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Lévy term structure models: No-arbitrage and completeness

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Abstract. The Lévy term structure model due to Eberlein and Raible is extended to non-homogeneous driving processes. The classes of equivalent martingale and local martingale measures for various filtrations are characterized. It turns out that in a number of standard situations the martingale measure is unique.

Key words: Term structures, Lévy processes, no-arbitrage, completeness

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1 Introduction

Fixed-income markets are an important sector of the global financial markets although their ups and downs are less perceived by the public than those of the more spectacular stock markets. Nevertheless from the point of view of volume they often outperform the latter. Mathematically the modeling of interest rate-sensitive instruments such as bonds, swaps, caps, and floors is quite demanding. A great variety of short-rate models with increasing degrees of sophistication were developed over the years. One of the earliest models was introduced by Vasiček [11]. At the high end of the list one could mention Sandmann and Sondermann [9]. In comparison with equity price models the characteristic property which has to be built in is mean reversion.

In 1992 Heath et al. [5] made a significant step forward, specifying exogenously the dynamics of instantaneous, continuously compounded forward rates. Equivalently the dynamics of bond prices for all maturities T $(0 \leq T \leq T_\star)$ are given

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simultaneously. Soon after this Sandmann et al. [10] and Brace et al. [2] succeeded to embed discretely compounded market rates into this framework of continuously compounded forward rates. In the line of the Heath–Jarrow–Morton forward rate approach a very general model on the level of semimartingale theory has been introduced by Björk et al. [1]. It is driven by a finite number of Wiener processes plus a random measure which allows to take jumps into account. A bit more restrictive but versatile enough for modeling purposes are the models driven by Lévy processes which were introduced in [3] and extended in [4]. Lévy driven interest rate models are strongly supported by empirical facts (see [8]). They can be calibrated using a few parameters only. In this paper we investigate questions of no-arbitrage and completeness of this model class. A somewhat surprising result is that in the case of a one-dimensional driving process a Black–Scholes type situation arises: there is a unique equivalent martingale measure.

Let us recall that a term structure model can be specified by the dynamics of the instantaneous forward rates $f(t,T)$, where the maturities T are all points of a bounded interval $I = [0, T_\star]$ (the maturity $T = 0$ has no financial meaning, but is included here for easier notation), and of course $0 \le t \le T$. For a Lévy driven term structure, the instantaneous forward rates satisfy the equations

$$f(t,T) = f(0,T) + \int_0^t \gamma(s,T)^\star dL_s, \qquad 0 \le t \le T, \;\; T \in I \tag{1.1}$$

where the driving term L is a d-dimensional Lévy process and the coefficients $\gamma(s,T)$ are d-dimensional as well, possibly random, with some measurability and integrability conditions to be specified later. $\gamma^\star$ stands for the transpose of γ.

The discounted price at time $t \le T$ of a default-free zero coupon bond with unit nominal value maturing at time T is then

$$P(t,T) = \exp\left(-\int_0^t r(s)ds - \int_t^T f(t,s)ds \right), \tag{1.2}$$

where $r(s) = f(s,s)$ is the spot rate.

In a bond market one can trade for all maturities $T \in J$, for some subset $J \subset I$, and in practice two cases are encountered: the set J is finite, or it is dense in I, in which case it is usually either I or $(0, T_\star]$.

When J is finite, say $J = \{T_1, \ldots, T_n\}$, the vector of all prices $(P(t,T_i) : 1 \le i \le n)$ is an n-dimensional process (semimartingale) driven by some Lévy process L, and the analysis is the same as for a family of n stock prices: we have (mild) conditions ensuring no-arbitrage, and the model is complete when L is continuous and $d \le n$, and it is in general incomplete in the other cases.

Much more interesting in our present situation is when J is dense in I: despite the fact that in reality J is always finite, one uses the infinite – idealized – case, even in practice, as a limiting model allowing to take care of large values of n. It also allows to explain the interactions between the $P(t,T)$s for various values of T, which are intrinsic to the model (1.1)–(1.2). Then two questions naturally arise:

1: No-arbitrage. This question essentially amounts to the existence of one or several *equivalent martingale measures*, that is a probability measure Q which is equivalent to the historical measure $\mathbb{P}$ and such that each process $t \mapsto P(t,T)$ for $T \in J$ is a martingale. This question requires first to define the filtration w.r.t. which those price processes are martingales: it can be the filtration $(\mathcal{F}_t)$ generated by the driving process L, or the smaller filtration $(\mathcal{G}_t)$ which is generated by the price processes $P(t,T)$ themselves, for $T \in J$.

2: Completeness. Mathematically, and if the first question is answered positively, completeness is in fact *uniqueness* of the equivalent martingale measure. This issue is important for two reasons:

a) The fair price of a claim, based on the price processes $P(.,T)$ for $T \in J$, is the expected value of the claim under an equivalent martingale measure, so completeness amounts to the fact that any claim can be priced in a unique way.

b) Completeness is also related to hedging: in view of "martingale representation theorems" which may be found for example in [6], it is in fact equivalent to the property that any bounded (or more generally integrable) claim Y can be written as

$$Y = \mathbb{E}_Q(Y) + \sum_{n \geq 1} \int_0^{T_n} H_s^n \, dP(s,T_n),$$

where $\mathbb{E}_Q$ is the expectation w.r.t. the (unique) equivalent martingale measure and H^n are predictable processes and the T_n's are maturities in J; of course the above series should be convergent in some sense: so completeness actually amounts to the possibility of hedging any claim based upon the price processes $P(.,T)$ for $T \in J$.

So below we concentrate on the case when J is dense in I. Then the answers to the above two questions are indeed quite different from the case when J is finite, or when we study a finite number of stock prices. And they are indeed a bit surprising. Heuristically we have:

- The answer to Question 1 is "no", unless we have a very special structure for the coefficients $\gamma(s,T)$. This was already observed by Heath–Jarrow–Morton [5] in the case when L is continuous.
- Assuming that there exists an equivalent martingale measure, the answer to Question 2 is "yes" when the dimension d of the driving process L is 1, and under some weak non-degeneracy condition. When $d \geq 2$, the answer is in general "no".

When we additionally assume that the coefficients γ in (1.1) are non-random, we have a complete and simple answer in the 1-dimensional case:

Theorem 1.1 *Assume that the driving process is* 1*-dimensional. Assume also that the coefficients* $\gamma(s,T)$ *are non-random, and that* J *is dense in* I*. Then the set of all equivalent martingale measures is either empty or is a singleton.*

This is a part of Theorem 6.4, which also contains a necessary and sufficient condition for the existence of an equivalent martingale measure.

2 Assumptions and notation

1) First we describe the driving process. For reasons to be seen later, it is convenient to take a driving process L that is not necessarily a Lévy process, but which has *independent increments and absolutely continuous characteristics*, called more shortly a PIIAC. Another natural name could be a "non-homogeneous Lévy process". That means that the d-dimensional process L is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and has independent increments and its law is characterized by the following characteristic functions, where $u \in \mathbf{R}^d$, the transpose is denoted by $u^\star$, and $|u|$ is the Euclidian norm:

$$E\left(e^{iu^\star L_t}\right) = \exp \int_0^t \left(iu^\star b_s - \frac{u^\star c_s u}{2} + \int_{\mathbf{R}^d} \left(e^{iu^\star x} - 1 - iu^\star x \, 1_{\{|x|\le 1\}}\right) F_s(dx)\right) ds. \tag{2.1}$$

Here $b_t \in \mathbf{R}^d$, and c_t is a symmetric nonnegative-definite $d \times d$ matrix, and the Lévy measure F_t is a measure on $\mathbf{R}^d$ with $F_t(\{0\}) = 0$, and we have the following integrability assumptions:

$$\int_0^{T_\star} \left(|b_t| + ||c_t|| + \int_{\mathbf{R}^d} (|x|^2 \wedge 1) F_t(dx)\right) dt < \infty \tag{2.2}$$

where $||c||$ denotes any norm on the set of $d \times d$ matrices. It is of course no restriction to assume further that $\int (|x|^2 \wedge 1) F_t(dx) < \infty$ for all t. We denote by $(\mathcal{F}_t)$ the filtration generated by L, and we assume that $\mathcal{F} = \mathcal{F}_{T_\star}$. Let us recall that these characteristics (b_t, c_t, F_t) are connected with the so-called "canonical decomposition" of L in the following way: we have

$$L_t = \int_0^t b_s ds + L_t^c + \int_0^t \int_{\mathbf{R}^d} x 1_{\{|x|\le 1\}} (\mu - \nu)(ds, dx) + \sum_{s \le t} \Delta L_s 1_{\{|\Delta L_s| > 1\}}; \tag{2.3}$$

here L^c is the "continuous martingale part" of L, which turns out to be of the form $L_t^c = \int_0^t c_s^{1/2} dW_s$ in our case, where W is a standard d-dimensional Brownian motion and $c_s^{1/2}$ is a measurable version of the square-root of the symmetric nonnegative-definite matrix c_s; μ is the random measure associated with the jumps of L, and $\nu(ds, dx) = ds F_s(dx)$ is its (non-random) compensator.

