each ν), which entails

$$\forall s, t \leq s \leq T, \forall A \in \mathcal{F}_t, \quad \mathbb{E}\bigl(R_{s\wedge\tau_n}^\nu \mathbf{1}_A\bigr) \leq \mathbb{E}\bigl(R_{t\wedge\tau_n}^\nu \mathbf{1}_A\bigr).$$

Using the definition of admissibility and the boundedness of Y, we obtain the uniform integrability of $(R_{t\wedge\tau_n}^\nu)$ and $(R_{s\wedge\tau_n}^\nu)$. Passing to the limit, we get $\mathbb{E}(R_s^\nu \mathbf{1}_A) \leq \mathbb{E}(R_t^\nu \mathbf{1}_A)$, which entails the supermartingale property of R^ν, as soon as $\nu \in \mathcal{A}_t$. Both this supermartingale property and the relation $R_t^\nu = U_\alpha(x - Y_t)$ imply

$$V_t^B(x) = \operatorname*{ess\,sup}_{\nu\in\mathcal{A}_t} \mathbb{E}^{\mathcal{F}_t}\bigl(U_\alpha\bigl(X_T^{\nu,x,t} - B\bigr)\bigr) \leq U_\alpha(x - Y_t).$$

Now, to obtain (4.1), we focus on the second point of Theorem 4.1. Since $z \longmapsto F^\alpha(s, z)$ is a continuous functional of z, which tends to $+\infty$ as $|z|$ goes to ∞, the infimum in the expression of F^α is attained. Furthermore, relying on the same selection argument as in Lemma 11 in [13] and thanks to the continuity of the functional and the predictability of the processes λ and Z, there exists a measurable choice of ν_s^* satisfying (4.2), i.e., $A^{\nu^*} \equiv 0$. To check that $\nu^* \in \mathcal{A}_t$, we argue that, by the choice of ν^* given in Theorem 4.1 and since 0 is in $\mathcal{C}$,

$$\forall s \in [0, T], \quad \left| m_s\left(\nu_s^* - \left(Z_s + \frac{\lambda_s}{\alpha}\right)\right)\right| \leq \left| m_s\left(Z_s + \frac{\lambda_s}{\alpha}\right)\right|.$$

Since $|m_s(\nu_s^* - Z_s)| \leq |m_s(\nu_s^* - (Z_s + \frac{\lambda_s}{\alpha}))| + |m_s \frac{\lambda_s}{\alpha}|$, we obtain a control of $|m(\nu^* - Z)|$ depending only on the processes Z and λ. Hence, thanks to Kazamaki's criterion (see [14]), $\mathcal{E}(-\alpha(\nu^* - Z)\cdot M)$ is a true martingale. The process R^{ν^*} such that, for all $s \geq t$,

$$R_s^{\nu^*} = -e^{-\alpha(x - Y_t)} \mathcal{E}_{t,s}\bigl(-\alpha(\nu^* - Z)\cdot M + \alpha L\bigr),$$

is a true martingale, which implies that $\nu^* \in \mathcal{A}_t$ and (4.1).

To recover the dynamic programming principle, we define the $\mathcal{F}_\tau$-measurable random variable $V_\tau^B(x)$ for any $\mathcal{F}$-stopping time τ in the same way as $V_t^B(x)$. The same procedure as the one used to prove (4.1) entails

$$V_\tau^B(x) = U_\alpha(x - Y_\tau) = U_\alpha\bigl(X_\tau^{\nu^*,\tau,x} - Y_\tau\bigr) = R_\tau^{\nu^*}.$$

Applying the optional sampling theorem between τ and σ to the martingale R^{ν^*} defined by $R^{\nu^*} = U_\alpha(X^{\nu^*,\tau,x} - Y)$, we get (4.3). □

4.2 Power and logarithmic utilities

As in [13], we introduce two other types of utility functions. The first one is the power utility defined for all $\gamma \in]0,1[$ by $U_\gamma(x) = \frac{1}{\gamma}x^\gamma$ (γ being fixed, we write U^1 instead of U_γ). The second one is the logarithmic utility given by $U^2(x) = \ln(x)$.

