Finance Stoch (2007) 11:3–27
DOI 10.1007/s00780-006-0027-z

Optimal dividend policy and growth option

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Received: 25 July 2005 / Accepted: 4 September 2006 /
Published online: 6 December 2006

Abstract We analyse the interaction between the dividend policy and the decision on investment in a growth opportunity of a liquidity constrained firm. This leads us to study a mixed singular control/optimal stopping problem for a diffusion that we solve quasi-explicitly by establishing a connection with an optimal stopping problem. We characterize situations where it is optimal to postpone the distribution of dividends in order to invest at a subsequent date in the growth opportunity. We show that uncertainty and liquidity shocks have an ambiguous effect on the investment decision.

Keywords Mixed singular control/optimal stopping problem · Local time · Dividend · Growth option

JEL subject classification G11 · C61 · G35

Mathematics Subject Classification (2000) 60G40 · 91B70 · 93E20

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1 Introduction

Research on optimal dividend payouts for a cash constrained firm is based on the premise that the firm wants to pay some of its surplus to the shareholders as dividends and therefore follows a dividend policy that maximizes the expected present value of all payouts until bankruptcy. This approach has been used in particular to determine the market value of a firm which, in line with Modigliani and Miller [23], is defined as the present value of the sum of future dividends. In diffusion models, the optimal dividend policy can be determined as the solution of a singular stochastic control problem. In two influential papers, Jeanblanc and Shiryaev [18] and Radner and Shepp [26] assume that the firm exploits a technology defined by a cash generating process that follows a drifted Brownian motion. They show that the optimal dividend policy is characterized by a threshold so that whenever the cash reserve goes above this threshold, the excess is paid out as dividend.

Models that involve singular stochastic controls or mixed singular/regular stochastic controls are now widely used in mathematical finance. Recent contributions have for instance emphasized restrictions imposed by a regulatory agency [25], the interplay between dividend and risk policies [1,3,14], or the analysis of hedging and insurance decisions [27]. A new class of models that combine features of both regular stochastic control and optimal stopping has recently emerged. Two recent papers in this line are Miao and Wang [22], who study the interactions between investment and consumption under incomplete markets, and Hugonnier et al. [16], who focus on irreversible investment for a representative agent in a general equilibrium framework. From a mathematical viewpoint, the problem we are interested in is different and combines features of both singular stochastic control and optimal stopping. Such models are less usual in corporate finance and, to the best of our knowledge, only Guo and Pham [13] dealt with such an issue. These authors consider a firm having to choose the optimal time to activate production and then control it by buying or selling capital. Their problem can be solved in a two-step formulation which consists in solving the singular control problem arising from the production activity after the exercise of the investment option.

The novelty of our paper is to consider the interaction between dividends and investment as a singular control problem. Specifically, we consider a firm with a technology in place and a growth option. The growth option offers the firm the opportunity to invest in a new technology that increases its profit rate. The firm has no access to external funding and therefore finances the opportunity cost of the growth option from its cash reserve. Our objective is then to study the interactions between dividend policy and investment decisions. Such an objective leads us to deal with a mixed singular control/optimal stopping problem that we solve by establishing a connection with an optimal stopping problem. Precisely, let us consider the two following alternative strategies: (i) never invest in the growth option (and follow the associated optimal dividend policy), (ii) defer dividend distributions, invest optimally in the growth option (and follow the associated optimal dividend policy). We show that the firm value under the optimal dividend/investment policy coincides with the value function of the optimal stopping problem whose payoff function is the maximum of the values of the firm computed under the above strategies (i) and (ii). The equivalence between the mixed singular control/optimal stopping problem and the stopping problem is proved in our main theorem and is based on a verification procedure for stochastic control. We compute quasi-explicitly the value function and show that it is piecewise C^2 and not necessarily concave as in standard singular control problems. Furthermore, from a detailed analysis based on properties of local time, we construct explicitly the optimal dividend/investment policy. Our model allows us to address several important questions in corporate finance. We explain when it is optimal to postpone dividend distribution, to accumulate cash and to invest at a subsequent date in the growth option. We analyse the effects of cash flow and uncertainty shocks on dividend policy and investment decision. We study the effects of financing constraints on dividend policy and investment decision with respect to a situation where the firm has unlimited cash.

Finally, our work helps to bridge the gap between the literature on optimal dividend payouts and the now well-established real option literature. The real option literature analyses optimal investment policies that can be mathematically determined as solutions of optimal stopping problems. The original model is due to McDonald and Siegel [21] and has been extended in various ways by many authors.[1] An important assumption of standard models is that the investment decision can be made independently of the financing decision. In contrast, in our paper, two interrelated features drive our investment problem. First, the firm is cash-constrained and must finance the investment using its cash reserve. Second, the firm must decide its dividend distribution policy in view of its growth opportunity. Such a perspective can be related to Boyle and Guthrie [2] who analyse, in a numerical model, the dynamic investment decision of a firm submitted to cash constraints. Two state variables drive their model: the cash process and a project value process for which the decision maker has to pay a fixed amount. Boyle and Guthrie [2] do not consider, however, the dividend distribution policy.

The outline of the paper is as follows. Section 2 describes the model, analyses some useful benchmarks, provides a formulation of our problem based on the dynamic programming principle, and derives a necessary and sufficient condition for the growth option to be worthless. Section 3 states and proves our main theorem, derives the optimal dividend/investment policy and presents financial implications. Section 4 concludes.

[1] See for instance Dixit and Pindyck [9] for an overview of this literature. Recent developments include for example the impact of asymmetric information in a duopoly model [5,20], the impact of agency conflicts and information asymmetries [11], regime switches [12], learning [6], incomplete markets and risk aversion [15], [17], or investment in alternative projects [7].

2 The model

2.1 Formulation of the problem

We consider a firm whose activities generate a cash process. The firm faces liquidity constraints that cause bankruptcy as soon as the cash process reaches the threshold 0. The manager of the firm acts in the best interest of its shareholders and maximizes the expected present value of dividends up to bankruptcy. At any time the firm has the option to invest in a new technology that increases the drift of the cash generating process from μ_0 to $\mu_1 > \mu_0$ without affecting its volatility σ. This growth opportunity requires a fixed investment cost I that must be financed using the cash reserve. Our purpose is to study the optimal dividend/investment policy of such a firm.

The mathematical formulation of our problem is as follows. We start with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration $(\mathcal{F}_t)_{t\geq 0}$ and a Brownian motion $W = (W_t)_{t\geq 0}$ with respect to $(\mathcal{F}_t)$. In the sequel, $\mathcal{Z}$ denotes the set of positive non-decreasing right-continuous processes and $\mathcal{T}$ the set of $\mathcal{F}_t$-stopping times. A control policy $\pi = (Z_t^\pi, \tau^\pi; t \geq 0)$ models a dividend/investment policy and is said to be admissible if Z_t^π belongs to $\mathcal{Z}$ and if τ^π belongs to $\mathcal{T}$. We denote the set of all admissible controls by Π. The control component Z_t^π therefore corresponds to the total amount of dividends paid out by the firm up to time t, and the control component τ^π represents the time of investment in the growth opportunity. A given control policy $(Z_t^\pi, \tau^\pi; t \geq 0)$ fully characterizes the associated investment process $(I_t^\pi)_{t\geq 0}$ which belongs to $\mathcal{Z}$ and is defined by the relation $I_t = I \mathbb{1}_{t\geq\tau^\pi}$. We denote by X_t^π the cash reserve of the firm at time t under a control policy $\pi = (Z_t^\pi, \tau^\pi; t \geq 0)$. The dynamics of the cash process X_t^π satisfies

$$\mathrm{d}X_t^\pi = (\mu_0 \mathbb{1}_{t<\tau^\pi} + \mu_1 \mathbb{1}_{t\geq\tau^\pi})\mathrm{d}t + \sigma \mathrm{d}W_t - \mathrm{d}Z_t^\pi - \mathrm{d}I_t^\pi, \quad X_{0-}^\pi = x.$$

Observe that at the investment time τ^π, the cash process jumps by an amount of $(\Delta X^\pi)_{\tau^\pi} \equiv X_{\tau^\pi}^\pi - X_{\tau^\pi-}^\pi = -I - (Z_{\tau^\pi}^\pi - Z_{\tau^\pi-}^\pi)$. This reflects the fact that we do not a priori exclude strategies that distribute some dividend at the investment time τ^π. For a given admissible control π, we define the time of bankruptcy by

$$\tau_0^\pi = \inf\{t \geq 0 \ : \ X_t^\pi \leq 0\},$$

and the firm value V_π by

$$V_\pi(x) = \mathbb{E}_x\left[\int_0^{\tau_0^\pi} \mathrm{e}^{-rs}\mathrm{d}Z_s^\pi\right].$$

The objective is to find the optimal return function which is defined as

$$V(x) = \sup_{\pi\in\Pi} V_\pi(x), \tag{2.1}$$

and the optimal policy $\pi^\star$ such that

$$V_{\pi^\star}(x) = V(x).$$

We thus consider in this paper the interaction between dividends and investment as a mixed singular control/optimal stopping problem. Our main theorem shows that problem (2.1) can be reduced to a stopping problem that we solve quasi-explicitly.