2) As mentioned before, for the model described by (1.1) there usually is no equivalent martingale measure. To see that more clearly we reformulate our model as follows:

$$f(t, T) = f(0, T) + \int_0^t \alpha(s, T) ds + \int_0^t \gamma(s, T)^\star dL_s, \qquad 0 \le t \le T, \ T \in I. \tag{2.4}$$

Here L and $\gamma(s, T)$ are d-dimensional, while $\alpha(s, T)$ is 1-dimensional. Some examples of "natural" γ's are provided later.

Observe that (2.4) and (1.1) are two formulations of the same model: if (2.4) holds with L, γ and α, then (1.1) holds with the $(d+1)$-dimensional PIIAC L' whose first d components are those of L and the $(d+1)$th component is a pure drift ($L_t'^{d+1} = t$), and the coefficient γ' whose first d components are those of γ, and the $(d+1)$th component is α. Conversely if we have (1.1) with L and γ, we also have (2.4) with the same L and γ, and with $\alpha(s,T) \equiv 0$. But the form (2.4) turns out to be much handier to deal with.

3) Now we state our assumptions on the coefficients. First, in (2.4) the initial values $f(0,T)$ are deterministic, and measurable and bounded in T. Moreover α and γ are respectively an $\mathbf{R}$-valued and an $\mathbf{R}^d$-valued function on $\Omega \times I \times I$ satisfying the following (below, $\mathcal{P}$ and $\mathcal{O}$ denote the predictable and the optional σ-fields on $\Omega \times I$):

$$\left.\begin{array}{l} s > T \quad \Rightarrow \quad \alpha(\omega,s,T) = 0, \quad \gamma(\omega,s,T) = 0, \\ (\omega,s,t) \mapsto \alpha(\omega,s,t),\ \gamma(\omega,s,t) \qquad \text{are } \mathcal{P} \otimes \mathcal{B}(I)\text{-measurable}, \\ S(\omega) := \sup_{s,t \le T_\star} (|\alpha(\omega,s,t)| + |\gamma(\omega,s,t)|) < \infty. \end{array}\right\} \tag{2.5}$$

Then (2.4) makes sense, and we can find a "joint" version of all $f(t,T)$ such that $(\omega,t,T) \mapsto f(t,T)(\omega)1_{\{t\le T\}}$ is $\mathcal{O} \otimes \mathcal{B}(I)$-measurable.

Often we will assume the following hypothesis:

Hypothesis (DET) The processes α and γ are *deterministic*. □

From time to time we will also need the following assumption on the process L:

Hypothesis (BJ) The process L has jumps bounded by a constant M. Equivalently, we have $F_t(|x| > M) = 0$ for all t. □

If this assumption is satisfied, we can always assume that it holds in fact with $M = 1$: indeed we can replace L by L/M (which has jumps bounded by 1) and $\gamma(s,T)$ by $M\gamma(s,T)$ in (2.4), so both the filtration $(\mathcal{F}_t)$ and the rates are unchanged.

4) The discounted bond price processes are given by (1.2), and it is convenient to write them as $P(T)_t = P(t,T)$. Each process $P(T)$ is a priori defined on the interval $[0,T]$, but it is also convenient to extend it (in a trivial way) to the whole interval I by setting $P(T)_t = P(T)_T$ for $t \ge T$. With this extension, and exactly as in [1], a simple calculation taking (2.4) into account shows that

$$\left.\begin{array}{l} P(T)_0 = \exp\left(-\int_0^T f(0,s)ds\right), \\ P(T)_t = P(T)_0\ \exp\left(\int_0^t A(T)_s ds + \int_0^t \Gamma(T)_s^\star dL_s\right), \end{array}\right\} \tag{2.6}$$

where

$$A(T)_t = -\int_{t\wedge T}^T \alpha(t,s)ds, \qquad \Gamma(T)_t = -\int_{t\wedge T}^T \gamma(t,s)ds. \tag{2.7}$$

Moreover, we also have $P(T)_t = P(T)_0\mathcal{E}(H(T))_t$, where $\mathcal{E}(H)$ denotes the Doléans exponential of H, and $H(T)$ is the following process (see for example [7], to which we refer for all facts concerning stochastic calculus):

$$H(T)_t = \frac{1}{2}\int_0^t \varGamma(T)_s^\star c_s \varGamma(T)_s\, ds + \int_0^t A(T)_s ds + \int_0^t \varGamma(T)_s^\star dL_s + \sum_{s\le t}\left(e^{\varGamma(T)_s^\star \varDelta L_s} - 1 - \varGamma(T)_s^\star \varDelta L_s\right). \tag{2.8}$$

With the notation of (2.3), the above processes can also be written as

$$\begin{aligned} H(T)_t = &\int_0^t a(T)_s ds + \int_0^t \varGamma(T)_s^\star dL_s^c \\ &+ \int_0^t \int_{\mathbf{R}^d} (\varGamma(T)_s^\star x\, 1_{\{|x|\le 1\}})(\mu-\nu)(ds,dx) \\ &+ \int_0^t \int_{\mathbf{R}^d} \left(e^{\varGamma(T)_s^\star x} - 1 - \varGamma(T)_s^\star x\, 1_{\{|x|\le 1\}}\right)\mu(ds,dx), \end{aligned} \tag{2.9}$$

where

$$a(T)_t = A(T)_t + \varGamma(T)_t^\star b_t + \frac{1}{2}\,\varGamma(T)_t^\star c_t \varGamma(T)_t. \tag{2.10}$$

5) Examples One of the most popular volatility structures $\varGamma(T)$ is the Vasiček structure, given by

$$\varGamma(T)_t = \frac{\widehat{\sigma}}{a}\left(1 - e^{-a(T-t)}\right)$$

for parameters $\widehat{\sigma} > 0$ and $a \neq 0$. Sometimes it is sufficient to consider the Ho–Lee structure, given by

$$\varGamma(T)_t = \widehat{\sigma}(T-t).$$

These two structures provide Markov short rates, see Theorem 4.4 of [3] for details. Both cases refer to $d = 1$.

Note that here we describe only $\varGamma(T)$, or equivalently $\gamma(t,s)$. These models say nothing about the coefficients $\alpha(s,t)$, which as we shall see later are essentially determined by $\gamma(s,t)$ if we are to have a martingale measure.

6) Now we turn to equivalent martingale measures. First, let us emphasize that we have here two "natural" filtrations: the *original filtration* $(\mathcal{F}_t)$ with respect to which all processes are adapted; it is the filtration generated by L. We also have the smaller *observed filtration* $(\mathcal{G}_t)$, which is the filtration generated by the prices $P(T)_t$ for all $T \in J$. The corresponding final σ-field is $\mathcal{G} = \mathcal{G}_{T_\star}$. And, even when $J = I$, the inclusion $\mathcal{G}_t \subset \mathcal{F}_t$ may be strict. Then we have different notions of equivalent martingale measures, according to which filtration we are interested in.

We denote by $\mathcal{Q}_{\mathcal{F}}$ (resp. $\mathcal{Q}_{\mathcal{G}}$) the set of all probability measures on $(\varOmega,\mathcal{F})$ (resp. $(\varOmega,\mathcal{G})$) which are equivalent to $\mathbb{P}$, and under which the processes $P(T)$ are martingales for all $T \in J$, and relative to $(\mathcal{F}_t)$ (resp. $(\mathcal{G}_t)$).

If we replace "martingale" by "local martingale", we get bigger sets denoted by $\mathcal{Q}_{\mathcal{F},loc}$ (resp. $\mathcal{Q}_{\mathcal{G},loc}$): in the context of a closed time interval, a local martingale is a process for which there is an increasing sequence (S_n) of stopping times with values in $[0, T_\star]$ and such that the process stopped at each S_n is a uniformly integrable martingale, and such that $\mathbb{P}(S_n = T_\star)$ goes to 1. These bigger sets are also the sets of equivalent probability measures under which the processes $H(T)$ of (2.9) are local martingales, and this is what makes them much easier to study than $\mathcal{Q}_{\mathcal{F}}$ and $\mathcal{Q}_{\mathcal{G}}$.

Finally we also introduce the sets $\mathcal{Q}'_{\mathcal{F}}$ and $\mathcal{Q}'_{\mathcal{F},loc}$ consisting of all measures Q in $\mathcal{Q}_{\mathcal{F}}$ and $\mathcal{Q}_{\mathcal{F},loc}$ respectively, and under which the driving process L is still a PIIAC.

So far we have altogether six different sets of equivalent martingale measures, but *the most meaningful one from the point of view of economics is the set* $\mathcal{Q}_{\mathcal{G}}$, for which we need to check whether it is empty or not, and if not whether it is a singleton or not. It is also a priori the most difficult set to study, so we look at the relations between those sets below. But before that we make use of Girsanov's theorem to describe the set $\mathcal{Q}_{\mathcal{F},loc}$.

