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An ODE approach for the expected discounted penalty at ruin in a jump-diffusion model

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Abstract Under the assumption that the asset value follows a phase-type jump-diffusion, we show that the expected discounted penalty satisfies an ODE and obtain a general form for the expected discounted penalty. In particular, if only downward jumps are allowed, we get an explicit formula in terms of the penalty function and jump distribution. On the other hand, if the downward jump distribution is a mixture of exponential distributions (and upward jumps are determined by a general Lévy measure), we obtain closed-form solutions for the expected discounted penalty. As an application, we work out an example in Leland's structural model with jumps. For earlier and related results, see Gerber and Landry [Insur. Math. Econ. 22:263–276, 1998], Hilberink and Rogers [Finance Stoch. 6:237–263, 2002], Asmussen et al. [Stoch. Proc. Appl. 109:79–111, 2004], and Kyprianou and Surya [Finance Stoch. 11:131–152, 2007].

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1 Introduction

In the classical model of ruin theory, the process

$$X_t = x + ct - Z_t, \quad t \geq 0, \tag{1.1}$$

stands for the surplus process of an insurance company. Here, $x > 0$ is the initial surplus, $c > 0$ is the rate at which premiums are received, and $Z = (Z_t; t \geq 0)$ is a compound Poisson process which represents the aggregate claims between time 0 and t. (Note that in the insurance context, X has only downward jumps.) Ruin is the event that $X_t \leq 0$ for some $t \geq 0$. Let τ be the time of ruin and X_τ the negative surplus when ruin occurs. Given a penalty scheme g, Gerber and Shiu [17] considered the expected discounted penalty

$$\Phi(x) = \mathbb{E}_x\big[e^{-r\tau} g(X_\tau)\big]. \tag{1.2}$$

(Here $r \geq 0$ is the risk-free rate, and we use the convention that $e^{-r\cdot(+\infty)} = 0$.) By taking $g \equiv 1$ and $r = 0$, the ruin probability is a special case of (1.2). For a general penalty scheme g, (1.2) represents the amount payable at ruin, and it depends on the deficit at ruin. For more results and related problems, see [1].

Gerber [15] extended the classical model (1.1) by adding an independent diffusion. Then the surplus process takes the form

$$X_t = x + ct + \sigma W_t - Z_t, \quad t \geq 0. \tag{1.3}$$

Here $\sigma > 0$, and $W = (W_t; t \geq 0)$ is a standard Brownian motion independent of Z. In this case, ruin may be caused by oscillation (that is, $X_\tau = 0$) or by a claim (that is, $X_\tau < 0$). Dufresne and Gerber [13] studied the probability of ruin caused by oscillation and the probability of ruin caused by a claim. Moreover, as in [16, 17] considered the expected discounted penalty (1.2). They heuristically derived the integro-differential equation for Φ and showed that Φ satisfies a renewal integral equation. As an application of the renewal equation, they determined the optimal exercise strategy for a perpetual American put option under the assumptions that the log price of the stock is of the form (1.3) and only downward jumps of X are allowed. On the other hand, by a different approach, Mordecki [25] considered optimal exercise strategies and perpetual options for exponential Lévy models. In terms of the supremum and infimum processes of the Lévy process, he derived closed formulas for the optimal exercise strategies and prices of perpetual American call and put options. In particular, if X is the independent difference of a spectrally positive Levy process and a compound Poisson process whose jump distribution is a mixture of exponential distributions, Mordecki [25] gave explicit formulas for optimal exercise strategies and prices of perpetual American put options. (Similar results also hold for call options.) For related results on American option pricing, see also [2, 5, 6, 8, 9], and others.

In addition to option pricing theory, the expected discounted penalty also is of importance in the structural form modeling of credit risk. The structural form modeling includes, based on (exogenous) default caused by an insufficiency of assets relative to liabilities, the classic Black–Scholes–Merton model of corporate debt pricing and Leland's structural model, for which (endogenous) default occurs when the issuer's assets reach a level so small that the issuer finds it optimal to declare bankruptcy. All the aforementioned models of credit risk have relied on diffusion processes to model the evolution of the assets. However, while the diffusion approach is mathematically tractable and has inputs and parameters of the models observable and estimable, it cannot capture the basic features of credit risk observed empirically.

There are many extensions of the Black–Scholes–Merton model and Leland's model. One example is [18] in which the model of Leland [22] or Leland and Toft [23] was extended by adding downward jumps in the dynamics of the firm's assets. If the log value of the assets follows a jump-diffusion as in (1.3) with only downward jumps, Hilberink and Rogers gave explicit formulas for the values of the debt and the value of the firm up to Fourier transforms. Without closed-form solutions for these two values, by imposing smooth-pasting condition, they surprisingly determined the optimal default boundary in terms of the Wiener–Hopf factors of a Lévy process. Also, by numerical inversion for Fourier transforms, they presented some interesting results and discussed their interpretation. For recent works and related results, see [4, 11, 21].

In this paper we study the function Φ defined in (1.2). Here X is a jump-diffusion of the form (1.3), and the jump distribution for X is a two-sided phase-type distribution. The main results of the paper are outlined as follows. We first show that Φ satisfies an integro-differential equation (Theorem 2.4) and then derive an ordinary differential equation for Φ (Theorem 3.5). Based on the ODE, we show in Proposition 3.6 that the function Φ can be written as a linear combination of known exponential functions. Moreover, if only downward jumps are allowed, we calculate any higher-order (right-hand) derivative of Φ at zero in terms of the penalty function g and jump distribution. As a consequence, we obtain an explicit formula for Φ when there are only downward jumps for X (Theorem 3.7). If the downward jumps have a mixed exponential distribution and the upward jumps have a general distribution, we also obtain an explicit formula for Φ in Theorem 4.3. As hints for possible finance and insurance applications of our results, we provide some examples. In particular, in the setup of Leland's model with jumps, we determine the optimal endogenous default and obtain the equity, debt, and firm values in closed-form formulas in Examples 3.9 and 4.6.

The plan of the rest of the paper is as follows. In Sect. 2 we introduce the process X and derive an integro-differential equation for Φ. Section 3 recalls the definition of phase-type distributions and presents our ODE approach for Φ. In Sect. 4 we consider a general Lévy process X which is a difference of a spectrally positive Lévy process and a compound Poisson process with only upward jumps. Moreover, if the jump distribution for the compound Poisson process is a mixture of exponential distributions, based on the results in Sect. 3, we conjecture the solution form of Φ and by using the Feynman-Kac formula we verify our conjecture. Section 5 concludes the paper. Notation, proofs of lemmas, propositions, and technical results are relegated to appendices.

2 Integro-differential equation

To start with, we specify a Lévy process that we consider in this paper unless otherwise stated. We are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which there are a standard Brownian motion $W = (W_t; t \geq 0)$ and a compound Poisson process $Z = (Z_t = \sum_{n=1}^{N_t} Y_n; t \geq 0)$. Here the Poisson process $N = (N_t; t \geq 0)$ has parameter $\lambda > 0$ and the random variables $(Y_n; n \in \mathbb{N})$ are independent and identically distributed. We assume further that the distribution F of Y_1 has a bounded density f that is continuous on $\mathbb{R}\backslash\{0\}$. In addition, W, N, and(Y_n) are assumed to be independent. For every $x \in \mathbb{R}$, let $\mathbb{P}_x$ be the law of the process

$$X_t = X_0 + ct + \sigma W_t - Z_t, \tag{2.1}$$

where $c \in \mathbb{R}$, $\sigma > 0$, and $X_0 = x$. Write $\mathbb{P}_0$ for $\mathbb{P}$ and $\mathbb{E}_x[Z] = \int Z(\omega)\, d\mathbb{P}_x(\omega)$ for a random variable Z. For every $\zeta \in i\mathbb{R}$, we have

$$\mathbb{E}_0\big[e^{\zeta X_1}\big] = e^{\psi(\zeta)}, \tag{2.2}$$

where

$$\psi(\zeta) = D\zeta^2 + c\zeta + \lambda \int e^{-\zeta y}\, dF(y) - \lambda \tag{2.3}$$

and

$$D = \frac{\sigma^2}{2}.$$

(ψ is called the *characteristic exponent of* X.) Moreover, the infinitesimal generator L of X has a domain containing $\mathcal{C}_0^2(\mathbb{R})$ and, for any $h \in \mathcal{C}_0^2(\mathbb{R})$,

$$Lh(x) = Dh''(x) + ch'(x) + \lambda \int h(x-y)\, dF(y) - \lambda h(x). \tag{2.4}$$

(For details, see [3].) On the other hand, let $(\mathcal{F}_t)$ be the usual augmentation of the natural filtration of X. Then for every Borel set A, the entry time of A by X,

$$\tau_A = \inf\{t \geq 0 : X_t \in A\}, \tag{2.5}$$

is an $(\mathcal{F}_t)$-stopping time. Let $\tau = \tau_{(-\infty,0]}$.

From now on, we fix a bounded Borel penalty function g on $(-\infty, 0]$ and let the function Φ be given by

$$\Phi(x) = \mathbb{E}_x\big[e^{-r\tau} g(X_\tau)\big], \quad x \in \mathbb{R}. \tag{2.6}$$

Note that $\Phi(x) = g(x)$ for $x \leq 0$ and, in the insurance literature, Φ is called the *expected discounted penalty* for the penalty function g.

In [16] and [30], the following regularities were used implicitly. For a rigorous proof and related works, see [7, 10].

Theorem 2.1 *The function* Φ *in* (2.6) *has the following properties*:

1. *For all* $r \geq 0$, $\Phi \in \mathcal{C}_b^1(\mathbb{R}_+) \cap \mathcal{C}^2(\mathbb{R}_{++})$.
2. *If* $r > 0$, *or* $r = 0$ *and* $\mathbb{E}[X_1] > 0$, *then* $\Phi \in \mathcal{C}_0^1(\mathbb{R}_+)$.

Lemma 2.2 *The process* $\{e^{-r(\tau\wedge t)}\Phi(X_{\tau\wedge t}); t \geq 0\}$ *is a* $(\mathbb{P}_x, \mathcal{F}_{t\wedge\tau})$*-martingale and, for all* t,

$$\mathbb{E}_x\left[e^{-r\tau}g(X_\tau)\middle|\mathcal{F}_{\tau\wedge t}\right] = e^{-r(t\wedge\tau)}\Phi(X_{\tau\wedge t}).$$

Hence, for any $(\mathcal{F}_{t\wedge\tau})$*-stopping time* η,

$$\mathbb{E}_x\left[e^{-r(t\wedge\eta\wedge\tau)}\Phi(X_{t\wedge\tau\wedge\eta})\right] = \Phi(x).$$

Proof Please refer to Appendix A. □

We define $Lh(x)$ by expression (2.4) for all functions h on $\mathbb{R}$ such that h', h'', and the integral in (2.4) exist at x.

Lemma 2.3 *The function* $L\Phi$ *is in* $\mathcal{C}(\mathbb{R}_{++})$.

Proof Please refer to Appendix A. □

Theorem 2.4 *The function* Φ *satisfies the integro-differential equation*

$$(L - r)\Phi(x) = 0, \quad x > 0. \tag{2.7}$$

If $r > 0$, *or* $r = 0$ *and* $\mathbb{E}[X_1] > 0$, *we have* $\Phi \in \mathcal{C}^2_{0,b}(\mathbb{R}_{++})$.

Proof Assume that $(L - r)\Phi(x) = 0$ for all $x > 0$. Then we have

$$\Phi''(x) = -\frac{c}{D}\Phi'(x) - \frac{\lambda}{D}\int \Phi(x-y)\,dF(y) + \frac{\lambda + r}{D}\Phi(x).$$

The second statement follows from this and from Theorem 2.1.

To establish (2.7), we fix $x > 0$ and let $\epsilon \in (0, x)$. By Theorem 2.1, we have that $\Phi \in \mathcal{C}^1_b(\mathbb{R}_+) \cap \mathcal{C}^2(\mathbb{R}_{++})$. Since the behavior of Φ'' near $0+$ and $+\infty$ is unclear, we stop the process X at the time $\tilde{\tau}$, where $\tilde{\tau}$ is the entry time of $(-\infty, \epsilon] \cup [x+1, \infty)$. Then $\tilde{\tau}$ is an $(\mathcal{F}_t)$-stopping time. Moreover, since $\tilde{\tau} \leq \tau$, we see that $\tilde{\tau} \wedge t$ is an $(\mathcal{F}_{\tau\wedge t})$-stopping time. By Lemma 2.2, we have $\mathbb{E}_x[e^{-r(t\wedge\tilde{\tau})}\Phi(X_{\tilde{\tau}\wedge t})] = \Phi(x)$. In Appendix A, we also prove that

$$\mathbb{E}_x\left[e^{-r(t\wedge\tilde{\tau})}\Phi(X_{\tilde{\tau}\wedge t})\right] = \mathbb{E}_x\left[\int_0^{\tilde{\tau}\wedge t} e^{-ru}(L - r)\Phi(X_u)\,du\right] + \Phi(x). \tag{2.8}$$

From these we get

$$\mathbb{E}_x\left[\int_0^{t\wedge\tilde{\tau}} e^{-ru}(L - r)\Phi(X_u)\,du\right] = 0. \tag{2.9}$$

On the other hand, by Lemma 2.3,

$$\mathbb{E}_x\left[\sup_{u<t\wedge\tilde{\tau}}\left|se^{-ru}(L - r)\Phi(X_u) - (L - r)\Phi(X_0)\right|\right] \to 0, \quad t \downarrow 0+. \tag{2.10}$$

Therefore, by (2.9), we obtain

$$
\begin{aligned}
\big|(L-r)\Phi(x)\big| &= \left|\frac{1}{t}\mathbb{E}_x\left[\int_0^{t\wedge\tilde{\tau}} e^{-ru}(L-r)\Phi(X_u)\,du\right] - (L-r)\Phi(x)\right| \\
&\le \mathbb{E}_x\left[\sup_{u<t\wedge\tilde{\tau}} \big|e^{-ru}(L-r)\Phi(X_u) - (L-r)\Phi(X_0)\big|\right] \\
&\quad + \big|(L-r)\Phi(x)\big|\frac{\mathbb{E}_x[t-\tilde{\tau}\wedge t]}{t}.
\end{aligned}
$$

Note that $\tilde{\tau} > 0$ $\mathbb{P}_x$-a.s. and hence $t - \tilde{\tau}\wedge t = 0$ for all sufficiently small t. Since further $0 \le \frac{t-\tilde{\tau}\wedge t}{t} \le 2$, we have $\mathbb{E}_x[\frac{t-\tilde{\tau}\wedge t}{t}] \to 0$ as $t \to 0+$. Together with (2.10) and the last inequality, we establish that

$$
\big|(L-r)\Phi(x)\big| \le \limsup_{t\to 0+}\left|\frac{1}{t}\mathbb{E}_x\left[\int_0^{t\wedge\tilde{\tau}} e^{-ru}(L-r)\Phi(X_u)\,du\right] - (L-r)\Phi(x)\right| = 0.
$$

The proof is complete. □

3 Explicit formula for Φ: the ODE method

In this section, we consider phase-type jump distributions and the boundary value problem

$$
\begin{cases} (L-r)\Phi = 0 & \text{in } \mathbb{R}_{++}, \\ \Phi = g & \text{on } \mathbb{R}_{-}, \end{cases} \tag{3.1}
$$

where L is defined by (2.4) and $r \ge 0$. Our main purpose is to show that if Φ satisfies (3.1), then it satisfies an ODE on $\mathbb{R}_{++}$. From this we obtain a general form of Φ and derive an explicit formula for Φ under some technical conditions. Based on these results, we consider an example in credit risk modeling (Example 3.9). To begin with, we recall the definition of phase-type distributions.

