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DISCUSSION FORUM

Accuracy of single measurements

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Abstract Single measurements are widely used in industry, trade and science, yet the problem of the estimation of the accuracy of this type of measurements is neither addressed nor even recognized in traditional Metrology. In particular, the *Guide to the Expression of Uncertainty in Measurement* is devoted to multiple measurements only and does not mention single measurements. This paper studies the problem of estimating the inaccuracy of single measurements and describes solutions to this problem. The proposed methods are based on metrological characteristics of measuring instruments rated in accordance with Recommendation R34 of International Organization of Legal Metrology (OIML). These characteristics are usually given in manufacturer certificates or provided by calibration laboratories. This paper treats single measurements as the basic type of measurement and multiple measurements as sets of successive single measurements.

Keywords Single measurement · Measuring instrument · Inaccuracy · Uncertainty · Limits of error

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Introduction

The quality of a measurement is determined by its accuracy, which also often determines the measurement cost. Accuracy is a ''positive'' characteristic of the measurement but in reality is usually expressed with the help of a ''negative'' characteristic—inaccuracy—of that measurement. The importance of knowing the accuracy of a measurement is universally recognized. A knowledge of measurement accuracy is required to compare the results of measurements of the same physical quantity obtained by different operators, to define safe ranges for parameters of technological processes, and to estimate the reliability of product quality control, as well as in almost all other applications of measurements. It is also necessary at the stage of planning a technological process and choosing proper measuring instruments for that process.

Single measurements are the most common type of measurements used in industry and trade. However, the problem of the estimation of the accuracy of this type of measurement is neither addressed nor even recognized in traditional metrology. In particular, the *Guide to the Expression of Uncertainty in Measurement* (GUM) [1] is devoted to multiple measurements only and does not mention single measurements.

Methods for calculating the inaccuracy of these measurements are described in *Measurement errors and uncertainties: theory and practice* [2] (and its earlier editions). These methods solve the problem of the estimation of the accuracy of simple measurements but would be more practical and more widely used in the form of step-by-step recommendations. Moreover, single measurements must also be considered as the basic type of measurement because every multiple measurement (with the exception of measurements in calibration) is in essence a set of repeated single measurements. Therefore, the inaccuracy of single measurements forms an inevitable part of the inaccuracy of multiple measurements, and must be taken into consideration in the calculation of the inaccuracy of the latter. Thus, a practical recommendation for estimating single measurement inaccuracy would be useful not only for applications in industry and trade but also in science, where multiple measurements are common.

The current paper presents first steps toward such a recommendation, based on relevant parts of the general theory of single measurements developed in my previous work. In doing so, the current paper demonstrates that such a recommendation is both necessary and possible. Of course, the actual recommendation needs to be developed by a standard body, and needs to be more detailed and illustrated with examples. This future recommendation could be incorporated into GUM [1].

Basic concepts

A *single measurement* is a measurement in which the measuring instrument comes into contact with a physical object only once, and only one reading is taken to obtain the result of a measurement. Sometimes the above-mentioned contact is repeated two or three times but the goal of these extra measurements is to avoid a blunder or to be sure that the model of the physical object under study (e.g., the assumption that the object is round so that it can be characterized by its diameter) is applicable within the required accuracy of the measurement.

The *inaccuracy* of a measurement is usually expressed as an interval that covers the true value of the measurand. The half-width of this interval is called *uncertainty* if it is expressed as a confidence interval (i.e., that the interval covers the true value with a certain probability) and as *limits of error* if it does not have connection with any probability.

The accuracy of a single measurement is determined mainly by the accuracy of the measuring instrument(s) involved. The latter is determined by the rated metrological characteristics of this instrument(s) as listed in the manufacturer's certificates, specifications, etc., and by the environmental conditions under which the measurement is made. These characteristics are rated according to rules given by International Organization of Legal Metrology (OIML) Recommendation R34 [3].

The environmental condition under which no influence quantity disturbs the indication of a measuring instrument is called the reference condition of the measuring instrument. This reference condition provides the possibility of realizing the highest accuracy of the measuring instrument. The error of a measuring instrument under the reference condition is called the *intrinsic error* of the instrument. The intrinsic error is rated in the form of its permissible limits, which may be verified or provided by calibration laboratories.

Besides intrinsic errors, rated metrological characteristics include additional errors that reflect the effect of *influence quantities*. Sometimes, instead of the additional error caused by an influence quantity, the *influence function* of this quantity is given. If the influence function is linear, it is usually replaced by the corresponding *influence coefficient*. The influence function is described as a nominal function with permissible deviation. Accordingly, the influence coefficient is also specified as a nominal value with its permissible deviation. The influence function or the influence coefficient and the limits of their inaccuracy along with the value of the influence quantity allows the experimenter to make corrections to the reading of a measuring instrument and to calculate the limits of error of this correction.

