



# Investment decisions under incomplete markets in the presence of wealth effects

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## Abstract

This paper investigates the impact of market incompleteness on investment decisions. In contrast to previous studies, we assume an entrepreneur has power utility and investment causes an additive increase in her wealth. By using the binomial theorem, we derive the analytical solutions for the option value and investment threshold. First, we show that an increase in wealth alleviates the over-investment problem caused by the market incompleteness. In addition, the marginal value of wealth is always more than one and decreases in the wealth level. Finally, the increase in wealth substantially enhances the value of the investment option and lowers the idiosyncratic risk premium.

**Keywords** Real options · Power utility · Additive structure · Wealth effects · Incomplete markets

**JEL Classification** E2 · G11 · G31

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## 1 Introduction

Since the seminal work of McDonald and Siegel (1986), the standard real options framework has become a basic tool with a number of applications.<sup>1</sup> Although the real options approach to investment has been developed substantially, most papers in this field rely on the complete markets (henceforth, CM) assumption, which is violated in many applications.<sup>2</sup> Recently, several papers extend the standard real options approach to analyze the implications of market incompleteness. To reduce the dimensions of their problem, the existing literature either adopts the exponential utility function or assumes exercising the option results in a proportional increase in the entrepreneur's wealth (e.g., Miao and Wang 2007; Choi et al. 2017).<sup>3</sup> Either assumption is made in order for researchers to single out and focus on the option value of waiting, but both assumptions give rise to the conclusion that the optimal exercise policy is invariant to the entrepreneur's wealth.

Albeit applying exponential utility can deal with real investment decision, it is made primarily for tractability reasons and not particularly relevant to the wealth effect. As documented by Miao and Wang (2007), although the power utility is more commonly used in economics, they adopt the exponential utility just for the tractability consideration. In fact, the exponential utility and power utility are two most commonly used utility specifications. The motivation of the use of power utility over exponential utility is that the power utility captures wealth effects, it should be more suitable to characterize people's preferences. Merton (1969) finds the exponential utility behaviorally less plausible than power utility. Merton (1992) states that optimal consumption-wealth ratio with power utility is stationary and thus consistent with real data. Driven by the strength of power utility, in this paper we consider an incomplete market real-option model (e.g., Henderson 2007; Miao and Wang 2007) by presuming that an entrepreneur has power utility and investment causes an additive increase in the entrepreneur's wealth.<sup>4</sup> Then we can investigate the impact of the entrepreneur's wealth on the investment decision. For simplicity, we assume that the entrepreneur's utility depends solely on her final wealth at the time of investment. When the market is complete, the entrepreneur's

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<sup>1</sup> Majd and Pindyck (1987) extend the basic model with a time-to-build feature. Dixit (1989) considers entry and exit decisions under uncertainty. Dixit and Pindyck (1994) provide a textbook treatment of important contributions to this literature. Grenadier (1996), Grenadier (2002), Grenadier and Malenko (2011) and Bensoussan et al. (2017) consider the real options problem in a game-theoretic environment.

<sup>2</sup> For example, consider entrepreneurial activities. Entrepreneurs might have valuable projects, but these projects might not be freely traded or their payoffs might not be spanned by existing assets. These capital market imperfections could be due to agency frictions or transactions costs. Thus, investment opportunities can have substantial non-diversifiable idiosyncratic risks. Moreover, entrepreneurs' well-being depends heavily on the outcome of their investments. As documented by Moskowitz and Vissing-Jorgensen (2002), about 3/4 of all private equity is owned by households for whom it constitutes at least half of their total net worth.

<sup>3</sup> Besides the preference-based approach, Thijssen (2011) interprets market incompleteness as a source of ambiguity over the appropriate no-arbitrage discount factor.

<sup>4</sup> An exception is Choi et al. (2017). In their paper, exercising the option results in a proportional increase in the entrepreneur's wealth. However, they make such an assumption just for the tractability consideration as well.

wealth has no influence on the investment decision, and the value of the investment option is identical to its counterpart in the standard real options model. When the market is incomplete, we derive the closed-form solutions for the option value and investment threshold by using the binomial theorem.