Contrary to the exponential case, we have to impose that the wealth process is positive. We focus our attention to the case where there is no liability any more (i.e., $B \equiv 0$ in the problem (1.1)). In this context, a constrained trading strategy is a d-dimensional process ρ which takes its values in the constraint set $\mathcal{C}$ and such that, for each i, ρ^i stands for the fraction of the wealth invested in stock i. The discounted price process S is again assumed to satisfy (2.1), and we denote by $X^\rho = X^{\rho,t,x}$ the wealth process associated with the strategy ρ and such that $X_t^\rho = x$. Its expression for any $s \in [t,T]$ is

$$X_s^\rho = x + \int_t^s X_u^\rho \rho_u \frac{dS_u}{S_u} = x + \int_t^s X_u^\rho \rho_u \, dM_u + \int_t^s X_u^\rho \rho_u' \, d\langle M\rangle_u \lambda_u.$$

The decomposition of the price process S is the same as in Sect. 2.1, and, in particular, λ is a predictable $\mathbb{R}^d$-valued process. For each case, we give in Sects. 4.2.1 and 4.2.2 a definition of the admissibility set for trading strategies (this set is always denoted by $\mathcal{A}_t$). Denoting by U the utility function, we are going to characterize the value process associated to the utility maximization problem with liability equal to zero; this process is defined at time t by

$$V_t(x) = \operatorname*{ess\,sup}_{\rho \in \mathcal{A}_t} \mathbb{E}^{\mathcal{F}_t}\left(U\left(x + \int_t^T X_u^\rho \rho_u \frac{dS_u}{S_u}\right)\right). \tag{4.4}$$

4.2.1 The power utility case

Definition 4.3 The set of *admissible strategies* $\mathcal{A}_t$ consists of all d-dimensional predictable processes $\rho = (\rho_s)_{s\in[t,T]}$ such that $\rho_s \in \mathcal{C}$ ($\mathbb{P}$-a.s. for all s) and

$$\int_t^T \rho_s' \, d\langle M\rangle_s \rho_s = \int_t^T |m_s \rho_s|^2 \, dC_s < \infty \quad \mathbb{P}\text{- a.s.}$$

This condition entails that the stochastic exponential $\mathcal{E}(\rho \cdot M)$ is a continuous local martingale. We can now solve the problem (4.4) for the power utility function U^1.

Theorem 4.4 *Let V^1 be the value process associated with the problem* (4.4) *and having the utility function $U = U^1$.*

(1) *Its expression is*

$$V_t^1(x) = \frac{x^\gamma}{\gamma} \exp(Y_t).$$

In this expression, (Y, Z, L) *stands for the unique solution of the* $\mathrm{BSDE}(f^1, \frac{1}{2}, 0)$ *of type* (2.4) *given by*

$$Y_t = 0 - \int_t^T f^1(s, Z_s)\, dC_s + \int_t^T \frac{1}{2}\, d\langle L\rangle_s - \int_t^T Z_s\, dM_s - (L_T - L_t),$$

the process L *is a real-valued martingale strongly orthogonal to* M, *and the generator* f^1 *is given by*

$$f^1(s, z) = \inf_{\rho\in\mathcal{C}} \frac{\gamma(1-\gamma)}{2}\left(\left|m_s\left(\rho - \left(\frac{z+\lambda_s}{1-\gamma}\right)\right)\right|^2\right) \\ - \frac{\gamma(1-\gamma)}{2}\left|m_s\left(\frac{z+\lambda_s}{1-\gamma}\right)\right|^2 - \frac{1}{2}|m_s z|^2.$$