2.2 Benchmarks

Assume for the moment that the firm has only access to one of the two technologies (say, technology $i = 0$ for drift μ_0 and technology $i = 1$ for drift μ_1). The cash process $X_i = (X_{i,t})_{t\geq 0}$ therefore satisfies

$$\mathrm{d}X_{i,t} = \mu_i \mathrm{d}t + \sigma \mathrm{d}W_t - \mathrm{d}Z_{i,t}.$$

The firm value $V_{i,t}$ at time t is defined by the standard singular control problem

$$V_{i,t} = \operatorname*{ess\,sup}_{Z_i \in \mathcal{Z}} \mathbb{E}_x \left[\int_{t\wedge\tau_{i,0}}^{\tau_{i,0}} \mathrm{e}^{-r(s-t\wedge\tau_{i,0})} \mathrm{d}Z_{i,s} \,\middle|\, \mathcal{F}_{t\wedge\tau_{i,0}} \right], \tag{2.2}$$

where $\tau_{i,0} = \inf\{t : X_{i,t} \leq 0\}$ is the time of bankruptcy. This is the standard model of optimal dividends proposed by Jeanblanc and Shiryaev [18] or Radner and Shepp [26]. It follows from these papers that the firm value satisfies $V_{i,t} = V_i(X_{i,t\wedge\tau_{i,0}})$, where

$$V_i(x) = \sup_{Z_i \in \mathcal{Z}} \mathbb{E}_x \left[\int_0^{\tau_{i,0}} \mathrm{e}^{-rs} \mathrm{d}Z_{i,s} \right]. \tag{2.3}$$

Moreover, there exists a threshold x_i such that the optimal dividend policy, the solution of problem (2.3), is the local time $L^{x_i}(\mu_i, W)$ defined by the increasing process

$$L_t^{x_i}(\mu_i, W) = \max\left[0, \max_{0\leq s\leq t} (\mu_i s + \sigma W_s - x_i)\right].$$

Computations are explicit and give

$$V_i(x) = \mathbb{E}_x \left[\int_0^{\tau_{i,0}} \mathrm{e}^{-rs} \mathrm{d}L_s^{x_i}(\mu_i, W) \right] = \frac{f_i(x)}{f_i'(x_i)}, \quad 0 \leq x \leq x_i, \tag{2.4}$$

with

$$f_i(x) = \mathrm{e}^{\alpha_i^+ x} - \mathrm{e}^{\alpha_i^- x} \quad \text{and} \quad x_i = \frac{1}{\alpha_i^+ - \alpha_i^-} \ln \frac{(\alpha_i^-)^2}{(\alpha_i^+)^2}, \tag{2.5}$$

where $\alpha_i^- < 0 < \alpha_i^+$ are the roots of the characteristic equation

$$\mu_i x + \frac{1}{2}\sigma^2 x^2 - r = 0.$$

If the firm starts with cash reserves x above x_i, the optimal dividend policy distributes immediately the amount $(x - x_i)$ as exceptional dividend and then follows the dividend policy defined by the local time $L^{x_i}(\mu_i, W)$. Thus, for $x \geq x_i$,

$$V_i(x) = x - x_i + V_i(x_i), \tag{2.6}$$

where

$$V_i(x_i) = \mathbb{E}_{x_i}\left[\int_0^{\tau_{i,0}} \mathrm{e}^{-rs}\mathrm{d}L_s^{x_i}(\mu_i, W)\right] = \frac{\mu_i}{r}.$$

It is worth noting that the function f_i defined on $[0,\infty)$ is nonnegative, increasing, concave on $[0, x_i]$, convex on $[x_i, \infty)$ and satisfies $f_i' \geq 1$ on $[0,\infty)$ together with $\mathcal{L}_i f_i - r f_i = 0$ on $[0, x_i]$ where $\mathcal{L}_i$ is the infinitesimal generator of the drifted Brownian motion $\mu_i t + \sigma W_t$. Observe also that V_i is concave on $[0, x_i]$ and linear above x_i. Finally, it is also important to note that there is no obvious comparison between x_0 and x_1 (see for instance [27], Proposition 2). We shall repeatedly use all these properties in the next sections.

Coming back to our problem (2.1), we deduce from the above standard results that the strategies

$$\pi^0 = \left(Z_t^0, 0\right) = \left((x - x_0)_+ 1\!\!1_{t=0} + L_t^{x_0}(\mu_0, W) 1\!\!1_{t>0}\,,\, \infty\right) \tag{2.7}$$

and

$$\pi^1 = \left(Z_t^1, 0\right) = \left((x - I) - x_1)_+ 1\!\!1_{t=0} + L_t^{x_1}(\mu_1, W) 1\!\!1_{t>0},\, 0\right) \tag{2.8}$$

lead to the inequalities $V(x) \geq V_0(x)$ and $V(x) \geq V_1(x - I)$. Strategy π^0 corresponds to the investment policy *"never invest in the growth option (and follow the associated optimal dividend policy)"*, while strategy π^1 corresponds to the investment policy *"invest immediately in the growth option (and follow the associated optimal dividend policy)"*. Finally, note that because the inequality $x - I \leq 0$ leads to immediate bankruptcy, the firm value $V_1(x - I)$ is given by

$$\begin{cases} V_1(x - I) = \max\left(0, \dfrac{f_1(x - I)}{f_1'(x_1)}\right), & 0 \leq x \leq x_1 + I, \\ V_1(x - I) = x - I - x_1 + \frac{\mu_1}{r}, & x \geq x_1 + I. \end{cases} \tag{2.9}$$

2.3 First results

In this section we prove that the value function V satisfies the dynamic programming principle. We then derive a necessary and sufficient condition under which the growth opportunity is worthless.

Proposition 2.1 *The value function V satisfies the dynamic programming principle:*

$$V(x) = \sup_{\pi \in \Pi} \mathbb{E}_x\left[\int_0^{(\tau^\pi \wedge \tau_0^\pi)-} \mathrm{e}^{-rs}\mathrm{d}Z_s^\pi + \mathrm{e}^{-r(\tau^\pi \wedge \tau_0^\pi)} V_1\left(X^\pi_{(\tau^\pi \wedge \tau_0^\pi)-} - I\right)\right]. \tag{2.10}$$

Proof Let us define

$$W(x) = \sup_{\pi\in\Pi} \mathbb{E}_x\left[\int_0^{(\tau^\pi\wedge\tau_0^\pi)-} \mathrm{e}^{-rs}\mathrm{d}Z_s^\pi \;+\; \mathrm{e}^{-r(\tau^\pi\wedge\tau_0^\pi)} V_1\left(X^\pi_{(\tau^\pi\wedge\tau_0^\pi)-} - I\right)\right].$$

We start by proving the inequality $V(x) \leq W(x)$. Consider a given admissible policy $\pi = (Z_t^\pi, \tau^\pi)$. Now, from (2.2) and (2.3), the firm value at the investment date τ^π satisfies

$$\begin{aligned} V_{1,\tau^\pi} &= \operatorname*{ess\,sup}_{Z\in\mathcal{Z}} \mathbb{E}\left[\int_{\tau^\pi\wedge\tau_0^\pi}^{\tau_0^\pi} \mathrm{e}^{-r(s-\tau^\pi\wedge\tau_0^\pi)}\mathrm{d}Z_s \,\middle|\, \mathcal{F}_{\tau^\pi\wedge\tau_0^\pi}\right] \\ &= V_1\left(X^\pi_{(\tau^\pi\wedge\tau_0^\pi)-} - I\right) \\ &= V_1\left(X^\pi_{\tau^\pi\wedge\tau_0^\pi}\right), \end{aligned} \tag{2.11}$$

where the first equality uses the relation $\tau_0^\pi = \tau_{1,0}$, which holds almost surely on the event $\{\tau^\pi \wedge \tau_0^\pi = \tau^\pi\}$. We then deduce

$$\begin{aligned} V_\pi(x) &= \mathbb{E}_x\left[\int_0^{\tau_0^\pi} \mathrm{e}^{-rs}\mathrm{d}Z_s^\pi\right] \\ &= \mathbb{E}_x\left[\int_0^{(\tau^\pi\wedge\tau_0^\pi)-} \mathrm{e}^{-rs}\mathrm{d}Z_s^\pi + \mathbb{E}\left[\int_{\tau^\pi\wedge\tau_0^\pi}^{\tau_0^\pi} \mathrm{e}^{-rs}\mathrm{d}Z_s^\pi \,\middle|\, \mathcal{F}_{\tau^\pi\wedge\tau_0^\pi}\right]\right] \\ &\leq \mathbb{E}_x\left[\int_0^{(\tau^\pi\wedge\tau_0^\pi)-} \mathrm{e}^{-rs}\mathrm{d}Z_s^\pi\right. \\ &\qquad \left. + \mathrm{e}^{-r(\tau^\pi\wedge\tau_0^\pi)} \operatorname*{ess\,sup}_{Z\in\mathcal{Z}} \mathbb{E}\left[\int_{\tau^\pi\wedge\tau_0^\pi}^{\tau_0^\pi} \mathrm{e}^{-r(s-\tau^\pi\wedge\tau_0^\pi)}\mathrm{d}Z_s \,\middle|\, \mathcal{F}_{\tau^\pi\wedge\tau_0^\pi}\right]\right] \\ &\leq \mathbb{E}_x\left[\int_0^{(\tau^\pi\wedge\tau_0^\pi)-} \mathrm{e}^{-rs}\mathrm{d}Z_s^\pi + \mathrm{e}^{-r(\tau^\pi\wedge\tau_0^\pi)} V_1\left(X^\pi_{(\tau^\pi\wedge\tau_0^\pi)-} - I\right)\right]. \end{aligned} \tag{2.12}$$

Taking the supremum over $\pi \in \Pi$ on both sides gives the desired inequality. The reverse inequality relies on the fact that Z_t^1 defined by (2.8) is the optimal dividend policy for problem (2.11). Indeed, consider the control