Definition 3.1 Assume $\boldsymbol{B}$ is an $N \times N$ nonsingular subintensity matrix, that is, $b_{ij} \ge 0$ for $i \ne j$, $b_{ii} \le 0$, and $\boldsymbol{b}^\top = -\boldsymbol{B}\boldsymbol{e}^\top \in \mathbb{R}_+^N \backslash \{\boldsymbol{0}\}$. Here, $\boldsymbol{e} = [1\ 1 \cdots\ 1]$ and $\boldsymbol{0} = [0\ 0 \cdots\ 0]$. Let $\boldsymbol{\alpha}$ be an N-dimensional probability function. The probability distribution function F with the density function

$$
f(x) = \begin{cases} \boldsymbol{\alpha} e^{x\boldsymbol{B}}\boldsymbol{b}^\top, & x > 0, \\ 0, & x \le 0, \end{cases}
$$

is called a *phase-type* distribution. We denote this distribution by $\mathrm{PH}(\boldsymbol{\alpha}, \boldsymbol{B})$. We say that the representation $(\boldsymbol{\alpha}, \boldsymbol{B})$ is *minimal* if there do not exist $N_0 < N$, $\boldsymbol{\alpha}'$ of dimension N_0, and a nonsingular subintensity matrix $\boldsymbol{B}'$ of dimension N_0 such that $f(x) = \boldsymbol{\alpha}' e^{x\boldsymbol{B}'}\boldsymbol{b}'^\top \mathbf{1}_{x>0}$.

We consider the process X defined in (2.1) and assume that its jump distribution F has the probability density function f given by

$$f(x) = \begin{cases} pf_{(+)}(x), & x > 0, \\ qf_{(-)}(-x), & x < 0, \end{cases} \tag{3.2}$$

where $p + q = 1$, $p, q \in \mathbb{R}_+$, and $f_{(\pm)}$ are of $\mathrm{PH}(\boldsymbol{\alpha}_\pm, \boldsymbol{B}_\pm)$. Here $\boldsymbol{B}_+$ and $\boldsymbol{B}_-$ are not necessarily of the same dimension. We write $\boldsymbol{I}$ for the identity matrices of the same dimensions as those of $\boldsymbol{B}_+$ and $\boldsymbol{B}_-$ if there is no confusion.

Remark 3.2

1. A summary of analytic facts of phase-type distributions is given in Appendix B. It is worth noting that the phase-type distributions are dense in the set of all probability distributions on $\mathbb{R}_{++}$. Special cases of phase-type distributions include exponential distributions, Gamma distributions with integer parameter, and mixtures of exponential distributions. For details, see [1] or [26].
2. Asmussen et al. [2] considered a Lévy process X as in (2.1) except that $Z = (Z_t; t \geq 0)$ is given by

$$Z_t = \sum_{n=1}^{N_t^+} Y_n^+ - \sum_{n=1}^{N_t^-} Y_n^-, \tag{3.3}$$

where $N^\pm$ are both Poisson processes, and (Y_n^+) (resp. (Y_n^-)) are independently and identically distributed with distribution $\mathrm{PH}(\boldsymbol{\alpha}_+, \boldsymbol{B}_+)$ (resp. $\mathrm{PH}(\boldsymbol{\alpha}_-, \boldsymbol{B}_-)$). They also assumed that W, N^+, N^-, (Y_n^+), and (Y_n^-) are independent. Their processes have the same characteristic exponent as ours, and therefore the finite-dimensional distributions of their processes coincide with those of ours. Since the finite-dimensional distributions determine the law of the process, our definition of X is sufficient.

By assumption (3.2) and Theorem B.2, the characteristic exponent ψ in (2.2) is given by

$$\psi(\zeta) = D\zeta^2 + c\zeta + \lambda\psi_1(\zeta) - \lambda, \quad \zeta \in i\mathbb{R}, \tag{3.4}$$

where $\psi_1(\zeta) = \int e^{-\zeta y} f(y)\, dy$ is of the form

$$\psi_1(\zeta) = p\boldsymbol{\alpha}_+(\zeta\boldsymbol{I} - \boldsymbol{B}_+)^{-1}\boldsymbol{b}_+^\top + q\boldsymbol{\alpha}_-(-\zeta\boldsymbol{I} - \boldsymbol{B}_-)^{-1}\boldsymbol{b}_-^\top. \tag{3.5}$$

Since the right-hand side of (3.5) is a rational function in ζ on $\mathbb{C}$ (see Theorem B.2), the right-hand side of (3.4) actually is a rational function on $\mathbb{C}$ with a finite number of poles in $\mathbb{C}\backslash i\mathbb{R}$. Accordingly, we consider ψ and ψ_1 on $\mathbb{C}$ as analytic functions except at the poles in $\mathbb{C}\backslash i\mathbb{R}$.

Let $\mathcal{P}_0(\zeta)$ be the "minimal" polynomial with leading coefficient 1 such that the zeros of $\mathcal{P}_0(\zeta)$ coincide with the poles of $\psi_1(\zeta)$, counting their multiplicity. Write

$$\mathcal{P}_1(\zeta) = \mathcal{P}_0(\zeta)\big(\psi(\zeta) - r\big). \tag{3.6}$$

Then the zeros of $\psi(\zeta) - r$ coincide with those of the polynomial $\mathcal{P}_1(\zeta)$, counting their multiplicity.

On the other hand, the infinitesimal generator of the process X takes the form

$$Lh(x) = Dh''(x) + ch'(x) + \lambda p \boldsymbol{\alpha}_+ T^+_{\boldsymbol{B}_+} h(x) \boldsymbol{b}_+^\top + \lambda q \boldsymbol{\alpha}_- T^-_{\boldsymbol{B}_-} h(x) \boldsymbol{b}_-^\top - \lambda h(x) \quad (3.7)$$

for all $h \in \mathcal{C}_0^2(\mathbb{R})$. Here, for a nonsingular subintensity matrix $\boldsymbol{B}$, the matrix-valued operators $T^\pm_{\boldsymbol{B}}$ are defined on the set of bounded measurable functions h by

$$T^\pm_{\boldsymbol{B}} h(x) = \int_{\mathbb{R}_\pm} h(x-y) e^{\pm \boldsymbol{B} y} \, dy. \quad (3.8)$$

(When we perform integration and differentiation with respect to a matrix of continuous parameter, these operations are meant to be performed termwise.)

The following is a further refinement of Theorem 2.4.

Proposition 3.3 *Assume that the jump density f is like in (3.2), and $\mathbb{E}[X_1] > 0$ if $r = 0$. Then Φ is in $\mathcal{C}^\infty_{0,b}(\mathbb{R}_{++})$ and, for $k \geq 0$, we have the recursive formula*

$$\Phi^{(k+2)}(x) = -\frac{c}{D}\Phi^{(k+1)}(x) + \frac{(\lambda + r)}{D}\Phi^{(k)}(x) - \frac{\lambda}{D} E_k(x), \quad (3.9)$$

where

$$E_k(x) = p\boldsymbol{\alpha}_+ \left(\boldsymbol{B}_+^k T^+_{\boldsymbol{B}_+} \Phi(x) + \sum_{j=0}^{k-1} \boldsymbol{B}_+^j \Phi^{(k-1-j)}(x) \right) \boldsymbol{b}_+^\top$$
$$+ q\boldsymbol{\alpha}_- \left((-\boldsymbol{B}_-)^k T^-_{\boldsymbol{B}_-} \Phi(x) + \sum_{j=0}^{k-1} (-1)^{j+1} \boldsymbol{B}_-^j \Phi^{(k-1-j)}(x) \right) \boldsymbol{b}_-^\top. \quad (3.10)$$

(Here $\Phi^{(0)}(x) = \Phi(x)$.)

Proof Please refer to Appendix A. □

Next we transform the integro-differential equations into ordinary differential equations when the two sides of the jump distribution are both phase-type distributions. Before giving the general result, we treat a simple case by direct calculation.

Example 3.4 We consider the model (2.1) with the jump distribution $dF(y) = \eta e^{-\eta y} \mathbf{1}_{y>0} \, dy$ for some $\eta > 0$. Then

$$\psi(\zeta) = D\zeta^2 + c\zeta + \lambda \frac{\eta}{\eta + \zeta} - \lambda$$

and

$$Lh(x) = Dh''(x) + ch'(x) + \lambda \int_0^\infty h(x-y) \eta e^{-\eta y} \, dy - \lambda h(x).$$

Note that $\int_0^\infty h(x-y)e^{-\eta y}\,dy = e^{-\eta x}\int_{-\infty}^x h(y)e^{\eta y}\,dy$ and

$$\left(\frac{d}{dx}+\eta\right)\left(e^{-\eta x}\int_{-\infty}^x h(y)e^{\eta y}\,dy\right) = h(x).$$

Hence, by Theorem 2.4, Φ satisfies the ODE

$$\begin{aligned}0 &= \left(\frac{d}{dx}+\eta\right)(L-r)\Phi(x)\\ &= D\Phi'''(x) + (D\eta + c)\Phi''(x) + (c\eta - \lambda - r)\Phi'(x) - r\eta\Phi(x).\end{aligned}$$

On the other hand, we have $\mathcal{P}_0(\zeta) = \eta + \zeta$ and

$$\begin{aligned}\mathcal{P}_1(\zeta) &= D\zeta^2(\zeta+\eta) + c\zeta(\zeta+\eta) + \lambda\eta - (\lambda+r)(\zeta+\eta)\\ &= D\zeta^3 + (D\eta+c)\zeta^2 + (c\eta - \lambda - r)\zeta - r\eta.\end{aligned}$$

Therefore Φ satisfies an ODE with the characteristic polynomial $\mathcal{P}_1$.

Theorem 3.5 *Let D_1 be the differential operator with the characteristic polynomial $\mathcal{P}_1$ given by* (3.6). *Then $D_1\Phi \equiv 0$ on $\mathbb{R}_{++}$.*

Proof Let $L_2 = L_2(\mathbb{R})$ be the space of square-integrable functions defined on $\mathbb{R}$, and set $\langle f_1, f_2\rangle = \int f_1(x) f_2(x)\,dx$. For an operator A on L_2, write A^* for its adjoint (i.e., $\langle Ak, h\rangle = \langle k, A^* h\rangle$ for all k, h in L_2). By integration by parts, we have $\langle k', h\rangle = -\langle k, h'\rangle$ whenever k or h has compact support. Write $Th(x) = \int h(x-y) f(y)\,dy$. By a change of variables and Fubini's theorem, we have

$$\begin{aligned}\langle Th, k\rangle &= \int k(x)\int h(x-y) f(y)\,dy\,dx\\ &= \int h(z)\int k(z-y) f(-y)\,dy\,dz = \big\langle h, T^* k\big\rangle,\end{aligned}$$

where $T^* k(x) \equiv \int k(x-y) f(-y)\,dy$.

Let D_0 be the differential operator with the characteristic polynomial $\mathcal{P}_0$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R}_{++})$ a test function. Recall that, by Theorem 2.4, $(L-r)\Phi \equiv 0$ on $\mathbb{R}_{++}$. Therefore we have

$$0 = \big\langle D_0(L-r)\Phi, \phi\big\rangle = \big\langle \Phi, \big(L^* - r\big) D_0^* \phi\big\rangle. \tag{3.11}$$

Here L^* is given by

$$L^* h(x) = Dh''(x) - ch'(x) + \lambda T^* h(x) - \lambda h(x).$$

Set $L_{\mathcal{D}} = L - r - \lambda T$. Then

$$\big(L^* - r\big) D_0^* \phi = \lambda T^* D_0^* \phi + L_{\mathcal{D}}^* D_0^* \phi = \lambda T^* D_0^* \phi + (D_0 L_{\mathcal{D}})^* \phi. \tag{3.12}$$

Write D_2 as the differential operator corresponding to the polynomial $\mathcal{P}_2(\zeta) = \mathcal{P}_0(\zeta)\psi_1(\zeta)$. We prove that $T^* D_0^* \phi = D_2^* \phi$ a.e. by showing that both $T^* D_0^* \phi$ and

$D_2^*\phi$ are in $L_2 \cap L_1$ and have the same Fourier transforms. (Here $L_1 = L_1(\mathbb{R})$ is the space of integrable functions on $\mathbb{R}$.)