(2) *There exists an optimal strategy* ρ_1^* *satisfying*, $\mathbb{P}$-*a.s. for all* s,

$$(\rho_1^*)(s) \in \arg\min_{\rho\in\mathcal{C}} \left|m_s\left(\rho - \left(\frac{Z_s+\lambda_s}{1-\gamma}\right)\right)\right|^2. \tag{4.5}$$

Remark 4.5 The expression of the optimal strategy ρ^* is already known in the Brownian setting and when there are no trading constraints. In that case, the wealth process X^π satisfies

$$dX_s^\pi = rX_s^\pi\, ds + X_s^\pi\big(\sigma_s\pi_s\, dW_s + (\mu - r)\pi_s\, ds\big). \tag{4.6}$$

In the elementary case of constant coefficients in (4.6), the optimal proportion is equal to $\frac{\mu-r}{(1-\gamma)\sigma^2}$ (this result can be found in [6] or also in the seminal paper of Merton [15]). In [21], the author generalizes those previous results assuming that the price process is a geometric Brownian motion and assuming that there exists an additional asset, which is driven by another Brownian motion correlated to the first one. In that case, the explicit formula for the optimal strategy incorporates the effect of the correlation factor.

Proof of Theorem 4.4 We just give a sketch of the proof, which is similar to the one given in the exponential case and relies on the same dynamic method as in [13]. To this end, we define the process R^ρ for all $s \in [t, T]$ by $R_s^\rho = X_s^\rho \exp(Y_s)$. We first write

$$X_s^\rho = x + \int_t^s X_u^\rho \rho_u\, dM_u + \int_t^s X_u^\rho (m_u\rho_u)'(m_u\lambda_u)\, dC_u,$$

and since Y is solution of the $\mathrm{BSDE}(f^1, \frac{1}{2}, 0)$, simple computations lead to

$$R_s^\rho = R_t^\rho \frac{1}{\gamma}\mathcal{E}_{t,s}\big((\gamma\rho + Z)\cdot M + L\big)\exp\big(\tilde{A}_s^\rho - \tilde{A}_t^\rho\big),$$

where the process $\tilde{A}^\rho$ is such that

$$\tilde{A}^\rho_s = \int_0^s \bigg(f^1(u, Z_u) + \frac{1}{2}|m_u Z_u|^2 + \frac{\gamma(\gamma-1)}{2}|m_u \rho_u|^2 + \gamma (m_u\rho_u)'\big(m_u(Z_u+\lambda_u)\big)\bigg)\, dC_u.$$

By the definition of f^1, we check that

- R^ρ is a supermartingale for any $\rho \in \mathcal{A}_t$,
- R^{ρ^*} is a martingale for any strategy ρ_1^* satisfying (4.5), taking into consideration that, for such a strategy, we have $|m\rho_1^*| \leq |m\frac{(Z+\lambda)}{(1-\gamma)}|$.

Besides, we obtain

$$V_t^1(x) = \mathbb{E}^{\mathcal{F}_t}\big(R_T^{\rho_1^*}\big) = R_t^{\rho_1^*} = \frac{x^\gamma}{\gamma}\exp(Y_t). \qquad \square$$

4.2.2 The logarithmic utility case

We again introduce the notion of *admissible strategy* adapted to our problem.

Definition 4.6 The set of *admissible strategies* $\mathcal{A}_t$ consists of all d-dimensional predictable processes ρ such that $\rho_s \in \mathcal{C}$ $\mathbb{P}$-a.s. for all s and such that

$$\mathbb{E}\bigg(\int_t^T \rho_s' \, d\langle M\rangle_s \rho_s\bigg) = \mathbb{E}\bigg(\int_t^T |m_s\rho_s|^2\, dC_s\bigg) < \infty.$$

Theorem 4.7 *Let V^2 be the value process associated with the problem* (4.4) *and having the utility function $U = U^2$.*

