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# A note on Wick products and the fractional Black-Scholes model

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**Abstract.** In some recent papers (Elliott and van der Hoek 2003; Hu and Øksendal 2003) a fractional Black-Scholes model has been proposed as an improvement of the classical Black-Scholes model (see also Benth 2003; Biagini et al. 2002; Biagini and Øksendal 2004). Common to these fractional Black-Scholes models is that the driving Brownian motion is replaced by a fractional Brownian motion and that the Itô integral is replaced by the Wick integral, and proofs have been presented that these fractional Black-Scholes models are free of arbitrage. These results on absence of arbitrage complelety contradict a number of earlier results in the literature which prove that the fractional Black-Scholes model (and related models) will in fact admit arbitrage. The objective of the present paper is to resolve this contradiction by pointing out that the definition of the *self-financing trading strategies* and/or the definition of the *value* of a portfolio used in the above papers does not have a reasonable economic interpretation, and thus that the results in these papers are not economically meaningful. In particular we show that in the framework of Elliott and van der Hoek (2003), a naive buy-and-hold strategy does not in general qualify as "self-financing". We also show that in Hu and Øksendal (2003), a portfolio consisting of a positive number of shares of a stock with a positive price may, with positive probability, have a negative "value".

**Key words:** Mathematical finance, fractional Brownian motion, arbitrage

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## 1 Introduction

The classical Black-Scholes model in mathematical finance consists of a stochastic process $S$ called a risky asset (e.g., a stock) and a risk free asset (e.g., a bank account). It is assumed that $S$ follows a geometric Brownian motion, i.e., $S$ is (under the objective measure $P$) the solution of (for simplicity we set the volatility to unity)

$$dS_t = \alpha S_t dt + S_t dW_t,$$

where $W$ is a standard $P$-Brownian motion. We consider the model on a fixed time interval $[0, T]$. The risk free asset is denoted $B$ and for notational simplicity we assume that the short rate equals zero, so $B_t \equiv 1$ or equivalently

$$B_0 = 1, \; dB_t = 0.$$

Based on some empirical studies it has been suggested (e.g. Mandelbrot 1997; Shiryaev 1999) that the driving noise of the risky asset should not be the Brownian motion $W$ but a fractional Brownian motion (henceforth FBM) $W^H$ with Hurst index $H \neq 1/2$. For $H \in (1/2, 1)$ it is possible to define the stochastic integral w.r.t. $W^H$ in the path-wise sense, and it has been shown (e.g. Cheridito 2003; Rogers 1997; Salopek 1998; Shiryaev 1998) that, using the path-wise integral concept, the Black-Scholes model (and related models) based on FBM is not free of arbitrage. See also Maheswaran and Sims (1993) for an early discussion on arbitrage with FBM in finance. On a more general level, the following result was proved by Delbaen and Schachermayer (1994, Theorem 7.2). Suppose we are restricted to use simple predictable integrands (i.e., piecewise buy-and-hold) as trading strategies and the risky asset $S$ is an adapted, locally bounded, càdlàg process. If $S$ is not a semi-martingale then there exists a *free lunch with vanishing risk* (which is slightly weaker than an arbitrage). It is well known that fractional Brownian motion with $H \neq 1/2$ is not a semimartingale.

On the other hand, Hu and Øksendal (2003) and Elliott and van der Hoek (2003) have suggested the use of a new stochastic integral concept (and a related calculus) based on the Wick product (see also Duncan et al. 2000). Using this machinery they have (with some variation in their respective frameworks) suggested fractional Black-Scholes models which are "free of arbitrage" in the sense induced by the use of the Wick integral. Similar ideas have also been carried out in Benth (2003), Biagini and Øksendal (2004) and Biagini et al. (2002).

Clearly, these results on absence of arbitrage are not compatible with the earlier literature cited above, and the reason is that the very definitions of *portfolio value* and/or *self-financing portfolios* are completely different from their standard counterparts. The proposed alternative definitions are instead as follows.