3 Description of equivalent local martingale measures

The description of the set $\mathcal{Q}_{\mathcal{F},loc}$ is classical and relatively simple. Let us consider the two measures m and $\overline{m}$ on $\Omega \times I$ and $\Omega \times I \times \mathbf{R}^d$ given by

$$m(d\omega, dt) = \mathbb{P}(d\omega)\, dt, \qquad \overline{m}(d\omega, dt, dx) = \mathbb{P}(d\omega)\, dt\, F_t(dx).$$

Then we consider pairs (u, Y), where

- $u = (u^i)_{i \le d}$ is a predictable $\mathbf{R}^d$-valued process such that

$$\int_0^{T_\star} u_t^\star\, c_t\, u_t\, dt < \infty \qquad \text{a.s.} \tag{3.1}$$

- Y is a $\mathcal{P} \otimes \mathcal{R}^d$-measurable $(0, \infty)$-valued function, such that

$$\int_0^{T_\star} dt \int_{\mathbf{R}^d} \left(|Y(t,x) - 1| \wedge (Y(t,x) - 1)^2\right) F_t(dx) < \infty \qquad \text{a.s.} \tag{3.2}$$

We denote by $\mathcal{Y}$ the set of all equivalence classes of such pairs (u, Y), for the equivalence relation $(u, Y) \sim (u', Y')$ defined as

$$(u - u')^\star c(u - u') = 0 \quad m\text{-a.e. and } \; Y' = Y \quad \overline{m}\text{-a.e.} \tag{3.3}$$

Next, we denote by $\mathcal{Y}(J)$ the set of all (equivalence classes of) pairs (u, Y) which satisfy

$$T \in J \;\Rightarrow \tag{3.4}$$
$$\int_0^{T_\star} dt \int_{\mathbf{R}^d} \left| Y(t,x) \left(e^{\Gamma(T)_t^\star x} - 1\right) - \Gamma(T)_t^\star x\, 1_{\{|x| \le 1\}} \right| F_t(dx) < \infty \text{ a.s.}$$

The following theorem is a consequence of Girsanov's theorem (see [7]), and it is also essentially contained in [1], but for completeness and because of its importance we provide a sketch of the proof.

Theorem 3.1 *There is a one-to-one correspondence between the probabilities in* $\mathcal{Q}_{\mathcal{F},loc}$ *and the set* $\mathcal{Y}_m(J)$ *of all pairs* $(u, Y) \in \mathcal{Y}(J)$ *which, for every* $T \in J$, *satisfy*

$$a(T)_t + \Gamma(T)_t^\star c_t u_t \\ + \int_{\mathbf{R}^d} \left(\left(e^{\Gamma(T)_t^\star x} - 1 \right) Y(t,x) - \Gamma(T)_t^\star x \, 1_{\{|x| \le 1\}} \right) F_t(dx) = 0 \tag{3.5}$$

for m*-almost all* (ω, t)*. Moreover, the density process of the measure in* $\mathcal{Q}_{\mathcal{F},loc}$ *associated with the pair* (u, Y) *is then the Doléans exponential* $\mathcal{E}(M)$ *of the* $\mathbb{P}$*-local martingale*

$$M_t = \int_0^t u_s^\star dL_s^c + \int_0^t \int_{\mathbf{R}^d} (Y(s,x) - 1)(\mu - \nu)(ds, dx). \tag{3.6}$$

In (3.5) the integral converges because of (3.4). The local martingale M in (3.6) is well defined as soon as (3.1) and (3.2) are satisfied, and two different pairs (u, Y) and (u', Y') give the same process M if and only if they satisfy (3.3): this is why we identify two equivalent pairs (u, Y).

Sketch of the proof. Let Q be an equivalent probability measure. Girsanov's theorem tells us that not only is the density process of Q w.r.t. $\mathbb{P}$ of the form $\mathcal{E}(M)$, where M is associated by (3.6) with some pair $(u, Y) \in \mathcal{Y}$, but also that the canonical decomposition of L under Q is (compare with (2.3)):

$$L_t = \int_0^t b'_s ds + L'^c_t + \int_0^t \int_{\mathbf{R}^d} x 1_{\{|x| \le 1\}} (\mu - \nu')(ds, dx) + \sum_{s \le t} \Delta L_s 1_{\{|\Delta L_s| > 1\}}; \tag{3.7}$$

here L'^c is the "continuous martingale part" of L w.r.t. Q, and ν' is the compensator of μ w.r.t. Q as well, which turns out to be $\nu'(\omega, dt, dx) = Y(\omega, t, x) \cdot \nu(dt, dx)$, and $b'_t = b_t + c_t u_t + \int_{\mathbf{R}^d} (Y(t,x) - 1) x 1_{\{|x| \le 1\}} F_t(dx)$.

Now, Q belongs to $\mathcal{Q}_{\mathcal{F},loc}$ if and only if the processes of (2.8) are Q-local martingales. Since the integral of a locally bounded predictable process w.r.t. a local martingale is a local martingale itself, using (3.7) yields that $Q \in \mathcal{Q}_{\mathcal{F},loc}$ if and only the processes $Z(T) + Z'(T)$ are Q-local martingales for all $T \in J$, where

$$Z(T)_t = \frac{1}{2} \int_0^t \Gamma(T)_s^\star c_s \Gamma(T)_s \, ds + \int_0^t A(T)_s ds + \int_0^t \Gamma(T)_s^\star b'_s ds,$$
$$Z'(T)_t = \sum_{s \le t} \left(e^{\Gamma(T)_s^\star \Delta L_s} - 1 - \Gamma(T)_s^\star \Delta L_s \, 1_{\{|\Delta L_s| \le 1\}} \right).$$

Note that $Z(T)$ and $Z'(T)$ are of finite variation, and $Z(T)$ is predictable, so this condition amounts saying that each $Z'(T)$ admits a predictable Q-compensator

$Z''(T)$, which further satisfies $Z(T)+Z''(T)=0$. The existence of this predictable Q-compensator is exactly condition (3.4), under which we obtain

$$Z''(T)_t = \int_0^t ds \int_{\mathbf{R}^d} \left(e^{\Gamma(T)_s^\star x} - 1 - \Gamma(T)_s^\star x \, 1_{\{|x|\le 1\}} \right) Y(s,x) F_s(dx).$$

At this point, and in view of the previous expression for b'_t and of (2.10), it is easy to check that the condition $Z(T)+Z''(T)=0$ is in fact (3.5), up to an m-null set. □

It is also useful to write the process $H(T)$ in a way which makes transparent its "martingality" under $Q \in \mathcal{Q}_{\mathcal{F},loc}$. Upon examining the previous proof, we see that $H(T)$ is the integral of $\Gamma(T)^\star$ w.r.t. the sum of L'^c (which is $L_t'^c = L_t^c - \int_0^t c_s u_s ds$) and of the third term on the right of (3.7), plus $Z'(T) - Z''(T)$. Putting all these together allows us to write

$$\begin{aligned} H(T)_t &= \int_0^t \Gamma(T)_s^\star dL_s'^c + \int_0^t \int_{\mathbf{R}^d} \left(e^{\Gamma(T)_s^\star x} - 1 \right) (\mu(ds,dx) - \nu'(ds,dx)) \\ &= \int_0^t \Gamma(T)_s^\star (dL_s^c - c_s u_s ds) + \int_0^t \int_{\mathbf{R}^d} \left(e^{\Gamma(T)_s^\star x} - 1 \right) (\mu(ds,dx) \\ &\quad - Y(s,x) \cdot \nu(ds,dx)). \end{aligned} \tag{3.8}$$

Since $\mathbb{P} \in \mathcal{Q}_{\mathcal{F},loc}$ amounts to saying that the pair $(u \equiv 0, Y \equiv 1)$ satisfies (3.5), the following is obvious:

Corollary 3.2 *The probability measure $\mathbb{P}$ itself belongs to $\mathcal{Q}_{\mathcal{F},loc}$ if and only if for every $T \in J$ we have*

$$\int_0^{T_\star} dt \int_{\mathbf{R}^d} \left| e^{\Gamma(T)_t^\star x} - 1 \right| 1_{\{|x|>1\}} \, F_t(dx) < \infty \qquad a.s., \tag{3.9}$$

and

$$a(T)_t + \int_{\mathbf{R}^d} \left(e^{\Gamma(T)_t^\star x} - 1 - \Gamma(T)_t^\star x \, 1_{\{|x|\le 1\}} \right) F_t(dx) = 0 \tag{3.10}$$

for m-almost all (ω,t).

Let us end this section with some comments on the fact that for the set $\mathcal{Q}_{\mathcal{F},loc}$ to be non-empty the coefficients α and γ should be related in a quite special way.

To see this, assume that $d=1$ and that L is a non-trivial *continuous* Lévy process, that is $L_t = bt + \sqrt{c} W_t$ with $c>0$ and W a standard Brownian motion. Then $F_t = 0$, and $\mathcal{Y}$ reduces to the set of all predictable processes u satisfying $\int_0^{T_\star} u_s^2 ds < \infty$ a.s., and (3.5) reads as

$$A(T)_t + b\Gamma(T)_t + \frac{c}{2}\,\Gamma(T)_t^2 + c\Gamma(T)_t u_t = 0. \tag{3.11}$$

If $J = I$, and taking advantage of the relations (2.7), we can "differentiate" (3.11) in T almost everywhere, to get:

Proposition 3.3 *Assume that* $d = 1$ *and that* L *is a non-trivial continuous Lévy process. If* $J = I$ *the set* $\mathcal{Q}_{\mathcal{F},loc}$ *is not empty if and only if there is a predictable process* u *satisfying* $\int_0^{T_\star} u_s^2 ds < \infty$ *a.s., such that*

$$\alpha(\omega, s, T) = -(b + c\Gamma(T)_s(\omega) + cu_s(\omega))\gamma(\omega, s, T) \tag{3.12}$$

for $\widetilde{m}$*-almost all* (ω, s, T)*, where* $\widetilde{m}$ *is the measure* $m \otimes dt$ *on* $\Omega \times [0, T_\star] \times [0, T_\star]$.