A measurement error may consist of a certain number of component errors, which in turn may be divided into finer components. The component errors that cannot be further subdivided are called *elementary errors*. A typical case, and the one we will most assume in this paper, is one-level components, where a measurement error consists of a certain number of elementary errors.

Accuracy of a single measurement involving a single measuring instrument

The great majority of measuring instruments were created for single measurements. Some of these instruments are so simple that the inaccuracy of the corresponding measurements can be estimated without calculation. For example, the inaccuracy of the length measurement performed with a ruler is determined simply by rounding off the readings on the ruler. Also, calculating the inaccuracy of a measurement is not necessary when it is known beforehand that the accuracy of that measurement will be ''good enough'' for the goal of this measurement. This includes most household measurements, such as weighing ingredients for a cooking recipe or measuring the voltage of a car battery with an industrial tester. In other measurements, the inaccuracy must be calculated. This can be accomplished by the following step-by-step procedure.

1. Identify all possible sources of elementary errors. This list always includes the intrinsic error of the measuring instrument and the time at which the last calibration of the instrument was performed. Other sources include influence quantities whose values fall outside the limits of the reference condition, the interaction between the measuring instrument and the object whose parameter

is being measured, the discrepancy between the object and its model, and so on.

2. Estimate all elementary errors. Note that for some measuring instruments, their intrinsic errors depend on the reading point, i.e., on the measurand value indicated on the device. Frequently, the intrinsic error is rated assuming a particular reading point, usually the one with the smallest error. In these cases, the intrinsic errors must be recalculated to the reading point of the measuring instrument in the actual measurement in question. Elementary errors are usually estimated in the form of their limits but, in some cases, point estimations can be found. For example, point estimations are possible when one knows both the influence function or influence coefficient of the measuring instrument and the actual value of the influence quantity. The concrete value of the error provides the possibility of correcting the reading of the instrument. However, the inaccuracy of the correction must then be taken into consideration as an elementary error; this inaccuracy is described by the limits of the residual error. As a result of the above, we obtain the limits of all elementary errors. Notably, this includes random elementary errors such as the rounding error or the variation in the indication of a measuring instrument, which are also expressed in this case as permissible limits rather than as standard deviation. Thus, the resulting error of a single measurement is the sum of all elementary errors presented by their limits.
3. Express the intrinsic error and all the elementary errors in the same form, either as absolute or relative errors.
4. Calculate the inaccuracy of the measurement result. This inaccuracy is usually calculated not for the actual measuring instrument involved but rather for any instrument of the same type.

In the simplest case, i.e., a measurement under reference conditions, the inaccuracy of the measurement result is equal to the intrinsic error θ_0 of the measuring instrument involved, which, in accordance with OIML Recommendation R34 [3] is expressed as the limits of error. In other cases, the inaccuracy must be calculated as the sum of all the elementary errors and is typically expressed as uncertainty. I will next describe the procedure for this calculation.

Let ζ be the measurement error, ζ_0 be the intrinsic error of the measuring instrument, and ζ_i, i = 1,...,m, be the additional errors. We have:

$$\zeta = \zeta_0 + \sum_{i=1}^{m} \zeta_i. \qquad (1)$$

The problem is to find the limits for ζ given that we know the limits θ_0 and θ_i of the elementary errors:

$$|\zeta_0| \leqslant \theta_0 \quad \text{and} \quad |\zeta_i| \leqslant \theta_i.$$

Formally, the worst-case limits of the resulting error would be obtained as the arithmetical sum of all elementary errors limits. However, this worst-case error could occur only if all elementary errors simultaneously reach their upper or lower limits, which is practically impossible. A realistic solution to this problem is provided by a probabilistic approach. For this purpose, a mathematical model of elementary errors is needed. It is possible to construct such a model because, in addition to the known limits of elementary errors, we also know that, even though *all* measuring instruments of the same type have the same *limits* of their error, the actual error of each particular instrument may be different. Therefore, we can consider these errors as random quantities. Unfortunately, the distribution function of these errors cannot be obtained experimentally because this function is unstable [2]. However, given that random quantities representing these errors have known limits, in accordance with Shannon's theory of information, we can assume that this random quantity has a uniform (rectangular) distribution function. This is a conservative assumption because it produces the distribution with the greatest uncertainty (in the common sense of the word). A random quantity with uniform distribution is now widely used as the mathematical model of errors rated by their limits.