First, we show that $T^* D_0^*\phi$ and $D_2^*\phi$ are in $L_1 \cap L_2$. Since $\phi \in \mathcal{C}_c^\infty(\mathbb{R}_{++})$, we obtain $D_2^*\phi \in L_1 \cap L_2$. Also,

$$\int \left|T^* D_0^*\phi(x)\right| dx \le \iint \left|D_0^*\phi(x-y)\right| dx\, f(-y)\, dy \le \left\|D_0^*\phi\right\|_{L_1} \|f\|_{L_1} < \infty,$$

and hence $T_0^* D_0^*\phi \in L_1$. We show that $T^* D_0^*\phi \in L_2$. Since $D_0^*\phi \in \mathcal{C}_c^\infty(\mathbb{R}_{++})$ and $T^*h(x) = \int h(x-y) f(-y)\, dy$, it suffices to show that $Tk \in L_2$ for any $k \in \mathcal{C}_c^\infty(\mathbb{R}_{++})$. Note that, by Theorem B.1, f has tails that decay exponentially and, hence, $\int f^2\, dx < \infty$. Let $k \in \mathcal{C}_c^\infty(\mathbb{R}_{++})$ and write H for the compact support of k. By the Cauchy–Schwarz inequality we get

$$\begin{aligned}\int \left[Tk(x)\right]^2 dx &= \int \left(\int k(x-y) f(y)\, dy\right)^2 dx = \int \left(\int_H k(y) f(x-y)\, dy\right)^2 dx \\ &\le \int \left(\int k(y)^2\, dy\right)\left(\int_H f(x-y)^2\, dy\right) dx \\ &\le \|k\|_{L_2}^2 \|f\|_{L_2}^2 \int_H dx < \infty.\end{aligned}$$

Next, we show that the Fourier transforms $\mathcal{F}(T^* D_0^*\phi)$ and $\mathcal{F}(D_2^*\phi)$ coincide, where $\mathcal{F}h(\theta) = \int e^{-2\pi i\theta x} h(x)\, dx$. Recall that $\psi_1(\zeta) = \int e^{-\zeta y} f(y)\, dy$ and notice that $\mathcal{F}(D_0^*\phi)(\theta) = \mathcal{P}_0(-2\pi i\theta)\mathcal{F}(\phi)(\theta)$. (See [28], Sect. 5.3.) Since $T^* D_0^*\phi \in L_1 \cap L_2$, we have, for all $\theta \in \mathbb{R}$,

$$\begin{aligned}\mathcal{F}(T^* D_0^*\phi)(\theta) &= \int e^{-2\pi i\theta x}\left(\int D_0^*\phi(x-y) f(-y)\, dy\right) dx \\ &= \int \left(\int D_0^*\phi(x-y) e^{-2\pi i\theta(x-y)}\, dx\right) e^{-2\pi i\theta y} f(-y)\, dy \\ &= \psi_1(-2\pi i\theta)\mathcal{P}_0(-2\pi i\theta)\mathcal{F}(\phi)(\theta) \\ &= \mathcal{P}_2(-2\pi i\theta)\mathcal{F}(\phi)(\theta) \\ &= \mathcal{F}\big(D_2^*\phi\big)(\theta).\end{aligned}$$

By the Fourier inversion formula, we deduce that $T^* D_0^*\phi = D_2^*\phi$ almost everywhere.

By (3.12) and the fact that $T^* D_0^*\phi = D_2^*\phi$ a.e., $(L^* - r)D_0^*\phi = \lambda D_2^*\phi + (D_0 L_{\mathcal{D}})^*\phi = D_1^*\phi$ a.e. Hence, by (3.11), we have

$$0 = \big\langle \Phi, (L^* - r)D_0^*\phi\big\rangle = \big\langle \Phi, D_1^*\phi\big\rangle = \langle D_1\Phi, \phi\rangle.$$

Since $\phi \in \mathcal{C}_c^\infty(\mathbb{R}_{++})$ is arbitrary, we have $D_1\Phi = 0$ a.s. on $(0,\infty)$. Since $\Phi \in \mathcal{C}_{0,b}^\infty(\mathbb{R}_{++})$, $D_1\Phi \equiv 0$ on $\mathbb{R}_{++}$. This completes the proof. □

Assume from now on that $r > 0$. We denote by $\mathcal{Z}_{(-)}^r = (\rho_i^r)_{i=1}^S$ the collection of zeros of $\psi(\zeta) - r$ (counting their multiplicity) with strictly negative real parts. We say that $\mathcal{Z}^r_{(-)}$ is *separable* if its members are distinct. We write $\mathcal{Z}_{(-)}$ for $\mathcal{Z}^r_{(-)}$ and ρ_i for ρ^r_i if these cause no confusion.

Proposition 3.6 *There exist polynomials $Q_i(x)$ such that $\Phi(x) = \sum_{i=1}^S \boldsymbol{Q}_i(x)e^{\rho_i x}$ for $x \geq 0$.*

Proof Please refer to Appendix A. □

Let $\boldsymbol{U}$ be the column vector with entries $U_j = \Phi^{(j-1)}(0)$ for $1 \leq j \leq S$ and $\boldsymbol{V}$ the $S \times S$ *Vandermonde* matrix with $\boldsymbol{V}_{ij} = \rho_j^{i-1}$. Here $\Phi^{(0)}(0) = \Phi(0)$ and, for $k \geq 1$, $\Phi^{(k)}(0)$ is the kth (right-hand) derivative of Φ at 0.

Theorem 3.7 *Suppose that $\mathcal{Z}_{(-)}$ is separable. Then we have $\Phi(x) = \sum_{i=1}^S \boldsymbol{Q}_i e^{\rho_i x}$, where $\boldsymbol{Q} = (\boldsymbol{Q}_1, \boldsymbol{Q}_2, \ldots, \boldsymbol{Q}_S)^T$ is the unique constant column vector satisfying the system of linear equations $\boldsymbol{V}\boldsymbol{Q} = \boldsymbol{U}$. Moreover, if there are no upward jumps for X (i.e., $q = 0$), then $\Phi^{(k)}(0)$ can be obtained explicitly by the recursive formula*

$$\Phi^{(k+2)}(0) = \frac{1}{D}\left[-c\Phi^{(k+1)}(0) + (\lambda + r)\Phi^{(k)}(0) - \lambda E_k(0)\right] \tag{3.13}$$

with $\Phi(0) = g(0)$,

$$E_k(0) = \boldsymbol{\alpha}_+\left(\boldsymbol{B}_+^k \int_0^\infty g(-y)e^{\boldsymbol{B}y}\,dy + \sum_{j=0}^{k-1} \boldsymbol{B}_+^j \Phi^{(k-1-j)}(0)\right)\boldsymbol{b}_+^\top, \tag{3.14}$$

and

$$\Phi'(0) = -\left(\frac{c}{D} + \rho_r^*\right)g(0) + \frac{\lambda}{D}\int_0^\infty dv \int_v^\infty dF(y)\,e^{-\rho_r^* v} g(v - y). \tag{3.15}$$

(Here ρ_r^* is a positive real number satisfying $\psi(\rho_r^*) = r$.)

Proof By Proposition 3.6, we know that $\Phi(x) = \sum_{i=1}^S \boldsymbol{Q}_i(x)e^{\rho_i x}$, where $\boldsymbol{Q}_j(x)$ are polynomials. Since $\mathcal{Z}_{(-)}$ is separable, standard theory of ordinary differential equations gives that $\boldsymbol{Q}_j(x)$ must be a constant for all j. For details, see [29], Lessons 20B and 20D.

Simple calculation shows that the constant vector $\boldsymbol{Q}$ satisfies the system of linear equations $\boldsymbol{V}\boldsymbol{Q} = \boldsymbol{U}$. Since $\mathcal{Z}_{(-)}$ is separable, the ρ_i are distinct and, hence, the Vandermonde matrix $\boldsymbol{V}$ is invertible (see, e.g., [14], p. 218). Since $\boldsymbol{V}$ is invertible, $\boldsymbol{Q}$ is the unique solution of the system of linear equations. By letting $x \to 0+$ in (3.9) and (3.10), we obtain (3.13) and (3.14). Formula (3.15) follows from Theorem C.1. □

Remark 3.8 It is interesting to compare our results with those of Asmussen et al. [2]. By martingale stopping and Wiener–Hopf factorization, they obtained similar results as ours.

As an application of Theorem 3.7, we consider Hilberink and Rogers' extension of Leland's model. We first recall the basic setup of Leland's model and then obtain the optimal default boundary in explicit form. Furthermore, we provide a procedure to calculate the values of bond, firm, and equity.

Example 3.9 (Optimal capital structure) We shall assume that, under a risk-neutral measure $\mathbb{Q}$, the value of the firm's assets is given by $V_t = Ve^{ct+\sigma W_t - Z_t}$. The setting of debt issuance and coupon payment follows the convention in [23]. In the time interval $(t, t+dt)$, the firm issues new debt with face value $a\,dt$ and maturity profile $k(t) = me^{-mt}$. With the exponential maturity profile, the face value of the debt maturing in $(t, t+dt)$ is the same as the face value of the newly issued debt. Thus the face value of all pending debt is equal to the constant $A = a\int_0^\infty e^{-mt}\,dt = a/m$. All bondholders will receive coupons at rate b until default. All debt is of equal seniority and, once default occurs, the bondholders get the rest of the value, βV_τ, after the bankruptcy cost $(1-\beta)V_\tau$. We assume that there is a corporate tax rate δ, and the coupons paid can be offset against tax. We further assume that the default occurs at time $\tau = \inf\{t \geq 0; V_t \leq L\}$, where L will be determined optimally later on. As in [18], we assume that the market has a risk-free rate $r > 0$, and $\mu > 0$ is the proportional rate at which profit is disbursed to investors. Then the total value at time 0 of all debt is

$$D(V, L) = \frac{bA + mA}{m+r}\mathbb{E}_{\mathbb{Q}}\big[1 - e^{-(m+r)\tau}\big] + \beta\mathbb{E}_{\mathbb{Q}}\big[V_\tau e^{-(m+r)\tau}\big], \tag{3.16}$$

and the value of the firm is

$$v(V, L) = V + \frac{bA\delta}{r}\mathbb{E}_{\mathbb{Q}}\big[1 - e^{-r\tau}\big] - (1-\beta)\mathbb{E}_{\mathbb{Q}}\big[V_\tau e^{-r\tau}\big]. \tag{3.17}$$

The value of equity of the firm is given by

$$S(V, L) = v(V, L) - D(V, L). \tag{3.18}$$

(See [18] or Remark 4.8 below.) By imposing the smooth-pasting condition, we calculate the optimal default-triggering level L^* that maximizes the value of equity of the firm.

Set $X_t = \log\frac{V_t}{L}$ and $x = \log\frac{V}{L}$. Note that $\tau = \inf\{t \geq 0; V_t \leq L\} = \inf\{t \geq 0; X_t \leq 0\}$ and, under the risk-neutral measure $\mathbb{Q}$, X is of the form (2.1). Then we have

$$D(V, L) = \frac{A(m+b)}{m+r}\big(1 - G_{m+r}(x)\big) + \beta L H_{m+r}(x) \tag{3.19}$$

and

$$v(V, L) = V + \frac{A\delta b}{r}\big(1 - G_r(x)\big) - (1-\beta)LH_r(x), \tag{3.20}$$

where

$$G_s(x) = \mathbb{E}_x\big[e^{-s\tau}\big] \tag{3.21}$$

and

$$H_s(x) = \mathbb{E}_x\big[e^{-s\tau}e^{X_\tau}\big]. \tag{3.22}$$

Hence, we obtain that

$$\begin{aligned}\frac{\partial S}{\partial V}(V,L) &= \frac{\partial}{\partial V}\big(v(V,L)-D(V,L)\big)\\ &= 1-\frac{\delta Ab}{r}G_r'(x)\frac{1}{V}-(1-\beta)LH_r'(x)\frac{1}{V}\\ &\quad +\frac{A(m+b)}{m+r}G_{m+r}'(x)\frac{1}{V}-\beta LH_{m+r}'(x)\frac{1}{V}.\end{aligned}$$

As in [18, 23], by imposing the smooth pasting condition (i.e., $\frac{\partial S}{\partial V}(V,L)|_{V=L}=0$), we get the optimal default-triggering level

$$L^*=\frac{\frac{\delta Ab}{r}G_r'(0)-\frac{A(m+b)}{m+r}G_{m+r}'(0)}{1-(1-\beta)H_r'(0)-\beta H_{m+r}'(0)}. \tag{3.23}$$

Assume further that X has only downward jumps and $\sigma>0$. (Indeed, in this case, Kyprianou and Surya [21] proved that the optimal default triggering level does satisfy the condition of smooth pasting.) By taking $g(y)=e^y 1_{y\le 0}$ in (3.15), we get

$$\begin{aligned}H_s'(0) &= \frac{d}{dx}H_s(x)\bigg|_{x=0}\\ &= -\frac{c}{D}-\rho_s^*+\frac{\lambda}{D}\int_0^\infty dv\int_v^\infty dF(y)\,e^{-\rho_s^* v}e^{v-y}\\ &= -\frac{c}{D}-\rho_s^*+\frac{\lambda}{D}\int_0^\infty dF(y)\int_0^y e^{(1-\rho_s^*)v}e^{-y}\,dv\\ &= -\frac{c}{D}-\rho_s^*+\frac{\lambda}{D(1-\rho_s^*)}\int_0^\infty \big(e^{-\rho_s^* y}-1+1-e^{-y}\big)\,dF(y)\\ &= -\frac{c}{D}-\rho_s^*+\frac{1}{D(1-\rho_s^*)}\big[-D\big(\rho_s^*\big)^2-c\rho_s^*+s-\big(\psi(1)-D-c\big)\big]\\ &= \frac{s-\psi(1)}{D(1-\rho_s^*)}+1.\end{aligned} \tag{3.24}$$

