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An exact analytical solution for discrete barrier options

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Abstract. In the present paper we provide an analytical solution for pricing discrete barrier options in the Black-Scholes framework. We reduce the valuation problem to a Wiener-Hopf equation that can be solved analytically. We are able to give explicit expressions for the Greeks of the contract. The results from our formulae are compared with those from other numerical methods available in the literature. Very good agreement is obtained, although evaluation using the present method is substantially quicker than the alternative methods presented.

Key words: Barrier options, discrete monitoring, Wiener-Hopf equation, Black-Scholes, z-transform

JEL Classification: G13, C63

Mathematics Subject Classification (2000): 45E10, 44A10, 60H30, 60G51.

1 Introduction

In this paper we study the valuation problem for discrete barrier options. Barrier options are common, extensively traded types of exotic derivatives. They are activated (knock-ins) or terminated (knock-outs) if a specific trigger is reached before the expiry date. There are now several papers dealing with the pricing of barrier options and a great number of valuation techniques have been proposed.
In practice, barrier options differ from those studied in the academic literature in many respects. One of the most important is the monitoring frequency of the underlying assets, i.e. the frequency of observation of the triggering event. With discrete monitoring the trigger is checked at fixed times (e.g. weekly or monthly).

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As a consequence a knock-out (knock-in) option becomes less (more) expensive as the number of monitoring dates increases. In the case of continuous monitoring, several pricing formulae in the Black-Scholes framework are known, [18,27,35, 31,23]. Unfortunately, the discrepancy between option prices under continuous and discrete monitoring can be huge. One of the first papers to notice the importance of this discrepancy was offered by Heynen and Kat [21]. Since then, several papers have proposed approximations based on a variety of different numerical approaches, [5,4,6,8,11–13,15,22,24–26,41,42]. A review can be found in [16].

In the present paper, we consider the pricing problem for a down-and-out barrier option under a geometric Brownian motion (GBM) process with discrete monitoring. We then show how to reduce its evaluation to an integral equation of Wiener-Hopf type. The latter problem admits an *analytical solution* in the standard Black-Scholes framework, i.e. when the underlying asset evolves according to a GBM and the knock-out clause is activated by a constant barrier. The paper is organised as follows. In the following section, the valuation problem is reduced to a scalar Wiener-Hopf integral equation of the second kind. The solution is obtained explicitly in Appendix A in terms of infinite sums of simple functions plus a single special function (12). These are summarised in Sect. 2.1 together with an alternative representation (derived in Appendix B) that is suitable for numerical computation in the cases of special parameter values. Section 2.2 offers the formal solution of the barrier option price as an inverse z-transform of the Wiener-Hopf solution, and also presents explicit formulae for the Greeks that makes our procedure really competitive with respect to Monte Carlo simulation. Section 3 relates the result in the present paper to the celebrated Spitzer identity and to other papers that use this for pricing discrete barrier options [9,30,32]. Numerical results are compared and contrasted with alternative numerical methods in Sect. 4, and final remarks are offered in Sect. 5.

2 The model

We work in a standard Black-Scholes framework where the underlying asset evolves, under the risk-neutral measure, according to a GBM process

$$dx_t = rx_t\,dt + \sigma x_t\,dW_t,$$

with initial stock price x_0 and where r and σ are respectively the constant risk-free rate and the constant instantaneous volatility. We want to price a down-and-out call option, i.e. a call option that expires worthless if a lower barrier has been hit at a monitoring date. The corresponding down-and-in call can be priced by subtracting from the price of a standard call the price of the down-and-out call. The barrier put option can be priced using the put-call transformation given in [20]. Let $0 = t_0 < t_1 < \ldots < t_n < \ldots < t_N = T$ be the monitoring dates, T the option maturity and l the constant lower barrier active at all times t_n. The nth time interval is defined as $t_n < t < t_{n+1}$ and we denote the price of the barrier option in this interval as $C(x, t, n) \equiv C(x, t, n; l)$. Then $C(x, t, n)$ satisfies the well-known

Black-Scholes partial differential equation (PDE)

$$-\frac{\partial C(x,t,n)}{\partial t}+rx\frac{\partial C(x,t,n)}{\partial x}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 C(x,t,n)}{\partial x^2}=rC(x,t,n). \quad (1)$$

Given that the trigger condition is checked only at fixed times, we need to update the initial condition at each of the monitoring dates t_n:

$$\begin{aligned} C(x,t_n,n) &= C(x,t_n,n-1)\,\mathbf{1}_{(x\geq l)},\\ C(x,t_0,0) &= (x-K)\,\mathbf{1}_{(x\geq \max(K,l))},\end{aligned}$$

where K is the exercise price of the option and $\mathbf{1}_{(x\geq l)}$ is the indicator, or Heaviside, function

$$\mathbf{1}_{(x\geq l)}=\begin{cases}1 & \text{if } \; x\geq l,\\ 0 & \text{if } \; x<l.\end{cases}$$

We can use the standard change of variables (see [44], p. 98)

$$C(x,t,n)=w(z,t,n),$$

where

$$z=\ln(x/l)\,; \quad k=\ln(K/l)\,; \quad m=r-\sigma^2/2,$$

to transform the partial differential equation (1) into

$$\begin{aligned} &-w_t+mw_z+\tfrac{\sigma^2}{2}w_{zz}=rw,\\ &w(z,t_n,n)=w(z,t_n,n-1)\,\mathbf{1}_{(z\geq 0)},\, n=1,2,3\ldots,\\ &w(z,t_0,0)=l\left(e^z-e^k\right)\mathbf{1}_{(z\geq\delta)},\end{aligned}$$

with

$$\delta=\max(k,0). \quad (2)$$

Employing the second transformation $w(z,t,n)=e^{\alpha z+\beta t}g(z,t,n)$, in which $\alpha=-m/\sigma^2$, $c^2=\sigma^2/2$, $\beta=\alpha m+\alpha^2\sigma^2/2-r$, the function $g(z,t,n)$ satisfies the heat equation

$$-g_t+c^2g_{zz}=0 \quad (3)$$

with initial conditions, for $n=1,2,3,\ldots,$

$$g(z,t_n,n)=g(z,t_n,n-1)\,\mathbf{1}_{(z\geq 0)},$$

whilst when $n=0$ the initial condition becomes

$$g(z,0,0)=le^{-\alpha z}\left(e^z-e^k\right)\mathbf{1}_{(z\geq\delta)}.$$

When $t_n < t < t_{n+1}$ and $z > 0$ the solution of the above partial differential equation (3) is given by, see e.g. [40], p. 47,

$$g(z,t,n) = \begin{cases} \int_0^{+\infty} S(z-\xi, t-t_n)\, g(\xi, t_n, n-1)\, d\xi, & n = 1, 2, \dots, \\ l \int_0^{+\infty} S(z-\xi, t-t_n)\, e^{-\alpha\xi} \left(e^{\xi} - e^{k}\right) \mathbf{1}_{(\xi \geq \delta)} d\xi, & n = 0, \end{cases} \tag{4}$$

where the kernel $S(z,t)$ is the Gaussian, $S(z,t) = e^{-z^2/\left(4c^2 t\right)}/\sqrt{4\pi c^2 t}$. Note that an iterative application of (4) provides an evaluation of discrete barrier options in terms of multivariate normal probabilities. This approach is followed by Heynen and Kat [21], but from a numerical point of view is hardly feasible when the number of observation points becomes large, say more than 10.

Let us consider the function $g(z,t,n)$ only at the monitoring times t_n, and let τ be the fixed period between the monitoring dates so that $t_n + \tau = t_{n+1}$. We set $f(z,n) = g(z, t_n, n-1)$, i.e. the value of g at the upper end, t_n, of the $(n-1)$-th time interval, so that we have

$$f(z,n) = \int_0^{+\infty} \frac{e^{-(z-\xi)^2/\left(4c^2\tau\right)}}{\sqrt{4\pi c^2 \tau}} f(\xi, n-1)\, d\xi, \quad n = 2, 3, \dots, \tag{5}$$

and for $n = 1$

$$f(z,1) = l \int_0^{+\infty} S(z-\xi, \tau)\, e^{-\alpha\xi} \left(e^{\xi} - e^{k}\right) \mathbf{1}_{(\xi \geq \delta)} d\xi. \tag{6}$$

The last equation may be absorbed into the set in (5) by defining the additional function

$$f(z,0) = l e^{-\alpha z} \left(e^{z} - e^{k}\right) \mathbf{1}_{(z \geq \delta)}. \tag{7}$$

We now take the *z-transform* of the above difference equation (5) by multiplying both sides of (5) by q^n, $q \in \mathbb{C}$, to get

$$q^n f(z,n) = q \int_0^{+\infty} S(z-\xi, \tau)\, q^{n-1} f(\xi, n-1)\, d\xi$$

and then summing over all $n \geq 1$. Assuming that we can interchange the order of integration and summation[1] (which may be proved *a posteriori*), we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} q^n f(z,n) &= q \int_0^{+\infty} S(z-\xi, \tau) \sum_{n=1}^{\infty} q^{n-1} f(\xi, n-1)\, d\xi \\ &= q \int_0^{+\infty} S(z-\xi, \tau) \sum_{n=0}^{\infty} q^{n} f(\xi, n)\, d\xi. \end{aligned}$$

[1] The interchange of integration and summation requires $\sum_{j=0}^{n} q^j f(\xi, j)$ to converge uniformly. The z-transform $\sum_{j=0}^{\infty} q^j f(\xi, j)$ is in fact a power series in q with coefficients $f(\xi, j)$ and radius of convergence given in (14). A power series converges uniformly in a closed and bounded interval contained in the interval of convergence.