- In Elliott and van der Hoek (2003) and Hu and Øksendal (2003) the price of the risky asset $S$ is modelled by a geometric fractional Brownian motion, which is the solution to the equation

$$dS_t = S_t \diamond dW_t^H, \quad S_0 = s_0,$$

  where $\diamond$ denotes the Wick product (see below for details).

- In Elliott and van der Hoek (2003) the *portfolio value* is defined in the standard way as

$$V_t = h_t^0 B_t + h_t^1 S_t, \tag{1}$$

where $h^0$ and $h^1$ are the respective numbers of units of the riskless and the risky asset held in the portfolio. In Hu and Øksendal (2003), however, this definition is changed into its formal Wick counterpart, and the portfolio value is thus defined as

$$V_t = h_t^0 B_t + h_t^1 \diamond S_t. \tag{2}$$

- In Elliott and van der Hoek (2003) the standard Itô-type *self-financing condition*

$$dV_t = h_t^1 dS_t$$

is translated to

$$dV_t = h_t^1 S_t \diamond dW_t^H, \tag{3}$$

whereas in Hu and Øksendal (2003) it is replaced by its formal Wick analogue

$$dV_t = h_t^1 \diamond dS_t. \tag{4}$$

We also recall the definition of an arbitrage. An *admissible* (i.e., there exists a constant $a > 0$ such that $V_t \geq -a$ a.s. for all $t \in [0, T]$) self-financing portfolio strategy is an *arbitrage* if it satisfies the conditions

$$V_0 = 0, \ P(V_T \geq 0) = 1, \ P(V_T > 0) > 0. \tag{5}$$

Since the purpose of the present paper is precisely to discuss whether these new concepts are reasonable from an economic point of view, we need to introduce some new terminology to keep the various concepts apart. For the new concepts in Elliott and van der Hoek (2003) and Hu and Øksendal (2003) we use the following terminology.

- A process $V$ defined by (2) will henceforth be referred to as a *Wick-value* process.
- A portfolio strategy that satisfies (3), where the process $V$ is defined in the standard way (1), is called a *Wick*$^{\mathbf{1}}$*-financing* portfolio. This is the setup in Elliott and van der Hoek (2003).
- An admissible Wick$^{\mathbf{1}}$-financing portfolio satisfying (5) is called a *Wickbitrage*$^{\mathbf{1}}$.
- A portfolio strategy that satisfies (4), where $V$ is defined as the Wick-value process in (2), is called a *Wick*$^{\mathbf{2}}$*-financing* portfolio. This is the setup in Hu and Øksendal (2003).
- An admissible Wick$^{\mathbf{2}}$-financing portfolio satisfying (5) is called a *Wickbitrage*$^{\mathbf{2}}$.

In this paper we claim that:

- The definition of Wick$^{\mathbf{1}}$-financing portfolios used in Elliott and van der Hoek (2003) has no economic interpretation as a self-financing condition.

- In fact, we construct a simple portfolio strategy, which is trivially self-financing from an intuitive accounting point of view, but which is not Wick[1]-financing.
- Thus, the arbitrage concept in Elliott and van der Hoek (2003) does not correspond to arbitrage in the intuitive sense.
- If one insists to use the pricing theory in Elliott and van der Hoek (2003) for trading purposes, then this will in some cases lead to easily implementable naive arbitrage opportunities for the counterpart.

We also claim that:

- Replacing the standard definition of value $V^h = h^0 B + h^1 S$ by the Wick-value $\mathcal{V}^h = h^0 B + h^1 \diamond S$, as in Hu and Øksendal (2003), cannot be motivated from an economic point of view.
- In fact, the definition of Wick-value of a portfolio is completely different from what "most people" would call value of the portfolio.
- In particular we construct a portfolio with zero amount in the risk free asset such that, on a set with positive probability, the asset price is positive, the number of units of the risky asset held in the portfolio is positive, but the Wick-value of the portfolio is negative!

Let us also mention the interesting recent work by Sottinen and Valkeila (2003) where the fractional Black-Scholes models based on the Wick integral and on the Riemann-Stieltjes integral are compared and the induced pricing relations are determined.