(1) *Its expression is $V_t^2(x) = \ln(x) + Y_t$. Here Y stands for the unique solution of the* BSDE($f^2, 0$) *of type* (2.7) *given by*

$$Y_t = 0 - \int_t^T f^2(s)\, dC_s - \int_t^T Z_s\, dM_s - \int_t^T dL_s,$$

and the generator f^2 is

$$f^2(s) = \inf_{\rho\in\mathcal{C}} \frac{1}{2}\big|m_s(\rho - \lambda_s)\big|^2 - \frac{1}{2}|m_s\lambda_s|^2.$$

(2) *There exists an optimal strategy ρ_2^* satisfying* ($\mathbb{P}$*-a.s. for all* s)

$$(\rho_2^*)(s) \in \arg\min_{\rho\in\mathcal{C}} \big|m_s(\rho-\lambda_s)\big|^2. \tag{4.7}$$

Remark 4.8 As in the power utility case, we recover the expression of the optimal proportion in the Brownian setting. Assuming that the coefficients μ, σ, and r are constant, this proportion is equal to $\rho^* \equiv \frac{(\mu-r)}{\sigma^2}$.

Proof of Theorem 4.7 The wealth process X^ρ satisfies again

$$X_s^\rho = x + \int_t^s X_u^\rho \rho_u \, dM_u + \int_t^s X_u^\rho (m_u \rho_u)'(m_u \lambda_u)\, dC_u.$$

Now, using both Itô's formula and the assumption that Y solves a BSDE of type (2.7) yields

$$R_s^\rho = \ln\big(X_s^\rho\big) + Y_s = \ln(x) + Y_t + \int_t^s \big((\rho_u + Z_u)\, dM_u + dL_u\big) + A_2^\rho(s) - A_2^\rho(t),$$

where the process A_2^ρ is such that

$$A_2^\rho(s) = \int_0^s \left(f^2(u) - \frac{1}{2}|m_u \rho_u|^2 + (m_u \rho_u)'(m_u \lambda_u) \right) dC_u.$$

From the definition of f^2 we obtain $A_2^\rho \le 0$ and we deduce that $\ln(X^\rho) + Y$ is a supermartingale for any $\rho \in \mathcal{A}_t$. If, besides, ρ_2^* satisfies (4.7) then $A_2^{\rho_2^*} = 0$, and hence $|m(\rho_2^* - \lambda)| \le |m\lambda|$. Since assumption (H_λ) on λ implies the uniform integrability of $R^{\rho_2^*}$, we can claim that $\ln(X^{\rho_2^*}) + Y$ is a martingale. Such a strategy ρ_2^* is optimal, and applying the optional sampling theorem to $R^{\rho_2^*}$, we get

$$V_t^2(x) = \mathbb{E}^{\mathcal{F}_t}\big(R_T^{\rho_2^*}\big) = R_t^{\rho_2^*} = \ln(x) + Y_t. \qquad \square$$

5 Conclusion

In this paper, we have solved the utility maximization problem by characterizing both the value process and the optimal strategies. The novelty of our study is that we have used a dynamic method in the context of a general (and not necessarily Brownian) continuous filtration and in the presence of portfolio constraints. This last assumption entails that the introduced BSDEs have quadratic growth.

Since we are not in a Brownian setting, the first part of our work consists in justifying new existence and uniqueness results for solutions of a specific type of quadratic BSDEs. This study leads to an expression of the value process in terms of a solution of a BSDE of the previous type. Relying on the dynamic programming principle, we are able to characterize this value process for three cases of utility functions. This type of BSDE has already been studied in a particular case in [17] in connection with the notion of indifference utility price. However, one of the main differences in [17] is that there no constraints are imposed on the portfolio. Furthermore and contrary to our setting, they refer to duality methods. Our study depends heavily on the assumption that the filtration is continuous, and we hope to study the case where jumps are allowed. Another perspective is to study the connection with the problem of utility indifference pricing.