So, by defining

$$F(z,q) = \sum_{n=0}^{\infty} q^n f(z,n) \tag{8}$$

and adding $f(z,0)$ to both sides, we arrive at the following integral equation for $F(z,q)$:

$$F(z,q) = q \int_0^{+\infty} S(z-\xi,\tau)\, F(\xi,q)\, d\xi + f(z,0) \tag{9}$$

defined over the interval $0 < z < \infty$.

2.1 Solution of the Wiener-Hopf equation

We can recognize in (9) an integral equation of the second kind with a semi-infinite range, and a convolution structure, i.e. the kernel S depends on the difference $z-\xi$. This integral equation has to be solved with respect to the unknown function $F(z,q)$. Fortunately, given the form of the kernel, this can be recognized as a Wiener-Hopf equation; see [33,28]. We remark that if the integral were extended to the whole real line it would be sufficient to use a Fourier transform method. However, since the integral range, and the variable z range, is the positive real line it is necessary here to employ the Wiener-Hopf method. For continuity of the text, and for brevity, all details of the solution procedure for Eq. (9) are given in Appendix A where we prove that the exact solution of the above integral equation is

$$\begin{aligned} F(z,q) = &-\tfrac{il\gamma}{2} e^{(1-\alpha)k} \sum_{n=-\infty}^{\infty} \frac{e^{i\mu_n |z-k|/\gamma}}{\mu_n(\mu_n - i\alpha\gamma\,\mathrm{sgn}(z-k))(\mu_n - i(\alpha-1)\gamma\,\mathrm{sgn}(z-k))} \\ &+ l e^{-\alpha z} \left\{ \frac{e^z}{1-qe^{(\alpha-1)^2\gamma^2}} - \frac{e^k}{1-qe^{(\alpha\gamma)^2}} \right\} \mathbf{1}_{(z\geq k)} \\ &-\tfrac{il\gamma}{4} e^{(1-\alpha)k} \sum_{n=-\infty}^{\infty} \frac{L_+(\mu_n) e^{i\mu_n z/\gamma}}{\mu_n} \\ &\times \sum_{m=-\infty}^{\infty} \frac{L_+(\mu_m) e^{i\mu_m k/\gamma}}{\mu_m(\mu_m + i\alpha\gamma)(\mu_m + i(\alpha-1)\gamma)(\mu_m+\mu_n)}, \end{aligned} \tag{10}$$

valid for $k > 0$, where the sign of x is denoted by

$$\mathrm{sgn}(x) = \begin{cases} 1, & x \geq 0, \\ -1, & x < 0, \end{cases} \tag{11}$$

and from (62) we have

$$L_+(u) = \exp\left\{ \frac{u}{\pi i} \int_0^{\infty} \frac{\ln\left(1 - qe^{-z^2}\right)}{z^2 - u^2} dz \right\}, \quad \Im(u) > 0. \tag{12}$$

Also, the complex coefficients are

$$\mu_m = \sqrt{\ln q + 2m\pi i}, \quad -\infty < m < \infty; \tag{13}$$

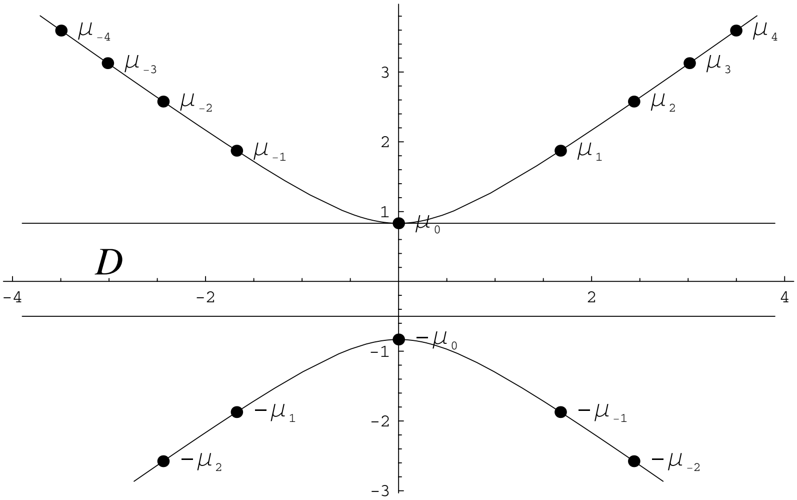

Fig. 1. The complex u-plane showing the locus of the zeros of $L\,(u)$, $\pm\mu_n$ $(-\infty < n < \infty)$, as $\arg(q)$ increases from 0 to 2π and with $|q| = 1/2$. The indicated positions on the path are when $\arg(q) = 0$. Also shown is the overlapping strip of analyticity, $\mathcal{D} \equiv \{u|\,(1-\alpha)\,\gamma < \Im(u) < \Im(\mu_0)\}$ (where in this example $(1-\alpha)\,\gamma > -\Im(\mu_0)$), inside which the Wiener-Hopf equation (33) is defined

they lie in the upper half-plane as shown in Fig. 1. Note that (10) is given by (27) and (61) with the constants there replaced by the original parameters $a = \alpha\gamma$, $b = \gamma$, and $\gamma = c\sqrt{\tau} = \sigma\sqrt{\tau/2}$. Finally, for (10) to offer a unique solution to the Wiener-Hopf equation (9) it is shown in (40) that the z-transform parameter q must satisfy

$$|q| < \exp\left\{-(1-\alpha)^2\gamma^2\mathbf{1}_{((1-\alpha)\gamma\geq 0)}\right\}. \tag{14}$$

In formula (10) we can identify the first two terms as the z-transform of the Black-Scholes call price, times a multiplicative term. This can be proved by solving the integral equation (9) over the entire real axis by means of Fourier transforms, and thence obtaining an algebraic equation in the complex domain. The solution of this equation admits an analytical Fourier inverse, given by the first two terms in (10), that can be computed by using residue calculus.

In the case $k \leq 0$, it can be demonstrated (see (64)) that the Wiener-Hopf solution has the alternative form, for $\alpha > 0$:

$$\begin{aligned} F(z,q) &= le^{-\alpha z}\left\{\frac{e^z}{1-qe^{(\alpha-1)^2\gamma^2}} - \frac{e^k}{1-qe^{(\alpha\gamma)^2}}\right\} \\ &-\frac{l}{2}\sum_{n=-\infty}^{\infty}\frac{L_+(\mu_n)e^{i\mu_n z/\gamma}}{\mu_n}\left(\frac{e^k}{L_+(i\alpha\gamma)(\mu_n - i\alpha\gamma)} - \frac{1}{L_+(i(\alpha-1)\gamma)(\mu_n - i(\alpha-1)\gamma)}\right). \end{aligned} \tag{15}$$

Note that one can show that (10) and (15) are consistent when $k = 0$. In the case $\alpha < 0$, the above formula will not work because the point $i\alpha\gamma$ is below the contour. However, we may replace $L_+(i\alpha\gamma)$ by $L(i\alpha\gamma)/L_-(i\alpha\gamma)$ where $L\,(u) = 1 - qe^{-u^2}$ is known explicitly and also $L_-(i\alpha\gamma) = L_+\,(-i\alpha\gamma)$ (L is an even function). The same procedure can be applied if $\alpha - 1 < 0$. As for the case $k > 0$, the z-transform inverse of the first term admits an analytical expression given by $le^{-\alpha z}\left\{e^{z+(\alpha-1)^2\gamma^2 t} - e^{k+(\alpha\gamma)^2 t}\right\}$. This is natural because if the barrier is above the strike, the condition of being in-the-money at expiry is redundant: if the option does not expire for the spot price touching the barrier then at maturity the spot will certainly be above the barrier, and hence the strike.

The only term in the solution $F(z,q)$ that does not have an explicit z-transform inverse is the last expression in (10) and (15). It is therefore convenient to study this quantity, defined as $\breve{F}(z,q)$ by

$$\begin{aligned}
&\breve{F}(z,q) = \\
&-\tfrac{il\gamma}{4}e^{(1-\alpha)k}\sum_{n=-\infty}^{\infty}\frac{L_+(\mu_n)e^{i\mu_n z/\gamma}}{\mu_n}\sum_{m=-\infty}^{\infty}\frac{L_+(\mu_m)e^{i\mu_m k/\gamma}}{\mu_m(\mu_m+i\alpha\gamma)(\mu_m+i(\alpha-1)\gamma)(\mu_m+\mu_n)}\mathbf{1}_{(k\geq 0)} \\
&-\tfrac{l}{2}\sum_{n=-\infty}^{\infty}\frac{L_+(\mu_n)e^{i\mu_n z/\gamma}}{\mu_n}\left(\frac{e^k}{L_+(i\alpha\gamma)(\mu_n-i\alpha\gamma)} - \frac{1}{L_+(i(\alpha-1)\gamma)(\mu_n-i(\alpha-1)\gamma)}\right)\mathbf{1}_{(k<0)}.
\end{aligned} \tag{16}$$