The paper is organised as follows. In Sect. 2 we recall the standard self-financing condition in the classical Black-Scholes model. In Sect. 3 we discuss the model presented in Elliott and van der Hoek (2003) and in Sect. 4 we consider the approach by Hu and Øksendal (2003).

## 2 The standard self-financing condition

Since a large part of the present paper concerns self-financing portfolio strategies we now give a brief recapitulation of the self-financing condition in the simple situation where all prices are driven by Wiener processes and where we use the Itô integral concept. Consider therefore a financial market with $n+1$ asset price processes $S^0, S^1, \ldots, S^n$, and denote the corresponding vector process by $S$. We consider an adapted portfolio process $h = (h^0, h^1, \ldots, h^n)$ and define the *value process* $V^h$ associated with $h$ by the standard formula

$$V_t^h = \sum_{i=0}^{n} h_t^i S_t^i = h_t S_t,$$

where equality between random variables always is interpreted as equality $P$-almost surely. We now want to define the concept of a *self-financing* portfolio. In discrete time this is trivial: it simply means that at any re-balancing point in time (for $P$-almost all $\omega$) the cost of your new portfolio has to equal the value of your old portfolio. The self-financing condition is thus a *budget constraint* that has to be imposed on the portfolio dynamics, and it is an easy exercise to see that in discrete time the self-financing condition takes the form

$$\Delta V_k = h_k \Delta S_k,$$

where we use the notation $\Delta V_k = V_{k+1} - V_k$ and $\Delta S_k = S_{k+1} - S_k$. On the right hand side we recognise a discrete differential of Itô type, i.e., a discrete version of the standard self-financing condition.

In continuous time the self-financing concept becomes more complicated. However, a minimal requirement seems to be that a buy-and-hold portfolio, i.e., a portfolio which is constant over a fixed time interval, should qualify as self-financing over that interval. Let us thus consider the time interval $[t_0, t_1]$ and a portfolio $h$ which is constant (but possibly stochastic and then $\mathcal{F}_{t_0}$ measurable) over that interval. At any time $t \in [t_0, t_1]$ the portfolio value will be $V_t = h_t S_t$, and in particular, since $h$ is constant, the change in value of the portfolio over the interval is given by

$$V_{t_1} - V_{t_0} = h_{t_1} S_{t_1} - h_{t_0} S_{t_0} = h_{t_0} \left(S_{t_1} - S_{t_0}\right) = \int_{t_0}^{t_1} h_t dS_t, \tag{6}$$

where the integral is defined trajectorywise. We thus again have the standard Itô value dynamics

$$dV_t = h_t dS_t. \tag{7}$$

It is also easy to see that for portfolios which are piecewise buy-and-hold (and which satisfy the obvious budget constraint at each rebalancing point) we again obtain the same standard value dynamics (7). The big problems appear when we try to define what we mean with a continuously rebalanced self-financing portfolio.

To exemplify: if we for simplicity assume that also the portfolio process $h$ is a continuous semi-martingale (it is sufficient to assume that $h$ is adapted càdlàg), then the formal (unconstrained) portfolio dynamics are given by Itô's formula as

$$dV_t = h_t dS_t + S_t dh_t + d\langle h, S\rangle_t \tag{8}$$

where we again use the Itô stochastic integral. The question is now what the budget restriction of a self-financing portfolio looks like in continuous time, and the problem is that in continuous time there is no such thing as "the next rebalancing point in time". To handle this problem, one can discretise time in the continuous model, write down the self-financing condition for the discretised model, and then let the length of the elementary time intervals of the discrete time model tend to zero. If this program is carried out, then it turns out (see Björk 2004, pp. 81–82) that the budget constraint for a self-financing portfolio takes the form

$$S_t dh_t + d\langle h, S\rangle_t = 0.$$

Substituting this expression into the general portfolio dynamics (8) we obtain the standard self-financing condition as

$$dV_t = h_t dS_t, \tag{9}$$

or equivalently in integral form as

$$V_t = V_0 + \int_0^t h_u dS_u.$$

In some textbooks, the self-financing condition is given directly by (9) with a remark that "it is intuitively obvious". As we have seen above, however, the condition is far from obvious; it relies on a *carefully formalised economic argument*. Furthermore, although the concept of a self-financing portfolio is easily understood in economic terms, *the formal appearance of the condition depends heavily upon the stochastic integral concept used*. If, for example, we insist on using the Stratonovich integral concept instead of the Itô concept, then the correct formalisation of the self-financing condition is given by

$$dV_t = h_t \circ dS_t - \frac{1}{2} d\langle h, S\rangle_t,$$

where $\circ$ denotes the Stratonovich integral. If we use still another integral concept then the self-financing condition will again take on another formal appearance.