With the above assumption that the components on the right-hand side of Eq. (1) are uniformly distributed, if the number of these components grows to infinity, the resulting distribution approaches the normal distribution according to Central Limit Theorem. In practice, the resulting distribution can be assumed to be normal when the number of components is five or more. Then, the estimate of its variance, S_{rd}^2, based on the rated elementary errors, is

$$S_{\mathrm{rd}}^2 = \theta_0^2/3 + 1/3\sum_{i=1}^{m} \theta_i^2.$$

The limits of the uncertainty of a measurement are equal to the limits of the confidence interval for the true value of the measurand. For the normal distribution function and the confidence probability P = 0.95, as is well known, these limits are $u_{0.95}$ = 1.96 S_{rd}. For P = 0.99, the uncertainty is $u_{0.99}$ = 2.58 S_{rd}. Those limits define the uncertainty of a measurement. If the number of items in Eq. 1 is less than 5, the resulting error limits are given by the formula in Eq. 2:

$$u_p = k_p\sqrt{\left(\theta_0^2 + \sum_{i=1}^{m} \theta_i^2\right)}. \qquad (2)$$

This formula was proposed and studied by Rabinovich [2]. It was shown there that, for the most common confidence probability $P = 0.95$, the coefficient $k_{0.95} = 1.1$ and, remarkably, its value is independent of the number of items $n = m + 1$. The inaccuracy of this calculation is less then 2%. On the other hand, for $P = 0.99$, if $k_{0.99} = 1.4$ and is assumed constant, the inaccuracy of the calculation is +9% for $n = 2$ and –6% for an infinite number of items. Thus, in this case, the number of items influences the coefficient k_{p}. This shortcoming can be easily removed as shown by Rabinovich [2]. However, when $n = 2$ and $P = 0.99$, the sum of elementary errors is actually better obtained simply as the arithmetical sum of their limits. This simplifies the calculation and avoids overestimation of the inaccuracy that results from the above calculations in this case, especially when the limits of the elementary errors are not equal. In this case, as in the case in which inaccuracy is determined by the intrinsic error only, the inaccuracy is called *limits of error* and is obtained as

$$\theta = \theta_0 + \theta_1.$$

The confidence probability used today is almost always $P = 0.95$. For this probability, as explained above, Eq. 2 can be used regardless of the number of items and thus can be considered as universal:

$$u_{0.95} = 1.1\sqrt{\left(\theta_0^2 + \sum_{i=1}^{m} \theta_i^2\right)}. \tag{3}$$

The above calculations were derived assuming that elementary errors, even if they are systematic errors and thus constant for a given instrument, differ randomly from one instrument to another. In some cases, however, the magnitude of an elementary error is the same for all measuring instruments of a given type. Although this error, as always, has permissible limits, the probabilistic model for it is unacceptable. For example, it is known that the dependence between temperature and the electromotive force (EMF) of a thermocouple is non-linear. This dependence is often replaced with a sum of line segments. The difference between the curve and the broken line for a specific type of thermocouple is constant and has some determined limits. The error caused by application of a certain type of thermocouple in measurement will be the same for all such thermocouples. Therefore, the probabilistic model cannot be taken for this error; it must be considered as an absolutely constant elementary error. Instead, the limits of an absolutely constant error must be arithmetically added to the calculated uncertainty of the single measurement. Of course, this addition to the uncertainty does not change its confidence probability.

It should be stressed that the rated characteristics of a measuring instrument apply to all measuring instruments of the same design. Therefore the measuring instrument in a measurement can be replaced by another instrument of the same design and the estimation of the inaccuracy of the measurement will not change. However, if this estimation is based on the individually rated characteristics of the measuring instrument (for example, obtained by a calibration laboratory), then this measuring instrument cannot be replaced. Of course, the measurement in the latter case is more accurate.

Accuracy of a single measurement involving a chain of measuring instruments

There are a number of direct single measurements performed by several measuring instruments connected in a chain. An example of such a measurement is the measurement of voltage with a potentiometer, a voltage divider and a standard cell. If the potentiometer indication is N_p, the divide coefficient of the divider is K_d and the EMF of the standard cell is U_{sc}, then the measured voltage is

$$U_x = K_d N_p U_{\mathrm{sc}}. \tag{4}$$

Let the measurement be performed under the reference condition; the inaccuracy of this measurement is then determined by the intrinsic errors of all the measuring instruments involved. These errors are rated by their limits. To sum them up, we must first transform their limits into limits of errors in the measurement result. This can be done in many ways, most commonly using Taylor's series. Rabinovich [2] describes this example in detail; in this case the transformation was performed simply by the differentiation of Eq. 4.

Let the intrinsic error of the voltage divider be ζ_{d}, the intrinsic error of potentiometer indication be ζ_{p}, and that of the standard cell be ζ_{sc}. After applying Taylor's series (which is widely used and not explained here), we obtain the expression for the measurement error:

$$\zeta = w_d\zeta_d + w_p\zeta_p + w_{\mathrm{sc}}\zeta_{\mathrm{sc}}, \tag{5}$$

where w_d, w_p and w_{sc} are the corresponding transformation coefficients.