Note that the convergence of the above expressions is very slow when z or k are small, and so in this case a more efficient formula is required for numerical purposes. Therefore, in Appendix B we obtain an alternative integral representation given here as

$$\begin{aligned}
&\breve{F}(z,q) = \\
&-\tfrac{l\gamma q}{2\pi}e^{(1-\alpha)k}\left\{\frac{(-1)^{s+1}}{2}\sum_{n=-\infty}^{\infty}\frac{L_+(\mu_n)e^{i\mu_n z/\gamma}}{\mu_n^{s+2}}\int_C\frac{e^{-\xi^2}e^{i\xi k/\gamma}\xi^{s+1}}{(\xi+i\alpha\gamma)(\xi+i(\alpha-1)\gamma)(\xi+\mu_n)L_-(\xi)}d\xi\right. \\
&\left.+\tfrac{q}{2\pi i}\sum_{p=0}^{s}(-1)^{p}\left(\int_{C_0}\frac{e^{-\zeta^2}e^{i\zeta z/\gamma}}{L_-(\zeta)\zeta^{p+1}}d\zeta\right)\left(\int_C\frac{e^{-\xi^2}e^{i\xi k/\gamma}\xi^p}{(\xi+i\alpha\gamma)(\xi+i(\alpha-1)\gamma)L_-(\xi)}d\xi\right)\right\}\mathbf{1}_{(k\geq 0)} \\
&+\tfrac{lq}{2\pi i}\int_C\frac{e^{-\xi^2}e^{-i\xi z/\gamma}}{L_+(\xi)}\left(\frac{e^k}{L_+(i\alpha\gamma)(\xi+i\alpha\gamma)} - \frac{1}{L_+(i(\alpha-1)\gamma)(\xi+i(\alpha-1)\gamma)}\right)d\xi\,\mathbf{1}_{(k<0)},
\end{aligned} \tag{17}$$

where the contour C runs from $-\infty$ to $+\infty$ along a line parallel to the real axis with $\gamma(1-\alpha) < \Im(\xi) < \Im(\mu_0)$, when $\alpha \leq 1$, and the contour C_0 runs from $-\infty$ to $+\infty$ along a line such that $0 < \Im(\zeta) < \Im(\mu_0)$. Alternatively, both integral paths could run along the real line except that C_0 must be indented above the pole of order $p+1$ at the origin, and when $\alpha \leq 1$ the contour C is indented to run between the poles at $\xi = i\gamma(1-\alpha)$ and $\xi = \mu_0$. The integer parameter s is free and is chosen large enough for rapid convergence of the sum.

2.2 *The analytical formula*

The solution of the Wiener-Hopf equation gives the function $\breve{F}(z,q)$, but we still have to invert the z-transform (and then add in the leading terms from (10) and (15)) in order to recover the original function $f(z,n)$. The inversion formula is

$$\mathcal{Z}^{-1}\left(\breve{F}(z,q)\right) = \frac{1}{2\pi\rho^n}\int_0^{2\pi}\breve{F}\left(z,\rho e^{iu}\right)e^{-inu}du, \quad n = 0,1,2,\ldots \tag{18}$$

where $\rho = |q|$ satisfies (14). Formally, the expression (18) with (16) or (17) gives an analytical solution for the pricing problem of discrete barrier options. The barrier option price $C(x, t_n, n-1)$ at a monitoring date t_n is given by

$$C(x, t_n, n-1) = \left(\frac{x}{l}\right)^{\alpha} e^{\beta t_n} f\left(\ln \frac{x}{l}, n\right), \quad n = 1, 2, 3, \ldots \tag{19}$$

As observed in the previous section, in (10) and (15) we can identify a term due to the Black-Scholes formula. Therefore, (19) may be written as

$$C(x, t_n, n-1) = C_{BS}(x) + \left(\frac{x}{l}\right)^{\alpha} e^{\beta t_n} \breve{f}\left(\ln \frac{x}{l}, n\right), \quad n = 1, 2, 3, \ldots, \tag{20}$$

where $\breve{f}(z, n) = \mathcal{Z}^{-1}\left(\breve{F}(z, q)\right)$ and $C_{BS}(x)$ is given by

$$C_{BS}(x) = \begin{cases} xN(d_1) - Ke^{-rt_n}N(d_2), & k \geq 0, \\ x - Ke^{-rt_n}, & k < 0, \end{cases}$$

$$d_1 = \frac{\ln(x/K) + \left(r + \frac{\sigma^2}{2}\right)(t_n - t_0)}{\sigma\sqrt{t_n - t_0}}; \quad d_2 = d_1 - \sigma\sqrt{t_n - t_0}.$$

The above formula can be used also for the computation of the Greeks, i.e., the derivatives with respect to x of the option price. Indeed, the function $\breve{F}\left(z, \rho e^{iu}\right)$ is analytic for all $0 \leq u \leq 2\pi$ and so may be differentiated without difficulty[2]. For the Delta, $\frac{\partial C}{\partial x}(x, t_n, n-1)$, we have

$$\Delta = \Delta_{BS}(x) + \frac{1}{l}\left(\frac{x}{l}\right)^{\alpha-1} e^{\beta t_n}\left[\alpha\breve{f}(z, n) + \frac{\partial \breve{f}}{\partial z}(z, n)\right] \tag{21}$$

with $z = \ln(\frac{x}{l})$, where $\Delta_{BS} = N(d_1)\mathbf{1}_{k\geq 0} + \mathbf{1}_{k<0}$ and

$$\frac{\partial \breve{f}}{\partial z}(z, n) = \frac{1}{2\pi\rho^n}\int_0^{2\pi} \frac{\partial \breve{F}}{\partial z}(z, \rho e^{iu})e^{-inu}du.$$

In a similar way we obtain for the Gamma, $\partial^2 C/\partial x^2(x, t_n, n-1)$, that

$$\Gamma = \Gamma_{BS}(x) + \frac{1}{l^2}\left(\frac{x}{l}\right)^{\alpha-2} e^{\beta t_n}\left[\alpha(\alpha-1)\breve{f}(z, n) + (2\alpha-1)\frac{\partial \breve{f}}{\partial z} + \frac{\partial^2 \breve{f}}{\partial z^2}\right] \tag{22}$$

with $z = \ln(\frac{x}{l})$, where $\Gamma_{BS}(x)$ is the Black-Scholes Gamma and

$$\frac{\partial^2 \breve{f}}{\partial z^2}(z, n) = \frac{1}{2\pi\rho^n}\int_0^{2\pi} \frac{\partial^2 \breve{F}}{\partial z^2}(z, \rho e^{iu})e^{-inu}du.$$

[2] In particular, the interchange of derivative and inversion integral can be justified using 33.7 p. 267 in [34]. The differentiation term by term of the series in (16) can be justified by the convergence of the series and by observing that the sequence of derivatives converges uniformly for $z > 0$.

Note that by carefully combining $\breve{f}(z,n)$ and its derivatives appearing in the expression for the Delta and Gamma, one can obtain Greeks that are computationally of the same cost as the option price itself. The above derivatives can be computed using (16) or the integral representation (17).

3 Relationship with the Spitzer identity

In a recent work Petrella and Kou [32], using the Spitzer identity [38], have developed an algorithm to price discrete barrier options. In particular, starting from the Spitzer identity and generalizing an idea by Ohgren [30] and Borovkov and Novikov [9], they obtain by recursion the Fourier transform of the solution, which at the final step must be inverted by a suitable numerical procedure. In the present paper, the inverse of the Fourier transform is obtained in analytical form and we need to perform the numerical z-transform inversion.

To understand the relationship between our approach and the one in [32] we have to recall the Spitzer identity [38]. This identity gives the z-transform of the characteristic function of the successive maxima of a sequence of i.i.d. random variables. In the literature, other important papers like [14,43,39] have examined the equivalence between the Spitzer identity and the solution of a Wiener-Hopf integral equation. With respect to the results in these papers, we provide the analytical solution to the Wiener-Hopf equation when the kernel is Gaussian; in other words we analytically invert the characteristic function appearing in the Spitzer identity. However, although our methodology is quite general and applicable to processes with independent and stationary increments, it is dubious if an analytical expression can be obtained in more general cases than the Brownian motion considered here.

The solution of the Wiener-Hopf equation presented in Appendix A requires that the transformed kernel of the integral equation (or here more precisely $\delta(\xi) - qS(\xi,\tau)$, where $\delta(\xi)$ is the generalised delta function) is decomposed into a product of two functions, one analytic in the upper half of the transform plane and the other analytic in an overlapping lower half-plane. The Spitzer contribution consists in giving a different representation to these functions and in their probabilistic interpretation in terms of the characteristic function of the maximum and minimum of a geometrically stopped random walk. The use of geometrically (exponentially) distributed random times as stopping times allows one to replace the use of the z-transform (Laplace transform) by considerations related only to the probabilistic structure of the process under consideration. For an expository discussion see [19].

In order to stress the relationship of our approach with the Spitzer identity in the Gaussian case, let us consider the process

$$S_n = Z_1 + ... + Z_j + ... + Z_n, \quad S_0 = 0$$

where Z_j are i.i.d. and let $M_n = \max(0, S_1, ..., S_n)$. Spitzer [38], using combinatorial arguments, derived the identity

$$\begin{aligned}\psi(u,q) &= \sum_{n=0}^{\infty}\mathbb{E}\left(e^{iuM_n}\right)q^n \\ &= \exp\left(\sum_{n=1}^{\infty}\frac{q^n}{n}\mathbb{E}\left(e^{iuS_n};S_n\ge 0\right)\right)\exp\left(\sum_{k=1}^{\infty}\frac{q^n}{n}\Pr\left(S_n<0\right)\right) \\ &= \psi_+(u,q)\,\psi_-(0,q) \end{aligned} \tag{23}$$

and where

$$\psi_+(u,q)\,\psi_-(u,q)=\frac{1}{1-q\mathbb{E}\left(e^{iuZ}\right)}.$$

In other words, Spitzer obtained an expression for the z-transform of the characteristic function of the discrete maximum. His factorization $\psi(u,q) = \psi_+(u,q)$ $\psi_-(u,q)$ is comparable to the factorization of the Fourier transform $L(u) = 1 - q\mathcal{F}(k(z)) = L_+(u)L_-(u)$. Moreover, $\psi_+(u,q)$ and $\psi_-(u,q)$ have a probabilistic interpretation as characteristic functions of the maximum and minimum stopped at a geometrically distributed stopping time.