We find the following main novel results. First, market incompleteness results in over-investment and the optimal investment threshold is increasing in the entrepreneur's wealth until it equals its counterpart in the standard real options model. While for previous literature, they are unable to study the effect of the entrepreneur's wealth on the investment decision.

In addition, we find that market incompleteness significantly lowers the option value, which is consistent with the existing literature. For the entrepreneur's wealth, we find that an increase in her wealth level substantially enhances the value of the investment option. As the entrepreneur's wealth increases, the value of the investment option increases until it equals its counterpart in the standard real options model. When the market is complete, the marginal option value of wealth is always zero, as the entrepreneur's wealth has no impact on the value of the investment option. In our model, the marginal option value of wealth is positive and decreases in the wealth level. Since the project's risk cannot be diversified, the entrepreneur naturally demands the idiosyncratic risk premium. We show that the increase in the entrepreneur's wealth significantly lowers the idiosyncratic risk premium.

Finally, we consider the effect of risk. First, market incompleteness weakens the positive effect of volatility on the investment timing. For instance, when the project's volatility increases from 0.3 to 0.4, the investment threshold in the standard real options model increases by 31.9%. However, the investment threshold in our model increases by only 4.2%. Furthermore, we find that market incompleteness also weakens the positive effect of risk on the value of the investment option.

This paper is related to the literature on dynamic corporate finance under incomplete markets. Henderson (2007) considers the impact of entrepreneurial risk aversion and incompleteness on investment timing and the value of the option to invest. Miao and Wang (2007) analyze the joint decisions of business investments, consumption/savings, and portfolio selection with undiversifiable idiosyncratic risks. In their model, there is no wealth effect since the agent is assumed to have a CARA preference. Chen et al. (2010) develop a dynamic incomplete-markets model of entrepreneurial firms and study the implications of non-diversifiable risk on the capital structure decision. Sorensen et al. (2014) study the impact of non-diversifiable risk on valuing private equity. One essential difference between the literature and this paper is that we solve a model of investment decisions for power utility to characterize the wealth effect whereas they focus on CARA utility and ignore the wealth effect. Another important point of departure is that we solve our model analytically whereas they solve their model numerically. Finally, we derive implications of wealth change for the effective risk aversion and idiosyncratic risk premium, which are not examined in these papers. There are also several studies related to investment under uncertainty by using a preference specification with power utility. For instance, Hugonnier and Morellec (2007) and Choi et al. (2017)

consider the real options problem under incomplete market by adopting the CRRA utility function. However, they are unable to study the effect of the entrepreneur's wealth on the investment decision since these papers depend on the hypothesis that exercising the option results in a proportional increase in the entrepreneur's wealth, rather than an additive one as our model.

Our model also relates to recent work on dynamic corporate finance with other frictions. Bolton et al. (2011) propose a unified model of dynamic investment, financing, and risk management for financially constrained firms. DeMarzo et al. (2012) integrate the dynamic agency framework (DeMarzo and Sannikov 2006) with the neoclassical  $q$  theory of investment (Hayashi 1982). Bolton et al. (2013) extend Bolton et al. (2011) to allow for time-varying investment and financing opportunities. Bolton et al. (2019a) formulate a model of dynamic investment with inalienable human capital. Bolton et al. (2019b) develop a model of investment timing under uncertainty for a financially constrained firm. In contrast to their financial frictions, this paper develops a model of investment timing under uncertainty with a wealth effect in incomplete markets.

The remainder of this paper is organized as follows: Sect. 2 describes the model setup, which includes the investment opportunity and the entrepreneur's preferences. The model solutions are provided in Sects. 3, and 4 considers the the complete-markets (CM) case. The quantitative results and discussions are provided in Sect. 5. Section 6 analyzes the robustness of the results. Finally, Sect. 7 concludes the paper.