$\pi = (Z_t^\pi \mathbb{1}_{t<\tau^\pi} + Z_t^1 \mathbb{1}_{t\geq\tau^\pi}, \tau^\pi)$ where Z_t^π and τ^π are arbitrarily chosen in $\mathcal{Z}$ and $\mathcal{T}$. Then we get

$$\begin{aligned} V(x) &\geq V_\pi(x) \\ &= \mathbb{E}_x\Bigg[\int_0^{(\tau^\pi\wedge\tau_0^\pi)-} \mathrm{e}^{-rs}\mathrm{d}Z_s^\pi \\ &\qquad + \mathrm{e}^{-r(\tau^\pi\wedge\tau_0^\pi)} \operatorname*{ess\,sup}_{Z\in\mathcal{Z}} \mathbb{E}\Bigg[\int_{\tau_0^\pi\wedge\tau^\pi}^{\tau_0^\pi} \mathrm{e}^{-r(s-\tau^\pi\wedge\tau_0^\pi)}\mathrm{d}Z_s^\pi \,\Bigg|\, \mathcal{F}_{\tau^\pi\wedge\tau_0^\pi}\Bigg]\Bigg] \\ &= \mathbb{E}_x\Bigg[\int_0^{(\tau^\pi\wedge\tau_0^\pi)-} \mathrm{e}^{-rs}\mathrm{d}Z_s^\pi + \mathrm{e}^{-r(\tau^\pi\wedge\tau_0^\pi)} V_1\left(X^\pi_{(\tau^\pi\wedge\tau_0^\pi)-} - I\right)\Bigg]. \end{aligned}$$

Taking the supremum over (Z^π, τ^π) on the right-hand side gives the result. □

We now establish a necessary and sufficient condition under which, for all current values of the cash process, the growth opportunity is worthless.

Proposition 2.2 *We have*

$$V(x) = V_0(x) \text{ for all } x \geq 0 \quad \text{if and only if} \quad \frac{\mu_1 - \mu_0}{r} \leq (x_1 + I) - x_0.$$

Proof It follows from the previous section that if $V(x) = V_0(x)$ for all $x \geq 0$, then $V_0(x) \geq V_1(x)$ which implies for $x \geq \max\{x_0, x_1 + I\}$ the inequality $\frac{\mu_1-\mu_0}{r} \leq (x_1+I) - x_0$. The sufficient condition in Proposition 2.2 is less obvious and relies on the following lemma.

Lemma 2.3 *If* $\left(\frac{\mu_1 - \mu_0}{r}\right) \leq (x_1+I) - x_0$, *then* $V_0(x) \geq V_1(x-I)$ *for all* $x \geq 0$.

Proof We distinguish three cases. First, if $x \in [0, I]$, then $V_1(x-I) = 0 \leq V_0(x)$. Second, if $x \geq x_0$, then

$$V_1(x-I) < x - x_1 + \frac{\mu_1}{r} \leq x - x_0 + \frac{\mu_0}{r} = V_0(x),$$

where the first inequality comes from the concavity of V_1, the second inequality is our assumption, and the last equality follows from the definition of V_0 for $x \geq x_0$. Third, fix $x \in [I, x_0]$ and consider the function k defined on $[I, x_0]$ by the relation $k(x) = V_0(x) - V_1(x-I)$. We already know that $k(I) > 0$ and $k(x_0) > 0$. Note also that $k'(x_0) = 1 - V_1'(x_0 - I) \leq 0$ and $k''(x_0) \geq 0$. Next, suppose that there exists $y \in (I, x_0)$ such that $k(y) = 0$. Because k is decreasing and convex in a left neighbourhood of x_0, there exists $z \in (y, x_0)$ such that $k'(z) = 0$ with k concave in a neighbourhood centred in z. We thus obtain

$$\mathcal{L}_0 k(z) - rk(z) = \frac{\sigma^2}{2} k''(z) - rk(z) < 0. \tag{2.13}$$

Using the equality $\mathcal{L}_0 V_0(x) - rV_0(x) = 0$, which holds for all $x \in (I, x_0)$, we get

$$\mathcal{L}_0 k(x) - rk(x) = -\mathcal{L}_0 V_1(x - I) + rV_1(x - I). \tag{2.14}$$

Now, because $\mu_1 > \mu_0$, the inequality $x_0 \geq x_1 + I$ holds by assumption and the relation $\mathcal{L}_1 V_1(x - I) - rV_1(x - I) = 0$ is therefore satisfied for $x \in (I, x_0)$. We then deduce for all $x \in (I, x_0)$ that

$$\mathcal{L}_0 V_1(x - I) - rV_1(x - I) = (\mathcal{L}_0 - \mathcal{L}_1)V_1(x - I) = (\mu_0 - \mu_1)V_1'(x - I) < 0,$$

where the last inequality follows because $V_1(\cdot - I)$ is increasing and from $\mu_1 > \mu_0$. It then follows from (2.14) that $\mathcal{L}_0 k(z) - rk(z) > 0$. This contradicts (2.13) and concludes the proof of Lemma 2.3. □

We now finish the proof of Proposition 2.2. By Eq. (2.12), for all fixed $\pi = (Z_t^\pi, \tau^\pi; t \geq 0) \in \Pi$, we have

$$\begin{aligned} V_\pi(x) &\leq \mathbb{E}_x \left[\int_0^{(\tau^\pi \wedge \tau_0^\pi)-} \mathrm{e}^{-rs} \mathrm{d}Z_s^\pi + \mathrm{e}^{-r(\tau^\pi \wedge \tau_0^\pi)} V_1\left(X^\pi_{(\tau^\pi \wedge \tau_0^\pi)-} - I \right) \right] \\ &\leq \mathbb{E}_x \left[\int_0^{(\tau^\pi \wedge \tau_0^\pi)} \mathrm{e}^{-rs} \mathrm{d}Z_s^\pi + \mathrm{e}^{-r(\tau^\pi \wedge \tau_0^\pi)} V_0\left(X^\pi_{(\tau^\pi \wedge \tau_0^\pi)-} \right) \right] \\ &\leq V_0(x), \end{aligned}$$

where the second inequality comes from Lemma 2.3 and the third from the dynamic programming principle applied to the value function V_0. It thus follows that $V(x) \leq V_0(x)$ which implies our result since the reverse inequality is always true. □

In the rest of the paper, condition (H1) will refer to the inequality

$$\frac{\mu_1 - \mu_0}{r} > (x_1 + I) - x_0.$$

Condition (H1) is therefore a necessary and sufficient condition for the growth option *not* to be worthless. Note that condition (H1) ensures the existence and uniqueness of a positive real number $\tilde{x}$ such that $V_0(x) \geq$ (resp. $\leq$) $V_1(x - I)$ for $x \leq$ (resp. $\geq$) $\tilde{x}$. This property will play a crucial role in the next section.

3 Main results

We derive in this section our main results. First, we present and comment in Sect. 3.1 our main theorem and prove it in Sect. 3.2. Next, we derive in Sect. 3.3 the optimal dividend/investment policy and develop in Sect. 3.4 the economic interpretations.

3.1 The main theorem

Let us denote by $R = (R_t)_{t\geq 0}$ the cash reserve process generated by the activity in place in the absence of dividend distribution, so that

$$\mathrm{d}R_t = \mu_0 \mathrm{d}t + \sigma \mathrm{d}W_t,$$

and consider the stopping problem with the value function

$$\phi(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[\mathrm{e}^{-r(\tau\wedge\tau_0)} \max\left(V_0(R_{\tau\wedge\tau_0}), V_1(R_{\tau\wedge\tau_0} - I)\right)\right], \tag{3.1}$$

where $\tau_0 = \inf\{t \geq 0 \ : \ R_t \leq 0\}$. We show

Theorem 3.1 *For all* $x \in [0, \infty)$, $V(x) = \phi(x)$.

The intuition of our result is as follows. Having in mind the properties derived in Sect. 2 and standard results on optimal stopping problems, one expects that the optimal dividend/investment policy is defined by a reflecting barrier for the dividend policy together with an investment threshold. Such a guess implies that only two alternative strategies remain available: (i) ignore the growth option and pay out any surplus above x_0 as dividend; (ii) postpone dividend distribution, invest at a certain threshold b in the growth opportunity and pay out any surplus above x_1 as dividend. Theorem 3.1 shows that this intuition is correct. In other words, Theorem 3.1 says that the manager fits his dividend policy to the option value to invest in the growth opportunity and everything happens as if he had simply to choose between paying dividends versus retaining the earnings for investment. The mixed singular control/optimal stopping problem (2.1) is therefore reduced to the stopping problem (3.1).

3.2 Proof of the main theorem

The proof follows the standard line of stochastic control which relies on the dynamic programming principle and Hamilton–Jacobi–Bellman (HJB) equation. We start with the following lemma.

Lemma 3.2 *For all* $x \in [0, \infty)$, $V(x) \geq \phi(x)$.