Obviously, in the above discussion we assumed that $h$ and $S$ were semimartingales. For a price process like FBM which is not a semimartingale, the argument above may not be applied directly, but must be replaced by a separate argument for the model under discussion. The fundamental moral, however, always remains the same:

- The self-financing condition is a *fundamental economic concept*.
- To derive the correct form of the self-financing condition one has to do a very careful analysis of the particular model under study.
- In particular, the formal appearance of the self-financing condition will depend crucially on the stochastic integral concept used.
- If we introduce a new stochastic integral concept then we are *not justified* to take the Itô self-financing condition and just replace the Itô integral with a new integral concept. Doing this may easily result in pure nonsense.

## 3 The fractional market model and the Wick[1]-financing condition

Consider the fractional Black-Scholes model in Elliott and van der Hoek (2003). For simplicity we always assume that $P$ is the "risk neutral measure". We assume the existence of a risky asset $S$ following a so-called geometric fractional Brownian motion on $[0, T]$. Let $W^H$ be a fractional Brownian motion with Hurst index $H \in (1/2, 1)$, i.e., a zero mean Gaussian process with covariance function

$$E(W_t^H W_s^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}),$$

and let $\mathbb{F} = (\mathcal{F}_t)$ be the natural filtration generated by $W^H$. Elliott and van der Hoek (2003) actually cover the case $H \in (0, 1)$ whereas Hu and Øksendal (2003) consider $H \in (1/2, 1)$. To give statements relevant to both papers we assume that

$H \in (1/2, 1)$. However, with slight modifications the claims made in this paper are applicable to all $H \in (0, 1/2) \cup (1/2, 1)$. The process $S$ is defined as the solution to the equation

$$dS_t = S_t \diamond dW_t^H,$$

where the stochastic differential is interpreted as a Wick integral (see Duncan et al. 2000). The solution to this equation is (cf. Elliott and van der Hoek 2003; Hu and Øksendal 2003)

$$S_t = s_0 \exp \left\{ W_t^H - \frac{1}{2} t^{2H} \right\}.$$

We also have the usual bank account $B_t \equiv 1$ with zero interest rate. We define the *portfolio process* $h = (h^0, h^1)$ as a bivariate $\mathbb{F}$-adapted process, where $h_t^0$ is the amount in the bank account at time $t$ and $h_t^1$ is the number of shares of the risky asset in the portfolio at time $t$. The *value process* associated with a portfolio is defined as $V^h$ where

$$V_t^h = h_t^0 B_t + h_t^1 S_t = h_t^0 + h_t^1 S_t \tag{10}$$

and there is an ordinary product between $h_t^1$ and $S_t$. We say that $h$ is *Wick*$^{\mathbf{1}}$*-financing* if

$$dV_t^h = h_t^0 dB_t + h_t^1 S_t \diamond dW_t^H,$$

i.e., if

$$dV_t^h = h_t^1 S_t \diamond dW_t^H.$$

In Elliott and van der Hoek (2003) a portfolio is said to be "self-financing" if it is Wick$^{\mathbf{1}}$-financing in the sense described above. However, there is no economic justification of the use of Wick$^{\mathbf{1}}$-financing portfolios. The Wick$^{\mathbf{1}}$-financing condition is simply introduced as the formal analogue of the standard Itô form of the self-financing condition (9). As was noted in the previous section, this is a dangerous way to go, and we will now investigate where it leads us.