## 2 Model setup

In this paper, we consider a real-options-based irreversible investment model. There are no traded financial assets other than a risk-free asset that pays interest at a constant rate  $r > 0$ . We first describe the investment opportunity. Then, we discuss the entrepreneur's objective function. We assume exercising the option results in an additive increase in the entrepreneur's wealth. Therefore, the investment decision in our model is affected by the entrepreneur's wealth.

### 2.1 The investment opportunity

Following previous literature on real options (e.g., McDonald and Siegel 1986; Grenadier and Wang 2007), we assume that the project payoff,  $X_t$ , at time  $t$  evolves over time according to the following geometric Brownian motion:

$$\frac{dX_t}{X_t} = \mu dt + \sigma dZ_t, \quad (1)$$

where  $\mu$  is the expected growth rate,  $\sigma$  is the volatility parameter, and  $Z$  is a standard Brownian motion. The entrepreneur has an opportunity to invest in this project and can undertake this project irreversibly at some endogenously chosen time  $\tau$ . Investment at any time costs  $I$ . The entrepreneur pays this cost only at the investment time  $\tau$ . This cost is financed from the entrepreneur's own wealth. If there is a

shortage of funds, the entrepreneur can borrow at the risk-free rate  $r$ . After the entrepreneur exercises her option at time  $\tau$ , the project generates a lump-sum payoff  $X_\tau$ .

## 2.2 Entrepreneur's preferences

Let  $\{W_t : t \geq 0\}$  denote the entrepreneur's wealth process. There only exists a risk-free asset, and its dynamics are given by

$$dW_t = rW_t dt. \quad (2)$$

That is, the entrepreneur accumulates wealth at the rate  $rW_t$ .

For simplicity, we assume that the entrepreneur's utility depends solely on her final wealth at instant  $\tau$ . At the investment time  $\tau$ , the entrepreneur pays the investment cost  $I$  and obtains the lump-sum payoff  $X_\tau$ , and hence, her wealth is raised by the amount  $X_\tau - I$ . Thus, the entrepreneur's wealth, immediately after the exercise time  $\tau$ , becomes

$$W_\tau = W_{\tau-} + X_\tau - I, \quad (3)$$

where  $W_{\tau-}$  and  $W_\tau$  denote the entrepreneur's wealth just before and immediately after she exercises the investment option, respectively. Integrating (2) from 0 to  $\tau$ , we obtain  $W_{\tau-} = e^{r\tau}W_0$ , then (3) becomes

$$W_\tau = e^{r\tau}W_0 + X_\tau - I. \quad (4)$$

Without loss of generality, we assume that the entrepreneur's time discount rate is equal to  $r$ , the risk-free interest rate.<sup>5</sup> Before the investment, we focus on the optimal exercise time  $\tau$  of the real option by writing the entrepreneur's optimization problem as follows:

$$\begin{aligned} J(W_0, X_0) &= \max_{\tau} \mathbb{E} \left[ \frac{(e^{-r\tau}W_\tau)^{1-\gamma}}{1-\gamma} \right] \\ &= \mathbb{E} \left\{ \frac{[W_0 + e^{-r\tau}(X_\tau - I)]^{1-\gamma}}{1-\gamma} \right\}, \end{aligned} \quad (5)$$

where the second equality follows from (4). One might be misled about the wealth structure (4), namely, that the wealth effect can completely disappear if he/she only focuses on the threshold level and re-scaling the investment cost. However, notice that the implied option value clearly depends on both  $W_t$  and  $X_t$ , which is explained in more detail in Sect. 5. Therefore, the wealth effect still exists in our model.

<sup>5</sup> We take the discounting factor inside the utility function, while it is common to have it separately out of the utility function. Consider the case with zero project value, then the choice of  $\tau$  should have no influence on the entrepreneur's value function. If the discounting factor stays outside the utility function, the entrepreneur's value will be affected by  $\tau$ , which is undesirable.