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Appendix: Proof of Proposition 3.2

Contrary to Lemma 3.1, where the process Y is supposed to be in S^∞, in this proposition, the process U^n is only assumed to be in S^2. First, we apply Itô's formula to $(e^{\Gamma C_t}|U_t^n|^2)$, Γ being a nonnegative constant, to get

$$d\big(e^{\Gamma C_t}\big|U_t^n\big|^2\big) = \Gamma e^{\Gamma C_t}\big|U_t^n\big|^2\,dC_t + e^{\Gamma C_t}\big(2U_t^n\,dU_t^n + d\langle U^n\rangle_t\big) \tag{A.1}$$

with

$$2U_t^n\,dU_t^n + d\langle U^n\rangle_t = -2U_t^n g^n\big(t, U_t^n, V_t^n\big)\,dC_t + \big|m_t V_t^n\big|^2\,dC_t + d\langle N^n\rangle_t + 2U_t^n\big(V_t^n\,dM_t + dN_t^n\big).$$

Since (U^n, V^n, N^n) is in $S^2 \times L^2(d\langle M\rangle \times d\mathbb{P}) \times \mathcal{M}^2([0,T])$, it follows that the process K defined by

$$\forall s \in [0,T], \quad K_s = \int_0^s 2e^{\Gamma C_u} U_u^n\big(V_u^n\,dM_u + dN_u^n\big),$$

is a true martingale. We now fix $t \in [0,T]$ and we rewrite Itô's formula (A.1) between s and T as

$$e^{\Gamma C_s}\big|U_s^n\big|^2 - e^{\Gamma C_T}\big|U_T^n\big|^2 = \int_s^T e^{\Gamma C_u} U_u^n\big(-\Gamma U_u^n + 2g^n\big(u, U_u^n, V_u^n\big)\big)\,dC_u - \int_s^T e^{\Gamma C_u}\big(\big|m_u V_u^n\big|^2\,dC_u + d\langle N^n\rangle_u\big) - (K_T - K_s).$$

Relying on the Lipschitz property of the generator g^n, we get

$$2\big|U_u^n\big|\big|g^n\big(u, U_u^n, V_u^n\big)\big| \le 2\big|U_u^n\big|\big|g^n(u,0,0)\big| + 2L_n\big(\big|U_u^n\big|^2 + \big|U_u^n\big|\big|m_u V_u^n\big|\big),$$

and using the inequality $|2L_n ab| \le (2(L_n)^2 a^2 + \frac{1}{2}b^2)$, we obtain

$$2L_n\big|U_u^n\big|\big|m_u V_u^n\big| \le 2(L_n)^2\big|U_u^n\big|^2 + \frac{1}{2}\big|m_u V_u^n\big|^2.$$

Combining these two last inequalities, setting $\Gamma = 2((L_n)^2 + L_n)$, and taking the conditional expectation with respect to $\mathcal{F}_t$ in Itô's formula applied to $e^{\Gamma C_s}|U_s^n|^2$ between t and T yields

$$\begin{aligned} e^{\Gamma}C_t\big|U_t^n\big|^2 &\le \mathbb{E}\big(e^{\Gamma C_T}\big|U_T^n\big|^2\big|\mathcal{F}_t\big) \\ &\quad + \mathbb{E}\bigg(\int_t^T e^{\Gamma C_u}\Big(2\big|U_u^n\big|\big|g^n(u,0,0)\big| + \frac{1}{2}\big(\big|m_u V_u^n\big|^2\big)\Big)\,dC_u\bigg|\mathcal{F}_t\bigg) \\ &\quad - \mathbb{E}\bigg(\int_t^T e^{\Gamma C_u}\big(\big|m_u V_u^n\big|^2\,dC_u + d\langle N^n\rangle_u\big)\bigg|\mathcal{F}_t\bigg). \end{aligned}$$

This leads to

$$\mathbb{E}\left(\int_t^T e^{\Gamma C_u}\left(\left|m_u V_u^n\right|^2 dC_u + d\langle N\rangle_u\right)\middle|\mathcal{F}_t\right) \\ \le 2\mathbb{E}\left(e^{\Gamma C_T}\left|U_T^n\right|^2 + 2\int_t^T e^{\Gamma C_u}\left|U_u^n\right|\left|g^n(u,0,0)\right| dC_u\middle|\mathcal{F}_t\right). \quad \text{(A.2)}$$