Proof According to Proposition 2.1 and (2.11), we have for every policy (Z_t^π, τ^π) and for all $x \geq 0$

$$\begin{aligned} V(x) &\geq \mathbb{E}\left[\int_0^{(\tau_0^\pi\wedge\tau^\pi)-} \mathrm{e}^{-rs}\,\mathrm{d}Z_s^\pi + \mathrm{e}^{-r(\tau_0^\pi\wedge\tau^\pi)} V_1\left(X_{\tau_0^\pi\wedge\tau^\pi}\right)\right] \\ &= \mathbb{E}\left[\int_0^{(\tau_0^\pi\wedge\tau^\pi)-} \mathrm{e}^{-rs}\,\mathrm{d}Z_s^\pi + \mathrm{e}^{-r(\tau_0^\pi\wedge\tau^\pi)} V\left(X_{\tau_0^\pi\wedge\tau^\pi}\right)\right]. \end{aligned}$$

The strategy with $Z^\pi_s = 0$ for $0 \leq s \leq t$ and $\tau^\pi = t$ leads to

$$V(x) \geq \mathbb{E}\left(\mathrm{e}^{-r(t\wedge\tau_0^\pi)} V(R_{t\wedge\tau_0^\pi})\right),$$

and it results from the Markov property that the process $(\mathrm{e}^{-r(t\wedge\tau_0^\pi)} V(R_{t\wedge\tau_0^\pi}))_{t\geq 0}$ is a supermartingale which dominates the function $\max(V_0(\cdot), V_1(\cdot - I))$. On the other hand, according to optimal stopping theory, our candidate value function ϕ is defined as the smallest supermartingale function which dominates $\max(V_0(\cdot), V_1(\cdot - I))$; hence the inequality $V(x) \geq \phi(x)$ follows. □

The proof of the reverse inequality $V(x) \leq \phi(x)$ is more involved and requires a verification result for the HJB equation associated to problem (2.10). One indeed expects, from the dynamic programming principle, that the value function satisfies the HJB equation

$$\max(1 - v', \mathcal{L}_0 v - rv, V_1(\cdot - I) - v) = 0. \tag{3.2}$$

The next proposition shows that any piecewise C^2 function which is a supersolution to the HJB equation (3.2) is a majorant of the value function V.

Proposition 3.3 (Verification result for the HJB equation) *Suppose we can find a positive function $\tilde{V}$, piecewise C^2 on $(0, +\infty)$ with bounded first derivatives*[2] *and such that for all $x > 0$,*

(i) $\mathcal{L}_0\tilde{V} - r\tilde{V} \leq 0$ *in the sense of distributions,*
(ii) $\tilde{V}(x) \geq V_1(x - I)$,
(iii) $\tilde{V}'(x) \geq 1$,

with the initial condition $\tilde{V}(0) = 0$. Then $\tilde{V}(x) \geq V(x)$ for all $x \in [0, \infty)$.

Proof We must show that for any control policy $\pi = (Z^\pi_t, \tau^\pi; t \geq 0)$, $\tilde{V}(x) \geq V_\pi(x)$ for all $x > 0$. Let us write $Z^\pi_t = Z^{\pi,c}_t + Z^{\pi,d}_t$ where $Z^{\pi,c}_t$ is the continuous part of Z^π_t and $Z^{\pi,d}_t$ is the purely discontinuous part of Z^π_t. Using a generalized Itô's formula (see [8], Theorem VIII-25 and Remark c, p. 349), we can write

$$\begin{aligned}
\mathrm{e}^{-r(\tau^\pi\wedge\tau_0^\pi)}\tilde{V}\left(X^\pi_{(\tau^\pi\wedge\tau_0^\pi)-}\right) = \tilde{V}(x) &+ \int_0^{(\tau^\pi\wedge\tau_0^\pi)-} \mathrm{e}^{-rs}\left(\mathcal{L}_0\tilde{V}\left(X^\pi_s\right) - r\tilde{V}\left(X^\pi_s\right)\right)\mathrm{d}s \\
&+ \int_0^{(\tau^\pi\wedge\tau_0^\pi)-} \mathrm{e}^{-rs}\tilde{V}'\left(X^\pi_s\right)\sigma\mathrm{d}W_s \\
&- \int_0^{(\tau^\pi\wedge\tau_0^\pi)-} \mathrm{e}^{-rs}\tilde{V}'\left(X^\pi_s\right)\mathrm{d}Z^c_s \\
&+ \sum_{s<\tau^\pi\wedge\tau_0^\pi} \mathrm{e}^{-rs}\left(\tilde{V}\left(X^\pi_s\right) - \tilde{V}\left(X^\pi_{s-}\right)\right).
\end{aligned}$$

[2] In the sense of Definition 4.8, p. 271 in Karatzas and Shreve [19].

Since $\tilde{V}$ satisfies (i), the second term of the right hand side is nonpositive. On the other hand, the first derivative of $\tilde{V}$ being bounded, the third term is a square integrable martingale. Taking expectations, we get

$$\mathbb{E}_x\left[\mathrm{e}^{-r(\tau^\pi\wedge\tau_0^\pi)}\tilde{V}\left(X^\pi_{(\tau^\pi\wedge\tau_0^\pi)-}\right)\right] \le \tilde{V}(x) - \mathbb{E}_x\left[\int_0^{(\tau^\pi\wedge\tau_0^\pi)-} \mathrm{e}^{-rs}\tilde{V}'\left(X_s^\pi\right)\,\mathrm{d}Z_s^{\pi,c}\right] + \mathbb{E}_x\left[\sum_{s<\tau\wedge\tau_0}\mathrm{e}^{-rs}\left(\tilde{V}\left(X_s^\pi\right)-\tilde{V}\left(X_{s-}^\pi\right)\right)\right].$$

Since $\tilde{V}'(x)\ge 1$ for all $x>0$, we have $\tilde{V}(X_s^\pi)-\tilde{V}(X_{s-}^\pi)\le X_s^\pi - X_{s-}^\pi$. Therefore, using the equality $X_s^\pi - X_{s-}^\pi = -(Z_s^\pi - Z_{s-}^\pi)$ for $s<\tau^\pi\wedge\tau_0^\pi$, we finally get

$$\begin{aligned}\tilde{V}(x) &\ge \mathbb{E}_x\left[\mathrm{e}^{-r(\tau^\pi\wedge\tau_0^\pi)}\tilde{V}\left(X^\pi_{(\tau^\pi\wedge\tau_0^\pi)-}\right)\right] + \mathbb{E}_x\left[\int_0^{(\tau^\pi\wedge\tau_0^\pi)-} \mathrm{e}^{-rs}\tilde{V}'\left(X_s^\pi\right)\,\mathrm{d}Z_s^{\pi,c}\right]\\ &\quad + \mathbb{E}_x\left[\sum_{s<\tau\wedge\tau_0}\mathrm{e}^{-rs}\left(Z_s^\pi - Z_{s-}^\pi\right)\right]\\ &\ge \mathbb{E}_x\left[\mathrm{e}^{-r(\tau^\pi\wedge\tau_0^\pi)}V_1\left(X^\pi_{(\tau^\pi\wedge\tau_0^\pi)-}-I\right)\right] + \mathbb{E}_x\left[\int_0^{(\tau^\pi\wedge\tau_0^\pi)-} \mathrm{e}^{-rs}\,\mathrm{d}Z_s^{\pi}\right]\\ &= V_\pi(x),\end{aligned}$$

where assumptions (ii) and (iii) have been used for the second inequality. □

We call thereafter supersolution to the HJB equation (3.2) any function $\tilde{V}$ satisfying Proposition 3.3. To complete the proof of Theorem 3.1, it thus remains to verify that our candidate value function ϕ is a supersolution to the HJB equation (3.2). This will clearly imply the inequality $V(x)\le\phi(x)$. It is worth pointing out that, contrary to a standard verification procedure, we do not need here to finish the proof of Theorem 3.1 by constructing a control policy whose performance functional coincides with the value function ϕ. The reason is that we proved in Lemma 3.2 that the inequality $V(x)\ge\phi(x)$ is always satisfied. Deriving the optimal control/stopping strategy is nevertheless crucial for a detailed analysis of economic interpretations and will be done in Sect. 3.3. We now turn to the last step of the proof of Theorem 3.1.

Proposition 3.4 *ϕ is a supersolution to the HJB equation* (3.2).

The proof of Proposition 3.4 requires to solve quasi-explicitly the optimal stopping problem (3.1), a task we achieve in the next paragraph.

Solution to the optimal stopping problem (3.1)

As a first remark, note that, from Lemma 3.2 and from the definition of the optimal stopping problem (3.1), we have $V(x)\ge\phi(x)\ge\theta(x)$ for all positive x, where θ is the value function of the optimal stopping problem

$$\theta(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[\mathrm{e}^{-r(\tau \wedge \tau_0)} V_1(R_{\tau \wedge \tau_0} - I) \right], \tag{3.3}$$

with $\tau_0 = \inf\{t \geq 0 \ : \ R_t \leq 0\}$. The value function θ therefore represents the value of the option to invest in the growth opportunity when the manager decides to postpone dividend payments until investment. In line with the intuition underlying Theorem 3.1, one anticipates that if for all positive x, the option value $\theta(x)$ is larger than $V_0(x)$, then we have the equalities $V(x) = \phi(x) = \theta(x)$. A crucial point will be to show that the inequality $\theta(x) > V_0(x)$ holds for all positive x if and only if it is satisfied at the threshold x_0 that triggers the distribution of dividends when the firm is run under the technology in place. In such a situation, the optimal dividend/investment policy will be to postpone dividend distribution, to invest at a certain threshold b in the growth opportunity and to pay out any surplus above x_1 as dividend. The next proposition makes all these points precise and derives the solution to the optimal stopping problem (3.1).

Proposition 3.5 *The following hold:*

(A) *If condition* (H1) *is satisfied, then:*
  (i) *If $\theta(x_0) > V_0(x_0)$, then the value function ϕ satisfies $\phi(x) = \theta(x)$ for all positive x.*
  (ii) *If $\theta(x_0) \leq V_0(x_0)$, then the value function ϕ has the structure*

$$\phi(x) = \begin{cases} V_0(x), \quad 0 \leq x \leq a, \\ V_0(a)\, \mathbb{E}_x[\mathrm{e}^{-r\tau_a} \mathbb{1}_{\tau_a < \tau_c}] + V_1(c-I)\mathbb{E}_x[\mathrm{e}^{-r\tau_c} \mathbb{1}_{\tau_a > \tau_c}] \\ \qquad = A\mathrm{e}^{\alpha_0^+ x} + B\mathrm{e}^{\alpha_0^- x}, \quad a \leq x \leq c, \\ V_1(x-I), \quad x \geq c, \end{cases}$$

where $\tau_a = \inf\{t \geq 0 \ : \ R_t \leq a\}$ and $\tau_c = \inf\{t \geq 0 \ : \ R_t \geq c\}$ and where A, B, a, c are determined by the continuity and smooth-fit C^1 conditions at a and c, i.e.,

$$\begin{aligned} \phi(a) &= V_0(a), \\ \phi(c) &= V_1(c-I), \\ \phi'(a) &= V_0'(a), \\ \phi'(c) &= V_1'(c-I). \end{aligned}$$

(B) *If condition* (H1) *is not satisfied, then $\phi(x) = V_0(x)$ for all positive x.*

Figures 1 and 2 illustrate cases (i) and (ii) of Proposition 3.5. We establish Proposition 3.5 through a series of lemmas. The first one derives quasi-explicitly the value function θ.