If Z_j is a Gaussian random variable with zero mean and variance 2, then $\mathbb{E}\left(e^{iuZ_j}\right) = e^{-u^2}$. Therefore it is easy to show that

$$\psi_+(u,q)=\exp\left(\sum_{n=1}^{\infty}\frac{q^n}{2n}e^{-nu^2}\left(1+erf\left(i\sqrt{n}u\right)\right)\right), \tag{24}$$

$$\psi_-(0,q)=\exp\left(\frac{1}{2}\sum_{k=1}^{\infty}\frac{q^n}{n}\right)=\exp\left(-\frac{1}{2}\ln(1-q)\right)=\frac{1}{(1-q)^{\frac{1}{2}}},$$

where $erf(u)$ is the error function, $erf(u) = (2/\sqrt{\pi})\int_0^u e^{-t^2}\,dt$. The factorization of $\psi(u,q)$ as a product of the functions ψ_+ and ψ_- is related to the factorization $L(u) = L_+(u)\,L_-(u)$ we give in (42). Indeed, there we write

$$\ln\left(L_+(u)\right)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\ln\left(1-qe^{-\xi^2}\right)}{\xi-u}d\xi=-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\sum_{n=1}^{\infty}\frac{\left(qe^{-\xi^2}\right)^n}{n(\xi-u)}d\xi \tag{25}$$

by using the Taylor expansion of $\ln(1+x)$. We note that

$$\int_{-\infty}^{+\infty}\frac{e^{-n\xi^2}d\xi}{\xi-u}=i\pi e^{-nu^2}\left(1+erf\left(i\sqrt{n}u\right)\right),\qquad \Im(u)>\Im(\xi).$$

Thus, interchanging the sum and integral in (25) and exponentiating yields

$$L_+(u)=\exp\left(-\frac{1}{2}\sum_{n=1}^{\infty}\frac{q^ne^{-nu^2}}{n}\left(1+erf\left(i\sqrt{n}u\right)\right)\right),\qquad \Im(u)>-\Im(\mu_0).$$

In other words, $\psi_+(u,q) = 1/L_+(u)$. As we know that $L_-(u) = L_+(-u)$, it immediately follows that

$$L_-(u) = \exp\left(-\frac{1}{2}\sum_{n=1}^{\infty}\frac{q^n e^{-nu^2}}{n}\left(1 - erf\left(i\sqrt{n}u\right)\right)\right), \qquad \Im(u) < \Im(\mu_0),$$

from which we have $L_-(0) = (1-q)^{\frac{1}{2}} = 1/\psi_-(0,q)$, as desired.

We should like to stress another important difference between our approach and that employed in [9,32] and [30]. This is that their procedure requires as input an expression for $\psi_+(u,q)$ and $\psi_-(0,q)$. They are able to compute these for some non-Gaussian models. However, the evaluation of the expected values appearing in (23) is not available for general Lévy process models. This is not a restriction for the present approach in which the functions L_+ and L_- are obtained using expressions (50) and (51); these just require knowledge of the characteristic function of the underlying variable, which is in general known in closed form. Finally, the Wiener-Hopf technique has recently been used successfully to solve a number of different evaluation problems, especially within the framework of exotic derivatives [29,10,36] and in connection with Lévy processes. However, the exotic contracts considered in these papers always assume continuous monitoring of the underlying asset and the Wiener-Hopf factorizations used there arise as continuous limits as the monitoring distance goes to zero. Note that this factorization has been obtained by Baxter and Donkser [7] using a probabilistic approach. Unfortunately, the analytical solutions obtained there do not admit simple numerical implementation (typically the solution is given as a complex multiple integral with difficult integrands).

4 Numerical results

In this section we offer numerical results. To do this, a solution to the barrier option price (19) can be computed by combining (10) with a numerical approximation to the z-transform inverse (18). A simple and accurate algorithm, based on the Fourier series method, can be found in Abate and Whitt [1]. They approximate the integral in (18) using the trapezoidal rule with a step size of π/n, and obtain the result

$$\breve{f}(z,n) \approx \tilde{f}(z,n) = \frac{1}{2n\rho^n}\sum_{j=1}^{2n}(-1)^j\,\Re\left(\breve{F}\left(z,\rho e^{ji\pi/n}\right)\right) \tag{26}$$

$$= \frac{1}{2n\rho^n}\left\{\breve{F}(z,\rho) + (-1)^n\breve{F}(z,-\rho) + 2\sum_{j=1}^{n-1}(-1)^j\,\Re\left(\breve{F}\left(z,\rho e^{ji\pi/n}\right)\right)\right\}.$$

Note that the last expression is valid for all $n > 0$, where the sum term is taken as zero for $n = 1$. Abate and Whitt are able to provide an error bound when, in our case, ρ satisfies the constraint (14); it is given by

$$\left|\breve{f}(z,n) - \tilde{f}(z,n)\right| \leq \frac{\rho^{2n}}{1-\rho^{2n}}.$$

Table 1. In the Table we price a single barrier down-and-out call option for different levels of l and different monitoring dates n. Parameters used are spot price = 100, strike = 100, $r = 0.10$, $\sigma = 0.3$, $T = 0.2$. The competing methods are the recursive integration method (RI) in [5] where a grid with 2000 nodes has been used; the continuous monitoring formula (CC) with a correction based on shifting the barrier level in [13]; the trinomial tree (TT) in [12]; the Simpson recursive quadrature method (SQ) (grid spacing 2000 points) in [16]; Monte Carlo (MC) with 10^8 simulations with Mersenne twister generator and antithetic variables according to the results reported in [8]. The Wiener-Hopf (WH) solution in (10) has been computed with 300x100 terms for the double sum (700x100 terms if $l = 95$ and $n = 5$). The inversion of the z-transform has been computed setting $\rho = 10^{-\gamma/2n}$, $\gamma = 8$ and n the number of monitoring dates

l	n	WH+ZT	RI	CC	TT	SQ	MC (st.error)
89	5	6.28076	6.2763	6.284	6.281	6.2809	6.28092 (0.00078)
95	5	5.67111	5.6667	5.646	5.671	5.6712	5.67124 (0.00076)
97	5	5.16725	5.1628	5.028	5.167	5.1675	5.16739 (0.00073)
99	5	4.48917	4.4848	4.050	4.489	4.4894	4.48931 (0.0007)
89	25	6.20995	6.2003	6.210	6.21	6.2101	6.21059 (0.00078)
95	25	5.08142	5.0719	5.084	5.081	5.0815	5.08203 (0.00073)
97	25	4.11582	4.1064	4.113	4.115	4.11598	4.11621 (0.00067)
99	25	2.81244	2.8036	2.673	2.812	2.8128	2.81261 (0.00057)

For practical purposes, this error bound, when ρ^{2n} is small, is approximately equal to ρ^{2n}. Hence, to have accuracy to $10^{-\gamma}$, say, we require $\rho = 10^{-\gamma/2n}$, which lies within the uniqueness constraint. Note that in practice it is important to adjust ρ in this way for each value of n, so that ρ^n stays small but bounded away from zero as $n \to \infty$. Otherwise small computational errors in the numerical evaluation of the denominator in (26) are magnified and will lead to gross errors in the evaluation of $\breve{f}(z, n)$.

In Table 1, we compare different numerical methods with the solution of the Wiener-Hopf equation. We consider a single barrier down-and-out call option for different barrier levels and different monitoring dates. The competing methods are the recursive integration method (RI) in [5] where a grid with 2000 nodes has been used, the continuous monitoring formula (CC) with a correction based on shifting the barrier level in [13], the trinomial tree (TT) in [12], the Simpson recursive quadrature method (SQ) (grid spacing 2000 points) in [16], Monte Carlo simulation with 10^8 simulations with Mersenne twister pseudo random generator and antithetic variables in [8][3]. The Wiener-Hopf solution (WH+ZT) in (10) has been computed using 20 terms in the first sum and 300×100 terms for the double sum appearing in the third term (except when $n = 5$ and $l = 99$ when 700×100 terms are required for the given number of decimal places; for 300×100 the formula gives the value 4.48911). As we can see, the solution gives results comparable to other methods and our values can be considered exact to the figures quoted; this is confirmed when comparing with the numerical results obtained by the integral formula (17).

[3] We should like to thank M. Bertoldi and M. Bianchetti (Caboto SIM, Banca Intesa Group) for kindly providing the results of the Monte Carlo simulation.