(Here in the fourth equation we use the fact that ρ_s^* is a positive real number satisfying $\psi(\rho_s^*)-s=0$.) Similarly, by taking $g(y)=1$ for all $y\le 0$ in (3.15), we get

$$\begin{aligned}G_s'(0) &= \frac{d}{dx}G_s(x)\bigg|_{x=0}\\ &= -\frac{c}{D}-\rho_s^*+\frac{\lambda}{D}\int_0^\infty dv\int_v^\infty dF(y)e^{-\rho_s^* v}\\ &= -\frac{c}{D}-\rho_s^*+\frac{\lambda}{D}\int_0^\infty dF(y)\int_0^y e^{-\rho_s^* v}\,dv\\ &= -\frac{c}{D}-\rho_s^*+\frac{\lambda}{D\rho_s^*}\int_0^\infty \big(1-e^{-\rho_s^* y}\big)\,dF(y)\end{aligned}$$

$$= -\frac{c}{D} - \rho_s^* + \frac{1}{D\rho_s^*}\left(D\left(\rho_s^*\right)^2 + c\rho_s^* - s\right)$$

$$= \frac{-s}{D\rho_s^*}. \tag{3.25}$$

Plugging (3.25) and (3.24) into (3.23) and using the fact that $\psi(1) = r - \mu$ (since $V = \mathbb{E}_{\mathbb{Q}}[e^{-(r-\mu)}V_1] = Ve^{-(r-\mu)+\psi(1)}$), we obtain the optimal default-triggering level

$$L^* = \frac{\frac{A(m+b)}{\rho_{m+r}^*} - \frac{\delta Ab}{\rho_r^*}}{\beta\frac{m+\mu}{\rho_{m+r}^*-1} + (1-\beta)\frac{\mu}{\rho_r^*-1}}. \tag{3.26}$$

Assume that the jump distribution F is of phase-type distribution PH$(\boldsymbol{\alpha}, \boldsymbol{B})$ on $(0, \infty)$. By using (3.18–3.20), we compute $D(V, L)$, $v(V, L)$ and $S(V, L)$ in terms of $G_s(x)$ and $H_s(x)$. Note that

$$G_s^{(k+2)}(0) = -\frac{c}{D}G_s^{(k+1)}(0) + \frac{\lambda + s}{D}G_s^{(k)}(0) - \frac{\lambda}{D}I_k(0) \tag{3.27}$$

and

$$H_s^{(k+2)}(0) = -\frac{c}{D}H_s^{(k+1)}(0) + \frac{\lambda + s}{D}H_s^{(k)}(0) - \frac{\lambda}{D}J_k(0), \tag{3.28}$$

where

$$I_k(0) = \boldsymbol{\alpha}\left(\boldsymbol{B}^k\int_0^\infty e^{\boldsymbol{B}y}\,dy + \sum_{j=0}^{k-1}\boldsymbol{B}^j G_s^{(k-1-j)}(0)\right)\boldsymbol{b}^T$$

and

$$J_k(0) = \boldsymbol{\alpha}\left(\boldsymbol{B}^k\int_0^\infty e^{-y}e^{\boldsymbol{B}y}\,dy + \sum_{j=0}^{k-1}\boldsymbol{B}^j H_s^{(k-1-j)}(0)\right)\boldsymbol{b}^T.$$

Using (3.24), (3.25), and $G_s(0) = H_s(0) = 1$ in (3.27) and (3.28), we get higher-order derivatives of G_s and H_s at zero. Then, by Theorem 3.7, we compute the constant vector $\boldsymbol{Q}$ and, hence, obtain solutions for $G_s(x)$ and $H_s(x)$.

To illustrate our method, we consider the particular case $dF(x) = \eta e^{-\eta x}1_{x>0}\,dx$. Then $\psi(\zeta) = D\zeta^2 + c\zeta + \lambda\frac{\eta}{\eta+\zeta} - \lambda$ and, for every $s > 0$, there exist $\rho_1 < \eta < \rho_2 < 0 < \rho^*$ such that $\psi(\rho_1) = \psi(\rho_2) = \psi(\rho^*) = s$ (i.e., $\mathcal{Z}_{(-)}^s = \{\rho_1, \rho_2\}$), and we write ρ^* for ρ_s^*. Recall that $G_s'(0) = \frac{-s}{D\rho^*}$. Hence $G_s(x) = \boldsymbol{Q}_1 e^{\rho_1 x} + \boldsymbol{Q}_2 e^{\rho_2 x}$, where $\boldsymbol{Q}$ is the unique column vector satisfying the equation

$$\begin{bmatrix} 1 & 1 \\ \rho_1 & \rho_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{Q}_1 \\ \boldsymbol{Q}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{-s}{D\rho^*} \end{bmatrix}.$$

Simple algebra shows that

$$\begin{bmatrix} \boldsymbol{Q}_1 \\ \boldsymbol{Q}_2 \end{bmatrix} = \frac{1}{\rho_2 - \rho_1}\begin{bmatrix} \rho_2 + \frac{s}{D\rho^*} \\ -\rho_1 - \frac{s}{D\rho^*} \end{bmatrix}.$$

Since $\rho_1\rho_2\rho^* = \frac{s\eta}{D}$, we get

$$G_s(x) = \frac{\rho_2 + \frac{s}{D\rho^*}}{\rho_2-\rho_1}e^{\rho_1 x} + \frac{-\rho_1 - \frac{s}{D\rho^*}}{\rho_2-\rho_1}e^{\rho_2 x}$$
$$= \frac{\rho_2(\eta+\rho_1)}{\eta(\rho_2-\rho_1)}e^{\rho_1 x} + \frac{-\rho_1(\eta+\rho_2)}{\eta(\rho_2-\rho_1)}e^{\rho_1 x}. \tag{3.29}$$

Recall that $H'(0) = \frac{s-\psi(1)}{D(1-\rho^*)} + 1$. Hence $H_s(x) = \boldsymbol{P}_1 e^{\rho_1 x} + \boldsymbol{P}_2 e^{\rho_2 x}$, where $\boldsymbol{P}$ is the unique column vector satisfying the equation

$$\begin{bmatrix} 1 & 1 \\ \rho_1 & \rho_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{P}_1 \\ \boldsymbol{P}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{s-\psi(1)}{D(1-\rho^*)} + 1 \end{bmatrix}.$$

Simple algebra shows that

$$\begin{bmatrix} \boldsymbol{P}_1 \\ \boldsymbol{P}_2 \end{bmatrix} = \frac{1}{\rho_2-\rho_1}\begin{bmatrix} \rho_2 - 1 - \frac{s-\psi(1)}{D(1-\rho^*)} \\ 1-\rho_1 + \frac{s-\psi(1)}{D(1-\rho^*)} \end{bmatrix}.$$

Using the fact that $(1+\eta)(\psi(1)-s) = D(1-\rho_1)(1-\rho_2)(1-\rho^*)$, we obtain

$$H_s(x) = \frac{\rho_2 - 1 - \frac{s-\psi(1)}{D(1-\rho^*)}}{\rho_2-\rho_1}e^{\rho_1 x} + \frac{1-\rho_1 + \frac{s-\psi(1)}{D(1-\rho^*)}}{\rho_2-\rho_1}e^{\rho_2 x}$$
$$= \frac{(\rho_2-1)(\eta+\rho_1)}{(1+\eta)(\rho_2-\rho_1)}e^{\rho_1 x} + \frac{(1-\rho_1)(\eta+\rho_2)}{(1+\eta)(\rho_2-\rho_1)}e^{\rho_1 x}. \tag{3.30}$$

Remark 3.10 The optimal level L^* in (3.26) coincides with that in [18]. Our ODE approach also works for the perpetual American put option. Indeed, we determine the optimal stopping times when only downward jumps are allowed. Also we calculate exactly the price of the perpetual American put option.

For general two-sided jump distributions, we get a necessary condition for the coefficient constant $\boldsymbol{Q}$. This necessary condition will play an important role in Sect. 4.

Proposition 3.11 *If* $(\boldsymbol{\alpha}_+, \boldsymbol{B}_+)$ *is minimal*, $\mathcal{Z}_{(-)}$ *is separable, and* $p > 0$, *then the constant vector* $\boldsymbol{Q}$ *for* Φ *satisfies* $\sum_{i=1}^S \boldsymbol{Q}_i = g(0)$ *and*

$$\boldsymbol{\alpha}_+\left(\int_0^\infty g(-y)e^{\boldsymbol{B}_+ y}dy\right)e^{x\boldsymbol{B}_+}\boldsymbol{b}_+^\top = \boldsymbol{\alpha}_+\left[\sum_{i=1}^S \boldsymbol{Q}_i(\rho_i \boldsymbol{I} - \boldsymbol{B}_+)^{-1}\right]e^{x\boldsymbol{B}_+}\boldsymbol{b}_+^\top, \quad \forall x > 0. \tag{3.31}$$

Proof Please refer to Appendix A. □

4 The constant vector $\boldsymbol{Q}$: the second method

In this section, we consider a more general process whose upward jumps are determined by a general Lévy measure and downward jumps by a compound Poisson process with a mixture of exponential jump distributions. We show that if a vector $\boldsymbol{Q}$ satisfies a system of linear equations, then a conjectured function must be the desired function Φ. To find a solution to the system of equations, instead of directly inverting the system of equations, we exploit the technique in [12] to obtain an explicit solution for Φ. With this explicit formula, we compute especially in Example 4.6 the optimal default-triggering level. We also provide an example (Example 4.7) which shows that we can calculate certain option prices with finite maturity up to a Fourier transform.

Let $X = (X_t, \{\mathbb{P}_x\}_{x\in\mathbb{R}})$ be the Lévy processes given by

$$X_t = X_0 + X_t^{(+)} - Z_t, \quad t \geq 0. \tag{4.1}$$

Here $X^{(+)} = (X_t^{(+)}; t \geq 0)$ is a Lévy process on $\mathbb{R}$ such that it starts at 0 and has a nontrivial diffusion part and no downward jumps, and Z is a compound Poisson process that is independent of $X^{(+)}$ with jump distribution given by $f_{(+)}$ =PH$(\boldsymbol{\alpha}_+, \boldsymbol{B}_+)$. As before, under $\mathbb{P}_x$, $X_0 = x$ a.s. Clearly, the process in (4.1) is a generalization of (2.1).

The characteristic exponent ψ of X is given by

$$\psi(\zeta) = D\zeta^2 + c\zeta + \int_0^\infty \left[e^{\zeta z} - 1 - \zeta z \mathbf{1}_{|z|\leq 1}\right]\nu(dz) + \lambda \int_0^\infty e^{-\zeta y} f_{(+)}(y)\, dy - \lambda,$$

where ν is an arbitrary Lévy measure on $(0, \infty)$ and $\int_0^\infty \min(1, z^2)\nu(dz) < \infty$. Moreover, the infinitesimal generator of X has a domain containing $\mathcal{C}_0^2(\mathbb{R})$ and is given by

$$\begin{aligned} Lh(x) =& Dh''(x) + ch'(x) + \int_0^\infty \left[h(x+z) - h(x) - h'(x)z\mathbf{1}_{|z|\leq 1}\right]\nu(dz) \\ &+ \lambda \int_0^\infty h(x-z) f_{(+)}(z)\, dz - \lambda h(x). \end{aligned}$$

Recall that $\Phi(x) = \mathbb{E}_x[e^{-r\tau} g(X_\tau)]$, where $\tau = \inf\{t \geq 0; X_t \leq 0\}$. The next proposition gives a converse to Theorem 2.4.

Proposition 4.1 *If* $\phi \equiv g$ *on* $(-\infty, 0]$, $\phi \in \mathcal{C}_0^2(\mathbb{R}_+)$, *and* $(L - r)\phi \equiv 0$ *on* $\mathbb{R}_{++}$, *then* $\phi \equiv \Phi$ *on* $\mathbb{R}$.

Proof Please refer to Appendix A. □

We consider below the special case where $f_{(+)}(y)$ is a mixture of exponential distributions. Namely, there exist constants $(p_j)_{j=1}^m$ and $(\eta_j)_{j=1}^m$ such that $p_j > 0$, $\eta_j > 0$, $\sum_{j=1}^m p_j = 1$, and

$$f_{(+)}(y) = \sum_{j=1}^m p_j \eta_j e^{-\eta_j y} \tag{4.2}$$

on $y>0$ and $f_{(+)}(y)=0$ otherwise. Without loss of generality, assume that the η_j are distinct.

It is worth noting that $\mathcal{Z}_{(-)}=(\rho_i)_{i=1}^S$ is separable and $S=m+1$ (see [2], Lemma 1(1) or [25], Corollary 2). Now, based on Theorem 3.7, we conjecture that $\Phi(x)=\mathbb{E}_x[e^{-r\tau}g(X_\tau)]=\sum_{i=1}^S \boldsymbol{Q}_i e^{\rho_i x}$ for some $\boldsymbol{Q}_i$. By Proposition 3.11, we further assume that $\boldsymbol{Q}$ satisfies (3.31). More precisely, for $f_{(+)}$ given by (4.2),

$$\sum_{j=1}^m p_j\eta_j e^{-\eta_j x}\int_{-\infty}^0 g(y)e^{\eta_j y}\,dy=\sum_{i=1}^{m+1}\boldsymbol{Q}_i\sum_{j=1}^m p_j\frac{\eta_j e^{-\eta_j x}}{\rho_i+\eta_j}. \tag{4.3}$$

(Note that $f_{(+)}=$PH$((p_1,\dots,p_m),\operatorname{diag}(-\eta_1,\dots,-\eta_m))$.) Recall that $\sum_{i=1}^S\boldsymbol{Q}_i=g(0)$. Then, by comparing the coefficients of $e^{-\eta_j x}$ in (4.3), we get the system of linear equations

$$\begin{cases}\displaystyle\sum_{i=1}^{m+1}\boldsymbol{Q}_i=g(0),\\ \displaystyle\sum_{i=1}^{m+1}\frac{\boldsymbol{Q}_i\eta_j}{\rho_i+\eta_j}=\int_{-\infty}^0 g(y)\eta_j e^{\eta_j y}\,dy,\quad 1\le j\le m.\end{cases} \tag{4.4}$$

The following proposition confirms our conjecture.