Lemma 3.6 *The value function θ is given by*

$$\begin{cases} \theta(x) = \dfrac{f_0(x)}{f_0(b)} V_1(b-I), \quad x \leq b, \\ \theta(x) = V_1(x-I), \quad x \geq b, \end{cases} \tag{3.4}$$

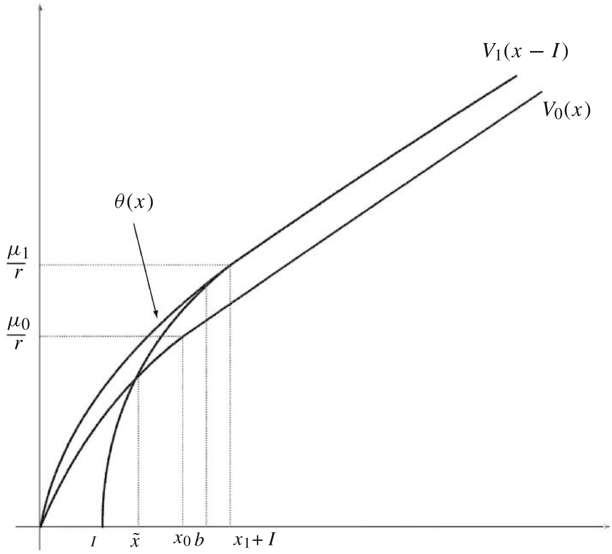

Fig. 1 $\theta(x_0) > V_0(x_0)$

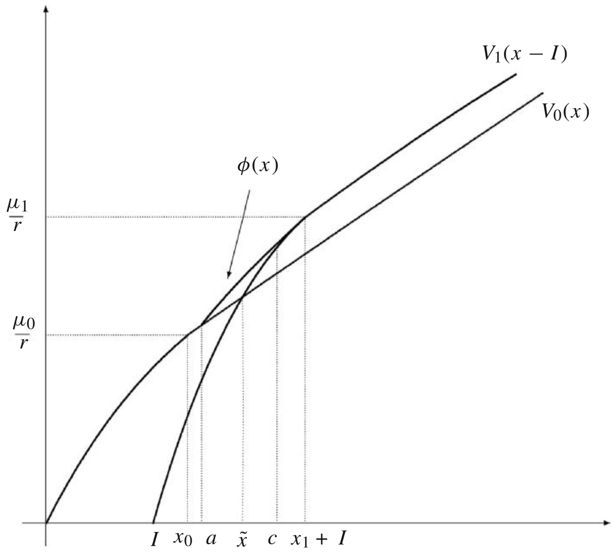

Fig. 2 $\theta(x_0) < V_0(x_0)$

where f_0 is defined in (2.5) *and where $b > I$ is defined by the smooth-fit condition*

$$\frac{V_1'(b - I)}{f_0'(b)} = \frac{V_1(b - I)}{f_0(b)}. \tag{3.5}$$

Proof It follows from Dayanik and Karatzas [4] (Corollary 7.1) that the optimal value function θ is C^1 on $[0,\infty)$, and furthermore from Villeneuve [29] (Theorem 4.2 and Proposition 4.6) that a threshold strategy is optimal. This allows us to use a standard verification procedure and to write the value function θ in terms of the free boundary problem

$$\begin{cases} \mathcal{L}_0\theta(x) - r\theta(x) = 0, \;\; 0 \leq x \leq b, \quad \text{and} \quad \mathcal{L}_0\theta(x) - r\theta(x) \leq 0, \;\; x \geq b, \\ \theta(b) = V_1(b-I), \quad \theta'(b) = V_1'(b-I). \end{cases} \tag{3.6}$$

Standard computations lead to the desired result. □

The next lemma characterizes the stopping region of the optimal stopping problem (3.1).

Lemma 3.7 *The stopping region S of problem* (3.1) *satisfies $S = S_0 \cup S_1$ with*

$$S_0 = \{0 < x < \tilde{x} \,|\, \phi(x) = V_0(x)\}$$

and

$$S_1 = \{x > \tilde{x} \,|\, \phi(x) = V_1(x-I)\},$$

where $\tilde{x}$ is the unique crossing point of the value functions $V_0(\cdot)$ and $V_1(x-\cdot)$.

Proof According to optimal stopping theory (see [10], or Theorems 10.1.9 and 10.1.12 in [24]), the stopping region S of problem ϕ satisfies

$$S = \{x > 0 \,|\, \phi(x) = \max(V_0(x), V_1(x-I))\}.$$

Now, from Proposition 5.13 and Corollary 7.1 by Dayanik-Karatzas [4], the hitting time $\tau_S = \inf\{t : R_t \in S\}$ is optimal and the optimal value function is C^1 on $[0,\infty)$. Moreover, it follows from Lemma 4.3 from Villeneuve [29] that $\tilde{x}$, defined as the unique crossing point of the value functions $V_0(\cdot)$ and $V_1(x-\cdot)$, does not belong to S. Hence, the stopping region can be decomposed into two subregions $S = S_0 \cup S_1$ with

$$S_0 = \{0 < x < \tilde{x} \,|\, \phi(x) = V_0(x)\}$$

and

$$S_1 = \{x > \tilde{x} \,|\, \phi(x) = V_1(x-I)\}.$$

□

We now obtain assertion (i) of Proposition 3.5 as a byproduct of the next lemma.

Lemma 3.8 *The following assertions are equivalent:*

(i) $\theta(x_0) > V_0(x_0)$.
(ii) $\theta(x) > V_0(x)$ *for all* $x > 0$.
(iii) $S_0 = \emptyset$.

Proof (i) $\Longrightarrow$ (ii). We start with $x \in (0, x_0)$. Let us define $\tau_{x_0} = \inf\{t : R_t < x_0\} \in \mathcal{T}$. The inequality $\theta(x_0) > V_0(x_0)$ together with the initial condition $\theta(0) = V_0(0) = 0$ implies

$$\mathbb{E}_x\left[\mathrm{e}^{-r(\tau_{x_0}\wedge\tau_0)}\left(\theta(R_{\tau_{x_0}\wedge\tau_0}) - V_0(R_{\tau_{x_0}\wedge\tau_0})\right)\right] > 0.$$

Itô's formula gives

$$\begin{aligned} 0 &< \mathbb{E}_x\left[\mathrm{e}^{-r(\tau_{x_0}\wedge\tau_0)}\left(\theta(R_{\tau_{x_0}\wedge\tau_0}) - V_0(R_{\tau_{x_0}\wedge\tau_0})\right)\right] \\ &= \theta(x) - V_0(x) + \mathbb{E}_x\left[\int_0^{\tau_{x_0}\wedge\tau_0} \mathrm{e}^{-rt}\left(\mathcal{L}_0\theta(R_t) - r\theta(R_t)\right)\,\mathrm{d}t\right] \\ &\le \theta(x) - V_0(x), \end{aligned}$$

where the last inequality follows from (3.6). Thus, $\theta(x) > V_0(x)$ for $0 < x \le x_0$. Assume now that $x > x_0$. We distinguish two cases. If $b > x_0$, it follows from (2.4) and (3.4) that $\theta(x) > V_0(x)$ for $x \le x_0$ is equivalent to $\theta'(x_0) > 1$. Then the convexity property of f_0 yields $\theta'(x) > 1$ for all $x > 0$. If on the contrary $b \le x_0$, then $\theta(x) = V_1(x - I)$ for all $x \ge x_0$. Since $V_1'(x - I) \ge 1$ for all $x \in [I, \infty)$, the smooth-fit principle implies $\theta'(x) \ge 1$ for all $x \ge x_0$. Therefore, the function $\theta - V_0$ is increasing for $x \ge x_0$ which ends the proof.

(ii) $\Longrightarrow$ (iii). Simply remark that (3.3) and (3.1) give $\phi \ge \theta$. Therefore, we have $\phi(x) \ge \theta(x) > V_0(x)$ for all $x > 0$ which implies the emptyness of S_0.

(iii) $\Longrightarrow$ (i). Suppose $S_0 = \emptyset$ and let us show that $\theta = \phi$. This will clearly imply $\theta(x_0) = \phi(x_0) > V_0(x_0)$ and thus (i). From optimal stopping theory, the process $\left(\mathrm{e}^{-r(t\wedge\tau_0\wedge\tau_S)}\phi(X_{t\wedge\tau_0\wedge\tau_S})\right)_{t\ge 0}$ is a martingale. Moreover, on the event $\{\tau_S < t\}$, we have $\phi(R_{\tau_S}) = V_1(R_{\tau_S} - I)$ a.s. It results that

$$\begin{aligned} \phi(x) &= \mathbb{E}_x\left[\mathrm{e}^{-r(t\wedge\tau_S)}\phi(R_{t\wedge\tau_S})\right] \\ &= \mathbb{E}_x\left[\mathrm{e}^{-r\tau_S}V_1(R_{\tau_S} - I)\mathbb{1}_{\tau_S<t}\right] + \mathbb{E}_x\left[\mathrm{e}^{-rt}\phi(R_t)\mathbb{1}_{t\le\tau_S}\right] \\ &\le \theta(x) + \mathbb{E}_x\left[\mathrm{e}^{-rt}\phi(R_t)\right]. \end{aligned}$$

Now, it follows from (2.6), (2.9) that $\phi(x) \le Cx$ for some positive constant C. This implies that $\mathbb{E}_x\left[\mathrm{e}^{-rt}\phi(R_t)\right]$ converges to 0 as t goes to infinity. We therefore deduce that $\phi \le \theta$ and thus that $\phi = \theta$. □

Assertion (ii) of Proposition 3.5 relies on the following lemma.