### 3 Complete markets (CM)

To isolate the effects of market incompleteness, we present the CM solution in this section. To make the market dynamically complete, we make the market dynamically complete by introducing a traded asset that is perfectly correlated with the shocks to project value (Wang et al. 2016). We apply the standard dynamic replicating portfolio argument (Black and Scholes 1973). CM can be achieved by dynamically and frictionlessly trading the risk-free asset and the introduced financial asset. Since the project's risk is assumed to be idiosyncratic, the introduced financial asset demands no risk premium. We summarize the main results for our CM framework in the following proposition. The details are provided in the appendix.

**Proposition 1** *The entrepreneur's value function under the CM assumption, denoted by  $J^*(W, X)$ , is given by*

$$J^*(W, X) = \frac{[P^*(W, X)]^{1-\gamma}}{1-\gamma}, \quad (6)$$

and the "total" wealth  $P^*(W, X)$  is

$$P^*(W, X) = W + V(X), \quad (7)$$

where

$$V(X) = \left(\frac{X}{X^*}\right)^\beta (X^* - I), \quad (8)$$

$$\beta = \frac{1}{\sigma^2} \left[ -\left(\mu - \frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2r\sigma^2} \right]. \quad (9)$$

The optimal investment threshold  $X^*$  is

$$X^* = \frac{\beta}{\beta - 1} I. \quad (10)$$

Next, we discuss the economic intuition for the CM solution. First, when the market is complete, the value function  $J^*(W, X)$  is increasing and homothetic of degree  $(1 - \gamma)$  in total wealth  $P^*(W, X)$  and the total wealth  $P^*$  is simply additive in wealth  $W$  and project value  $V$ . More importantly, the optimal investment threshold  $X^*$  is not affected by the entrepreneur's wealth  $W$ . Since the project risk is purely idiosyncratic, it can be diversified at no risk premium. The risk-averse entrepreneur optimally chooses zero risk exposure for her total wealth  $P^*$  by dynamic hedging. Thus,  $P^*$  evolves as

$$\frac{dP_t^*}{P_t^*} = rdt. \quad (11)$$

#### 4 Incomplete markets

By far, the only relevant state variables going forward are the entrepreneur wealth  $W_t$  and the project value  $X_t$ . Therefore, the corresponding value function is given by  $J(W_t, X_t)$ . By a standard dynamic programming argument, one can show that  $J(W, X)$  satisfies the following partial differential equation (PDE):

$$r(1 - \gamma)J(W, X) = rWJ_W(W, X) + \mu XJ_X(W, X) + \frac{1}{2}\sigma^2 X^2 J_{XX}(W, X), \quad (12)$$

where  $J_X$  denotes the first-order derivative of the function  $J$  with respect to  $X$ ,  $J_W$  denotes the first-order derivative of the function  $J$  with respect to  $W$ , and  $J_{XX}$  denotes the second-order partial derivative of  $J$  with respect to  $X$ . The left-hand side of this equation represents the required rate of return for commencing the project. The right-hand side is the expected change in value function in the region where the entrepreneur waits. The first two terms capture the effects of changes in wealth ( $rW$ ) and project value ( $\mu X$ ). The last term captures the effects of volatility in payoff process. Before characterizing the solution for the general case, we first discuss the solution for the case with zero project value,  $X = 0$ .

*No project:*  $X = 0$ . Because the project value  $X$  is assumed to follow a GBM process, both the drift and volatility of  $X$  are zero and hence  $X = 0$  is an absorbing state, in that  $X_t = 0$  for all  $t \geq v$  if  $X_v = 0$ . Therefore, the last two terms in the PDE (12) involving  $J_X(W, 0)$  and  $J_{XX}(W, 0)$  are zero. Precisely, the value function  $J(\cdot, 0)$  has the following closed-form solution:

$$J(W, 0) = \frac{W^{1-\gamma}}{1-\gamma}. \quad (13)$$

For any given level of wealth  $W$ , we have

$$J(W, 0) = J^*(W, 0), \quad (14)$$

which implies that utils are the same as the CM case as all risks are idiosyncratic and are completely diversified away.