Relation (3.12) is quite restrictive, because $u_s(\omega)$ should *not* depend on T.

When L has jumps, the situation is even more complicated. Under (DET) and (BJ) we can however state the following result, which is a trivial consequence of Theorem 4.1 and of the fact that a measure $Q \in \mathcal{Q}_{\mathcal{F},loc}$ actually belongs to $\mathcal{Q}'_{\mathcal{F},loc}$ if and only if there is a version of the associated pair (u, Y) which is non-random.

Proposition 3.4 *Assume (DET) and (BJ) and that* J *is dense in* I*. Then the set* $\mathcal{Q}'_{\mathcal{F},loc}$ *(or equivalently the set* $\mathcal{Q}_{\mathcal{F},loc}$*, as we will see later) is not empty if and only if there is a non-random pair* (u, Y) *satisfying (3.1) and (3.2) such that the functions* α *and* γ *satisfy the following relation:*

$$\begin{aligned}\alpha(t, T) + \gamma(t, T)^\star \Big(b_t + c_t\Gamma(T)_t + c_t u_t \\ + \int x \left(e^{\Gamma(T)_t^\star x} Y(t, x) - 1_{\{|x| \leq 1\}} \right) F_t(dx) \Big) = 0\end{aligned} \tag{3.13}$$

for $dt \times dT$*-almost all* (t, T) *in* $I \times I$.

4 Relations between sets of martingale measures for $(\mathcal{F}_t)$

Now we proceed to comparing the various sets $\mathcal{Q}_{\mathcal{F},loc}$, $\mathcal{Q}'_{\mathcal{F},loc}$, $\mathcal{Q}_{\mathcal{F}}$ and $\mathcal{Q}'_{\mathcal{F}}$.

Theorem 4.1 *Assume (DET), and either (BJ) or that* $d = 1$.

a) We have $\mathcal{Q}_{\mathcal{F},loc} = \emptyset$ *if and only if* $\mathcal{Q}'_{\mathcal{F},loc} = \emptyset$.
b) If the set $\mathcal{Q}_{\mathcal{F},loc}$ *contains more than one point, then so does the set* $\mathcal{Q}'_{\mathcal{F},loc}$.

For proving this result, we need two preliminary lemmas.

Lemma 4.2 *Let* $(u, Y) \in \mathcal{Y}$*, and* $U(T)_t = \int_{\{|x|>1\}} e^{\Gamma(T)^\star x} Y(t, x) F_t(dx)$*. Then we have* $(u, Y) \in \mathcal{Y}(J)$ *if and only if*

$$T \in J \quad \Rightarrow \quad \int_0^{T_\star} U(T)_t \, dt < \infty \quad a.s. \tag{4.1}$$

Proof Let us set

$$\psi_t(x) = |Y(t, x) - 1| \wedge (Y(t, x) - 1)^2 + |x|^2 \wedge 1.$$

Then (2.2) and (3.2) yield

$$\int_0^{T_\star} dt \int F_t(dx)\, \psi_t(x) < \infty \quad \text{a.s.} \tag{4.2}$$

The integrand in (3.4) is $|V(T,t,x) + W(T,t,x)|$, where

$$V(T,t,x) = e^{\Gamma(T)^\star x} Y(t,x)\, 1_{\{|x|>1\}},$$

$$W(T,t,x) = -Y(t,x)\, 1_{\{|x|>1\}} + \Big(Y(t,x)\left(e^{\Gamma(T)_t^\star x} - 1\right) - \Gamma(T)_t^\star x\Big)\, 1_{\{|x|\le 1\}}.$$

In view of (2.5), the variables $|\Gamma(T)_t|$ are smaller than some constant M (since we have (DET)) for all t, T. Then after some computations we arrive at

$$\begin{aligned}|W(T,t,x)| &\le e^M \left(M^2\, |x|^2 + M|Y(t,x) - 1|\, |x|\right) 1_{\{|x|\le 1\}} + Y(t,x)\, 1_{\{|x|>1\}} \\ &\le (2 + (M + M^2)e^M)\, \psi_t(x). \end{aligned} \tag{4.3}$$

Observe now that

$$\begin{aligned}\int_0^{T_\star} U(T)_s ds &= \int_0^{T_\star} ds \int_{\mathbf{R}^d} V(T,s,x) F_s(dx) \\ &\le \int_0^{T_\star} ds \int_{\mathbf{R}^d} |V(T,s,x) + W(T,s,x)| F_s(dx) \\ &\qquad + \int_0^{T_\star} ds \int_{\mathbf{R}^d} |W(T,s,x)| F_s(dx),\end{aligned}$$

and we conclude the result by (3.4) and (4.2). □

Let J' be a subset of J which is at most countable and such that the closures of J and J' agree. When J itself is at most countable we may take $J' = J$ and the next lemma is obvious.

Lemma 4.3 *Under the assumptions of Theorem 4.1, we have* $\mathcal{Y}_m(J) = \mathcal{Y}_m(J')$.

Proof It is clearly enough to prove that if $(u, Y) \in \mathcal{Y}_m(J')$, then we also have $(u, Y) \in \mathcal{Y}_m(J)$. So we pick $(u, Y) \in \mathcal{Y}_m(J')$, and we use the notation $U(T)_t$, $\psi_t(x)$ and $W(T,t,x)$ of the previous lemma and set

$$U'(T)_t = a(T)_t + \Gamma(T)_t^\star c_t u_t + \int_{\mathbf{R}^d} W(T,t,x) F_t(dx).$$

Since J' is countable we deduce from (4.2) and Lemma 4.2 and (3.5) the existence of a $\mathbb{P}$-full set Ω_0 such that all $\omega \in \Omega_0$ satisfy

$$\int_0^{T_\star} dt \int_{\mathbf{R}^d} \psi_t(\omega, x) F_t(dx) < \infty, \tag{4.4}$$

$$T \in J' \quad \Rightarrow \quad \int_0^{T_\star} U(T)_t(\omega)\, dt < \infty. \tag{4.5}$$

$$D(\omega) := \{t \in I : U(T)_t(\omega) \neq -U'(T)_t(\omega) \text{ for some } T \in J'\} \tag{4.6}$$

is a Lebesgue-null set.

In the rest of the proof, the point $\omega \in \Omega_0$ is fixed and not mentioned.

By (2.5) and (2.7), the functions $T \mapsto a(T)_t$ and $T \mapsto \Gamma(T)_t$ are continuous on I. This, combined with (4.3) and (4.4), yields that

$$\int_0^{T_\star} \sup_{T \in I} |U'(T)_t|\, dt < \infty, \qquad T \mapsto U'(T)_t \text{ is continuous.} \tag{4.7}$$

Let now $T \in J \backslash J'$, and let T_n be a sequence in J', which converges to T. Since $\Gamma(T_n)_t \to \Gamma(T)_t$ Fatou's Lemma and $U(T_n)_t = -U'(T_n)_t \geq 0$ and (4.7) give us

$$\int_0^{T_\star} U(T)_t\, dt \leq \liminf_n \int_0^{T_\star} U(T_n)_t\, dt = \liminf_n \int_0^{T_\star} |U'(T_n)_t|\, dt < \infty. \tag{4.8}$$

Then Lemma 4.2 yields that $(u, Y) \in \mathcal{Y}(J)$.

It remains to prove that if $T \in J$, the set $\{t \in I : U(T)_t \neq -U'(T)_t\}$ has Lebesgue measure 0, and this is where the assumptions of Theorem 4.1 come in (recall that $\omega \in \Omega_0$ is fixed).

a) Suppose first that (BJ) holds, so in fact one can assume that the jumps of L are bounded by 1. In this case we have $U(T)_t = 0$ identically, and the result is a trivial consequence of (4.6) and (4.7).

b) Suppose next that (BJ) fails, but that $d = 1$. We will prove that $U(T)_t = -U'(T)_t$ for all $t \notin D(\omega)$, and this will be enough. So we fix $t \notin D(\omega)$ and $T \in J$ and as above we take a sequence $T_n \in J'$ converging to T. Up to taking a subsequence, we can assume that the sequence $\Gamma(T_n)_t$ is either increasing, or decreasing, to $\Gamma(T)_t$. Suppose for example that $\Gamma(T_n)_t \uparrow \Gamma(T)_t$, the other case is treated in a similar way. On the one hand we have by (4.7) and $t \notin D(\omega)$:

$$U(T_n)_t = -U'(T_n)_t \to -U'(T)_t. \tag{4.9}$$

On the other hand we have

$$U(T_n)_t = \int_{\{x<-1\}} e^{\Gamma(T_n)_t x}\, Y(t,x) F_t(dx) + \int_{\{x>1\}} e^{\Gamma(T_n)_t x}\, Y(t,x) F_t(dx).$$