Table 2. In the table we price a single barrier down-and-out call option for different levels of l and different monitoring dates n. Parameters used are spot price = 100, strike = 100, r = 0.10, σ = 0.2, T = 0.5. The competing numerical approximations are the Wiener-Hopf solution computed using the integral representation (IR) in (17) combined with Padé approximant in [17]; Markov chain (MCh) with a grid with 1001 points in [15]; Monte Carlo (MC) with 10^8 simulations with Mersenne twister generator and antithetic variables according to the results reported in [8] (in brackets we have the standard errors); the trinomial tree (TT) in [12]; the Simpson recursive quadrature method (SQ) (grid spacing 2000 points) in [16]. The Wiener-Hopf (WH) solution in (10) has been computed with 300x100 terms for the double sum. The inversion of the z-transform has been computed setting $\rho = 10^{-\gamma/2n}$, γ = 8 and n the number of monitoring dates

l	n	WH+ZT		MCh	TT	SQ	MC (st.error)
		Formula (10)	IR (17)				
95	25	6.63156	6.63156	6.6307	6.6181	6.6317	6.63204 (0.0009)
99.5	25	3.35558	3.35558	3.3552	3.3122	3.3564	3.35584 (0.00068)
99.9	25	2.95073	3.00887	3.0095	2.9626	3.0098	3.00918 (0.00064)
95	125	6.16864	6.16864	6.1678	6.1692	6.1687	6.16879 (0.00088)
99.5	125	1.9613	1.9613	1.9617	1.9624	1.9628	1.96142 (0.00053)
99.9	125	1.51031	1.51068	1.5138	1.5116	1.5123	1.5105 (0.00046)

In Table 2 we consider a numerically more difficult example, that is taking the barrier level approaching the spot price. The competing methods are the Markov Chain method (MCh) in [15], the trinomial tree method in [13], the Simpson recursive quadrature method in [16] and the Monte Carlo simulation with 10^8 simulations with Mersenne twister pseudo random generator and antithetic variables in [8]. For the case of 25 monitoring dates and for a spot price far from the barrier (e.g. $l = 95$ in Table 2) we get an accurate solution. But as we move the barrier level closer to the spot price, the accuracy of our numerical solution in (10) deteriorates. This is due to the very slow convergence of the double summation when z and k are very small. The error also increases as we decrease the number of monitoring dates (from 125 to 25) because $1/\gamma = \sqrt{2n/T}/\sigma$ appears in the exponent of each sum (which is why we required 700$\times$100 terms for $n = 5$ and $l = 99$ in Table 1). In general, to yield acceptable values when the barrier is close to the spot and strike prices, a number of terms greater than the 300 $\times$ 100 actually used in Table 2 are required (indeed for $n = 25$ and $l = 99.9$ using 700$\times$100 terms we obtain 3.00562). For this reason, we need an improved representation for the exact solution when k or z approach zero; a useful alternative exact form is given in (17). Note that it is actually valid for all k, z values, and the parameter s is free to be adjusted in order to balance accuracy against computational effort. For example, if we take, say, $s = 11$ in (17) then the summand of the second term is

$$\frac{L_+(\mu_n)e^{i\mu_n z/\gamma}}{\mu_n^{s+2}} le^{-\alpha z} \int_C \frac{e^{-\xi^2} e^{-i\xi k/\gamma} \xi^{s+1}}{(\xi + i\alpha\gamma)(\xi + i(\alpha - 1)\gamma)(\xi + \mu_n) L_-(\xi)} d\xi,$$

which behaves as $O\left(n^{-(s+3)/2}\right)$ as $n \to \infty$, or $O\left(1/n^7\right)$, $\mu_n \sim n^{1/2}$, and therefore converges very rapidly. Hence, in the sum we only need a truncation number of 10 terms or so if $s = 10$ or 11. However, in (17), three of the four

Table 3. In the table we give the Delta and the Gamma for a single barrier down-and-out call option for different levels of l and different monitoring dates n. Spot price = 100, strike = 100, r = 0.10, σ = 0.2, T = 0.5. The WH+ZT solution has been computed using the same parameters as in Table 2 (except if l = 99.9, when in the double sum we have used 700x100 terms). The competing numerical approximations are the Wiener-Hopf solution computed using the integral representation (IR) in (17) combined with Padé approximant in [17]; Markov chain (MCh) with a grid with 1001 points in [15]; the modified finite difference explicit scheme (EFD) in [11] (the figures are taken from [15]); and Monte Carlo (MC) with 10^8 simulations with Mersenne twister generator and antithetic variables according to the results reported in [8] (in brackets we have the standard errors)

		Option Delta Δ				
l	n	WH+ZT		MCh	EFD	MC (st.error)
		Formula (21)	IR (17)			
95	25	0.92911	0.92912	0.9289	0.9291	0.92906 (0.00031)
99.5	25	1.07192	1.07115	1.0709	1.0714	1.07118 (0.00027)
99.9	25	1.66652	1.03757	1.0374	1.0378	1.03755 (0.00027)
95	125	0.98963	0.98963	0.9897	0.9895	0.98889 (0.00070)
99.5	125	1.27373	1.27373	1.2740	1.2761	1.27368 (0.00044)
99.9	125	1.14565	1.165562	1.1668	1.1674	1.16572 (0.00043)
		Option Gamma Γ				
95	25	−0.01171	−0.01277	−0.0129	−0.0129	−0.01285 (0.00035)
99.5	25	0.12429	0.12274	0.1226	0.1229	0.12274 (0.00015)
99.9	25	−48.40667	0.14827	0.1481	0.1484	0.14824 (0.00015)
95	125	−0.02068	−0.02078	−0.0209	−0.0208	−0.02040 (0.00078)
99.5	125	0.26083	0.26073	0.2601	0.2621	0.26078 (0.00054)
99.9	125	2.25320540	0.39302	0.3916	0.3944	0.39297 (0.00053)

integrals to be evaluated have $1/L_-(\xi)$ in their integrands. This term would make the integrals very expensive computationally if we use the integral form (51) for $L_-(\xi)$. Thus, considerable savings in time can be made if $L_\pm(u)$ are approximated by analytical expressions using Padé approximants. Full details of the validity and accuracy of this approach can be found in Abrahams [3,2]. Numerical results for the integral form of the solution are provided in Table 2 under the heading IR (integral representation). As we can see from the Table, use of this alternative solution representation eliminates the slow convergence problem experienced when using Eq. (10). The Wiener-Hopf results in the second column of Table 2 can be considered exact to the decimal places given.

In Table 3, using the same parameters as in Table 2, we compare the Greeks, that is Delta and Gamma, with those obtained by other methods, i.e. the Wiener-Hopf method using the integral representation (IR) (17), the Markov Chain method (MCh) in [15], the explicit finite difference (EFD) in [11] and Monte Carlo simulation results reported in [8]. Our formula agrees very well with the results reported for EFD and MC methods, except when the spot price is very close to the barrier.

Finally, we note that the computational cost in calculating $f(z, n)$ from (26) increases linearly with monitoring frequency n. Moreover, we can compare, for example, the case of l=95 and N=25 in Table 2. Using C code and a Pentium 600

MHz PC with 256 Mb RAM, the Monte Carlo simulation required 3.7 seconds with 10^4 runs (yielding a price estimate 6.603 and standard error 0.090), 37 seconds with 10^6 runs (price estimate 6.634, standard error 0.029) and 37000 seconds with 10^8 runs (price estimate 6.6320, standard error 0.0009). In contrast, the Wiener-Hopf solution coded in C required 50 seconds to give a three-digit accurate result (except the difficult case of $l = 99.9$ where at least 120 seconds are required to obtain a two-digits accurate result). This difference in computational cost, for a given level of accuracy, makes the proposed Wiener-Hopf method really competitive with respect to Monte Carlo simulation. Moreover, note that by carefully combining $f\,(z, n)$ and its derivatives appearing in the expression for Δ and Γ, one can obtain Greeks that are computationally of the same cost as the option price itself, whilst in the MC simulation the computational cost has been approximately equal to 40000 seconds (approximately 11 hours) with 10^8 runs, which is very expensive.

5 Conclusion

In this paper we have introduced a Wiener-Hopf and z-transform approach to obtain an analytical solution to the single barrier problem under the hypothesis of a geometric Brownian motion evolution for an underlying asset. As shown, the solution thus derived is in a form suitable for numerical evaluation, and the results were compared and contrasted with other numerical/approximate techniques. The present method could also be used to find the solution under different assumptions for the evolution. For example, this approach applies in all cases in which the process for the underlying asset has a Markov feature with a stationarity assumption (e.g., Lévy process). A wide class of evolution processes could then be analysed with the present technique, although the possibility of analytically solving the Wiener-Hopf equation is not so obvious. An additional advantage of the exact solution for discrete barrier options consists in the derivation of explicit expressions for the Greeks.

Finally, in the case of double barrier options, the pricing problem can be reduced to the solution of a Fredholm integral equation of the second kind with a difference kernel, which it is possible to solve in an L^2-framework. It is straightforward to pose the corresponding eigenvalue problem for the integral operator, and the solution admits a representation in terms of their eigenvalues and eigenfunctions. However, the exact calculation of the latter is sometimes involved and requires suitable numerical approximation. An alternative approach for double barrier options is to employ the modified Wiener-Hopf technique (or Jones' method) [28], and this is currently being investigated by the authors.

Appendix A: The solution of the Wiener-Hopf equation

The purpose of this Appendix is to obtain an exact analytical solution of the Wiener-Hopf equation written in (9). There are several key steps in obtaining an analytical solution to Wiener-Hopf equations. First, one must apply a Fourier transform to the integral equation, which converts it into a Riemann-Hilbert equation defined in a strip in the complex transform parameter plane. To solve this equation, the transformed kernel of the integral equation (or here more precisely $\delta(\xi) - qS(\xi, \tau)$, where $\delta(\xi)$ is the generalised delta function) must be decomposed into a product of two functions, one analytic in the upper half of the transform plane and the other analytic in an overlapping lower half-plane. By this means, the equation can be rearranged so that the left (right) side has similar upper (lower) analyticity properties, and hence analytical continuation arguments, and Liouville's theorem, can be applied to obtain an explicit solution.