Our claim is that the use of the word "self-financing" for the Wick$^{\mathbf{1}}$-financing condition defined above is severely misleading in the sense that the concept of a Wick$^{\mathbf{1}}$-financing portfolio has no natural economic interpretation. As a starting point we notice that the buy-and-hold portfolio discussed in connection with (6) will generically *not* be Wick$^{\mathbf{1}}$-financing. As we saw in (6) the buy-and-hold portfolio will satisfy

$$V_{t_1} - V_{t_0} = h_{t_0} \left( S_{t_1} - S_{t_0} \right). \tag{11}$$

However, in order to qualify as Wick$^{\mathbf{1}}$-financing the portfolio should instead satisfy the condition

$$V_{t_1} - V_{t_0} = \int_{t_0}^{t_1} h_{t_0} S_u \diamond dW_u^H, \tag{12}$$

and this does not in general coincide with (11) since (12) does not in general coincide with

$$h_{t_0} \int_{t_0}^{t_1} S_u \diamond dW_u^H.$$

To illustrate further, we now construct an explicit example of a portfolio strategy which is obviously self-financing in the standard sense, but which is not Wick$^{\mathbf{1}}$-financing. We will also see that the use of "risk neutral" pricing formulas based on the Wick$^{\mathbf{1}}$-financing concept, which has been suggested by Elliott and van der Hoek (2003), will in fact lead to easily implementable arbitrage possibilities in the standard naive sense.

*Example 1* Consider the following portfolio strategy with initial capital $x > 0$. At $t = 0$ we put all our money into the bank account and wait until $t = 1$. Since the short rate is equal to zero we still have the amount $x$ in the account at $t = 1$. At $t = 1$ we put all our money into the risky asset, i.e., we buy $x/S_1$ shares at the price $S_1$ and hold this position until $t = 2$. The value of our portfolio at $t = 2$ is of course given by

$$V_2 = \frac{x}{S_1} S_2.$$

Since no capital has been added or withdrawn during $[0, 2]$, we claim that any reasonable definition of a self-financing portfolio must include this one. However, as we will see below, this strategy is not Wick$^{\mathbf{1}}$-financing, i.e., it is not self-financing in the language of Elliott and van der Hoek (2003).

**Lemma 2** *Let $T = 2$ and $h$ be the portfolio described above, i.e., the portfolio defined by*

$$h_t^0 = 1_{(0,1]}(t), \quad h_t^1 = \frac{x}{S_1} 1_{(1,2]}(t).$$

*Then $h$ is not Wick$^{\mathbf{1}}$-financing.*

*Proof* By the above argument the value of the portfolio at $t = 2$ is $V_2 = xS_2/S_1$. Hence, to prove the result we have to show that

$$x \frac{S_2}{S_1} \neq x + \int_0^2 h_u^1 S_u \diamond dW_u^H.$$

In fact, not even their expected values are equal. We have

$$E\left(x + \int_0^2 h_u^1 S_u \diamond dW_u^H\right) = x,$$

whereas

$$\begin{aligned} E\left(\frac{x}{S_1}S_2\right) &= xE\left(\exp\left\{W_2^H - \frac{1}{2}2^{2H}\right\}\exp\left\{-W_1^H + \frac{1}{2}1^{2H}\right\}\right) \\ &= x\exp\left\{-\frac{1}{2}(2^{2H}-1)\right\}E\left(\exp\left\{W_2^H - W_1^H\right\}\right) \\ &= x\exp\left\{-\frac{1}{2}(2^{2H}-1)\right\}\exp\left\{\frac{1}{2}|2-1|^{2H}\right\} \\ &= x\exp\{1-2^{2H-1}\} \neq x \end{aligned} \tag{13}$$

for $H \neq 1/2$. □

If you insist on treating the Wick$^{\mathbf{1}}$-financing condition as a self-financing condition, then you will not regard the portfolio above as self-financing. This means that at $t = 2$ you will book the value of the portfolio (including financial costs) at a different number from $xS_1^{-1}S_2$. Since the portfolio is naively self-financing, your proposed book value will however entail a violation of corporate law (and you may be prosecuted).