The last integral above increases to $\int_{\{x>1\}} e^{\Gamma(T)_t x}\, Y(t,x) F_t(dx)$ by the monotone convergence theorem. If $h_T(x) = e^{\Gamma(T)_t x}\, Y(t,x)\, 1_{\{x<-1\}}$, then the functions h_{T_n} decrease pointwise to h_T and $h_{T_n} \leq h_{T_1}$, and $\int h_{T_1}(x) F_t(dx) \leq U(T_1)_t < \infty$. So it remains to apply the dominated convergence theorem to get that $U(T_n)_t \to U(T)_t$: comparing with (4.9) gives us $U(T)_t = -U'(T)_t$, and we are done. □

Proof of Theorem 4.1 From the previous lemma we can assume that J is at most countable. Suppose that $\mathcal{Q}_{\mathcal{F},loc}$ contains at least one point Q, associated with a pair $(u, Y) \in \mathcal{Y}_m(J)$. We have a $\mathbb{P}$-full set Ω_0 such that (4.4), (4.5) and (4.6) hold for all $\omega \in \Omega_0$. Then we pick an arbitrary ω_0 in Ω_0, and we set

$$u'_t(\omega) = u_t(\omega_0), \qquad Y'(\omega, t, x) = Y(\omega_0, t, x). \tag{4.10}$$

It is then obvious that the pair (u', Y') belongs to $\mathcal{Y}_m(J)$. Hence (u', Y') is associated with some $Q' \in \mathcal{Q}_{\mathcal{F},loc}$ whose density process w.r.t. $\mathbb{P}$ is $\mathcal{E}(M')$, where M' is given by (3.6) with u' and Y'. But now u' and Y' are non-random, hence M' is a PIIAC and an application of Girsanov's theorem readily yields that the characteristics of L under Q' are non-random as well: hence L is a PIIAC under Q', and we have $Q' \in \mathcal{Q}'_{\mathcal{F},loc}$.

Since $\mathcal{Q}'_{\mathcal{F},loc} \subset \mathcal{Q}_{\mathcal{F},loc}$, we deduce (a). As for (b), assume that there are two different measures Q and $\widetilde{Q}$ in $\mathcal{Q}_{\mathcal{F},loc}$. As above they are associated with two pairs (u, Y) and $(\widetilde{u}, \widetilde{Y})$ in $\mathcal{Y}_m(J)$, and two $\mathbb{P}$-full sets Ω_0 and $\widetilde{\Omega}_0$. Furthermore these two pairs are not equivalent in the sense of (3.3), otherwise the two measures Q and $\widetilde{Q}$ would agree. This means that we can find an $\omega_0 \in \Omega_0 \cap \widetilde{\Omega}_0$ such that (u', Y') given by (4.10) and $(\widetilde{u}', \widetilde{Y}')$ associated in a similar way with $(\widetilde{u}, \widetilde{Y})$ are not equivalent: so they give rise to two measures Q' and $\widetilde{Q}'$ in $\mathcal{Q}'_{\mathcal{F},loc}$ which are different: thus (b) is proved. □

Next, we have the:

Proposition 4.4 *Under (DET) we have $\mathcal{Q}'_{\mathcal{F},loc} = \mathcal{Q}'_{\mathcal{F}}$.*

Proof Let Q be in $\mathcal{Q}'_{\mathcal{F},loc}$. For each $T \in J$ the process $H(T)$ is a local martingale under Q, for the filtration $(\mathcal{F}_t)$. The formula (2.8) stays valid under Q, and L under Q is still a PIIAC. Since $\Gamma(T)$ and $A(T)$ are deterministic, it follows that $H(T)$ is also a PIIAC.

Recalling that $P(T)_t = P(T)_0 \mathcal{E}(H(T))_t$ and that $P(T)_0$ is deterministic, in order to prove the result it is clearly enough to show that if H is any PIIAC which is also a local martingale (on some probability basis $(\Omega, (\mathcal{F}_t), Q)$), then its Doléans exponential $Z = \mathcal{E}(H)$, which is a local martingale, is indeed a martingale.

Using the canonical decomposition of H, we can write it as $H = M + A + N$ where M is a square-integrable martingale and

$$\langle M, M\rangle_t = \int_0^t \rho_s ds, \quad A_t = \int_0^t \delta_s ds, \quad N_t = \sum_{s \le t} \Delta H_s 1_{\{|\Delta H_s|>1\}}. \quad (4.11)$$

Here ρ_s and δ_s are non-random locally integrable functions, and M and N are independent. Furthermore the compensator of the jump measure of H has the form $dt \times G_t(dx)$, and since H, hence $A + N$, are local martingales, we have in fact $\delta_s = -\int_{\{|x|>1\}} x G_s(dx)$ and that $\delta' = \int_0^{T^\star} ds \int_{\{|x|>1\}} |x| G_s(dx)$ is finite.

Now we let $H' = M + A$ and $Z' = \mathcal{E}(H')$ and $Z'' = \mathcal{E}(N)$. First, Z' is the solution of the SDE $dZ' = Z'_- dH'$ starting at $Z_0 = 1$, and in view of (4.11) it is well known (using Gronwall's Lemma for example) that $\mathbb{E}(\sup_{t \le T^\star} |Z'_t|) < \infty$. Next, we have $Z''_t = \prod_{s \le t, \Delta N_s \neq 0} (1 + \Delta N_s)$, so the fact that N is a non-homogeneous compound Poisson process with Lévy measure $dt \times G_t(dx) 1_{\{|x|>1\}}$ yields, if $\lambda_t = G_t(|x| > 1)$ and $\Lambda = \int_0^{T^\star} \lambda_s ds$:

$$\begin{aligned}\mathbb{E}(\sup_{t\leq T^\star}|Z''_t|) &\leq \mathbb{E}(\prod_{t\leq T^\star}|1+\Delta N_t|)\\ &= e^{-\Lambda}\sum_{n=0}^{\infty}\frac{\Lambda^n}{n!}\left(\frac{1}{\Lambda}\int_0^{T^\star}\lambda_s ds\left(1+\frac{1}{\lambda_s}\int_{\{|x|>1\}}|x|G_s(dx)\right)\right)^n\\ &= e^{-\Lambda}\sum_{n=0}^{\infty}\frac{\Lambda^n}{n!}\left(1+\frac{\delta'}{\Lambda}\right)^n = e^{\delta'}.\end{aligned}$$

Finally, we have $Z = Z'Z''$ and the processes Z' and Z'' are independent, so

$$\mathbb{E}(\sup_{t\leq T^\star}|Z_t|) \leq \mathbb{E}(\sup_{t\leq T^\star}|Z'_t|)\,\mathbb{E}(\sup_{t\leq T^\star}|Z''_t|) < \infty.$$

Therefore the local martingale Z is a martingale, and we are finished. □

Combining this with Theorem 4.1 gives us the following:

Theorem 4.5 *Assume (DET) and either (BJ) or* $d = 1$.

a) We have $\mathcal{Q}_{\mathcal{F},loc} = \emptyset$ *if and only if* $\mathcal{Q}'_{\mathcal{F}} = \emptyset$.
b) If the set $\mathcal{Q}_{\mathcal{F},loc}$ *contains more than one point, then so does the set* $\mathcal{Q}'_{\mathcal{F}}$.

5 Changing the filtration

As said at the beginning of the paper, the really meaningful set of martingale measures is $\mathcal{Q}_{\mathcal{G}}$, and we need to compare it with the previously studied sets of martingale measures like $\mathcal{Q}_{\mathcal{F}}$. Of course $\mathcal{Q}_{\mathcal{F}}$ and $\mathcal{Q}_{\mathcal{G}}$ are not immediately comparable, because they are sets of probability measures defined on different σ-fields. However, if we denote by $Q_{|\mathcal{G}}$ the restriction to $(\Omega,\mathcal{G})$ of any measure Q on $(\Omega,\mathcal{F})$, we have the property

$$Q \in \mathcal{Q}_{\mathcal{F}} \quad \Rightarrow \quad Q_{|\mathcal{G}} \in \mathcal{Q}_{\mathcal{G}}, \tag{5.1}$$

because any martingale w.r.t. some filtration is also a martingale w.r.t. any smaller filtration w.r.t. which it is adapted. Observe that the similar statement $Q \in \mathcal{Q}_{\mathcal{F},loc} \Rightarrow Q_{|\mathcal{G}} \in \mathcal{Q}_{\mathcal{G},loc}$ is not necessarily true. One can show, however, that it is true under (DET), or more generally when the processes $\Gamma(T)$ are predictable w.r.t. $(\mathcal{G}_t)$ for all $T \in J$.