For simplicity of notation we drop explicit mention of the parametric dependence on q, τ etc., and write the dependent variable as

$$h\left(z\right) \equiv \frac{1}{l} F\left(z, q\right), \tag{27}$$

where l is the (constant) lower barrier value, and the kernel as

$$k(z) \equiv S\left(z, \tau\right) = \frac{1}{\sqrt{4\pi c^2 \tau}} e^{-z^2/\left(4c^2\tau\right)}. \tag{28}$$

We also slightly generalise the forcing term $f(z, 0)$ from (7) arising in (9) by writing

$$v(z) = e^{-az/\gamma}\left(e^{bz/\gamma} - e^{bk/\gamma}\right)\mathbf{1}_{(z \geq \delta)}, \tag{29}$$

where δ is given in (2) and we can recover the original forcing $f(z, 0)$ by taking $lv(z)$ with $a \equiv \alpha\gamma$ and $b \equiv \gamma$. The constant γ is chosen for algebraic convenience a little later. The Wiener-Hopf equation to solve is now

$$h\left(z\right) = q\int_0^{+\infty} k\left(z - \xi\right) h\left(\xi\right) d\xi + v(z), \tag{30}$$

defined over $0 < z < \infty$, and we shall insist on a solution for which $h(z)$ is bounded for all finite z values. As will be shown, a unique solution to (30) is obtained if $\rho = |q|$ is sufficiently small. We commence the solution procedure by defining the Fourier transform of the unknown $h\left(z\right)$ as follows:

$$H\left(u\right) = \mathcal{F}\left(h\left(z\right)\right) = \int_{-\infty}^{+\infty} h\left(z\right) e^{izu/\gamma} dz$$

with its inverse

$$h\left(z\right) = \mathcal{F}^{-1}\left(H\left(u\right)\right) = \frac{1}{2\pi\gamma}\int_{-\infty}^{+\infty} H\left(u\right) e^{-izu/\gamma} du. \tag{31}$$

Here γ is chosen as $\gamma = c\sqrt{\tau}$ so that the Fourier transform of the kernel (28) is the simple expression $\mathcal{F}\left(k\left(z\right)\right) = e^{-u^2}$. We further extend the range of $h(z)$ to all z by defining

$$h\left(z\right) = 0 \text{ for } z < 0,$$

and also write

$$q\int_0^{+\infty} k(z-\xi)\,h(\xi)\,d\xi = -m(z) \text{ for } z<0,$$

in which $m(z)$ is an as yet unknown function. Then (30) becomes

$$h(z) - q\int_0^{+\infty} k(z-\xi)\,h(\xi)\,d\xi = \begin{cases} v(z), & z>0, \\ m(z), & z<0, \end{cases} \tag{32}$$

and if we now apply the Fourier transform to this equation (and employ the convolution relation) we get

$$H_+(u) - qe^{-u^2}H_+(u) = V_+(u) + M_-(u) \tag{33}$$

where

$$H_+(u) = \int_0^{+\infty} h(z)\,e^{izu/\gamma}dz, \tag{34}$$

$$V_+(u) = \int_0^{+\infty} v(z)\,e^{izu/\gamma}dz, \tag{35}$$

$$M_-(u) = \int_{-\infty}^{0} m(z)\,e^{izu/\gamma}dz. \tag{36}$$

From the properties of semi-infinite Fourier integrals (see [28]), the subscript "+" denotes a function analytic in the upper half of the complex u-plane; and to ensure convergence of (34) and (35), given the exponential form of the forcing (29), it is easy to show that $\Im(u) > b-a$ as long as b is positive. Similarly, the subscript "–" denotes a function analytic in a lower half-plane, and clearly the functional equation (33) only holds for u in the overlap region. For a *unique* solution to the integral equation (30) it can be shown that this strip must lie within the interval around the origin between the (infinite) set of zeros of $1-qe^{-u^2}$ in the upper half-plane and the set in the lower half-plane. This strip of analyticity is indicated in Fig. 1. The location of the zeros of the function

$$L(u) = 1 - qe^{-u^2} \tag{37}$$

are easily determined. These occur when $qe^{-u^2+2n\pi i}=1$ or, taking the logarithm of both sides and roots, for

$$u = \pm\sqrt{\ln q + 2n\pi i} = \pm\mu_n, \quad -\infty < n < \infty, \tag{38}$$

where we define $+\mu_n$ ($-\mu_n$) to lie in the upper (lower) half-plane. It is easy to show that

$$\Im(\mu_n) = \sqrt{\frac{-\ln\rho + \sqrt{\ln^2\rho + (2n\pi+\theta)^2}}{2}},$$

in which

$$q = \rho e^{i\theta} \tag{39}$$

and so the root in the upper half-plane which lies closest to the real line is μ_0. For the configuration shown in Fig. 1 we therefore have the constraint $\Im(\mu_0) > b - a$, or rearranging we require

$$\rho = |q| < \exp\left\{-(b-a)^2 \mathbf{1}_{(b-a \geq 0)}\right\}. \tag{40}$$

Finally, the strip of analyticity of the governing equation (33), i.e. the overlap region between the "+" and "−" regions, can be defined as

$$\mathcal{D} \equiv \{u|\max(-\Im(\mu_0), b-a) < \Im(u) < \Im(\mu_0)\}. \tag{41}$$

To proceed with the solution of the transformed equation, we must first decompose $L(u)$ into a product of two functions, analytic in the overlapping half-planes as indicated, so that

$$L(u) = L_+(u)\, L_-(u) \tag{42}$$

where we can show that

$$L_\pm(u) \sim 1 \tag{43}$$

as $u \to \infty$ in the strip $\mathcal{D}$. Note that $L_+(u) = L_-(-u)$ from the symmetry of the function $L(u)$. Thus, equation (33) can be rearranged as

$$L_+(u)\, H_+(u) = \frac{V_+(u)}{L_-(u)} + \frac{M_-(u)}{L_-(u)} \tag{44}$$

and if we perform an additive decomposition of the forcing term,

$$\frac{V_+(u)}{L_-(u)} = P_+(u) + P_-(u), \tag{45}$$

say, we obtain

$$L_+(u)\, H_+(u) - P_+(u) = P_-(u) + \frac{M_-(u)}{L_-(u)} \equiv \Omega(u). \tag{46}$$

We can observe that the left-(right-)hand side is analytic in the upper (lower) half-plane, and so both sides give analytic continuations from $\mathcal{D}$ into the whole complex plane. Thus, both sides are equal to an entire function $\Omega(u)$, say. We have taken from the outset that $h(z)$ and $v(z)$ are bounded functions near $z = 0$, and so it can be shown (see [28]) that

$$V_+(u) = O\left(\frac{1}{u}\right),\ H_+(u) = O\left(\frac{1}{u}\right) \quad \text{as } |u| \to \infty \tag{47}$$

in the upper region. Further, from (43) and (45) we can deduce $P_\pm(u) = O\left(\frac{1}{u}\right)$ as $|u| \to \infty$ in respective half-planes. So, both sides of (46) behave like $O(1/u)$

as $|u| \to \infty$ in their respective half-planes, which, by Liouville's theorem, implies that $\Omega(u) \equiv 0$. Therefore, the solution of the Wiener-Hopf problem is given by

$$H_+(u) = \frac{P_+(u)}{L_+(u)} \tag{48}$$

which, inverting via (31), yields

$$h(z) = \frac{1}{2\pi\gamma} \int_{-\infty}^{+\infty} \frac{P_+(u)}{L_+(u)} e^{-izu/\gamma} du. \tag{49}$$

Note that the functions that appear in the solution have straightforward integral representations as

$$L_+(u) = \exp\left\{\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\ln(L(\xi))}{\xi - u} d\xi\right\}, \quad \Im(u) > \Im(\xi), \quad \xi \in \mathcal{D}, \tag{50}$$

$$L_-(u) = \exp\left\{-\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\ln(L(\xi))}{\xi - u} d\xi\right\}, \quad \Im(u) < \Im(\xi), \quad \xi \in \mathcal{D}, \tag{51}$$

$$P_\pm(u) = \pm\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{V_+(\xi)}{L_-(\xi)} \frac{d\xi}{\xi - u}, \quad \Im(u) \gtrless \Im(\xi), \quad \xi \in \mathcal{D}. \tag{52}$$

A.1 Simplification of the solution

In this section we analyse and simplify the solution given in (49). Let us consider the quantity $V_+(u)/L_-(u) = V_+(u)L_+(u)/L(u)$. As discussed above in (37) – (38), the function $L(u)$ has simple zeros only, at $\pm\mu_n, -\infty < n < \infty$. Using the Mittag-Leffler expansion [37] we can write its (meromorphic) inverse as

$$\frac{1}{L(u)} = 1 + \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{\mu_n(u - \mu_n)} - \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{\mu_n(u + \mu_n)} \tag{53}$$

where we have used the fact that the residue at $u = \pm\mu_n$ is

$$\lim_{u \to \pm\mu_n} \frac{(u \mp \mu_n)}{L(u)} = \frac{1}{L'(\pm\mu_n)} = \frac{1}{2(\pm\mu_n) q e^{-\mu_n^2}} = \pm\frac{1}{2\mu_n}$$

in which the relation $qe^{-\mu_n^2} = 1$ is employed. As a consequence, we have

$$\begin{aligned} \frac{V_+(u)L_+(u)}{L(u)} = &\left(1 - \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{\mu_n(u + \mu_n)}\right) V_+(u)L_+(u) \\ &+ \left(\frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{\mu_n(u - \mu_n)}\right) V_+(u)L_+(u). \end{aligned}$$

The first term on the right-hand side is a function regular and nonzero in the upper half-plane, whereas the second has simple poles at $u = \mu_n$ in the upper region. The latter can be separated, by inspection, to give the split functions $P_{\pm}(u)$ introduced in (45), as

$$P_+(u) = \left(1 - \frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{1}{\mu_n (u+\mu_n)}\right) V_+(u) L_+(u) \qquad (54)$$

$$+\frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{V_+(u) L_+(u) - V_+(\mu_n) L_+(\mu_n)}{\mu_n (u-\mu_n)},$$

$$P_-(u) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{V_+(\mu_n) L_+(\mu_n)}{\mu_n (u-\mu_n)}. \qquad (55)$$

We now use the above in the rearranged solution (49) to get

$$h(z) = \frac{1}{2\pi\gamma}\int_{-\infty}^{+\infty} \left\{\frac{V_+(u)}{L(u)} - \frac{P_-(u) L_-(u)}{L(u)}\right\} e^{-izu/\gamma} du. \qquad (56)$$