There are even more problems connected with the use of the Wick$^{\mathbf{1}}$-financing concept, since it also has implications on the pricing of derivatives. We recall that for the Wick$^{\mathbf{1}}$-financing analogue of arbitrage we introduced the term *Wickbitrage*$^{\mathbf{1}}$. It then follows from the theory developed in Elliott and van der Hoek (2003) that for any claim $X \in \mathcal{F}_T$, the "risk neutral" or "Wickbitrage$^1$ free" price (in the present setting) at $t = 0$ of $X$ is given by (since $P$ is the "risk neutral" measure)

$$\Pi(0;X) = E(X). \tag{14}$$

Let us now consider the claim $xS_1^{-1}S_2$ above, paid out at $t = 2$. From (14) and (13) we see that we would then have a pricing formula

$$\Pi(0;X) = x\exp\{1-2^{2H-1}\},$$

and in our case, with $H \in (1/2, 1)$, we thus have

$$\Pi(0;X) < x.$$

Suppose now that we try to use this price in a position as a market maker, i.e., we are prepared to sell and buy at the price $x\exp\{1-2^{2H-1}\}$. Then an arbitrageur can easily create an arbitrage strategy against us by buying the claim $X$ from us at the price $x\exp\{1-2^{2H-1}\}$ and going short in the portfolio described above. This will leave the arbitrageur with a net of $x - x\exp\{1-2^{2H-1}\} > 0$ today, and this amount can be put into the bank. Since the portfolio strategy is naively self-financing, it can in fact be carried forward without additional costs, and at $t = 2$ the arbitrageur's positions will net. He/she will thus have made an arbitrage profit of $x - x\exp\{1-2^{2H-1}\} > 0$.

## 4 The fractional market model and the Wick-value process

The fractional market model studied in Hu and Øksendal (2003) is constructed as in the previous section. The risky asset $S$ is given by

$$S_t = s_0 \exp\left\{ W_t^H - \frac{1}{2} t^{2H} \right\},$$

where $W^H$ is a fractional Brownian motion with Hurst index $H \in (1/2, 1)$, and there is a bank account $B$ with deterministic short rate $r$, which we for simplicity put equal to zero so that $B_t \equiv 1$. We note that the risky asset $S$ can be represented as (see e.g. Hu and Øksendal 2003, for details)

$$S_t = \mathcal{E}(1_{[0,t]}),$$

where the Wick exponential

$$\mathcal{E}(f) = \exp\left\{ \int_R f_t dW_t^H - \frac{1}{2}|f|_\phi^2 \right\},$$

is defined for measurable deterministic functions $f : R \to R$ such that

$$|f|_\phi^2 = \int_R \int_R f(t) f(s) \phi(t,s) dt ds < \infty,$$
$$\phi(t,s) = H(2H-1)|t-s|^{2H-2}.$$

A portfolio process $h = (h^0, h^1)$ is also given as in the previous section, but the value of the portfolio is defined using a Wick product between $h_t^1$ and $S_t$ and therefore we use the word *Wick-value process* for the process $\mathcal{V}^h$ given by

$$\mathcal{V}_t^h = h_t^0 B_t + h_t^1 \diamond S_t = h_t^0 + h_t^1 \diamond S_t.$$

A portfolio $h$ with Wick-value process $\mathcal{V}^h$ is called *Wick*$^{\mathbf{2}}$*-financing* if

$$d\mathcal{V}_t^h = h_t^0 dB_t + h_t^1 \diamond dS_t,$$

i.e., if

$$d\mathcal{V}_t^h = h_t^1 \diamond dS_t.$$

In the paper by Hu and Øksendal (2003) a portfolio $h$ with Wick-value process $\mathcal{V}^h$ satisfying the Wick$^{\mathbf{2}}$-financing condition is said to be "self-financing". If we use the Wick-value process to define the value of a portfolio, then simple buy-and-hold strategies will in general be Wick$^{\mathbf{2}}$-financing. In particular the portfolio described in Example 1 will be Wick$^{\mathbf{2}}$-financing (we showed that it is not Wick$^{\mathbf{1}}$-financing). However, the definition of the Wick-value process $\mathcal{V}^h$ is difficult to motivate as a definition of value from an economic perspective, and we have not found any precise argument for it in Hu and Øksendal (2003).