We come back to Itô's formula (A.1) for the process $e^{\Gamma C_\cdot}|U_\cdot^n|^2$ between s and T. Taking the supremum over $s \in [t,T]$, it follows that

$$\sup_{t\le s\le T} e^{\Gamma C_s}\left|U_s^n\right|^2 \le e^{\Gamma C_T}\left|U_T^n\right|^2 + 2\int_t^T e^{\Gamma C_u}\left|U_u^n\right|\left|g^n(u,0,0)\right| dC_u \\ + \sup_{t\le s\le T}|K_T - K_s|.$$

Applying the Burkholder–Davis–Gundy inequality to the supremum of the square-integrable martingale K and using the relation $Cab \le \frac{C^2}{2}a^2 + \frac{1}{2}b^2$, we deduce the existence of a constant C such that

$$\mathbb{E}\left(\sup_{t\le s\le T} e^{\Gamma C_s}\left|U_s^n\right|^2\middle|\mathcal{F}_t\right) \\ \le \mathbb{E}\left(e^{\Gamma C_T}\left|U_T^n\right|^2 + 2\int_t^T e^{\Gamma C_u}\left|U_u^n\right|\left|g^n(u,0,0)\right| dC_u\middle|\mathcal{F}_t\right) \\ + \frac{C^2}{2}\mathbb{E}\left(\int_t^T e^{\Gamma C_u}\left(\left|m_u V_u^n\right|^2 dC_u + d\langle N\rangle_u\right)\middle|\mathcal{F}_t\right) \\ + \frac{1}{2}\mathbb{E}\left(\sup_{t\le s\le T} e^{\Gamma C_s}\left|U_s^n\right|^2\middle|\mathcal{F}_t\right),$$

where the constant C is generic and may vary from line to line. Combining this last inequality with (A.2), we deduce

$$\mathbb{E}\left(\sup_{t\le s\le T} e^{\Gamma C_s}\left|U_s^n\right|^2 + \int_t^T e^{\Gamma C_u}\left(\left|m_u V_u^n\right|^2 dC_u + d\langle N\rangle_u\right)\middle|\mathcal{F}_t\right) \\ \le C\mathbb{E}\left(e^{\Gamma C_T}\left|U_T^n\right|^2 + \int_t^T e^{\Gamma C_u}\left|U_u^n\right|\left|g^n(u,0,0)\right| dC_u\middle|\mathcal{F}_t\right).$$

To obtain the desired relation, we use the last estimate of the last term on the right-hand side of the previous inequality,

$$C\mathbb{E}\left(\int_t^T e^{\Gamma C_u}\left|U_u^n\right|\left|g^n(u,0,0)\right| dC_u\middle|\mathcal{F}_t\right) \\ \le \frac{1}{2}\mathbb{E}\left(\sup_{t\le u\le T} e^{\Gamma C_u}\left|U_u^n\right|^2\middle|\mathcal{F}_t\right) + \frac{C^2}{2}\mathbb{E}\left(\left(\int_t^T e^{\frac{\Gamma}{2}C_u}\left|g^n(u,0,0)\right| dC_u\right)^2\middle|\mathcal{F}_t\right).$$

We can now claim that relation (3.9) given in Proposition 3.2 holds true, using that

$$e^{\Gamma C_t}\left|U_t^n\right|^2 \le \mathbb{E}\Big(\sup_{t\le u\le T} e^{\Gamma C_u}\left|U_u^n\right|^2\Big|\mathcal{F}_t\Big).$$