It is important to note that the relative form of errors simplifies all calculations. In our example, in this case all transformation coefficients will be the same and equal to 1:

$$w'_d = w'_p = w'_{\mathrm{sc}} = 1.$$

Thus, Eq. 5 with errors of measuring instruments in relative form becomes

$$\zeta' = \zeta'_{\mathrm{d}} + \zeta'_{\mathrm{p}} + \zeta'_{\mathrm{sc}}$$

In general, the measuring chain may consist of m instruments. Let us first consider the case where the measurement is performed under reference conditions for all instruments involved. The inaccuracy of all instruments in this case is given by their intrinsic errors, which are represented as limits of permissible errors. The resulting measurement error is expressed as

$$\zeta = \sum_{j=1}^{m} w_{j,0}\zeta_{j,0} \qquad (6)$$

where $\zeta_{j,0}$ is the intrinsic error of the jth instrument (j = 1... m), and $w_{j,0}$ is the corresponding transformation coefficient. As discussed above in the section on Accuracy of a single measurement involving a single measuring instrument, knowing the limits of all the elementary errors, we are able to calculate the uncertainty of the measurement result. Again, the approach uses the uniform distribution for random quantities representing the instrument errors. If the number of items in Eq. 6 is five or more, we can use the normal distribution for the resulting distribution and its variance can be estimated as

$$S_{\mathrm{rd}}^2 = \frac{1}{3}\sum_{j=1}^{m} w_{j,0}^2\theta_{j,0}^2.$$

Knowing the variance, the uncertainly of the measurement result can then be calculated as described above for single measurements involving a single measuring instrument. The universal (i.e., applicable to any number of items) formula now is

$$u_{0.95} = 1.1\sqrt{\sum_{j=1}^{m} w_{j,0}^2\theta_{j,0}^2}.$$

If the measurement condition cannot be considered as the reference condition, the errors of the measuring instruments can no longer be viewed as elementary errors. Each instrument error in this case may consist of its own elementary errors due to the influence quantities. Instead of calculating the error limits of each instrument, it is useful to transform the elementary errors of each instrument directly into the elementary errors of the overall measurement error. Doing so is beneficial because it increases the number of elementary errors and thus provides more grounds to consider the resulting error to be normally distributed. Besides, this transformation allows us to consider the components to be uniformly distributed, therefore allowing use of the universal formula for uncertainty similar to Eq. 3. As the result, Eq. 6 becomes

$$\zeta = \sum_{j=1}^{m} w_{\mathrm{instr},j}\left\{\sum_{i=0}^{n_j} w_{\mathrm{elem},j,i}\zeta_{j,i}\right\},$$

where $\zeta_{j,i}$ is the ith elementary error of the jth instrument, $w_{\mathrm{elem},j,i}$ is the transformation coefficient of the above elementary error, n_j is the number of elementary errors of the jth instrument, and $w_{\mathrm{instr},j}$ is the transformation coefficient of the jth instrument.

Using again the approach for single measurements involving a single measuring instrument discussed above, it becomes possible to calculate the uncertainty of the measurement result by assuming the elementary errors to be uniformly distributed. The uncertainty in this case is

$$u_{0.95} = 1.1\sqrt{\sum_{j=1}^{m} w_{\mathrm{instr},j}^2\left\{\sum_{i=0}^{n_j} w_{\mathrm{elem},j,i}^2\theta_{j,i}^2\right\}}.$$

With at least five items (which in practice is always the case in this situation), we can use the normal distribution for the resulting distribution to estimate its variance as

$$S_{\mathrm{rd}}^2 = \frac{1}{3}\sum_{j=1}^{m} w_{\mathrm{instr},j}^2\left\{\sum_{i=0}^{n_j} w_{\mathrm{elem},j,i}^2\theta_{j,i}^2\right\},$$

and calculate the uncertainty from this variance as already discussed.

Similar calculations can be also used for single indirect measurements. While most indirect measurements are multiple measurements, single indirect measurements do occur. An example of a single indirect measurement is the measurement of the area of a plot of land. If the model of this plot is a rectangle, the estimation of this area is found as the product of measurements of the length of both sides of this plot. As an interesting side note, the inaccuracy of this result is often determined not by the errors in the length measurements but by the difference between the model of the plot and its actual form.

Conclusion

This paper presents methods for estimating the inaccuracy of single measurements based on the rated metrological characteristics of the measuring instruments involved. The paper formulates the view that single measurements constitute the basic, fundamental type of measurement. This position is supported not only by the fact that single measurements are the most common measurements in trade and industry but also because multiple measurements are essentially repeated single measurements.

This paper formulates practical methods for the estimation of the accuracy of single measurements and sketches a step-by-step procedure for the calculations involved. It could become the starting point for a detailed recommendation for the treatment of single measurements. I believe it would be beneficial to develop this recommendation as part of the next edition of GUM.

References

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