Proposition 4.2 *If $\boldsymbol{Q}$ satisfies (4.4) for some bounded Borel-measurable function g, then*

$$\Phi(x)=\mathbb{E}_x\left[e^{-r\tau}g(X_\tau)\right]=\sum_{i=1}^S\boldsymbol{Q}_i e^{\rho_i x},\quad \textit{for } x\ge 0. \tag{4.5}$$

Proof Please refer to Appendix A. □

In fact, the coefficient matrix of the system of equations (4.4) is invertible. Hence we can write $\boldsymbol{Q}$ in a general matrix form. Instead of the matrix form, we exploit the technique in [12] to get an explicit formula for $\boldsymbol{Q}$.

Theorem 4.3 *Assume that X satisfies (4.1) and $f_{(+)}(y)$ is given by (4.2). Then, for any bounded Borel function $g:\mathbb{R}_-\to\mathbb{R}$, we have $\Phi(x)=\mathbb{E}_x[e^{-r\tau}g(X_\tau)]=\sum_{i=1}^S\boldsymbol{Q}_i e^{\rho_i x}$ on $\mathbb{R}_+$, where, for $1\le h\le m+1$,*

$$\boldsymbol{Q}_h=\frac{1}{\rho_h\prod_{\ell=1,\ell\ne h}^{m+1}(-\rho_h+\rho_\ell)}\sum_{j=1}^{m+1}\boldsymbol{R}_j\left(\prod_{k=1}^{m+1}(\eta_j+\rho_k)\prod_{i=1,i\ne j}^{m+1}\frac{-\rho_h-\eta_i}{\eta_j-\eta_i}\right) \tag{4.6}$$

and

$$\boldsymbol{R}_j=g(0)-\int_0^\infty g(-y)\eta_j e^{-\eta_j y}\,dy. \tag{4.7}$$

(*Here we set $\eta_{m+1}=0$.*)

Proof Clearly, the system (4.4) is equivalent to the system of linear equations

$$\sum_{i=1}^{m+1} \frac{\boldsymbol{Q}_i \rho_i}{\rho_i + \eta_j} = \boldsymbol{R}_j, \quad 1 \le j \le m+1, \tag{4.8}$$

where the $\boldsymbol{R}_j$ are given by (4.7). Since the system (4.4) admits at most one solution, so does (4.8).

Consider the rational function

$$H(x) = \sum_{j=1}^{m+1} \boldsymbol{R}_j \prod_{k=1}^{m+1} \frac{\eta_j + \rho_k}{x + \rho_k} \prod_{i=1, i \neq j}^{m+1} \frac{x - \eta_i}{\eta_j - \eta_i}. \tag{4.9}$$

Note that, for each summand in (4.9), the numerator is a polynomial of degree m, and the denominator is a polynomial of degree $m+1$. By the principle of partial fraction decomposition, there exist constants $\boldsymbol{D}_h$, $1 \le h \le m+1$, such that

$$\sum_{j=1}^{m+1} \boldsymbol{R}_j \prod_{k=1}^{m+1} \frac{\eta_j + \rho_k}{x + \rho_k} \prod_{i=1, i \neq j}^{m+1} \frac{x - \eta_i}{\eta_j - \eta_i} = H(x) = \sum_{i=1}^{m+1} \frac{\boldsymbol{D}_i \rho_i}{\rho_i + x}. \tag{4.10}$$

By multiplying both sides of (4.10) by $x + \rho_h$ and then setting $x = -\rho_h$, we get that $\boldsymbol{D}_h$ is given by the right-hand side of (4.6). On the other hand, since $\prod_{i=1, i \neq j}^{m+1} \frac{\eta_h - \eta_i}{\eta_j - \eta_i} = 0$ for all $h \neq j$,

$$\sum_{i=1}^{m+1} \frac{\boldsymbol{D}_i \rho_i}{\rho_i + \eta_h} = H(\eta_h) = \boldsymbol{R}_h \prod_{k=1}^{m+1} \frac{\eta_h + \rho_k}{\eta_h + \rho_k} \prod_{i=1, i \neq h}^{m+1} \frac{\eta_h - \eta_i}{\eta_h - \eta_i} = \boldsymbol{R}_h, \quad 1 \le h \le m+1.$$

That is, $\boldsymbol{D}$ is a solution to (4.8). Since the solution to (4.8) is unique and (4.4) and (4.8) are equivalent, $\boldsymbol{Q} = \boldsymbol{D}$ is the unique solution of (4.4). By Proposition 4.2, we have $\Phi \equiv \sum_{i=1}^{S} \boldsymbol{Q}_i e^{\rho_i x}$ on $\mathbb{R}_+$. □

Remark 4.4 In fact, by approximation in (4.6) and (4.7), one sees that the conclusion of Theorem 4.3 still holds if g is any Borel function on $\mathbb{R}_-$ such that $\int_{-\infty}^{0} |g(y)| e^{\eta_j y}\, dy < \infty$ for all $1 \le j \le m$.

Example 4.5 We consider some special g.

(1) Assume that $g \equiv \mathbf{1}_{(-\infty, y)}$ for some $y \le 0$. Then $\boldsymbol{R}_j = -e^{\eta_j y}$ for $1 \le j \le m$ and $\boldsymbol{R}_{m+1} = 0$. Hence,

$$\mathbb{E}_x\left[e^{-r\tau} \mathbf{1}_{X_\tau < y}\right] = \sum_{h=1}^{m+1} \left[\frac{-1}{\rho_h \prod_{\ell=1, \ell \neq h}^{m+1} (-\rho_h + \rho_\ell)} \times \sum_{j=1}^{m} e^{\eta_j y} \left(\prod_{k=1}^{m+1} (\eta_j + \rho_k) \prod_{i=1, i \neq j}^{m+1} \frac{-\rho_h - \eta_i}{\eta_j - \eta_i} \right) \right] e^{\rho_h x}.$$

(2) Assume that $g \equiv 1_{\{0\}}$. Then $\boldsymbol{R}_j = 1$ for all j. Hence,

$$\mathbb{E}_x\big[e^{-r\tau}\mathbf{1}_{\{0\}}(X_\tau)\big] = \sum_{h=1}^{m+1}\Bigg[\frac{1}{\rho_h\prod_{\ell=1,\ell\neq h}^{m+1}(-\rho_h+\rho_\ell)} \times \sum_{j=1}^{m+1}\Bigg(\prod_{k=1}^{m+1}(\eta_j+\rho_k)\prod_{i=1,i\neq j}^{m+1}\frac{-\rho_h-\eta_i}{\eta_j-\eta_i}\Bigg)\Bigg]e^{\rho_h x}.$$

(Cf. [20].)

Example 4.6 (Optimal capital structure, continued) We use the notation and setting of Example 3.9, except that the process X is of the form (4.1) (also assume that $\sigma > 0$) and Z has a mixed exponential jump distribution (4.2). For every $s > 0$, let ρ_i^s, $1 \le i \le m+1$, be the negative real solutions to $\psi(z) - s = 0$. As before, we write ρ_i for ρ_i^s if this causes no confusion. By taking $g \equiv 1$ in (4.7), we get $\boldsymbol{R}_j = 0$ for $1 \le j \le m$ and $\boldsymbol{R}_{m+1} = 1$. By Theorem 4.3, we get

$$G_s(x) = \prod_{k=1}^{m+1}\rho_k\Bigg[\sum_{h=1}^{m+1}\Bigg(\frac{1}{\rho_h\prod_{\ell=1,\ell\neq h}^{m+1}(-\rho_h+\rho_\ell)}\prod_{i=1}^{m}\frac{\rho_h+\eta_i}{\eta_i}\Bigg)e^{\rho_h x}\Bigg] \tag{4.11}$$

and, hence,

$$G_s'(0) = \prod_{k=1}^{m+1}\rho_k\Bigg[\sum_{h=1}^{m+1}\frac{1}{\prod_{\ell=1,\ell\neq h}^{m+1}(-\rho_h+\rho_\ell)}\Bigg(\prod_{i=1}^{m}\frac{\rho_h+\eta_i}{\eta_i}\Bigg)\Bigg]. \tag{4.12}$$

Similarly, by taking $g(y) = e^y$ for $y \le 0$ in (4.7), we get $\boldsymbol{R}_j = \frac{1}{1+\eta_j}$ for $1 \le j \le m+1$. By Theorem 4.3 again, we get

$$H_s(x) = \sum_{h=1}^{m+1}\frac{1}{\rho_h\prod_{\ell=1,\ell\neq h}^{m+1}(-\rho_h+\rho_\ell)} \times \Bigg\{\sum_{j=1}^{m+1}\frac{1}{1+\eta_j}\Bigg[\prod_{k=1}^{m+1}(\eta_j+\rho_k)\prod_{i=1,i\neq j}^{m+1}\frac{-\rho_h-\eta_i}{\eta_j-\eta_i}\Bigg]\Bigg\}e^{\rho_h x} \tag{4.13}$$

and, hence,

$$H_s'(0) = \sum_{h=1}^{m+1}\frac{1}{\prod_{\ell=1,\ell\neq h}^{m+1}(-\rho_h+\rho_\ell)} \times \Bigg\{\sum_{j=1}^{m+1}\frac{1}{1+\eta_j}\Bigg[\prod_{k=1}^{m+1}(\eta_j+\rho_k)\prod_{i=1,i\neq j}^{m+1}\frac{-\rho_h-\eta_i}{\eta_j-\eta_i}\Bigg]\Bigg\}. \tag{4.14}$$

Plugging formulas (4.12) and (4.14) into (3.23), we get the explicit solution for the optimal default triggering level L^*(under the smooth pasting assumption). Similarly, plugging (4.11) and (4.13) into (3.19), (3.20), and (3.18) respectively, we get closed formulas for the value of debt, the value of the firm, and hence the value of equity.

Example 4.7 (Option pricing) A *first-touch digital* is a digital contract paying \$1 when and if a prespecified event occurs. Consider the first-touch digital, or *touch-and-out option*, which pays \$1 if the stock price S falls below the level H from above. If the stock price turns out to stay above the level H up to the expiration time $T > 0$, the claim expires worthless. It can be interpreted as an American-put-like option with a digital payoff. Since a payoff is the same for all levels of the stock price below the barrier, it is clearly optimal to exercise the option once the level H is crossed. We consider a general framework. Let S be the price process of a security satisfying $S = He^X$ under a risk-neutral probability measure, where X is given by (4.1), and $H > 0$ is a given constant threshold. Assume the existence of a constant risk-free rate $r > 0$. Consider a derivative with maturity T on this security whose payoff structure is given by

1. a bounded function $k(S_{\tau_H})$ at the time $\tau_H \le T$, where $\tau_H = \inf\{t; S_t \le H\}$,
2. a constant payment A at the time T if S does not cross the boundary H till maturity.

Let $x = \log(S_0/H)$, $g(y) = k(He^y)$ for $y \le 0$, and $\tau = \inf\{t; X_t \le 0\}$ as usual. Then by the risk-neutral pricing formula the risk-neutral price of this derivative is given by

$$\begin{aligned}\mathsf{V}(S_0, T) =&\mathbb{E}_x\big[e^{-r\tau} g(X_\tau)\mathbf{1}_{[\tau \le T]}\big] + \mathbb{E}_x\big[e^{-rT} A\mathbf{1}_{[\tau > T]}\big] \\ =&\mathbb{E}_x\big[e^{-r\tau} g(X_\tau)\mathbf{1}_{[\tau \le T]}\big] + e^{-rT} A\big(1 - \mathbb{P}_x[\tau \le T]\big). \end{aligned} \tag{4.15}$$

To find the unknown quantities $\mathbb{E}_x[e^{-r\tau} g(X_\tau)\mathbf{1}_{[\tau \le T]}]$ and $\mathbb{P}_x[\tau \le T]$, we take their Laplace transforms with respect to time T. By Fubini's theorem, for every $\beta > 0$,

$$\int_0^\infty e^{-\beta T}\mathbb{E}_x\big[e^{-r\tau} g(X_\tau)\mathbf{1}_{[\tau \le T]}\big]\,dT = \frac{1}{\beta}\mathbb{E}_x\big[e^{-(r+\beta)\tau} g(X_\tau)\big] \tag{4.16}$$

and

$$\int_0^\infty e^{-\beta T}\mathbb{P}_x[\tau \le T]\,dT = \frac{1}{\beta}\mathbb{E}_x\big[e^{-\beta\tau}\big]. \tag{4.17}$$

The right-hand sides of both equations can be written explicitly using Theorem 4.3. Hence we have an expression for the value $\mathsf{V}(S_0, T)$ up to a Fourier transform.

For the *first-touch digital*, we have $k \equiv 1$ and $A = 0$. Hence, using (4.16) and (4.11) with $s = r + \beta$, we deduce that the Laplace transform of the option price with respect to T is given by

$$\begin{aligned}&\int_0^\infty e^{-\beta T}\mathsf{V}(S_0, T)\,dT \\ &= \frac{1}{\beta}\mathbb{E}_x\big[e^{-(r+\beta)\tau}\big] \\ &= \frac{1}{\beta}\prod_{k=1}^{m+1}\rho_k\left\{\sum_{h=1}^{m+1}\left[\frac{1}{\rho_h\prod_{\ell=1,\ell\neq h}^{m+1}(-\rho_h+\rho_\ell)}\prod_{i=1}^{m}\frac{\rho_h+\eta_i}{\eta_i}\right]\left(\frac{S_0}{H}\right)^{\rho_h}\right\}. \end{aligned} \tag{4.18}$$

Remark 4.8 It is worth noting that the time zero value of debt in Example 3.9 is of the form $A\int_0^\infty me^{-mT}B(V,L,T)\,dT$, where $B(V,L,T)$ is the time zero value of a bond issued at zero with face value 1 and maturity T. The price formula (3.16) follows by similar arguments as in Example 4.7.

5 Concluding remarks

The expected discounted penalty is a generalized notion of the ruin probability in the insurance literature. It has been widely studied and generalized since Gerber and Shiu [17]. On the other hand, this function has been a major concern in the pricing of perpetual financial securities and the Laplace transforms of securities with finite maturity, and examples include option theory and credit risk modeling.

While empirical studies have been indicating the failure of diffusion models, the seminal paper of Merton [24] has received more attention in recent years. However, unlike the diffusion case in which many functionals are available, this is no longer the case for jump-diffusion processes, especially in the first-passage models. This is where the difficulty in the pricing of securities under a jump-diffusion process arises.