Now (5.1) does not provide us with enough information on $\mathcal{Q}_{\mathcal{G}}$; for example, it says that if $\mathcal{Q}_{\mathcal{F}}$ is not empty, so is $\mathcal{Q}_{\mathcal{G}}$; but it does not say that if $\mathcal{Q}_{\mathcal{F}}$ is a singleton, then the same is true of $\mathcal{Q}_{\mathcal{G}}$ (because a measure in $\mathcal{Q}_{\mathcal{G}}$ might very well not be extendable as a measure on $(\Omega,\mathcal{F})$). However, we presently see that under (DET), and up to changing the driving process L, the two filtrations $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ can be considered as the same, and all the previous results apply. To be more precise, consider the (non-random, because of (DET)) linear subspace of $\mathbf{R}^d$:

$$E_t = \text{span}(\Gamma(T)_t : T \in J). \tag{5.2}$$

We denote by the same symbol Π_t the orthogonal projection over E_t (in $\mathbf{R}^d$) and the associated $d\times d$ matrix. Since $\Gamma(T)_t$ is continuous in T and measurable in t, the space E_t is also the linear space spanned by $\Gamma(T)_t$ where T varies in an at most countable subset J' of J, hence it is "measurable" in t, so the function $t\mapsto \Pi_t$, which can be considered as taking its values in the set of $d\times d$ matrices, is measurable as well. Then we can set

$$\overline{L}_t=\int_0^t \Pi_s dL_s.$$

The process $\overline{L}$ is again a PIIAC. Since $\Gamma(T)_t=\Pi_t\Gamma(T)_t$ for all $T\in J$ and $\Pi_t=\Pi_t^\star$, we can rewrite (2.6) as

$$P(T)_t=P(T)_0\ \exp\left(\int_0^t A(T)_s ds+\int_0^t \Gamma(T)_s^\star d\overline{L}_s\right). \tag{5.3}$$

Proposition 5.1 *Under (DET), the filtration $(\mathcal{G}_t)$ is equal to the filtration $(\overline{\mathcal{F}}_t)$ generated by the process $\overline{L}$ above.*

Proof The inclusion $\mathcal{G}_t\subset\overline{\mathcal{F}}_t$ follows from (5.3) (recall that $\Gamma(T)_t$ is non-random, here).

For the converse, and as seen before, the process $\overline{L}$ and hence the filtration $(\overline{\mathcal{F}}_t)$ do not change if we replace J by a countable subset J' which is dense in J, while going from J to J' just decreases each $\mathcal{G}_t$. So we can assume that J itself is countable, and we denote by $T_1,T_2,\dots$ an enumeration of the points T in J.

Note that the filtration $(\mathcal{G}_t)$ is also the filtration generated by the processes

$$Q(T)_t=\int_0^t \Gamma(T)_s^\star\, dL_s$$

for $T\in J$. Let $\kappa(t,i)$ be the smallest $j\geq 1$ such that the vector space spanned by $(\Gamma(T_k)_t: 1\leq k\leq j)$ has dimension i, with $\kappa(t,i)=\infty$ if there is none. Then $d_t=\inf(i:\kappa(t,i+1)=\infty)$ is the dimension of the space E_t. We write $G_{t,i}=(G^j_{t,i})_{1\leq j\leq d}=\Gamma(T_{\kappa(t,i)})_t$ for $i=1,\dots,d_t$. We also denote by B_t the symmetric $d_t\times d_t$ matrix with entries $b_t^{ij}=G^\star_{t,i}G_{t,j}$. Since E_t is generated by the d_t independent vectors $G_{i,t}$ for $i=1,\dots,d_t$, each B_t is positive-definite and we denote by $b_t^{\prime ij}$ the (i,j)th entry of its inverse. Then a simple computation shows that the matrix Π_t is

$$\Pi_t=\sum_{i,j=1}^{d_t} b_t^{\prime ij}\, G_{t,i}\, G^\star_{t,j}.$$

At this stage, we can write

$$\begin{aligned}\overline{L}^i_t&=\sum_{j=1}^d\int_0^t \Pi_s^{ij} dL_s^j\\ &=\sum_{j=1}^d\sum_{r=1}^d\int_0^t 1_{\{d_s=r\}}\sum_{k,l=1}^r b_s^{\prime kl}G^i_{s,k}G^j_{s,l}\, dL^j_s\end{aligned}$$

$$= \sum_{r=1}^{d} \sum_{k,l=1}^{r} \sum_{m=l}^{\infty} \sum_{j=1}^{d} \int_0^t 1_{\{d_s=r\}} b_s'^{kl} G^i_{s,k} 1_{\{\kappa(s,l)=m\}} \Gamma(T_m)^j_s \, dL^j_s$$

$$= \sum_{r=1}^{d} \sum_{k,l=1}^{r} \sum_{m=l}^{\infty} \int_0^t 1_{\{d_s=r\}} b_s'^{kl} G^i_{s,k} 1_{\{\kappa(s,l)=m\}} \, dQ(T_m)_s.$$

Since the integrands above are all deterministic, it is clear that $\overline{L}^i_t$ is adapted to the filtration generated by the processes $Q(T_m)$ for $m \geq 1$, that is to $(\mathcal{G}_t)$. □

This proposition allows for a comparison between $\mathcal{Q}_{\mathcal{F}}$ and $\mathcal{Q}_{\mathcal{G}}$, under (DET). Indeed, let $L' = L - \overline{L}$. If L' is trivial (identically 0, or more generally non-random) then $\mathcal{F} = \mathcal{G}$. Otherwise the inclusion $\mathcal{G} \subset \mathcal{F}$ is strict and, as soon as $\mathcal{Q}_{\mathcal{G}}$ contains at least one point Q one can use Girsanov's theorem to construct other measures Q' which coincide with Q on $\mathcal{G}$ and change L' in an arbitrary way: then we have a whole family of measures in $\mathcal{Q}_{\mathcal{F}}$ whose restrictions to the σ-field $\mathcal{G}$ equal Q. That is, even when $\mathcal{Q}_{\mathcal{G}}$ is a singleton (the ideal case where we have no-arbitrage and completeness), then $\mathcal{Q}_{\mathcal{F}}$ is *not* a singleton.

On the other hand, this proposition and (5.3) show that under (DET) we can consider that the model is driven by a PIIAC which generates the same filtration as the price processes $P(T)$ for $T \in J$ (but of course, even if L is a Lévy process under $\mathbb{P}$, the new process $\overline{L}$ is a PIIAC but not necessarily Lévy: this is why we have considered above models driven by PIIAC's). Therefore, denoting by $\mathcal{Q}'_{\mathcal{G},loc}$ and $\mathcal{Q}'_{\mathcal{G}}$ the sets of all equivalent martingales measures on $(\Omega, (\mathcal{G}_t), \mathcal{G})$ such that our new driving process $\overline{L}$ is still a PIIAC, Theorem 4.5 gives us:

Theorem 5.2 *Assume (DET) and either (BJ) or $d = 1$, and also that J is dense in I.*

a) We have $\mathcal{Q}_{\mathcal{G},loc} = \emptyset$ if and only if $\mathcal{Q}'_{\mathcal{G}} = \emptyset$.
b) If the set $\mathcal{Q}_{\mathcal{G},loc}$ contains more than one point, then so does the set $\mathcal{Q}'_{\mathcal{G}}$.

In fact one can do a little bit better: namely this theorem holds without (BJ), even when $d \geq 2$, provided the dimension of the vector space E_t is 0 or 1 for Lebesgue-almost all t.

6 Uniqueness of the equivalent martingale measures

Now we take up the problem of uniqueness of the equivalent (local) martingale measure, assuming of course that there exists at least one. As said before, we can assume without restriction that $\int_{\mathbf{R}^d} (|x|^2 \wedge 1) F_t(dx) < \infty$ for all t.

If $u \in \mathbf{R}^d$ and $t \in I$, we denote by $\mathcal{Y}_{t,u}$ the set of all pairs (v, f), where $v \in \mathbf{R}^d$ and f is a Borel function on $\mathbf{R}^d$ satisfying the following three conditions:

$$\rho_t(f) := \int \left(|f(x)| \wedge f(x)^2\right) F_t(dx) < \infty, \tag{6.1}$$

$$\eta_t(u,f) := \int_{\{|x|>1\}} |f(x)| e^{u^\star x} F_t(dx) < \infty, \tag{6.2}$$

$$u^\star\, c_t\, v + \int \left(e^{u^\star x} - 1\right) f(x) F_t(dx) = 0. \tag{6.3}$$

Observe that under (6.1) we have

$$\int |f(x)| \left(|x| 1_{\{|x|\le 1\}} + 1_{\{|x|>1\}}\right) F_t(dx) < \infty, \tag{6.4}$$

hence (6.1) and (6.2) imply

$$\int |f(x)(e^{u^\star x} - 1)| F_t(dx) < \infty, \tag{6.5}$$

and the integral in (6.3) converges.

Denote by $\mathcal{Y}'_{t,u}$ the subset of $\mathcal{Y}_{t,u}$ consisting of all pairs (v,f) satisfying

$$v^\star c_t v + \int |f(x)| F_t(dx) > 0. \tag{6.6}$$

For any subset $J \subset I$ we put

$$\mathcal{U}_J = \{(\omega,t) : \bigcap_{T\in J} \mathcal{Y}'_{t,\Gamma(T)_t(\omega)} \neq \emptyset\}. \tag{6.7}$$

The following can be viewed as a version of Proposition 6.4 of [1]; the formulation is a bit different, because we do not impose Assumption 6.3 of [1] (here, Assumption 6.1 of [1] is automatically fulfilled).