Lemma 3.9 *Assume $\theta(x_0) \le V_0(x_0)$. Then there are two positive real numbers $a \ge x_0$ and $c \le x_1 + I$ such that*

$$S_0 = \,]0, a] \quad \textit{and} \quad S_1 = [c, +\infty[.$$

Proof From the previous lemma we know that the inequality $\theta(x_0) \le V_0(x_0)$ implies $S_0 \ne \emptyset$. We start the proof with the shape of the subregion S_0. Taking $x \in S_0$, we have to prove that any $y \le x$ belongs to S_0. As a result, we shall then define $a = \sup\{x < \tilde{x} \mid x \in S_0\}$. Now, according to Proposition 5.13 by Dayanik and Karatzas [4], we have

$$\phi(y) = \mathbb{E}_y\left[\mathrm{e}^{-r(\tau_S\wedge\tau_0)}\max\left(V_0(R_{\tau_S\wedge\tau_0}), V_1(R_{\tau_S\wedge\tau_0} - I)\right)\right].$$

Since $x \in S_0$, $x < \tilde{x}$ and thus $\tau_S = \tau_{S_0}\mathbb{P}^y$-a.s. for all $y \le x$. Hence,

$$\phi(y) = \mathbb{E}_y\left[\mathrm{e}^{-r(\tau_{S_0}\wedge\tau_0)}V_0(R_{\tau_{S_0}\wedge\tau_0})\right] \le V_0(y),$$

where the last inequality follows from the supermartingale property of the process $\left(\mathrm{e}^{-r(t\wedge\tau_0)}V_0(R_{t\wedge\tau_0})\right)_{t\ge 0}$. Now, assuming that $a < x_0$, (i.e., $\phi(x_0) > V_0(x_0)$) yields a contradiction since

$$\begin{aligned}\phi(a) &= V_0(a)\\ &= \mathbb{E}_a\left[\mathrm{e}^{-r\tau_{x_0}}\mathbb{1}_{\tau_{x_0}<\tau_0}V_0(R_{\tau_{x_0}})\right]\\ &\le \mathbb{E}_a\left[\mathrm{e}^{-r\tau_{x_0}}V_0(R_{\tau_{x_0}})\right]\\ &< \mathbb{E}_a\left[\mathrm{e}^{-r\tau_{x_0}}\phi(R_{\tau_{x_0}})\right]\\ &\le \phi(a),\end{aligned}$$

where the second equality follows from the martingale property of the process $\left(\mathrm{e}^{-r(t\wedge\tau_{x_0}\wedge\tau_0)}V_0(R_{t\wedge\tau_{x_0}\wedge\tau_0})\right)_{t\ge 0}$ under $\mathbb{P}^a$ and the last inequality follows from the supermartingale property of the process $\left(\mathrm{e}^{-r(t\wedge\tau_0)}\phi(R_{t\wedge\tau_0})\right)_{t\ge 0}$.

The shape of the subregion S_1 is a direct consequence of Lemma 4.4 by Villeneuve [29]. The only difficulty is to prove that $c \le x_1 + I$. Let us consider $x \in (a, c)$, and let us introduce the stopping times $\tau_a = \inf\{t : R_t = a\}$, and $\tau_c = \inf\{t : R_t = c\}$. Then we have

$$\begin{aligned}\phi(x) &= \mathbb{E}_x\left[\mathrm{e}^{-r(\tau_a\wedge\tau_c)}\max\left(V_0(R_{\tau_a\wedge\tau_c}), V_1(R_{\tau_a\wedge\tau_c} - I)\right)\right]\\ &\le \mathbb{E}_x\left[\mathrm{e}^{-r(\tau_a\wedge\tau_c)}\left(R_{\tau_a\wedge\tau_c} - (x_1 + I) + \frac{\mu_1}{r}\right)\right]\\ &= x - (x_1 + I) + \frac{\mu_1}{r} + \mathbb{E}_x\left[\int_0^{\tau_a\wedge\tau_c}\mathrm{e}^{-rs}(\mu_0 - r(R_s - (x_1 + I)) - \mu_1)\,\mathrm{d}s\right].\end{aligned}$$

Observe that on the stochastic interval $[0, \tau_a \wedge \tau_c]$, $R_s \ge a \ge x_0$ $\mathbb{P}^x$-a.s. and thus

$$\mu_0 - r(R_s - (x_1 + I)) - \mu_1 \le \mu_0 - r(x_0 - (x_1 + I)) - \mu_1 < 0,$$

by condition (H1). Therefore, $\phi(x) \le x - (x_1 + I) + \frac{\mu_1}{r}$ for $x \in (a, c)$. We conclude by noting that assuming the inequality $c > x_1 + I$ would yield the contradiction

$$\frac{\mu_1}{r} = V_1(x_1) < \phi(x_1 + I) \le \frac{\mu_1}{r}.$$

□

We now finish the proof of Proposition 3.5. It follows from Lemma 3.9 that the structure of the value function ϕ in assertion (ii) of Proposition 3.5 is a direct consequence of continuity and smooth-fit C^1 properties. Finally, consider case (B) of Proposition 3.5 and therefore assume that condition (H1) is not satisfied. Similar arguments to those used for studying the optimal stopping problem (3.3) easily yield the relation

$$V_0(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[e^{-r(\tau \wedge \tau_0)} V_0(R_{\tau \wedge \tau_0} - I) \right].$$

The equality $V(x) = \phi(x)$ follows then from Proposition 2.2. □

As a final remark note that if $\theta(x_0) = V_0(x_0)$, then we have that $a = x_0, c = b$ and the value functions ϕ and θ coincide. Indeed, using the same argument as in the first part of the proof of Lemma 3.8, we easily deduce from $\theta(x_0) = V_0(x_0)$ that $\theta(x) = V_0(x) = \phi(x)$ for $x \leq x_0$. Furthermore, (2.4) and (3.4) imply that $\theta(x_0) = V_0(x_0)$ is equivalent to $\theta'(x_0) = V'(x_0) = 1$, which implies that $a = x_0$. The equality $c = b$ follows then from relations (3.4) and (3.5). To summarize, if $\theta(x_0) = V_0(x_0)$, then $\left(e^{-r(t\wedge\tau_0)}\theta(R_{t\wedge\tau_0}\right)$ is the smallest supermartingale that majorizes $\left(e^{-r(t\wedge\tau_0)} \max(V_0(R_{t\wedge\tau_0}), V_1(R_{t\wedge\tau_0} - I)\right)$, from which it results that $\theta = \phi$.

We are now ready to prove Proposition 3.4, namely that ϕ is a supersolution to the HJB equation (3.2). This will complete the proof of Theorem 3.1.

Proof of Proposition 3.4 The result clearly holds if $\phi(x) = V_0(x)$ for all positive x (that is, if condition (H1) is not satisfied). Assume now that condition (H1) is satisfied. Two cases have to be considered.

(i) $\theta(x_0) > V_0(x_0)$.
In this case, $\phi = \theta$ according to part (i) of Proposition 3.5. It remains to check that the function θ satisfies the assumptions of Proposition 3.3. But according to optimal stopping theory, $\theta \in C^2[(0,\infty)\backslash b)]$, $\mathcal{L}_0\theta - r\theta \leq 0$ and clearly $\theta \geq V_1(\cdot - I)$. Moreover, it is shown in the first part of the proof of Lemma 3.8 that $\theta'(x) \geq 1$ for all $x > 0$. Finally, let us check that θ' is bounded above in the neighbourhood of zero. Clearly we have that

$$\theta(x) \leq \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[e^{-r(\tau \wedge \tau_0)} V_1(R_{\tau \wedge \tau_0}) \right],$$

and furthermore, the process $\left(e^{-r(t\wedge\tau_0)} V_1(R_{t\wedge\tau_0})\right)_{t\geq 0}$ is a supermartingale since $\mu_1 > \mu_0$. Therefore $\theta \leq V_1$, and the boundedness of the first derivative of θ then follows from (2.9).