The second term is straightforward; we deform its contour into the lower half-plane, picking up poles at $u = -\mu_n$, and then employ symmetry and expression (54) to get for $z > 0$

$$\begin{aligned} -\tfrac{1}{2\pi\gamma}\int_{-\infty}^{+\infty} \tfrac{P_-(u)L_-(u)}{L(u)} e^{-izu/\gamma} du &= \tfrac{-i}{2\gamma}\sum_{n=-\infty}^{+\infty} \tfrac{P_-(-\mu_n)L_-(-\mu_n)}{\mu_n} e^{i\mu_n z/\gamma} \\ = \tfrac{i}{4\gamma}\sum_{n=-\infty}^{+\infty} \tfrac{L_+(\mu_n)}{\mu_n} e^{i\mu_n z/\gamma} &\sum_{m=-\infty}^{+\infty} \tfrac{V_+(\mu_m)L_+(\mu_m)}{\mu_m(\mu_m+\mu_n)}. \end{aligned} \qquad (57)$$

The first term of (56) requires specific knowledge of $V_+(u)$. Employing (29) in (35) we find

$$V_+(u) = \int_{\delta}^{+\infty} e^{-az/\gamma}\left(e^{bz/\gamma} - e^{bk/\gamma}\right) e^{iuz/\gamma} dz$$

$$= \gamma e^{(iu-a)\delta/\gamma}\left\{\frac{e^{bk/\gamma}}{(iu-a)} - \frac{e^{b\delta/\gamma}}{(iu+b-a)}\right\}, \qquad (58)$$

where $\delta = \max(k, 0)$. There are two distinct cases to deal with, namely when $k > 0$ ($\delta = k$), or $k \leq 0$ ($\delta = 0$). For ease of exposition we now restrict attention to the slightly more complicated case $k > 0$, and quote the result when the strike is lower than the barrier at the end of this section. By inspection, the integrand in the first term of (56) is exponentially decaying in the upper half of the complex u-plane for $z < k$, and in the lower half-plane for $z > k$. So, for $0 < z < k$, we deform the contour of the first integral upwards to yield for $0 < z < k$

$$\tfrac{i}{\gamma}\sum_{n=-\infty}^{+\infty} \tfrac{V_+(\mu_n)}{2\mu_n} e^{-iz\mu_n/\gamma} = \tfrac{i}{2} e^{(b-a)k/\gamma}\sum_{n=-\infty}^{+\infty} \tfrac{e^{-i\mu_n(z-k)/\gamma}}{\mu_n}\left\{\tfrac{-1}{(i\mu_n+b-a)} + \tfrac{1}{(i\mu_n-a)}\right\}$$

$$= -\tfrac{i}{2} b e^{(b-a)k/\gamma}\sum_{n=-\infty}^{+\infty} \tfrac{e^{-i\mu_n(z-k)/\gamma}}{(\mu_n+ia)(\mu_n+i(b-a))\mu_n}. \qquad (59)$$

For $z > k$, deformation into the lower half-plane results in the picking up of residue contributions from the poles at $u = -\mu_n$ from $1/L(u)$, and poles in $V_+(u)$ at $u = -ia, u = -i(a-b)$. For simplicity here we initially assume that $ia, i(a-b) \neq \mu_n, \forall n$, but this is *not* a necessary requirement and the final result offered at the end of this Appendix does permit a coincidence of poles. So, we obtain

$$\begin{aligned}
&\frac{i}{\gamma}\sum_{n=-\infty}^{+\infty}\frac{V_+(-\mu_n)}{2\mu_n}e^{iz\mu_n/\gamma} - \left\{\frac{e^{bk/\gamma}e^{-za/\gamma}}{L(-ia)} - \frac{e^{-z(a-b)/\gamma}}{L(-i(a-b))}\right\}\\
&= \frac{i}{2}e^{(b-a)k/\gamma}\sum_{n=-\infty}^{+\infty}\frac{e^{i(z-k)\mu_n/\gamma}}{\mu_n}\left\{\frac{1}{-i\mu_n-a} - \frac{1}{(-i\mu_n-a+b)}\right\}\\
&\quad+\left\{\frac{e^{-z(a-b)/\gamma}}{L(i(a-b))} - \frac{e^{-(za-bk)/\gamma}}{L(ia)}\right\}\\
&= -\frac{ib}{2}e^{(b-a)k/\gamma}\sum_{n=-\infty}^{+\infty}\frac{e^{i\mu_n(z-k)/\gamma}}{\mu_n(\mu_n-ia)(\mu_n-i(a-b))}\\
&\quad+\left\{\frac{e^{zb/\gamma}}{\left(1-qe^{(a-b)^2}\right)} - \frac{e^{kb/\gamma}}{\left(1-qe^{a^2}\right)}\right\}e^{-za/\gamma}
\end{aligned} \tag{60}$$

for $z > k$. Finally, we can combine expressions (57), (59) and (60), and employ the relation

$$V_+(\mu_n) = -b\gamma e^{(b-a)k/\gamma}\frac{e^{i\mu_n k/\gamma}}{(\mu_n+ia)(\mu_n+i(a-b))}$$

to give the solution of the Wiener-Hopf equation (30) in $z > 0$, and for $k > 0$, as

$$\begin{aligned}
h(z) &= -\frac{ib}{2}e^{(b-a)k/\gamma}\sum_{n=-\infty}^{+\infty}\frac{e^{i\mu_n|z-k|/\gamma}}{\mu_n\left(\mu_n-ia\ \mathrm{sgn}(z-k)\right)\left(\mu_n-i(a-b)\ \mathrm{sgn}(z-k)\right)}\\
&+\left\{\frac{e^{zb/\gamma}}{\left(1-qe^{(a-b)^2}\right)} - \frac{e^{kb/\gamma}}{\left(1-qe^{a^2}\right)}\right\}e^{-za/\gamma}\mathbf{1}_{(z\geq k)}\\
&-\frac{ib}{4}e^{(b-a)k/\gamma}\sum_{n=-\infty}^{+\infty}\frac{L_+(\mu_n)e^{i\mu_n z/\gamma}}{\mu_n}\sum_{m=-\infty}^{+\infty}\frac{L_+(\mu_m)e^{i\mu_m k/\gamma}}{\mu_m(\mu_m+ia)(\mu_m+i(a-b))(\mu_m+\mu_n)},
\end{aligned} \tag{61}$$

where the sign function $\mathrm{sgn}(x)$ was defined in (11). We can easily show that the symmetry of the transformed kernel $L(u)$ enables the product factor (50) to be expressed as

$$L_+(u) = \exp\left\{\frac{u}{\pi i}\int_0^{+\infty}\frac{\ln\left(1-qe^{-z^2}\right)}{z^2-u^2}dz\right\}, \quad \Im(u) > 0, \tag{62}$$

which is routinely computable at all μ_n values, where $\mu_n = \sqrt{\ln q + 2n\pi i}$, with the branch chosen such that they lie in the upper half-plane, $\Im(\mu_n) > 0\ \forall n$. It is simple matter to repeat the procedure just performed when $k \leq 0$. It can be shown that the function $h\,(z)$ in this case is

$$h(z) = -\tfrac{1}{2}\sum_{n=-\infty}^{\infty}\left(\tfrac{e^{bk/\gamma}}{\mu_n - ia} - \tfrac{1}{\mu_n - i(a-b)}\right)\tfrac{e^{i\mu_n z/\gamma}}{\mu_n} + \left\{\tfrac{e^{zb/\gamma}}{1-qe^{(a-b)^2}} - \tfrac{e^{kb/\gamma}}{1-qe^{a^2}}\right\}e^{-za/\gamma} + \tfrac{1}{4}\sum_{n=-\infty}^{\infty}\tfrac{L_+(\mu_n)e^{i\mu_n z/\gamma}}{\mu_n}\sum_{m=-\infty}^{\infty}\left(\tfrac{e^{kb/\gamma}}{\mu_m + ia} - \tfrac{1}{\mu_m + i(a-b)}\right)\tfrac{L_+(\mu_m)}{\mu_m(\mu_m+\mu_n)}. \tag{63}$$

However, it can be substantially simplified by noting that the inner sum in the last term may be written as

$$\sum_{m=-\infty}^{\infty}\left(\frac{e^{kb/\gamma}}{\mu_m + ia} - \frac{1}{\mu_m + i\,(a-b)}\right)\frac{L_+(\mu_m)}{\mu_m(\mu_m+\mu_n)} = \frac{1}{\pi i}\int_C \frac{1}{L_-(\xi)(\xi+\mu_n)}\left(\frac{e^{kb/\gamma}}{\xi + ia} - \frac{1}{\xi + i(a-b)}\right)d\xi,$$

where the contour C runs along the real line but is indented, if necessary, above the pole at $\xi = -i(a-b)$ and below $\xi = \mu_0$. This is easily proved by deforming the contour into the upper half-plane, thereby collecting residues at $+\mu_m, -\infty < m < \infty$. Now, this contour may be deformed instead into the lower half-plane, which yields the alternative form

$$\frac{2}{L_+(\mu_n)}\left(\frac{e^{bk/\gamma}}{\mu_n - ia} - \frac{1}{\mu_n - i(a-b)}\right) - 2\left(\frac{e^{bk/\gamma}}{L_+(ia)(\mu_n - ia)} - \frac{1}{L_+(i(a-b))(\mu_n - i(a-b))}\right).$$

Substituting into (63) and cancelling the first term gives, for $k \leq 0$, that

$$h(z) = \left\{\tfrac{e^{zb/\gamma}}{1-qe^{(a-b)^2}} - \tfrac{e^{kb/\gamma}}{1-qe^{a^2}}\right\}e^{-za/\gamma} - \tfrac{1}{2}\sum_{n=-\infty}^{\infty}\tfrac{L_+(\mu_n)e^{i\mu_n z/\gamma}}{\mu_n}\left(\tfrac{e^{bk/\gamma}}{L_+(ia)(\mu_n - ia)} - \tfrac{1}{L_+(i(a-b))(\mu_n - i(a-b))}\right). \tag{64}$$

Finally, it is necessary to prove that the solution (61) remains valid if $\mu_p = ia$ or $\mu_p = i(a-b)$ for any p. On inspection it appears that the first two terms become singular at these values, and so we must show that the leading terms vanish. We can easily prove that this is the case and so the above expression (61) for $h(z)$ is always valid; restrictions on space prevent to give here this proof, which is available upon request.