In this paper, we consider the jump-diffusion process as in [2]. In the case where the jump distribution is a two-sided phase-type one, by a Fourier transform argument, we transform the integro-differential equation of the expected discounted penalty into a homogeneous ordinary differential equation. The present method could possibly be extended to transform more integro-differential equations into ordinary differential equations and hence give an alternative approach to compute prices in jump-diffusion models.

Next, by ODE theory, we know that the solution for the expected discounted penalty is a linear combination of some known exponential functions. Moreover, by using the limit behavior of the expected discounted penalty and the integro-differential equation, we can determine these coefficients (under some conditions). All these distinguish not only our approach from that of Asmussen et al. [2] but also itself from the classical method to solve differential equations in which the knowledge of boundary values is required. This could be applied to solve other functionals once we have transformed their integro-differential into an ordinary differential equation.

Our results in fact provide explicit solutions to a large amount of existing pricing problems in finance. Using our closed-form solutions, especially Theorem 4.3, the Laplace transforms of securities with finite maturity have explicit solutions as mentioned in Example 4.7, and we show an example by considering a touch-and-out option. In addition, the pricing problems of perpetual securities in jump-diffusion models have improved answers. For example, both the value of shareholders and the optimal bankruptcy level (by using the smooth-pasting condition) in the optimal capital structure problem have exact solutions. We worked out these in Example 3.9 under the setup of Hilberink and Rogers [18], which is an extension of Leland and Toft [23]. Moreover, in Example 4.6, we obtained a closed-form solution even in a two-sided jump case. By using our closed-form solution as criterion, many structural form models in credit risk can be reconsidered and modified to see whether more phenomena in empirical studies can be captured by a jump-diffusion model.

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Appendix A Notation and proofs

Notation We write $\mathbb{R}_+ = \{x \in \mathbb{R}; x \geq 0\}$, $\mathbb{R}_{++} = \{x \in \mathbb{R}; x > 0\}$, and analogously $\mathbb{R}_-$ and $\mathbb{R}_{--}$.

Let I be an interval in $\mathbb{R}$ and $n \in \mathbb{N}$. We introduce the following function spaces:

1. $\mathcal{C}(I)$ is the space of real-valued continuous functions f on I.
2. $\mathcal{C}_b(I)$ is the space of bounded continuous functions f on I.
3. $\mathcal{C}_0(I)$ is the space of continuous functions f on I with $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$, provided that I is not bounded above (resp. below). Also, $\mathcal{C}_{0,b}(I) = \mathcal{C}_0(I) \cap \mathcal{C}_b(I)$.
4. $\mathcal{C}_c(I)$ is the space of continuous functions f on I with compact supports.
5. $\mathcal{C}^n(I)$ is the space consisting of $f \in \mathcal{C}(I)$ with $f^{(n)} \in \mathcal{C}(I)$. Here, on I°, $f^{(n)}$ is the usual nth derivative. If x is the left- (resp. right-)hand boundary point of I and is in I, $f^{(n)}(x)$ is the nth right- (resp. left-)hand derivative at x. $\mathcal{C}_0^n(I)$, $\mathcal{C}_b^n(I)$, $\mathcal{C}_c^n(I)$, and $\mathcal{C}_{0,b}^n(I)$ are defined similarly.
6. $\mathcal{C}_c^\infty(I) = \bigcap_k \mathcal{C}_c^k(I)$, $\mathcal{C}_{0,b}^\infty(I) = \bigcap_k \mathcal{C}_{0,b}^k(I)$, and $\mathcal{C}_0^\infty(I) = \bigcap_k \mathcal{C}_0^k(I)$.

Proof of Lemma 2.2 Fix $t \geq 0$. On $\{t \geq \tau\}$, using the local property of conditional expectation (see [19], Lemma 6.2) and the fact that $\Phi = g$ on $\mathbb{R}_-$, we have

$$\mathbb{E}_x\big[e^{-r\tau} g(X_\tau)\big|\mathcal{F}_{\tau\wedge t}\big] = \mathbb{E}_x\big[e^{-r\tau} g(X_\tau)\big|\mathcal{F}_\tau\big] = e^{-r\tau}\Phi(X_\tau).$$

Similarly, on $\{t < \tau\}$,

$$\mathbb{E}_x\big[e^{-r\tau} g(X_\tau)\big|\mathcal{F}_{\tau\wedge t}\big] = \mathbb{E}_x\big[e^{-r\tau} g(X_\tau)\big|\mathcal{F}_t\big].$$

By the strong Markov property of X, on $\{t < \tau\}$,

$$\mathbb{E}_x\big[e^{-r\tau} g(X_\tau)\big|\mathcal{F}_{\tau\wedge t}\big] = e^{-rt}\mathbb{E}_{X_t}\big[e^{-r\tau} g(X_\tau)\big] = e^{-rt}\Phi(X_t).$$

The proof is complete. □

Proof of Lemma 2.3 Since $\Phi \in \mathcal{C}^2(\mathbb{R}_{++})$, it suffices to show that, as $y \to x$,

$$\int \Phi(y-z)\,dF(z) \to \int \Phi(x-z)\,dF(z).$$

For any $y > 0$, write

$$\int \Phi(y-z)\,dF(z) = \int_y^\infty g(y-z)\,dF(z) + \int_{-\infty}^y \Phi(y-z)\,dF(z).$$

Let $\epsilon > 0$ and find $M > x$ such that $\int_{(-\infty,x-M+1]\cup[x+M-1,\infty)} dF(z) < \epsilon$. Then, for all $y > 0$ such that $|x-y| \leq 1/2$,

$$\left|\int \Phi(y-z)\,dF(z) - \int \Phi(x-z)\,dF(z)\right|$$
$$\le 2\|g\|_\infty \epsilon + \left|\int_y^{y+M} g(y-z)\,dF(z) - \int_x^{x+M} g(x-z)\,dF(z)\right|$$
$$+ \left|\int_{y-M}^{y} \Phi(y-z)\,dF(z) - \int_{x-M}^{x} \Phi(x-z)\,dF(z)\right|$$
$$\le 2\|g\|_\infty \epsilon + \left|\int_{-M}^{0} g(z)\big[f(y-z) - f(x-z)\big]\,dz\right|$$
$$+ \left|\int_0^M \Phi(z)\big[f(y-z) - f(x-z)\big]\,dz\right|,$$

where f is the density function for F. Since g and Φ are bounded and f is continuous on $\mathbb{R}\backslash\{0\}$, by dominated convergence, we have

$$\limsup_{y\to x}\left|\int \Phi(y-z)\,dF(z) - \int \Phi(x-z)\,dF(z)\right| \le 2\|g\|_\infty \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the proof is completed. □

To prove (2.8), we need the following lemmas and Dynkin's formula.

Lemma A.1 *For every bounded Borel-measurable function g on $\mathbb{R}_-$, there exists a sequence of uniformly bounded functions (g_n) in $\mathcal{C}_0^2(\mathbb{R}_-)$ such that $g_n \to g$ Lebesgue-a.e. on $\mathbb{R}_-$ and $g_n(0) = g(0)$ for all n.*

Proof Write dm for Lebesgue measure on $\mathbb{R}$. Let h be a normal density, and set $dG(y) = h(y)dm$. Then, by the strict positivity of h, dG and dm are equivalent.

Fix $\epsilon > 0$. By dominated convergence, we can find N large enough such that we have $\int_{-\infty}^0 |g\mathbf{1}_{[-N,0]} - g|\,dG < \epsilon$. Also, since $g\mathbf{1}_{[-N,0]}$ is bounded and in $L_1(\mathbb{R}, dm)$, a modification of the proof of Theorem 2.4 in [28], Chap. 2, shows that we can find a sequence of uniformly bounded $\mathcal{C}_0^2(\mathbb{R}_-)$-functions (g'_n) such that $g'_n \to g\mathbf{1}_{[-N,0]}$ dm-a.e. Since dG and dm are equivalent, $g'_n \to g\mathbf{1}_{[0,N]}$ dG-a.e. Hence, by dominated convergence, we can pick n large such that $\int |g'_n - g\mathbf{1}_{[-N,0]}|\,dG < \epsilon$. We have obtained that $\int_{-\infty}^0 |g'_n - g|\,dG < 2\epsilon$. And this implies that there exists a sequence of $\mathcal{C}_0^2(\mathbb{R}_-)$-functions (g_n) such that $g_n \to g$ dG-a.e. on $\mathbb{R}_-$. Since dG and dm are equivalent, we see that $g_n \to g$ dm-a.e. on $\mathbb{R}_-$. In addition, it is clear that we can pick (g_n) to be uniformly bounded.

Now, for each n, define a $\mathcal{C}_0^2(\mathbb{R}_-)$-function h_n such that $h_n = g_n$ on $(-\infty, 1/n)$, $h_n(0) = g(0)$ and (h_n) is uniformly bounded. Then it is obvious that (h_n) is the desired sequence. This completes the proof. □

Lemma A.2 *The following distributions are absolutely continuous with respect to Lebesgue measure*:

(1) *The distribution* $\mathbb{P}_x[X_{\tilde{\tau}} < 0, X_{\tilde{\tau}} \in \cdot]$, *where* X *is defined by* (2.1), *and* $\tilde{\tau}$ *is defined as in the proof of Theorem* 2.4.
(2) *The distribution* $\mathbb{P}_x[X_\tau < 0, X_\tau \in \cdot]$ *for* $x > 0$, *where* X *is defined by* (4.1).

Proof We only show (2), and the proof of (1) follows similarly. Let $\mathcal{N}$ be a Borel set in $(-\infty, 0)$ with Lebesgue measure zero. Write J_n for the nth jump time of the compound Poisson process Z. Observe that, for $x > 0$,

$$\mathbb{P}_x[X_\tau < 0, X_\tau \in \mathcal{N}] = \sum_{n=1}^{\infty} \mathbb{P}_x[X_\tau < 0, X_\tau \in \mathcal{N}, \tau = J_n].$$

We first show that $\mathbb{P}_x[X_\tau \in \mathcal{N}, X_\tau < 0, \tau = J_1] = 0$. Note that since $X^{(+)}$, J_1 and Y_1 are independent, so are $X^{(+)}_{J_1}$ and Y_1. Hence, we have

$$\begin{aligned}\mathbb{P}_x[X_\tau \in \mathcal{N}, X_\tau < 0, \tau = J_1] &= \mathbb{P}_x\big[X^{(+)}_{J_1} - Y_1 \in \mathcal{N}, X^{(+)}_{J_1} - Y_1 < 0, \tau = J_1\big] \\ &\le \mathbb{P}_x\big[X^{(+)}_{J_1} - Y_1 \in \mathcal{N}\big] \\ &= \int \mathbb{P}_x[z - Y_1 \in \mathcal{N}]\, dG(z),\end{aligned}$$

where G is the distribution of $X^{(+)}_{J_1}$. Since Y_1 has an absolutely continuous distribution, from the last inequality we deduce that $\mathbb{P}_x[X_\tau \in \mathcal{N}, X_\tau < 0, \tau = J_1] = 0$.

In general, for each $n \ge 2$, we have by the strong Markov property that

$$\mathbb{P}_x[X_\tau < 0, X_\tau \in \mathcal{N}, \tau = J_n] = \mathbb{E}_x\big[\mathbf{1}_{\tau > J_{n-1}} \mathbb{P}_{X_{J_{n-1}}}[X_\tau < 0, X_\tau \in \mathcal{N}, \tau = J_1]\big] = 0,$$

since J_{n-1} is an $(\mathcal{F}_t)$-stopping time and $X_{J_{n-1}} > 0$ on $\{\tau > J_{n-1}\}$. This implies that $\mathbb{P}_x[X_\tau < 0, X_\tau \in \cdot]$ is absolutely continuous with respect to Lebesgue measure for every $x > 0$. □

Theorem A.3 (Dynkin's formula) *Let* $X = (X(t))$ *be an* $\mathbb{R}^n$*-valued jump-diffusion, and let* $k \in \mathbf{C}^2_\mathbf{0}(\mathbb{R}^\mathbf{n})$. *If* η *is a stopping time such that* $\mathbb{E}_x(\eta) < \infty$, *then*

$$\mathbb{E}_x\big[k(X_\eta)\big] = k(x) + \mathbb{E}_x\left[\int_0^\eta Ak\big(X(s)\big)\, ds\right]$$

where A *is the generator of* X *defined by*

$$Ak(x) = \lim_{t \to 0^+} \frac{1}{t}\big\{\mathbb{E}_x\big[k\big(X(t)\big)\big] - k(x)\big\} \text{ (if the limit exists).}$$

(*See*, *for example*, [27], *Exercise* 44.22.)

Proof of (2.8) As in the proof of Theorem 2.4, let $\epsilon > 0$ and $\tilde{\tau}$ the first exit time for X from $(\epsilon, x+1)$. Pick a sequence of uniformly bounded functions $(g_n) \subset \mathcal{C}^2_0((-\infty, 0])$ that converges to g, except on a set $\mathcal{N}$ of Lebesgue measure 0 in $\mathbb{R}_{--}$, and $g_n(0) = g(0)$ for all n. (See Lemma A.1.) Therefore, we may consider a sequence of uniformly bounded functions $(\Phi_n; n \ge 1)$ that satisfies the following conditions:

C1 $\Phi_n = g_n$ on $(-\infty, 0]$;
C2 $\Phi_n = \Phi$ on $(1/n, n)$;
C3 $\Phi_n \in \mathcal{C}_0^2(\mathbb{R})$.

By (C1), (C2), and the fact that $g_n \to g$ except on $\mathcal{N}$, we have that $\Phi_n \to \Phi$ pointwise on $\mathbb{R}$ except on $\mathcal{N}$.