Theorem 6.1 *If $m(\mathcal{U}_{J'}) = 0$ for some countable subset $J' \subset J$, the set $\mathcal{Q}_{\mathcal{F},loc}$ of equivalent local martingale measures contains at most one point.*

Proof Assume that there are two different equivalent local martingale measures, corresponding to the two pairs (u,Y) and (u',Y') in $\mathcal{Y}_m(J)$, and set

$$C_T = \{(\omega,t) : (u_t(\omega) - u'_t(\omega), Y(\omega,t,.) - Y'(\omega,t,.)) \in \mathcal{Y}_{t,\Gamma(T)_t(\omega)}\},$$

$$B = \Big\{(\omega,t) : (u_t(\omega) - u'_t(\omega))^\star c_t (u_t(\omega) - u'_t(\omega)) + \int F_t(dx) |Y(\omega,t,x) - Y'(\omega,t,x)| > 0\Big\}.$$

By Theorem 3.1 we have $m(C_T^c) = 0$ for all T, hence the complement of $C = \bigcap_{T\in J'} C_T$ is m-negligible. On the other hand the fact that the two equivalent martingale measures are distinct yields $m(B) > 0$, hence $m(B \cap C) > 0$.

Now if $(\omega,t) \in B \cap C$, we obviously have $(\omega,t) \in \mathcal{U}_{J'}$, hence $m(\mathcal{U}_{J'}) > 0$. □

Lemma 6.2 *Let U be a subset of $\mathbf{R}^d$ whose closure has a nonempty interior, and let $t \in I$. Then the set $\bigcap_{u\in U} \mathcal{Y}'_{t,u}$ is empty.*

Proof Let $(v,f) \in \bigcap_{u\in U} \mathcal{Y}'_{t,u}$.

Let u_0 be in U and in the interior of its closure $\overline{U}$. Then $V = \{u - u_0 : u \in U\}$ contains 0 and its closure $\overline{V}$ is a neighborhood of 0. From (6.2) and (6.4) we get for all $\lambda \in V$:

$$\int \left| f(x)\, e^{u_0^\star x} \left(|x| 1_{\{|x|\le 1\}} + e^{\lambda^\star x}\, 1_{\{|x|>1\}} \right) \right| F_t(dx) < \infty$$

Therefore, if f^+ and f^- denote the positive and negative parts of f, the two measures F_+ and F_- having densities $f^+(x)e^{u_0^\star x}$ and $f^-(x)e^{u_0^\star x}$ respectively w.r.t. F_t are positive measures on $\mathbf{R}^d \backslash \{0\}$, such that

$$\int \left(|x| 1_{\{|x|\le 1\}} + e^{\lambda^\star x} 1_{\{|x|>1\}} \right) F_\pm(dx) < \infty \tag{6.8}$$

for all $\lambda \in V$. Moreover, (6.3) for $u = u_0 + \lambda$ can be written as

$$u_0^\star c_t v + \lambda^\star c_t v + \int \left(e^{\lambda^\star x} - 1 \right) f(x) e^{u_0^\star x}\, F_t(dx) + \int \left(e^{u_0^\star x} - 1 \right) f(x) F_t(dx) = 0,$$

where the two integrals above converge, from what precedes. This holds in particular for $\lambda = 0$. Since $f(x)e^{u_0^\star x}\, F_t(dx) = F_+(dx) - F_-(dx)$, and if we set $\delta_\pm = \int x 1_{\{|x|\le 1\}}\, F_\pm(dx)$, we thus deduce

$$\begin{aligned} \lambda^\star(c_t v + \delta_+ - \delta_-) &+ \int \left(e^{\lambda^\star x} - 1 - \lambda^\star x 1_{\{|x|\le 1\}} \right) F_+(dx) \\ &= \int \left(e^{\lambda^\star x} - 1 - \lambda^\star x 1_{\{|x|\le 1\}} \right) F_-(dx) \end{aligned} \tag{6.9}$$

for all $\lambda \in V$.

Now we can find 2^d vectors $\lambda_1, \ldots, \lambda_{2^d}$ in V, each one being in one of the 2^d quadrants of $\mathbf{R}^d$, and such that each coordinate of each λ_i is bigger in absolute value than ε, for some $\varepsilon > 0$. Then obviously if x is in the same quadrant as λ_i we have $\varepsilon|x| \le \lambda_i^\star x$, hence $e^{\varepsilon|x|} \le \sum_{i=1}^{2^d} e^{\lambda_i^\star x}$, which by (6.8) is integrable on the set $\{|x| > 1\}$ w.r.t. both F_+ and F_-. That is, we have

$$\int \left(|x| 1_{\{|x|\le 1\}} + e^{\varepsilon|x|} 1_{\{|x|>1\}} \right) F_\pm(dx) < \infty. \tag{6.10}$$

Now let W be the ball centered at 0 and with radius ε. By virtue of (6.10) and of the dominated convergence theorem, both sides of the identity (6.9) are continuous functions of λ over W, and they agree on the set $V \cap W$ which is dense in W. Therefore (6.9) holds for all $\lambda \in W$.

Finally the left (resp. right) side of (6.9), as a function of λ, is the logarithm of the moment generating function of an infinitely divisible law with Lévy measure F_+ (resp. F_-) and drift $c_t v + \delta_+ - \delta_-$ (resp. 0) and vanishing Gaussian component. The moment generating function restricted to a neighborhood of 0 on which it is finite completely characterizes the law. Hence we deduce that $F_+ = F_-$ and $c_t v + \delta_+ - \delta_- = 0$. This implies $f = 0$ F_t-a.e., hence $\delta_+ = \delta_- = 0$, hence $c_t v = 0$. So (6.6) fails, implying that indeed there is no pair (v,f) in $\cap_{u\in U} \mathcal{Y}'_{t,u}$. □

Now we are ready to give a final answer for the 1-dimensional case.

Theorem 6.3 *Assume that $d = 1$ and that J is dense in I, and set*

$$H = \{(\omega, t) \in \Omega \times I : \int_t^{T_\star} |\gamma(\omega, t, s)| \, ds = 0, \ c_t + F_t(\mathbf{R}) > 0\}. \tag{6.11}$$

Then the set $\mathcal{Q}_{\mathcal{F},loc}$ of equivalent local martingale measures contains at most one point if and only if we have $m(H) = 0$.

Proof Assume first that $m(H) = 0$. Let J' be a countable subset of J which is also dense in I, and pick any (ω, t) in the complement of H, with $t < T_\star$. Let us show that

$$\bigcap_{T \in J'} \mathcal{Y}'_{t, \Gamma(T)_t(\omega)} = \emptyset. \tag{6.12}$$

There are indeed two possibilities:

a) Either $c_t + F_t(\mathbf{R}) = 0$: then obviously $\mathcal{Y}'_{t,u} = \emptyset$ for all $u \in \mathbf{R}$; therefore (6.12) holds.

b) Or $c_t + F_t(\mathbf{R}) > 0$ and $\int_t^{T_\star} |\gamma(\omega, t, s)| ds > 0$. Since $T \mapsto \Gamma(T)_t(\omega)$ is continuous by (2.5) and (2.7), the closure $\overline{U}$ of the set $U = \{\Gamma(T)_t(\omega) : T \in J'\}$ is a closed interval of $\mathbf{R}$, which necessarily contains 0 because $\Gamma(t)_t = 0$. Now, if $\overline{U} = \{0\}$ we have $\Gamma(T)_t(\omega) = 0$ for all $T \leq T_\star$, which in view of (2.7) implies $\int_t^{T_\star} |\gamma(\omega, t, s)| ds = 0$. This contradicts the assumption, hence the interval $\overline{U}$ has a positive length, so its interior is nonempty. By virtue of Lemma 6.2 we deduce again that (6.12) holds.

In other words, in all cases we have (6.12), and the fact that $\mathcal{Q}_{\mathcal{F},loc}$ contains at most one point follows from Theorem 6.1.

Conversely, assume $m(H) > 0$ and also that the set $\mathcal{Q}_{\mathcal{F},loc}$ contains at least a measure Q, which is associated with a pair $(u, Y) \in \mathcal{Y}_m(J)$. Then we put

$$u'_t(\omega) = \begin{cases} u_t(\omega) & \text{if } (\omega, t) \notin H \\ u_t(\omega) + 1 & \text{if } (\omega, t) \in H \end{cases}$$

$$Y'(\omega, t, x) = \begin{cases} Y(\omega, t, x) & \text{if } (\omega, t) \notin H \\ Y(\omega, t, x) + f_t(x) & \text{if } (\omega, t) \in H \end{cases}$$

where f_t is a collection of functions, measurable in the pair (t, x), and satisfying $\rho_t(f_t) \leq 1$ (see (6.1)), and also $\int |f_t(x)| F_t(dx) > 0$ whenever $F_t \neq 0$.

If $(\omega, t) \in H$ then $\Gamma(T)_t(\omega) = 0$ for all T, so one can check that the pair (u', Y') also belongs to $\mathcal{Y}_m(J)$ (actually, (3.4) and (3.5) are trivially fulfilled, while (3.1) and (3.2) follow easily from (2.2) and from the corresponding statements for u and Y). Furthermore if $(\omega, t) \in H$ we also have $c_t + F_t(\mathbf{R}) > 0$, so if $m(H) > 0$ the two pairs (u, Y) and (u', Y') will not satisfy (3.3), and (u', Y') is thus associated with a $Q' \in \mathcal{Q}_{\mathcal{F},loc}$ which does not agree with Q. □

The condition $m(H) = 0$ is a sort of *non-degeneracy condition* which is indeed very weak.

Note that this theorem is concerned with the filtration $(\mathcal{F}_t)$. Under (DET), and using the results of the previous section, we also get results concerning the (more interesting) filtration $(\mathcal{G}_t)$, and surprisingly those results are stronger: actually we need no non-degeneracy condition at all. The next theorem can be considered as the main result of this paper.