Appendix B: Improving the convergence of the Wiener-Hopf solution

The purpose of this Appendix is to obtain an alternative representation for the exact solution (10). As stated in Sect. 4, the infinite sums are very slowly convergent when z or k tend to zero, and so a more efficient formula is required for numerical purposes. We shall derive such a representation for all values of the strike and price. We start by recalling that q is defined as

$$|q| < \exp\left\{-(1-\alpha)^2\gamma^2 \mathbf{1}_{((1-\alpha)\gamma\geq 0)}\right\}$$

and so $\Im\left(\mu_n\right) > \gamma\left(1-\alpha\right), \forall n$. Now we concentrate on the double sum, I say, appearing in (10) and given by

$$I = \sum_{n=-\infty}^{\infty} \frac{L_+(\mu_n)e^{i\mu_n z/\gamma}}{\mu_n} \sum_{m=-\infty}^{\infty} \frac{L_+(\mu_m)e^{i\mu_m k/\gamma}}{\mu_m(\mu_m+i\alpha\gamma)(\mu_m+i(\alpha-1)\gamma)(\mu_m+\mu_n)}. \tag{65}$$

The functional dependence of I, and all following integrals, on z, k, q etc. is omitted for clarity. First, look at the integral

$$I_1 = \int_C \frac{L_+\left(\xi\right)e^{i\xi k/\gamma}}{\left(\xi+i\alpha\gamma\right)\left(\xi+i\left(\alpha-1\right)\gamma\right)\left(\xi+\mu_n\right)L\left(\xi\right)}d\xi, \tag{66}$$

where C runs from $-\infty$ to $+\infty$ along the real line except that it is indented above the pole at $\xi = i\gamma\left(1-\alpha\right)$, but below $\xi = \mu_0$, when $\alpha \leq 1$. Alternatively, the contour could be taken as a line parallel to the real line but with $\gamma\left(1-\alpha\right) < \Im(\xi) < \Im(\mu_0)$. The poles in the integrand in the lower half-plane occur at $\xi = -i\alpha\gamma$, $\xi = -i\left(\alpha-1\right)\gamma$, $\xi = -\mu_m, -\infty < m < +\infty$, the latter arising from the zeros of $L\left(\xi\right)$, and in the upper half-plane the poles lie at $\xi = +\mu_m$, $-\infty < m < +\infty$. Restricting attention to $k > 0$ for now, we deform the contour in the upper half-plane. This gives, on employing the Mittag-Leffler expansion (53), that

$$I_1 = \pi i \sum_{m=-\infty}^{+\infty} \frac{L_+\left(\mu_m\right)e^{i\mu_m k/\gamma}}{\mu_m\left(\mu_m+i\alpha\gamma\right)\left(\mu_m+i\left(\alpha-1\right)\gamma\right)\left(\mu_m+\mu_n\right)}.$$

Therefore I may be written as

$$I = \frac{1}{\pi i}\sum_{n=-\infty}^{\infty} \frac{L_+(\mu_n)e^{i\mu_n z/\gamma}}{\mu_n} \int_C \frac{L_+\left(\xi\right)e^{i\xi k/\gamma}}{L\left(\xi\right)\left(\xi+i\alpha\gamma\right)\left(\xi+i\left(\alpha-1\right)\gamma\right)\left(\xi+\mu_n\right)}d\xi.$$

We can improve the convergence of this integral by writing $1/L\left(\xi\right) = qe^{-\xi^2}/L\left(\xi\right) + 1$, where we have used $L\left(\xi\right) = 1 - qe^{-\xi^2}$, and so

$$\begin{aligned} I &= \frac{1}{\pi i}\sum_{n=-\infty}^{\infty} \frac{L_+(\mu_n)e^{i\mu_n z/\gamma}}{\mu_n} \int\limits_C \frac{L_+(\xi)e^{i\xi k/\gamma}}{(\xi+i\alpha\gamma)(\xi+i(\alpha-1)\gamma)(\xi+\mu_n)}\left(\frac{qe^{-\xi^2}}{L(\xi)}+1\right)d\xi \\ &= \frac{q}{\pi i}\sum_{n=-\infty}^{\infty} \frac{L_+(\mu_n)e^{i\mu_n z/\gamma}}{\mu_n} \int\limits_C \frac{e^{-\xi^2}e^{i\xi k/\gamma}}{L_-(\xi)(\xi+i\alpha\gamma)(\xi+i(\alpha-1)\gamma)(\xi+\mu_n)}d\xi, \end{aligned} \tag{67}$$

employing the relation $L_+\left(\xi\right)L_-\left(\xi\right) = L\left(\xi\right)$, and the fact that the second term on the top line is zero because there are no poles in the upper half-plane. Although the integral on the second line in (67) is now rapidly convergent, the sum in n is still slow to converge when z is small. To improve matters, we expand $1/\left(\xi+\mu_n\right)$ as

$$\begin{aligned}\frac{1}{\xi+\mu_n} &= \frac{1}{\mu_n}\frac{1}{\left(1+\frac{\xi}{\mu_n}\right)} = \frac{1}{\mu_n}\sum_{p=0}^{\infty}(-1)^p\,\frac{\xi^p}{\mu_n^p} \\ &= \frac{1}{\mu_n}\sum_{p=0}^{s}(-1)^p\,\frac{\xi^p}{\mu_n^p} + \frac{1}{\mu_n}\sum_{p=s+1}^{\infty}(-1)^p\,\frac{\xi^p}{\mu_n^p} \\ &= (-1)^{s+1}\,\frac{\xi^{s+1}}{\mu_n^{s+1}\left(\xi+\mu_n\right)} + \sum_{p=0}^{s}(-1)^p\,\frac{\xi^p}{\mu_n^{p+1}}.\end{aligned}$$

Note that s is an integer parameter that can be chosen for convenience. Hence

$$\begin{aligned}I = {}& \frac{(-1)^{s+1}q}{\pi i}\sum_{n=-\infty}^{\infty}\frac{L_+(\mu_n)e^{i\mu_n z/\gamma}}{\mu_n^{s+2}} \\ & \times\int_C \frac{e^{-\xi^2}e^{i\xi k/\gamma}\xi^{s+1}}{L_-\left(\xi\right)\left(\xi+i\alpha\gamma\right)\left(\xi+i\left(\alpha-1\right)\gamma\right)\left(\xi+\mu_n\right)}d\xi \\ & +\frac{q}{\pi i}\sum_{p=0}^{s}(-1)^p\sum_{n=-\infty}^{\infty}\frac{L_+(\mu_n)e^{i\mu_n z/\gamma}}{\mu_n^{p+2}} \\ & \times\int_C \frac{e^{-\xi^2}e^{i\xi k/\gamma}\xi^{p}}{L_-\left(\xi\right)\left(\xi+i\alpha\gamma\right)\left(\xi+i\left(\alpha-1\right)\gamma\right)}d\xi.\end{aligned}$$

Finally, we can show, for $z/\gamma>0$, that the following identity holds:

$$I_2 = \int_{C_0}\frac{L_+\left(\zeta\right)e^{i\zeta z/\gamma}}{L\left(\zeta\right)\zeta^{p+1}}d\zeta = \pi i\sum_{n=-\infty}^{+\infty}\frac{L_+\left(\mu_n\right)e^{i\mu_n z/\gamma}}{\mu_n^{p+2}},$$

where C_0 again runs from $-\infty$ to $+\infty$ along real values of ζ, but is indented above the pole of order $p+1$ at the origin. Again using $1/L\left(\xi\right)=qe^{-\xi^2}/L\left(\xi\right)+1$ gives

$$I_2 = q\int_{C_0}\frac{L_+\left(\zeta\right)e^{-\zeta^2}e^{i\zeta z/\gamma}}{L\left(\zeta\right)\zeta^{p+1}}d\zeta,$$

and so the modified form of I from (65) is

$$\begin{aligned}I = {}& \frac{(-1)^{s+1}q}{\pi i}\sum_{n=-\infty}^{\infty}\frac{L_+(\mu_n)e^{i\mu_n z/\gamma}}{\mu_n^{s+2}} \\ & \times\int_C \frac{e^{-\xi^2}e^{i\xi k/\gamma}\xi^{s+1}}{L_-\left(\xi\right)\left(\xi+i\alpha\gamma\right)\left(\xi+i\left(\alpha-1\right)\gamma\right)\left(\xi+\mu_n\right)}d\xi \\ & +\frac{q^2}{\left(\pi i\right)^2}\sum_{p=0}^{s}(-1)^p\int_{C_0}\frac{e^{-\zeta^2}e^{i\zeta z/\gamma}}{L_-\left(\zeta\right)\zeta^{p+1}}d\zeta \\ & \times\int_C \frac{e^{-\xi^2}e^{i\xi k/\gamma}\xi^{p}}{L_-\left(\xi\right)\left(\xi+i\alpha\gamma\right)\left(\xi+i\left(\alpha-1\right)\gamma\right)}d\xi,\end{aligned}$$

where we choose s large enough for rapid convergence of the infinite sum. This is discussed in Sect. 4.

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