To deduce the boundedness of U^n in S^∞, we use the two following properties. On the one hand, $|g^n(u,0,0)| \le \bar{\alpha}_u$ with the process $\bar{\alpha}$ satisfying $\int_0^T \bar{\alpha}_s\,dC_s \le a < \infty$ $\mathbb{P}$-a.s., and, on the other hand and for all n, the random variable $U_T^n = e^{\beta B}$ is bounded. □

References

1. Ansel, J.-P., Stricker, C.: Lois de martingale, densités et décomposition de Föllmer–Schweizer. Ann. Inst. H. Poincaré Probab. Stat. **28**, 375–392 (1992)
2. Becherer, D.: Bounded solutions to backward SDEs with jumps for utility optimization and indifference hedging. Ann. Appl. Probab. **16**, 2027–2054 (2006)
3. Biagini, S., Frittelli, M.: Utility maximization in incomplete markets for unbounded processes. Finance Stoch. **9**, 493–517 (2005)
4. Briand, P., Hu, Y.: BSDE with quadratic growth and unbounded terminal value. Probab. Theory Relat. Fields **136**, 604–618 (2006)
5. Briand, P., Coquet, F., Hu, Y., Mémin, J., Peng, S.: A converse comparison theorem for BSDEs and related properties of g-expectation. Electron. Commun. Probab. **5**, 101–117 (2000)
6. Cox, J.C., Huang, C.-F.: Optimal consumption and portfolio policies when asset prices follow a diffusion process. J. Econ. Theory **49**, 33–83 (1989)
7. Delbaen, F., Schachermayer, W.: The existence of absolutely continuous local martingale measures. Ann. Appl. Probab. **5**, 926–945 (1995)
8. El Karoui, N., Huang, S.-J.: A general result of existence and uniqueness of backward stochastic differential equations. In: El Karoui, N., Mazliak, L. (eds.) Backward Stochastic Differential Equations. Pitman Res. Notes Math. Ser., vol. 364, pp. 27–36. Longman, Harlow (1997)
9. El Karoui, N., Rouge, R.: Pricing via utility maximization and entropy. Math. Finance **10**, 259–276 (2000)
10. El Karoui, N., Peng, S., Quenez, M.C.: Backward stochastic differential equations in finance. Math. Finance **7**, 1–71 (1997)
11. Heath, D., Platen, E., Schweizer, M.: A comparison of two quadratic approaches to hedging in incomplete markets. Math. Finance **11**, 385–413 (2001)
12. Hodges, S.D., Neuberger, A.: Optimal replication of contingent claims under transaction costs. Rev. Futures Mark. **8**, 223–239 (1989)
13. Hu, Y., Imkeller, P., Müller, M.: Utility maximization in incomplete markets. Ann. Appl. Probab. **15**, 1691–1712 (2005)
14. Kazamaki, N.: Continuous Exponential Martingales and BMO. Lecture Notes in Math., vol. 1579. Springer, Berlin (1994)
15. Merton, R.C.: Optimum consumption and portfolio rules in a continuous-time model. J. Econ. Theory **3**, 373–413 (1971)
16. Kobylanski, M.: Backward stochastic differential equations and partial differential equations with quadratic growth. Ann. Probab. **28**, 558–602 (2000)
17. Mania, M., Schweizer, M.: Dynamic exponential utility indifference valuation. Ann. Appl. Probab. **15**, 2113–2143 (2005)
18. Owen, M.P.: Utility based optimal hedging in incomplete markets. Ann. Appl. Probab. **12**, 691–709 (2002)
19. Pardoux, É.: BSDEs, weak convergence and homogenization of semilinear PDEs. In: Clarke, F.H., Stern, R.J. (eds.) Nonlinear Analysis, Differential Equations and Control, pp. 503–549. Kluwer Academic, Dordrecht (1999)
20. Schachermayer, W.: Utility Maximization in Incomplete Markets. Lecture Notes in Math., vol. 1856, pp. 255–293. Springer, Berlin (2004)
21. Zariphopoulou, T.: A solution approach to valuation with unhedgeable risk. Finance Stoch. **5**, 61–82 (2001)