Theorem 6.4 *Assume (DET) and $d = 1$ and that J is dense in I. Then:*

(a) The three sets $\mathcal{Q}_{\mathcal{G},loc}$, $\mathcal{Q}_{\mathcal{G}}$ and $\mathcal{Q}'_{\mathcal{G}}$ are equal, and they are either empty, or a singleton $\{Q\}$.

(b) In order that $\mathcal{Q}_{\mathcal{G}}$ be not empty, it is necessary and sufficient that there exists a non-random *pair (u, Y) in $\mathcal{Y}_m(J)$ (that is, which satisfies (3.1), (3.2), (3.4) and (3.5)). In this case, Q can be extended (not necessarily in a unique way) to a measure in $\mathcal{Q}'_{\mathcal{F}}$, and the price processes $P(T)$ are the exponentials of processes with independent increments under Q as well as under $\mathbb{P}$.*

Finally (a) remains true if $d \geq 2$, as soon as the dimension d_t of the vector space E_t of (5.2) has $d_t \leq 1$ for all t.

Proof 1) The sufficient condition in (b) needs (DET), but neither $d = 1$ nor the denseness of J, and its proof goes as follows: if $\mathcal{Y}_m(J)$ contains a *non-random* pair (u, Y), the set $\mathcal{Q}_{\mathcal{F},loc}$ contains an associated measure Q; in the decomposition (3.7) we see that b' and ν' are deterministic, so L is a PIIAC under Q and thus $Q \in \mathcal{Q}'_{\mathcal{F},loc}$. By Proposition 4.4 we have $Q \in \mathcal{Q}_{\mathcal{F}}$, and (5.1) yields $Q_{|\mathcal{G}} \in \mathcal{Q}_{\mathcal{G}}$.

2) Assume now $d = 1$. Let us consider the PIIAC $\overline{L}$ of the previous section. Since $d = 1$, it is clear that the matrix Π_t, which here is a number, is equal to $1_D(t)$, where D is the set of all $t \in I$ such that $\int_t^{T_\star} |\gamma(t, s)| ds > 0$. So the characteristics of $\overline{L}$ are

$$(\overline{b}_t, \overline{c}_t, \overline{F}_t) = \begin{cases} (b_t, c_t, F_t) & \text{if } t \in D \\ (0, 0, 0) & \text{otherwise} \end{cases} \tag{6.13}$$

Proposition 5.1 implies that $(\mathcal{G}_t)$ is the filtration generated by $\overline{L}$. Then Theorem 4.5 applied with $\overline{L}$ instead of L gives that either $\mathcal{Q}_{\mathcal{G},loc} = \mathcal{Q}_{\mathcal{G}} = \mathcal{Q}'_{\mathcal{G}} = \emptyset$, or $\mathcal{Q}_{\mathcal{G},loc} = \mathcal{Q}_{\mathcal{G}} = \mathcal{Q}'_{\mathcal{G}} = \{Q\}$ is a singleton, or $\mathcal{Q}'_{\mathcal{G}}$ contains at least two measures. However, the latter case would contradict Theorem 6.3 with $\overline{L}$ instead of L, because in view of (6.13) and of the definition of D the set H associated by (6.11) is actually empty. Therefore we have (a).

Next we prove the necessary condition in (b): assume that $\mathcal{Q}'_{\mathcal{G}}$ contains a measure Q. By Theorem 3.1 we have a pair $(\overline{u}, \overline{Y})$ associated with Q and belonging to the set $\overline{\mathcal{Y}}_m(J)$ corresponding to the characteristics of $\overline{L}$. Since $\overline{L}$ is a PIIAC w.r.t. Q, we have that $(\overline{u}, \overline{Y})$ is non-random. Then if we set $u_t = \overline{u}_t$ and $Y(t, x) = \overline{Y}(t, x)$ when $t \in D$ and $u_t = 0$ and $Y(t, x) = 0$ elsewhere, it is obvious (because $\gamma(t, T) = 0$ for dT-almost all T if $t \notin D$) that the pair (u, Y) belongs to $\mathcal{Y}_m(J)$, and is obviously non-random: so we have the necessary condition. Furthermore by 1) above, (u, Y) is associated with some $Q' \in \mathcal{Q}'_{\mathcal{F}}$, and it is obvious from the construction of (u, Y) and from the fact that $\overline{L}_t = \int_0^t 1_D(s) dL_s$ that the process $\overline{L}$ has the same characteristics, hence also the same law, under Q and Q': this means that $Q = Q'_{|\mathcal{G}}$: therefore we have the first part of the second claim in (b), while the second part of it is trivial.

3) It remains to prove the last claim, when $d \geq 2$ and $d_t \leq 1$ for all t. We can find vectors $\delta_t = (\delta_t^i)_{1 \leq i \leq d}$ such that $\delta_t = 0$ if $d_t = 0$ and $|\delta_t| = 1$ if $d_t = 1$, and $\delta_t \in E_t$ for all t, and moreover such that $t \mapsto \delta_t$ is measurable. Since $\Gamma(T)_t \in E_t$ for all T we can write $\Gamma(T)_t = \Gamma(T)'_t \delta_t$ for some 1-dimensional processes $\Gamma(T)'_t$. Next, we define a 1-dimensional PIIAC L' by

$$L'_t = \sum_{i=1}^d \int_0^t \delta_s^i d\overline{L}_s^i.$$

In view of (5.3) we have

$$P(T)_t = P(T)_0 \ \exp\left(\int_0^t A(T)_s ds + \int_0^t \Gamma(T)'_s dL'_s \right).$$

So our model reduces to a 1-dimensional model, and the last claim follows from the statement in (a). □

Now in the multi-dimensional case ($d \geq 2$) things are more complicated because the set $\{\Gamma(T)_t(\omega) : T > t\}$ is a continuous curve in $\mathbf{R}^d$ and the interior of its closure is in general empty: so Lemma 6.2 does not apply here. The “rule” in that case is that the sets $\mathcal{Q}_{\mathcal{G},loc}$ and $\mathcal{Q}_{\mathcal{F},loc}$ are infinite as soon as they are not empty. To see this, we will give a simple example.

An example We assume here that $d = 2$ and $T_\star = 1$ and $c_t = 0$ and that $F_t = F$ is the probability measure on $\mathbf{R}^2$ having the following density:

$$f(x_1, x_2) = \frac{1}{\sqrt{\pi x_1}} \ e^{-2x_1 - x_2^2/4x_1} \ 1_{\mathbf{R}_+}(x_1).$$

Finally we take $b_t = \int x 1_{\{|x| \leq 1\}} F(dx)$. That is, under $\mathbb{P}$ the process L is a compound Poisson process with rate 1 and F is the law of the jumps of L.

More generally, for any $\beta > 0$ we consider the probability measure F_β having the density

$$f_\beta(x_1, x_2) = \frac{\beta + 1}{2\sqrt{\pi \beta x_1}} \ e^{-(\beta+1)x_1 - x_2^2/4\beta x_1} \ 1_{\mathbf{R}_+}(x_1).$$

We denote by $\mathbb{P}_\beta$ the law which makes L a compound Poisson process with rate 1 and law of jumps F_β. Of course $f = f_1$ and $\mathbb{P} = \mathbb{P}_1$. We observe that the measures $\mathbb{P}_\beta$ are all equivalent on the σ-field $\mathcal{F} = \mathcal{F}_{T_\star}$. It is clear that F_β is the law of a 2-dimensional variable (X_1, X_2) with X_1 being exponential with parameter $1 + \beta$, and X_2 being, conditionally on X_1, centered normal with variance $2\beta X_1$. Then

$$\int e^{u^2 x_1 + u x_2} F_\beta(dx_1, dx_2) = \frac{1}{1 - u^2} \tag{6.14}$$

whenever $u \in (-1, 1)$.

Next, we need to specify the coefficients. We take

$$\gamma(t,T)^i = \begin{cases} -(T-t)^3 1_{\{T>t\}} & \text{if } i=1 \\ -(T-t)1_{\{T>t\}} & \text{if } i=2, \end{cases}$$

which implies that $\Gamma(T)^1_t = (\Gamma(T)^2_t)^2 = \frac{1}{4}(T-t)^4$ if $t < T$ (recall (2.7)). Therefore the vector spaces E_t are all equal to $\mathbf{R}^2$ (when $t < T_\star$), so the filtration $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ are the same.

In view of (6.14) we see that (3.9) holds for any $\beta > 0$. It remains to choose the coefficient α, in such a way that there exists an equivalent martingale measure. In fact we will impose (3.10), which indeed holds simultaneously for all $\beta > 0$ as soon as $A(T)_t = -1/(1 - \frac{1}{4}(T-t)^4)$, that is if

$$\alpha(t,T) = \frac{(T-t)^3}{(1-\frac{1}{4}(T-t)^4)^2} 1_{\{T>t\}}.$$

In other words, we have $\mathbb{P}_\beta \in \mathcal{Q}_{\mathcal{G},loc}$ for all $\beta > 0$. Finally, it is easy to check that in this case the measures $\mathbb{P}_\beta$ are indeed in $\mathcal{Q}_\mathcal{G}$ (and even in $\mathcal{Q}'_\mathcal{G}$) here.

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