The lack of economic meaning of the Wick-value seems (for us) to be obvious, but to emphasise our point, let us study the definition of Wick-value in more detail.

Suppose we fix a time $t$ and study the value and the Wick-value of a portfolio at time $t$. It is important to distinguish between the random variables $h_t^0 : \Omega \to R$, $h_t^1 : \Omega \to R$, $S_t : \Omega \to R$ and the observed values $h_t^0(\omega)$, $h_t^1(\omega)$, $S_t(\omega)$, $\omega \in \Omega$. The usual definition of value can be written as

$$V_t^h(\omega) = h_t^0(\omega) + h_t^1(\omega)S_t(\omega)$$

for $P$-almost all $\omega \in \Omega$. This means that to compute the value of the portfolio at time $t$ we only need the observed amount $h_t^0(\omega)$ in the bank account, the observed number $h_t^1(\omega)$ of shares of the risky asset and the observed price $S_t(\omega)$ of the risky asset. All these can be observed at time $t$. If we want to buy $h_t^1(\omega)$ shares of the risky asset, say $h_t^1(\omega) = 10$, we simply have to instruct our broker to buy ten shares of the risky asset for us. Observing the market price $S_t(\omega)$ the broker can then tell us how much this will cost, namely $h_t^1(\omega)S_t(\omega)$. On the other hand, the situation is quite different if we insist to use the Wick-value to define the value of a portfolio. The Wick-value is given by

$$\mathcal{V}_t^h(\omega) = h_t^0(\omega) + (h_t^1 \diamond S_t)(\omega)$$

for $P$-almost all $\omega \in \Omega$. This means that to compute the Wick-value, it is not sufficient to know for a fixed $\omega$ the observed values of $h_t^0(\omega)$, $h_t^1(\omega)$ and $S_t(\omega)$. We have to plug in the entire random variables $h_t^1 : \Omega \to R$ and $S_t : \Omega \to R$. Hence if we want to buy, say, $h_t^1(\omega) = 10$ shares of the risky asset at time $t$ it is not sufficient to instruct our broker to buy ten shares of the risky asset. We must specify the entire random variable $h_t^1 : \Omega \to R$, i.e., we must let him know how many shares we would buy in $P$-almost all possible states of the world. Otherwise he cannot compute the Wick-value of the portfolio. This hardly seems practical, nor does it reflect what is actually going on in the real world. In reality we do not have to specify which actions we would take, had we observed a different scenario; only the actually observed information is relevant. Moreover, as the Wick-value typically differs from the usual definition of value of a portfolio anyone who insists to book the Wick-value would be violating corporate law.

It is our opinion that the number we call the value of the portfolio must be computable simply by considering observed values of the amount in the bank account, the number of shares of the risky asset and the price of the risky asset. We claim that any reasonable definition of value of a portfolio must be given by a deterministic function $F : R^3 \to R$ of the observed amount in the bank account, $h_t^0(\omega)$, the number of shares $h_t^1(\omega)$ of the risky asset and its observed price $S_t(\omega)$, such that

$$V_t^h(\omega) = F\left(h_t^0(\omega), h_t^1(\omega), S_t(\omega)\right)$$

for $P$-almost all $\omega \in \Omega$. The function $F$ could for instance include transaction costs, costs for liquidating the position etc. and need not neccessarily be thought of as the expression given by (10). The Wick-value $\mathcal{V}^h$ cannot be written in this form. Consider for instance the following example.