(ii) $\theta(x_0) \leq V_0(x_0)$.
In this case, the function ϕ is characterized by part (ii) of Proposition 3.5. Thus, $\phi = V_0$ on $(0,a)$, $\phi = V(\cdot - I)$ on $(c, +\infty)$ and $\phi(x) = Ae^{\alpha_0^+ x} + Be^{\alpha_0^- x}$ on (a,c). Hence, ϕ will be a supersolution if we prove that $\phi'(x) \geq 1$ for all $x > 0$. In fact, it is enough to prove that $\phi'(x) \geq 1$ for $x \in (a,c)$ because $V_0' \geq 1$ and $V_1'(\cdot - I) \geq 1$. The smooth-fit principle gives $\phi'(a) = V_0'(a) \geq 1$ and $\phi'(c) = V_1'(c - I) \geq 1$. Clearly, ϕ is convex in a right neighbourhood of a. Therefore, if ϕ is convex on (a,c), the proof is over. If not, the second

derivative of ϕ, given by $A(\alpha_0^+)^2 e^{\alpha_0^+ x} + B(\alpha_0^-)^2 e^{\alpha_0^- x}$, vanishes at most once on (a, c), say in d. Therefore,

$$1 \leq \phi'(a) \leq \phi'(x) \leq \phi'(d) \quad \text{for } x \in (a, d),$$

and

$$1 \leq \phi'(c) \leq \phi'(x) \leq \phi'(d) \quad \text{for } x \in (d, c),$$

which completes the proof of Proposition 3.4 and thus concludes the proof of Theorem 3.1. □

3.3 Optimal policy

We give here a construction of the optimal dividend/investment policy. Theorem 3.1 and Proposition 3.5 drive the intuition. For instance, one expects that if condition (H1) is satisfied together with the inequality $\theta(x_0) < V_0(x_0)$ then, for a current value of the cash reserve between the thresholds a and c, the optimal strategy is to delay any decision until the cash reserve process hits threshold a or threshold c. Two cases can then happen. If the cash reserve process rises to c before hitting a, the optimal strategy is to invest in the growth option and then to deliver any surplus above x_1 as dividend. On the contrary, if the cash reserve process falls to a before hitting c, the optimal strategy is to deliver as exceptional dividend the amount $a - x_0$ and never to invest in the growth opportunity. Assertion (ii) of the next proposition encompasses this particular case. We now state our result.

Proposition 3.10 *The following holds:*

(A) *If condition* (H1) *is satisfied, then:*

(i) *If $\theta(x_0) > V_0(x_0)$, then the policy $\pi^\star = (Z_t^{\pi^\star}, \tau^{\pi^\star})$ defined by the increasing right-continuous process*

$$Z_t^{\pi^\star} = ((R_{\tau_b} - I) - x_1)_+ 1\!\!1_{t=\tau_b} + L_t^{x_1}(\mu_1, W) 1\!\!1_{t>\tau_b}$$

and by the stopping time

$$\tau^{\pi^\star} = \tau_b$$

satisfies for all positive x the relation $\phi(x) = V_{\pi^\star}(x)$.

(ii) *If $\theta(x_0) \leq V_0(x_0)$, then the policy $\pi^\star = (Z_t^{\pi^\star}, \tau^{\pi^\star})$ defined by the increasing right-continuous process*

$$Z_t^{\pi^\star} = \left[(R_{\tau_a} - x_0)_+ 1\!\!1_{t=\tau_a} + (L_t^{x_0}(\mu_0, W) - L_{\tau_a}^{x_0}(\mu_0, W)) 1\!\!1_{t>\tau_a}\right] 1\!\!1_{\tau_a<\tau_c}$$
$$+ \left[((R_{\tau_c} - I) - x_1)_+ 1\!\!1_{t=\tau_c} + L_t^{x_1}(\mu_1, W) 1\!\!1_{t>\tau_c}\right] 1\!\!1_{\tau_c<\tau_a}$$

and by the stopping time

$$\tau^{\pi^\star} = \begin{cases} \tau_c, & \text{if } \tau_c < \tau_a \\ \infty, & \text{if } \tau_c > \tau_a \end{cases}$$

satisfies for all positive x the relation $\phi(x) = V_{\pi^\star}(x)$.

(B) *If condition* (H1) *is not satisfied, then the policy* $\pi^\star = (Z_t^{\pi^\star}, \tau^{\pi^\star})$ *defined by the increasing right-continuous process*

$$Z_t^{\pi^\star} = (x - x_1)_+ \mathbb{1}_{t=0} + L_t^{x_0}(\mu_0, W)\mathbb{1}_{t>0}$$

and by the stopping time

$$\tau^{\pi^\star} = \infty$$

satisfies for all positive x the relation $\phi(x) = V_{\pi^\star}(x)$.

Proof Part (i) is immediate from (2.8) and part (i) of Proposition 3.5. We start the proof of part (ii) by some helpful remarks on the considered policy $\pi^\star$. On the event $\{\tau_a < \tau_c\}$, the investment time $\tau^{\pi^\star}$ is infinite a.s. Moreover, denoting by $X^{\pi^\star}$ the cash process generated by the policy $\pi^\star$, we have that $X_{\tau_a}^{\pi^\star} = x_0$ a.s. and for $t \geq 0$, we have the equality

$$X_{\tau_a+t}^{\pi^\star} = x_0 + \mu_0 t + \sigma(W_{\tau_a+t} - W_{\tau_a}) - \left(L_{\tau_a+t}^{x_0}(\mu_0, W) - L_{\tau_a}^{x_0}(\mu_0, W)\right). \quad (3.7)$$

Now, introduce the process $B_t^{(a)} = W_{\tau_a+t} - W_{\tau_a}$. We know that $B^{(a)}$ is a Brownian motion independent of $\mathcal{F}_{\tau_a}$ (Theorem 6.16 in [19]), and from the uniqueness of the solution to the Skorohod equation (Chap. IX, Exercise 2.14 in [28]) follows by (3.7) the identity in law

$$L_{\tau_a+t}^{x_0}(\mu_0, W) - L_{\tau_a}^{x_0}(\mu_0, W) \overset{\text{law}}{=} L_t^{x_0}(\mu_0, B^{(a)}). \quad (3.8)$$

Keeping in mind these remarks, we now turn to the proof of (ii). According to the structure of the value function ϕ in Proposition 3.5, three cases have to be considered.

α) If $x \leq a$, then we have $\tau_a = 0$, $\tau^{\pi^\star} = \infty$ a.s. and

$$Z_t^{\pi^\star} = (x - x_0)_+ \mathbb{1}_{t=0} + L_t^{x_0}(\mu_0, W)\mathbb{1}_{t>0}.$$

We get

$$\begin{aligned} V_{\pi^\star}(x) &= \mathbb{E}_x\left[\int_0^{\tau_0^{\pi^\star}} e^{-rs} dZ_s^\star\right] \\ &= (x - x_0)_+ + \mathbb{E}_{\min(x,x_0)}\left[\int_0^{\tau_0^{\pi^\star}} e^{-rs} dL_s^{x_0}(\mu_0, W)\right] \\ &= V_0(x) = \phi(x). \end{aligned}$$

β) If $x \geq c$, then we have $\tau^{\pi^\star} = \tau_c = 0$ a.s.,

$$Z_t^{\pi^\star} = ((x - I) - x_1)_+ \mathbb{1}_{t=0} + L_t^{x_1}(\mu_1, W)\mathbb{1}_{t>0}$$

and $X_{\tau_c}^{\pi^\star} = x - I$ a.s. We thus obtain $V_{\pi^\star}(x) = V_1(x - I) = \phi(x)$.

γ) Finally, assume that $a < x < c$. We have

$$V_{\pi^\star}(x) = \mathbb{E}_x\left[\mathbb{1}_{\tau_a<\tau_c}\int_0^{\tau_0^{\pi^\star}} \mathrm{e}^{-rs}\mathrm{d}Z_s^{\pi^\star}\right] + \mathbb{E}_x\left[\mathbb{1}_{\tau_a>\tau_c}\mathrm{e}^{-r\tau_c}V_1(c-I)\right].$$

Now,

$$\begin{aligned}\mathbb{E}_x&\left[\mathbb{1}_{\tau_a<\tau_c}\int_0^{\tau_0{}^{\pi^\star}} \mathrm{e}^{-rs}\mathrm{d}Z_s^\star\right]\\ &= \mathbb{E}_x\left[\mathbb{1}_{\tau_a<\tau_c}\left(\mathrm{e}^{-r\tau_a}(a-x_0) + \int \mathbb{1}_{]\tau_a,\tau_0^{\pi^\star}]}(s)\mathrm{e}^{-rs}\mathrm{d}L_s^{x_0}(\mu_0,W)\right)\right]\\ &= \mathbb{E}_x\left[\mathbb{1}_{\tau_a<\tau_c}\mathrm{e}^{-r\tau_a}(a-x_0)\right] + A. \end{aligned} \tag{3.9}$$

On the other hand, on the event $\{\tau_a < \tau_c\}$, we have the equality

$$\tau_0^{\pi^\star} \equiv \inf\{s\ :\ X_s^{\pi^\star} \le 0\} = \tau_a + \inf\{s\ :\ X_{s+\tau_a}^{\pi^\star} \le 0\}\ \text{ a.s.}$$

It then follows from (3.7) and (3.8) that

$$\tau_0^{\pi^\star} - \tau_a \overset{\text{law}}{=} T_0 \equiv \inf\left\{s \ge 0\ :\ x_0 + \mu_0 s + \sigma B_s^{(a)} - L_s^{x_0}(\mu_0, B^{(a)}) \le 0\right\}.$$