Now, pick n large such that $(\epsilon, x+\epsilon) \subset (1/n, n)$. Then $\Phi_n(x) = \Phi(x)$ by (C2). By Dynkin's formula, for all $t > 0$,

$$\mathbb{E}_x\big[e^{-r(\tilde{\tau}\wedge t)}\Phi_n(X_{\tilde{\tau}\wedge t})\big] = \mathbb{E}_x\bigg[\int_0^{\tilde{\tau}\wedge t} e^{-ru}(L-r)\Phi_n(X_u)\,du\bigg] + \Phi(x). \tag{A.1}$$

Since (Φ_n) is uniformly bounded, $\Phi_n \to \Phi$ pointwise on $\mathbb{R}_+$, and $\Phi_n \to \Phi$ a.e. on $(-\infty, 0)$, by Lemma A.2(1) and dominated convergence,

$$\lim_{n\to\infty} \mathbb{E}_x\big[e^{-r(t\wedge\tilde{\tau})}\Phi_n(X_{t\wedge\tilde{\tau}})\big] = \mathbb{E}_x\Big[e^{-r(t\wedge\tilde{\tau})}\Phi(X_{\tilde{\tau}\wedge t})\Big]. \tag{A.2}$$

On the other hand, for every $u < \tilde{\tau}\wedge t$, $X_u \in (\epsilon, x+\epsilon)$, and hence $\Phi_n(X_u) = \Phi(X_u)$ for all large n. These give

$$(L-r)\Phi(X_u) - (L-r)\Phi_n(X_u) = \int \big[\Phi(X_u - y) - \Phi_n(X_u - y)\big]\,dF(y) \tag{A.3}$$

and hence

$$\big|(L-r)\big[\Phi(X_u) - \Phi_n(X_u)\big]\big| \le \sup_n \|\Phi_n\|_\infty + \|g\|_\infty. \tag{A.4}$$

By dominated convergence, (A.3) and the absolute continuity of F imply, for all $u < t\wedge\tilde{\tau}$,

$$(L-r)\Phi_n(X_u) \to (L-r)\Phi(X_u), \quad n\to\infty.$$

By (A.4) and dominated convergence, we have

$$\lim_{n\to\infty} \mathbb{E}_x\bigg[\int_0^{\tilde{\tau}\wedge t} e^{-ru}(L-r)\Phi_n(X_u)\,du\bigg] = \mathbb{E}_x\bigg[\int_0^{t\wedge\tilde{\tau}} e^{-ru}(L-r)\Phi(X_u)du\bigg]. \tag{A.5}$$

By (A.2) and (A.5), letting $n\to\infty$ for both sides of (A.1), we get (2.8). □

Proof of Proposition 3.3 Fix $x > 0$. Since $(L-r)\Phi(x) = 0$, by (3.7), we get that (3.9) holds for $k = 0$ and, hence, $\Phi \in \mathcal{C}_{0,b}^2(\mathbb{R}_{++})$. Since Φ is continuous at x (Theorem 2.1), we obtain, by Theorem B.3, that $T_{\boldsymbol{B}_+}^+\Phi$ is differentiable at x and

$$\frac{d}{dx}T_{\boldsymbol{B}_+}^+\Phi(x) = \boldsymbol{B}_+T_{\boldsymbol{B}_+}^+\Phi(x) + \Phi(x)\boldsymbol{I}.$$

Similarly, we have $\frac{d}{dx}T_{\boldsymbol{B}_-}^-\Phi(x) = -\boldsymbol{B}_-T_{\boldsymbol{B}_-}^-\Phi(x) - \Phi(x)\boldsymbol{I}$. So, by the differentiability of $T_{\boldsymbol{B}_\pm}^\pm\Phi$ at x and (3.9) for $k=0$, Φ'' is differentiable at x, and (3.9) holds for

$k = 1$. Since $\Phi \in \mathcal{C}^2_{0,b}(\mathbb{R}_{++})$ and $T^{\pm}_{\boldsymbol{B}_\pm}\Phi \in \mathcal{C}_{0,b}(\mathbb{R}_{++})$, $\Phi''' \in \mathcal{C}_{0,b}(\mathbb{R}_{++})$ and hence $\Phi \in \mathcal{C}^3_{0,b}(\mathbb{R}_{++})$. A similar argument holds for general k. □

Proof of Proposition 3.6 Denote by $\mathcal{Z}_{(+)}$ the collection of all solutions to $\psi(\zeta) - r = 0$ with nonnegative real parts, counting their multiplicity. Since $D_1\Phi \equiv 0$ on $\mathbb{R}_{++}$ and $\psi(\zeta) - r$ and $\mathcal{P}_1(\zeta)$ have the same zero set, Φ is of the form

$$\Phi(x) = \sum_{\rho_i \in \mathcal{Z}_{(-)}} Q_i(x)e^{\rho_i x} + \sum_{\rho' \in \mathcal{Z}_{(+)}} R_{\rho'}(x)e^{\rho' x}$$

for some polynomials $\boldsymbol{Q}_i$ and $R_{\rho'}$. (See [29], Theorem 19.3 and Lesson 20.) We show that $R_\rho = 0$ for all $\rho \in \mathcal{Z}_{(+)}$. Assume that $R_\rho(x) \neq 0$ for some $\rho \in \mathcal{Z}_{(+)}$. Let $a \geq 0$ be the maximum of all real parts of members in $\mathcal{Z}_{(+)}$, and let $\{\rho'_1, \ldots, \rho'_k\}$ be the set of all elements in $\mathcal{Z}_{(+)}$ such that $\Re(\rho'_j) = a$. Now, let $m \geq 0$ be the smallest integer such that the order of $R_{\rho'_j}$ is $\leq m$ for all j, and select the ρ'_j such that the order of $R_{\rho'_j}$ is equal to m. Still call the selected members $\{\rho'_1, \rho'_2, \ldots, \rho'_k\}$. Let $a_j \neq 0$ be the coefficient of x^m in $R_{\rho'_j}$ for all j. Then

$$\Phi(x)x^{-m}e^{-ax} = \sum_j a_j e^{i\Im(\rho'_j)x} + h(x), \quad \text{where } h(x) \to 0 \text{ as } x \to \infty.$$

However, $\sum_j a_j e^{i\Im(\rho'_j)x}$ is the discrete Fourier transform of a nonzero function and hence is not identically zero, and it has period 2π. So, $\Phi(x)x^{-m}e^{-ax} \nrightarrow 0$ as $x \to \infty$. Since $\Phi \in \mathcal{C}_0(\mathbb{R}_+)$, this is impossible. Hence, $R_{\rho'} \equiv 0$ for all $\rho' \in \mathcal{Z}_{(+)}$. □

Proof of Proposition 3.11 The first statement follows from the fact that $\sum_{i=1}^S \boldsymbol{Q}_i = \Phi(0) = g(0)$.

Consider the second statement. First, note that the minimality of the representation $(\boldsymbol{\alpha}_+, \boldsymbol{B}_+)$ guarantees that the collection of all zeros $\eta \in \mathbb{C}$ of the rational function $\frac{1}{r-\psi(\eta)}$ is equal to the collection of all eigenvalues of $\boldsymbol{B}_+$. (For details, see [2], Lemma 1 and the paragraph above it.) Hence, ρ_i is not an eigenvalue of $\boldsymbol{B}_+$ for each i, and $\rho_i\boldsymbol{I} - \boldsymbol{B}_+$ is invertible for all i. By Theorem B.1(4) and B.2(2), we have

$$\begin{aligned}\int_{-\infty}^x \Phi(x-y)f(y)\,dy &= \int_0^x \boldsymbol{Q}^\top \boldsymbol{e}^{\boldsymbol{\rho}}(x-y)f(y)\,dy + \int_{-\infty}^0 \boldsymbol{Q}^\top \boldsymbol{e}^{\boldsymbol{\rho}}(x-y)f(y)\,dy \\ &= \sum_{i=1}^S \boldsymbol{Q}_i e^{\rho_i x} p\boldsymbol{\alpha}_+(\rho_i\boldsymbol{I} - \boldsymbol{B}_+)^{-1}\big(\boldsymbol{I} - e^{x(\boldsymbol{B}_+ - \rho_i\boldsymbol{I})}\big)\boldsymbol{b}_+^\top \\ &\quad + \sum_{i=1}^S \boldsymbol{Q}_i e^{\rho_i x} q\boldsymbol{\alpha}_-(-\rho_i\boldsymbol{I} - \boldsymbol{B}_-)^{-1}\boldsymbol{b}_-^\top.\end{aligned}$$

Therefore, by (3.5),

$$\int_{-\infty}^{x} \Phi(x-y) f(y)\, dy = \sum_{i=1}^{S} \boldsymbol{Q}_i e^{\rho_i x} \psi_1(\rho_i) - \sum_{i=1}^{S} \boldsymbol{Q}_i e^{\rho_i x} p \boldsymbol{\alpha}_+ (\rho_i \boldsymbol{I} - \boldsymbol{B}_+)^{-1} e^{x(\boldsymbol{B}_+ - \rho_i \boldsymbol{I})} \boldsymbol{b}_+^\top.$$

Recall that $\psi(\rho_i) = D\rho_i^2 + c\rho_i + \lambda\psi_1(\rho_i) - (\lambda + r) = 0$ for all i. Since $\Phi(x) = \sum_{i=1}^{S} \boldsymbol{Q}_i e^{\rho_i x}$, we have

$$\begin{aligned} 0 &= (L-r)\Phi(x) = D\Phi''(x) + c\Phi'(x) + \lambda \int \Phi(x-y) f(y)\, dy - (\lambda + r)\Phi(x) \\ &= \sum_{i=1}^{S} \boldsymbol{Q}_i \psi(\rho_i) e^{\rho_i x} + \lambda \int_x^\infty g(x-y) f(y)\, dy \\ &\quad - \lambda \sum_{i=1}^{S} \boldsymbol{Q}_i e^{\rho_i x} p \boldsymbol{\alpha}_+ (\rho_i \boldsymbol{I} - \boldsymbol{B}_+)^{-1} e^{x(\boldsymbol{B}_+ - \rho_i \boldsymbol{I})} \boldsymbol{b}_+^\top \\ &= \lambda p \boldsymbol{\alpha}_+ \left(\int_0^\infty g(-y) e^{\boldsymbol{B}_+ y} dy \right) e^{x \boldsymbol{B}_+} \boldsymbol{b}_+^\top \\ &\quad - \lambda p \sum_{i=1}^{S} \boldsymbol{Q}_i e^{\rho_i x} \boldsymbol{\alpha}_+ (\rho_i \boldsymbol{I} - \boldsymbol{B}_+)^{-1} e^{x(\boldsymbol{B}_+ - \rho_i \boldsymbol{I})} \boldsymbol{b}_+^\top. \end{aligned}$$

So, the conclusion of the proposition follows. □

Proof of Proposition 4.1 Let $x > 0$. Similarly to the proof of (2.8), we can pick a sequence of uniformly bounded functions $(\phi_n) \subset \mathcal{C}_0^2(\mathbb{R})$ such that $\phi_n \equiv \phi$ on $\mathbb{R}_+$ and $\phi_n \to \phi$ a.e. on $(-\infty, 0)$. By Dynkin's formula, we have

$$\mathbb{E}_x\left[e^{-r(t\wedge\tau)} \phi_n(X_{\tau\wedge t})\right] = \mathbb{E}_x\left[\int_0^{t\wedge\tau} e^{-ru} (L-r)\phi_n(X_u)\, du\right] + \phi(x). \tag{A.6}$$

Note that ϕ_n and ϕ only differ on $(-\infty, 0)$ and $X_u > 0$ for all $u < t \wedge \tau$. Moreover, the jump distribution of Z is absolutely continuous. Hence, similarly to the proof of (A.5), we have

$$\lim_{n\to\infty} \mathbb{E}_x\left[\int_0^{t\wedge\tau} e^{-ru} (L-r)\phi_n(X_u)\, du\right] = \mathbb{E}_x\left[\int_0^{t\wedge\tau} e^{-ru} (L-r)\phi(X_u)\, du\right].$$

By dominated convergence and Lemma A.2(1), letting $n \to \infty$ on both sides of (A.6) gives

$$\mathbb{E}_x\left[e^{-r(t\wedge\tau)} \phi(X_{\tau\wedge t})\right] = \mathbb{E}_x\left[\int_0^{t\wedge\tau} e^{-ru} (L-r)\phi(X_u)\, du\right] + \phi(x) = \phi(x). \tag{A.7}$$

Note that the last equality follows from the assumption that $(L-r)\phi \equiv 0$ on $\mathbb{R}_{++}$. Let $t \to \infty$ on both sides of the last equality. Since $r > 0$, the result follows by the fact that $e^{-rt}\mathbf{1}_{[\tau > t]} \to 0$ as $t \to \infty$. This completes the proof. □

Proof of Proposition 4.2 Set $\phi(x) = \sum_{i=1}^{S} \boldsymbol{Q}_i e^{\rho_i x}$ for $x > 0$, and $\phi(x) = g(x)$ for $x \le 0$. Then, for every $x > 0$,

$$\begin{aligned}
(L-r)\phi(x) &= \sum_{i=1}^{m+1} \boldsymbol{Q}_i \left[D\rho_i^2 + c\rho_i + \int_0^\infty \left(e^{z\rho_i} - 1 - \rho_i z \mathbf{1}_{|z|\le 1}\right)\nu(dz) - (\lambda + r)\right] \\
&\quad \times e^{\rho_i x} \\
&\quad + \lambda \left[\sum_{i=1}^{m+1} \boldsymbol{Q}_i \int_0^x e^{\rho_i (x-y)} f_{(+)}(y) dy + \int_x^\infty g(x-y) f_{(+)}(y)\, dy \right] \\
&= \sum_{i=1}^{m+1} \boldsymbol{Q}_i \left[D\rho_i^2 + c\rho_i + \int_0^\infty \left(e^{z\rho_i} - 1 - \rho_i z \mathbf{1}_{|z|\le 1}\right)\nu(dz) \right.\\
&\quad \left. + \lambda \sum_{j=1}^{m} \frac{p_j \eta_j}{\eta_j + \rho_i} - (\lambda + r)\right] e^{\rho_i x} + \lambda \sum_{j=1}^{m} e^{-\eta_j x} p_j \\
&\quad \times \left[- \sum_{i=1}^{m+1} \frac{\eta_j \boldsymbol{Q}_i}{\eta_j + \rho_i} + \eta_j \int_{-\infty}^0 g(y) e^{\eta_j y}\, dy \right].
\end{aligned}$$

Since $\psi(\rho_i) - r = 0$ for all i and $\boldsymbol{Q}$ satisfies (4.4) and hence (4.3), we conclude that $(L-r)\phi(x) = 0$. By Proposition 4.1, $\phi \equiv \Phi$ on $\mathbb{R}$. □

Appendix B Toolbox for phase-type distributions

We first give a summary of some facts of matrix algebra.