Suppose we, at $t = 0$, fix a number $u \in (0, 1)$ and, at $t = 1$, we buy $S_u$ number of shares of the risky asset at the price $S_1$. In a reasonable model we claim that the cost of buying the portfolio at $t = 1$ is $S_u S_1$ or at least given by the function $F(0, S_u, S_1)$. However, if the Wick-value is used this results in

$$\begin{aligned}
\mathcal{V}_1^h &= S_u \diamond S_1 \\
&= \mathcal{E}(1_{[0,u]}) \diamond \mathcal{E}(1_{[0,1]}) \\
&= \mathcal{E}(1_{[0,u]} + 1_{[0,1]}) \\
&= \exp\left\{ W_u^H + W_1^H - \frac{1}{2} |1_{[0,u]} + 1_{[0,1]}|_\phi^2 \right\} \\
&= \exp\left\{ W_u^H + W_1^H - \frac{1}{2} \left( u^{2H} + 1 + \left( u^{2H} + 1 - (1-u)^{2H} \right) \right) \right\} \\
&= S_u S_1 \left\{ -\frac{1}{2} \left( u^{2H} + 1 - (1-u)^{2H} \right) \right\}.
\end{aligned}$$

That is,

$$\mathcal{V}_1^h(\omega) = S_u(\omega) S_1(\omega) \exp\left\{ -\frac{1}{2} \left( u^{2H} + 1 - (1-u)^{2H} \right) \right\}$$

for $P$-almost all $\omega \in \Omega$. This is quite remarkable. The Wick-value of the portfolio does not only depend on the observed number of shares $S_u(\omega)$ of the risky asset and the observed price $S_1(\omega)$ of the risky asset but also explicitly on the time $u$ which was arbitrarily chosen in $(0, 1)$.

There are even more problems if we want to interpret the Wick-value as the value of a portfolio. Consider a portfolio investing in the risky asset only. At $t = 1$ the portfolio consist of $h_1^1$ shares of the risky asset and the amount $h_1^0 = 0$ in the bank account. Then there is a set $\Omega' \in \mathcal{F}_1$ with $P(\Omega') > 0$ and a portfolio $h = (0, h^1)$ such that $h_1^1 > 0$ on $\Omega'$, but $\mathcal{V}_1^h < 0$ on $\Omega'$. That is, on $\Omega'$ a portfolio that contains a positive number of shares of the risky asset and zero amount in the bank account has negative Wick-value. This appears to be very unrealistic. The portfolio is easy to construct. In fact, we may take $h_1^1 = S_1 - s_0$. The result is summarised in the next lemma.

**Lemma 3** *There exist a set $\Omega' \in \mathcal{F}_1$ with $P(\Omega') > 0$ and a portfolio with $h_1 = (0, h_1^1)$ such that $h_1^1 > 0$ and $h_1^1 \diamond S_1 < 0$ on $\Omega'$.*

*Proof* Let $\Omega' = \{\omega \in \Omega \mid W_1^H(\omega) \in (\frac{1}{2}, \frac{3}{2})\}$ and $h_1^1 = S_1 - s_0$. Then $P(\Omega') > 0$ and since $S_1 = s_0 \exp\{W_1^H - \frac{1}{2}\}$ it follows that $h_1^1 > 0$ on $\Omega'$. Moreover,

$$\begin{aligned}
h_1^1 \diamond S_1 &= (S_1 - s_0) \diamond S_1 \\
&= S_1 \diamond S_1 - s_0 S_1 \\
&= s_0^2 \exp\{2W_1^H - 2\} - s_0^2 \exp\left\{ W_1^H - \frac{1}{2} \right\} \\
&= s_0^2 \left( \exp\{2W_1^H - 2\} - \exp\left\{ W_1^H - \frac{1}{2} \right\} \right).
\end{aligned}$$

Hence $h_1^1 \diamond S_1 < 0$ on $\Omega'$ and the conclusion follows. □

*Remark 4* We wish to emphasize that our criticism against the use of FBM and the Wick product in finance is only directed against the particular applications discussed in the present note. There is of course no a priori reason why there should not be other, perfectly valid, applications of these mathematical objects in finance. We wish to mention the recent paper Øksendal (2004). In that paper the process $S$ is no longer interpreted as the observed stock price. Instead it is given the interpretation of an unobserved "value" process, and the actual stock price is then produced through an "observer" in a quantum mechanical fashion. This theory is qualitatively very different from the ones discussed above, it is not affected by the particular criticism raised against the papers cited above, but it does in fact lead to arbitrage. It also raises completely different interpretational problems which are yet to be discussed.

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