Coming back to (3.9), we thus obtain

$$\begin{aligned} A &= \mathbb{E}_x\left[\mathbb{1}_{\tau_a<\tau_c}\mathbb{E}\left[\int \mathbb{1}_{]\tau_a,\tau_0^{\pi^\star}]}(s)\mathrm{e}^{-rs}\mathrm{d}L_s^{x_0}(\mu_0,W)\,\middle|\,\mathcal{F}_{\tau_a}\right]\right]\\ &= \mathbb{E}_x\left[\mathbb{1}_{\tau_a<\tau_c}\mathbb{E}\left[\int \mathbb{1}_{]0,\tau_0^{\pi^\star}-\tau_a]}(u)\mathrm{e}^{-r(u+\tau_a)}\mathrm{d}L_{u+\tau_a}^{x_0}(\mu_0,W)\,\middle|\,\mathcal{F}_{\tau_a}\right]\right]\\ &= \mathbb{E}_x\left[\mathbb{1}_{\tau_a<\tau_c}\mathrm{e}^{-r\tau_a}\mathbb{E}_{x_0}\left[\int \mathbb{1}_{]0,T_0]}(u)\mathrm{e}^{-ru}\mathrm{d}L_u^{x_0}(\mu_0,B^{(a)})\right]\right]\\ &= \mathbb{E}_x\left[\mathbb{1}_{\tau_a<\tau_c}\mathrm{e}^{-r\tau_a}V_0(x_0)\right]\end{aligned}$$

where the third equality follows from the independence of $B^{(a)}$ with respect to $\mathcal{F}_{\tau_a}$ and from (3.8) together with the fact that $L^{x_0}(\mu_0, B^{(a)})$ is an additive functional. We therefore obtain

$$\mathbb{E}_x\left[\mathbb{1}_{\tau_a<\tau_c}\int_0^{\tau_0^{\pi^\star}} \mathrm{e}^{-rs}\mathrm{d}Z_s^{\pi^\star}\right] = \mathbb{E}_x\left[\mathbb{1}_{\tau_a<\tau_c}\mathrm{e}^{-r\tau_a}V_0(a)\right]$$

which leads to

$$V_{\pi^\star}(x) = \mathbb{E}_x\left[\mathbb{1}_{\tau_a<\tau_c}\mathrm{e}^{-r\tau_a}V_0(a)\right] + \mathbb{E}_x\left[\mathbb{1}_{\tau_a>\tau_c}\mathrm{e}^{-r\tau_c}V_1(c-I)\right] = \phi(x).$$

The proof of the proposition is completed by remarking that assertion (B) follows directly from relation (2.7). □

3.4 Discussion

Our mathematical analysis addresses several important issues in corporate finance. We first characterize situations where it is optimal to postpone dividend distribution in order to invest later in the growth opportunity. We then investigate the effect of liquidity shocks on the optimal dividend/investment policy. In particular, we show that a liquidity shock can result in an inaction region in which the manager waits to see whether or not the growth opportunity is valuable. In a third step we analyse the effect of positive uncertainty shocks. In stark difference with the standard real options literature, we explain why a sufficiently large positive uncertainty shock can make worthless the option to invest in a growth opportunity. Finally, we identify situations where a cash constrained firm may want to accumulate cash in order to invest in the growth opportunity whereas an unconstrained firm will definitively decide not to invest.

When to postpone dividend distribution? Intuitively, delaying dividend distribution is optimal when the growth option is "sufficiently" valuable. Our model allows to make this point precise. Let us describe the optimal dividend/investment policy assuming the current value x of the cash reserve to be lower than the threshold level x_0 that triggers the distribution of dividends when the firm is run under the initial technology. Two cases arise. If, evaluated at the threshold x_0, the value of the option to invest in the new project is larger than the value of the firm under the technology in place $\big(\text{that is, } \theta(x_0) > V_0(x_0)\big)$, then the manager postpones dividend distribution in order to accumulate cash and to invest in the new technology at threshold b. Any surplus above x_1 will be then distributed as dividends. If, on the contrary, $\theta(x_0) < V_0(x_0)$, then the manager optimally ignores the growth option, runs the firm under the technology in place and pays out any surplus above x_0 as dividends.

The effect of liquidity shocks. Our model emphasizes the value of cash for optimal dividend/investment timing. Consider indeed the case where the current value x of the cash reserve is lower than the threshold x_0 and where $\theta(x_0) \leq V_0(x_0)$. Assume that an exogenous positive shock on the cash reserve occurs so that the current value x is now larger than x_0. Three possibilities must be considered. First, if $x > c$, then, according to Proposition 3.10, the manager optimally invests immediately in the new project (and pays out any surplus above $I + x_1$ as dividends). Second, if x lies in (x_0, a), then the manager pays out $x - x_0$ as "exceptional dividend", never invests in the new technology, and pays out any surplus above x_0 as dividends. Finally, if x lies in (a, c), then two scenarios can occur. If the cash reserve rises to c before hitting a, the manager invests in the new project (and pays out any surplus above x_1 as dividends). By contrast, if the cash reserve falls to a before hitting c, the manager pays $a - x_0$ as "exceptional dividend", never invests in the new technology, and pays out any surplus above x_0 as dividends. The region (a, c) is therefore an inaction region where the manager has not enough information to decide whether or not the growth option is valuable. He therefore chooses neither to distribute dividends nor to invest in the new technology. His final decision depends on which of the bounds a or c will be reached first by the cash flow process. As a result, our model suggests that a given cash injection does not always provoke or accelerate investment decision.

The effect of uncertainty shocks. In the standard real options literature as well as in the optimal dividend policy literature, increasing the volatility of the cash process has an unambiguous effect: greater uncertainty increases both the value of the option to invest (see [21]), and the threshold that triggers the distribution of dividends (see [27]). In our setting, because the dividend and the investment policies are interrelated, the effect of an uncertainty shock is ambiguous. Consider for instance a situation where initially $\theta(x_0) < V(x_0)$ with a current value x of the cash reserve lower than x_0 and assume that a positive shock on the volatility of the cash process occurs. The volatility shock increases the trigger x_0, but does not affect $V(x_0)$ which is by construction equal to $\frac{\mu_0}{r}$. A volatility shock, however, increases $\theta(x_0)$, the value of the option to invest in the new project, and therefore the inequality $\theta(x_0) < V(x_0)$ can happen to be reversed. In this case, the manager who initially ignores the growth opportunity will decide after a positive shock on uncertainty to accumulate cash and to exercise the growth opportunity at threshold b. Here, in line with the standard real options literature, a positive volatility shock makes the growth option valuable. An interesting feature of our model is that the opposite can also occur; more precisely, a sudden increase of the volatility can kill the growth option. The crucial remark here is that the difference $x_1 - x_0$ considered as a function of the volatility σ tends to $\frac{\mu_1-\mu_0}{r}$ when σ tends to infinity. This implies that for large volatility, condition (H1) is never satisfied and thus the growth opportunity is worthless. As a matter of fact, think of an initial situation where $\theta(x_0) > V(x_0)$ (and thus condition (H1) holds) and consider a shock on the volatility such that (H1) is no longer satisfied. In such a case, before the shock occurs, the optimal strategy is to postpone dividend distribution and to invest in the new technology at threshold b, whereas after the uncertainty shock, the growth option is worthless and will thus no more be considered by the manager.

The effect of liquidity constraints. As a last implication of our model, we now investigate the role of liquidity constraints. In the absence of liquidity constraints, the manager has unlimited cash holdings. The firm is never in bankruptcy, the manager injects money whenever needed and distributes any cash surplus in the form of dividends. In this setting, for a current cash reserve x, we thus have that $V_0(x) = x + \frac{\mu_0}{r}$ while $V_1(x - I) = x + \frac{\mu_1}{r} - I$. It follows that the manager invests in the growth option if and only if $\frac{\mu_1-\mu_0}{r} > I$, a decision that is furthermore immediate. We point out here that liquidity constraints have an ambiguous effect on the decision to exercise the growth opportunity. Indeed it can happen that, in the absence of liquidity constraints, exercising the growth option is optimal (that is, $\frac{\mu_1-\mu_0}{r} > I$), whereas it is never the case when there are liquidity constraints because condition (H1) does not hold. On the contrary, the growth opportunity can be worthless in the absence of liquidity constraints, whereas this is not the case with liquidity constraints. Such a situation occurs when $\frac{\mu_1-\mu_0}{r} < I$, condition (H1) holds and $\theta(x_0)>V_0(x_0)$ $\big($that is,[3] $r(x_1+I-x_0) < \mu_1-\mu_0 < rI$ and $\theta(x_0) > \frac{\mu_0}{r}\big)$. The reason is that investing in the growth option for a liquidity constrained firm will increase the drift of the cash generating process, and therefore will lower the probability of failure. An unconstrained firm, however, is not threatened by bankruptcy and will ignore the growth opportunity because the drift μ_1 driving the new technology is not large enough $(\mu_1 < I + r\mu_0)$.

4 Conclusion

In this paper, we consider the implications of liquidity for the dividend/investment policy of a firm that owns the perpetual right to invest in a new, profit rate increasing technology. The mathematical formulation of our problem leads to a mixed singular control/optimal stopping problem that we solve quasi-explicitly by using a connection with an auxiliary stopping problem. A detailed analysis based on the properties of local time gives the precise optimal dividend/investment policy. This type of problem is non standard and does not seem to have attracted much attention in the corporate finance literature. Our analysis follows the lines of stochastic control and relies on the choice of a drifted Brownian motion for the cash reserve process in the absence of dividend distribution. This modelling assumption guarantees the quasi-explicit nature of the value function ϕ. We use this feature for instance in Proposition 3.4 where we show that ϕ is a supersolution. Furthermore, the property of independent increments for Brownian motion plays a central role for deriving the optimal policy (Proposition 3.10). Clearly, future work is needed to examine the robustness of our results to more general diffusions than a drifted Brownian motion.

Acknowledgments We thank Monique Jeanblanc and two anonymous referees for thoughtful comments and suggestions. We also thank seminar participants at CREST and Bachelier for their comments. Financial support from the Fonds National de la Science is gratefully acknowledged. We remain solely responsible for the content of this paper.

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[3] These conditions are indeed compatible. Keeping in mind that the threshold x_0 is a single peaked function of μ_0 (see [27]), consider μ_0 large, I small and μ_1 in a left neighbourhood of $rI+\mu_0$. It then follows that $r(x_1+I-x_0) < 0 < \mu_1-\mu_0 < rI$ and $\tilde{x} < x_1+I$ which implies $\theta(x_0) \geq V_1(x_0-I) > V_0(x_0) = \frac{\mu_0}{r}$.

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