Theorem B.1

(1) *The matrix exponential function $e^{h\boldsymbol{A}}$ is differentiable on $\mathbb{R}$ and*

$$\frac{de^{h\boldsymbol{A}}}{dh} = \boldsymbol{A} e^{h\boldsymbol{A}} = e^{h\boldsymbol{A}} \boldsymbol{A}.$$

(2) *Let $\theta_1, \ldots, \theta_k$ be the eigenvalues of a matrix $\boldsymbol{A}$. Then, for each $s > \max_i \Re(\theta_i)$,* $\lim_{t\to\infty} e^{-st} \exp(t\boldsymbol{A}) = 0$.

(3) *Let $\boldsymbol{B}$ be a subintensity matrix. Then $\boldsymbol{B}$ is nonsingular if and only if every eigenvalue of $\boldsymbol{B}$ has a strictly negative real part.*

(4) *If $\boldsymbol{A}$ is nonsingular, then $\int_v^t e^{x\boldsymbol{A}}\, dx = \boldsymbol{A}^{-1}(e^{t\boldsymbol{A}} - e^{v\boldsymbol{A}})$; if further all eigenvalues of $\boldsymbol{A}$ have strictly negative real parts, then $\int_0^\infty e^{x\boldsymbol{A}}\, dx = -\boldsymbol{A}^{-1}$.*

The following summarizes the main analytic properties of phase-type distributions.

Theorem B.2 *Suppose that F is PH($\boldsymbol{\alpha}, \boldsymbol{B}$). Then*:

(1) *The n-th moment of F is $(-1)^n n! \boldsymbol{\alpha} \boldsymbol{B}^{-n} \boldsymbol{e}^\top$.*
(2) *For any $s \in \mathbb{C}$ with $\Re(s) \geq 0$, $s \mapsto \int_0^\infty e^{-sx}\, dF(x) = \boldsymbol{\alpha}(s\boldsymbol{I} - \boldsymbol{B})^{-1}\boldsymbol{b}^\top$ is a rational function.*

Assume that $\boldsymbol{B}$ is a nonsingular subintensity matrix. Theorem B.1(2) and (3) imply that

$$\lim_{y\to\infty} e^{\boldsymbol{B}y} e^{\delta y} = 0$$

for some $\delta > 0$. Hence, $T_{\boldsymbol{B}}^{\pm}$ given by (3.8) is well defined.

Theorem B.3 *Let h be a bounded Borel-measurable function. If h is continuous at x, then $T_{\boldsymbol{B}}^{\pm} h$ are both differentiable at x and $(\frac{d}{dx} - \boldsymbol{B})T_{\boldsymbol{B}}^{+} = -(\boldsymbol{B} + \frac{d}{dx})T_{\boldsymbol{B}}^{-} = \boldsymbol{I}$. Moreover, if $h \in \mathcal{C}_{0,b}(\mathbb{R}_{++})$, each entry of $T_{\boldsymbol{B}}^{\pm} h$ is in $\mathcal{C}_{0,b}(\mathbb{R}_{++})$.*

Proof Assume that h is bounded and continuous at x. By Theorem B.1(1), $\frac{d}{dx} e^{\boldsymbol{B}x} = \boldsymbol{B} e^{\boldsymbol{B}x}$. On the other hand, since h is continuous at x, $\frac{d}{dx}\int_{-\infty}^x h(y) e^{-\boldsymbol{B}y} dy = h(x) e^{-x\boldsymbol{B}}$. Therefore,

$$\frac{d}{dx} T_{\boldsymbol{B}}^{+} h(x) = \frac{d}{dx}\left(e^{\boldsymbol{B}x} \int_{-\infty}^x h(y) e^{-\boldsymbol{B}y}\, dy\right) = \boldsymbol{B} T_{\boldsymbol{B}}^{+} h(x) + h(x)\boldsymbol{I}.$$

This shows that $(\frac{d}{dx} - \boldsymbol{B})T_{\boldsymbol{B}}^{+} h(x) = h(x)\boldsymbol{I}$. Similarly, $-(\boldsymbol{B} + \frac{d}{dx})T_{\boldsymbol{B}}^{-} h(x) = h(x)\boldsymbol{I}$. Finally, if $h \in \mathcal{C}_{0,b}(\mathbb{R}_{++})$, then it is clear that $T_{\boldsymbol{B}}^{\pm} h$ is bounded on $\mathbb{R}$. Moreover, by dominated convergence, $T_{\boldsymbol{B}}^{\pm} h \in \mathcal{C}_{0,b}(\mathbb{R}_{++})$. This proves the last statement. □

Appendix C First-order derivative of Φ at zero

We consider the process (2.1) and set

$$\Delta = \sup\left\{\zeta \in \mathbb{R}_+; \int e^{-\xi y}\, dF(y) < \infty, \forall \xi \in [0, \zeta]\right\}. \tag{C.1}$$

Throughout this section, we assume that $\Delta > 0$ and $\psi(\Delta-) > 0$. Let $r > 0$. Since $\psi(0) - r < 0$ and ψ is strictly convex on $[0, \Delta)$, there exists a *unique* number $\rho^* \in (0, \Delta)$ such that $\psi(\rho^*) - r = 0$. (ρ^* is called the *Lundberg constant* in the actuarial literature.) We write

$$\beta = \frac{c}{D} + \rho^*, \quad \alpha = \beta + \rho^* \tag{C.2}$$

and $h_{\rho^*}(x) = e^{-\rho^* x} h(x)$ for any function h. Recall that $\Phi(x) = \mathbb{E}_x[e^{-r\tau} g(X_\tau)]$ for some fixed bounded Borel-measurable function $g : (-\infty, 0] \to \mathbb{R}$ and $\tau = \inf\{t \geq 0; X_t \leq 0\}$. Following similar arguments as in Sect. 3 of [16], we calculate the first-order derivative of Φ at 0.

Theorem C.1 *Suppose $r > 0$. The derivative of Φ at 0 is given by*

$$\Phi'(0+) = \vartheta_0 - \frac{\lambda}{D}\int_{-\infty}^{0} dF(y)\int_0^{-y} dv\, \Phi_{\rho^*}(v)e^{-\rho^* y}, \tag{C.3}$$

where

$$\vartheta_0 = -\beta g(0) + \frac{\lambda}{D}\int_0^{\infty} dv\int_v^{\infty} dF(y)\, e^{-\rho^* y} g_{\rho^*}(v-y). \tag{C.4}$$

In particular, if $\int_{(-\infty,0]} dF = 0$, then $\Phi'(0+) = \vartheta_0$.

Proof By Theorem 2.4, we have $(L-r)\Phi(v) = 0$ for all $v > 0$. Multiplying both sides of this equation by $e^{-\rho^* v}$ gives

$$e^{-\rho^* v}\left[D\frac{d^2}{dv^2} + c\frac{d}{dv} - (\lambda + r)\right]\Phi(v) + \lambda\int_{-\infty}^{\infty}\Phi(v-y)e^{-\rho^* v}\, dF(y) = 0. \tag{C.5}$$

Note that $\Phi(v) = e^{\rho^* v}\Phi_{\rho^*}(v)$ for $v \in \mathbb{R}$. Then

$$\Phi'(v) = \rho^* e^{\rho^* v}\Phi_{\rho^*}(v) + e^{\rho^* v}\Phi'_{\rho^*}(v) \tag{C.6}$$

and

$$\Phi''(v) = \left(\rho^*\right)^2 e^{\rho^* v}\Phi_{\rho^*}(v) + 2\rho^* e^{\rho^* v}\Phi'_{\rho^*}(v) + e^{\rho^* v}\Phi''_{\rho^*}(v).$$

Hence, (C.5) becomes

$$0 = D\Phi''_{\rho^*}(v) + \left(c + \rho^*\sigma^2\right)\Phi'_{\rho^*}(v) + \lambda\int_{-\infty}^{\infty}\Phi_{\rho^*}(v-y)e^{-\rho^* y}\, dF(y)$$
$$+ \left[D\left(\rho^*\right)^2 + c\rho^* - (\lambda + r)\right]\Phi_{\rho^*}(v).$$

Recall that $\psi(\rho^*) = r$. Then

$$0 = D\Phi''_{\rho^*}(v) + \left(c + \rho^*\sigma^2\right)\Phi'_{\rho^*}(v) + \lambda\int_{-\infty}^{\infty}\Phi_{\rho^*}(v-y)e^{-\rho^* y}\, dF(y)$$
$$- \lambda\Phi_{\rho^*}(v)\int e^{-\rho^* y}\, dF(y). \tag{C.7}$$

By Theorem 2.1, $\Phi'(0+)$ exists as a right-hand side derivative. Therefore, integrating (C.7) from $v = 0$ to $v = z$ gives

$$D\left[\Phi'_{\rho^*}(z) - \Phi'_{\rho^*}(0+)\right] + \left(c + \rho^*\sigma^2\right)\left[\Phi_{\rho^*}(z) - \Phi_{\rho^*}(0)\right]$$
$$+ \lambda\int_0^z dv\int_{-\infty}^{v} dF(y)\,\Phi_{\rho^*}(v-y)e^{-\rho^* y}$$
$$- \lambda\int_0^z dv\int dF(y)\,\Phi_{\rho^*}(v)e^{-\rho^* y} + \lambda\int_0^z dv\int_v^{\infty} dF(y) g_{\rho^*}(v-y)e^{-\rho^* y} = 0. \tag{C.8}$$

Note that we have, by Fubini's theorem, a change of variables; again, by Fubini's theorem,

$$\begin{aligned}
&\int_0^z dv \int_{-\infty}^v dF(y)\,\Phi_{\rho^*}(v-y)e^{-\rho^* y} \\
&\quad = \int_{-\infty}^0 dF(y) \int_0^z dv\,\Phi_{\rho^*}(v-y)e^{-\rho^* y} + \int_0^z dF(y) \int_y^z dv\,\Phi_{\rho^*}(v-y)e^{-\rho^* y} \\
&\quad = \int_{-\infty}^0 dF(y) \int_0^z dv\,\Phi_{\rho^*}(v-y)e^{-\rho^* y} + \int_0^z dF(y) \int_0^{z-y} dv\,\Phi_{\rho^*}(v)e^{-\rho^* y} \\
&\quad = \int_{-\infty}^0 dF(y) \int_0^z dv\,\Phi_{\rho^*}(v-y)e^{-\rho^* y} + \int_0^z dv \int_0^{z-v} dF(y)\,\Phi_{\rho^*}(v)e^{-\rho^* y}.
\end{aligned}$$

So, (C.8) is equivalent to

$$\begin{aligned}
0 = {} & D\big[\Phi'_{\rho^*}(z) - \Phi'_{\rho^*}(0)\big] + \big(c+\rho^*\sigma^2\big)\big[\Phi_{\rho^*}(z) - \Phi_{\rho^*}(0)\big] \\
& + \lambda\bigg[\int_{-\infty}^0 dF(y) \int_0^z dv\,\Phi_{\rho^*}(v-y)e^{-\rho^* y} \\
& - \int_0^z dv \bigg(\int_{z-v}^\infty + \int_{-\infty}^0\bigg) dF(y)\,\Phi_{\rho^*}(v)e^{-\rho^* y} \\
& + \int_0^z dv \int_v^\infty dF(y)e^{-\rho^* y} g_{\rho^*}(v-y)\bigg].
\end{aligned} \tag{C.9}$$

By Theorem 2.1, $\Phi'(z)$ and $\Phi(z)$ tend to zero as $z \to \infty$. Hence, letting $z \to \infty$ in the last equation gives

$$\begin{aligned}
\Phi'_{\rho^*}(0) = \frac{1}{D}\bigg[& -\big(c+\rho^*\sigma^2\big)g(0) - \lambda \int_{-\infty}^0 dF(y) \int_0^{-y} dv\,\Phi_{\rho^*}(v)e^{-\rho^* y} \\
& + \lambda \int_0^\infty dv \int_v^\infty dF(y)e^{-\rho^* y} g_{\rho^*}(v-y)\bigg].
\end{aligned} \tag{C.10}$$

Since $\Phi'(0) = \rho^* g(0) + \Phi'_{\rho^*}(0)$, we get (C.3). □

Proposition C.2 *Assume that* (3.2) *holds, and* $r > 0$. *Then the number* Δ *defined by* (*C*.1) *is in* $(0, \infty]$, *and* $\psi(\Delta-) = +\infty$.

Proof First, we show that $\Delta > 0$. Since all eigenvalues of $\boldsymbol{B}_-$ have strictly negative real parts by Theorem B.1(3), so do the eigenvalues of $\boldsymbol{B}_- + \zeta \boldsymbol{I}$ for all ζ in an open interval in $(0, \infty)$. For all these ζ, from Theorem B.1(4) we obtain that

$$\int_0^\infty e^{\zeta y} f_{(-)}(y)\,dy = \boldsymbol{\alpha}_-(-\zeta \boldsymbol{I} - \boldsymbol{B}_-)^{-1}\boldsymbol{b}_-^\top < \infty$$

and hence $\int e^{-\zeta y}\,dF(y) < \infty$.

Next, we show that $\psi(\Delta-) = +\infty$. If $\Delta < \infty$, Δ must be a pole of $\psi(\zeta)$ on $\mathbb{C}$ so that $\psi(\Delta-) = +\infty$ or $-\infty$. Since ψ is strictly convex on $[0, \Delta)$, we have $\psi(\Delta-) = +\infty$. If $\Delta = \infty$, we have $\psi(\Delta-) = +\infty$ by the fact that $D > 0$. This completes the proof. □

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