For the general case  $X > 0$ , the value function is given by

$$J(W, X) = \frac{[P(W, X)]^{1-\gamma}}{1-\gamma}, \quad (15)$$

where  $P(W, X)$  can be interpreted as the certainty equivalent wealth, the amount of wealth for which the entrepreneur is willing to permanently give up taking the project and liquid wealth,  $J(W, X) = J(P(W, X), 0) = J^*(P(W, X), 0)$ .<sup>6</sup>

<sup>6</sup> There are several reasons why we work with certainty equivalent wealth  $P(W, X)$  rather than directly with the value function  $J(W, X)$ . First, certainty equivalent wealth is an intuitive concept and is measured

Then the entrepreneur’s certainty equivalent wealth satisfies the following PDE:

$$rP(W, X) = rWP_W(W, X) + \mu XP_X(W, X) + \frac{1}{2}\sigma^2 X^2 P_{XX}(W, X) - \frac{1}{2}\gamma\sigma^2 X^2 \frac{P_X(W, X)^2}{P(W, X)}, \tag{16}$$

with four boundary conditions

$$P(W, 0) = W, \tag{17}$$

$$P(W, \bar{X}(W)) = W + \bar{X}(W) - I, \tag{18}$$

$$P_X(W, \bar{X}(W)) = 1, \tag{19}$$

$$P_W(W, \bar{X}(W)) = 1. \tag{20}$$

Here  $\bar{X}(W)$  is the optimal investment threshold, which depends on the wealth  $W$ . The first boundary condition can be directly derived from (13) and (15). The second condition is the value-matching condition. The third and final boundary conditions are the smooth-pasting conditions. Due to the existence of the investment cost, our model does not preserve the homogeneity property. Thus, we cannot reduce the dimension of our problem by treating  $w = W/X$  as the effective state variable. In addition, since we adopt the power utility specification, we cannot directly separate the option value from the wealth. As documented in Miao and Wang (2007), “while power utility is more commonly used in economics, this utility will complicate our analysis substantially since it will lead to a much more difficult two dimensional free-boundary problem.”

Fortunately, we derive the closed-form solution for the above PDE (16) by the binomial theorem. The following proposition summarizes the main results on the entrepreneur’s certainty equivalent wealth  $P(W, X)$  and the optimal investment threshold  $\bar{X}(W)$ . Proofs are provided in the appendix.

**Proposition 2** *The entrepreneur’s certainty equivalent wealth  $P(W, X)$  is*

$$P(W, X) = W \left[ \sum_{n=0}^{\infty} \binom{1-\gamma}{n} W^{-n} V_n(X) \right]^{\frac{1}{1-\gamma}}, \tag{21}$$

where

$$V_n(X) = \left( \frac{X}{\bar{X}} \right)^{\beta_n} (\bar{X} - I)^n, \tag{22}$$

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Footnote 6 continued

in units of wealth, while the unit for value function  $J(W, X)$  is utils, which cannot be directly measured. Second,  $P(W, X)$  is analytically convenient to work with. Third, the marginal (certainty equivalent) value of wealth,  $P_W(W, X)$  is a natural measure for the impact of financial frictions.



$$\beta_n = \frac{1}{\sigma^2} \left[ -\left( \mu - \frac{1}{2} \sigma^2 \right) + \sqrt{\left( \mu - \frac{1}{2} \sigma^2 \right)^2 + 2nr\sigma^2} \right]. \quad (23)$$

The optimal investment threshold  $\bar{X}(W)$  is the solution of the following equation:

$$\frac{(W + \bar{X} - I)^\gamma}{1 - \gamma} \sum_{n=1}^{\infty} \binom{1 - \gamma}{n} \beta_n W^{1 - \gamma - n} (\bar{X} - I)^n = \bar{X}. \quad (24)$$

To guarantee convergence, we need

$$W > \bar{X} - I. \quad (25)$$

As shown in (21), the entrepreneur's certainty equivalent wealth can be treated as a collection of real options  $V_n(X)$  for  $n = 1, 2, \dots, \infty$ . Here,  $V_n(X)$  is the value of an investment option with the discount rate  $nr$  and lump-sum payoff  $(X_\tau - I)^n$ . More importantly, the liquid wealth  $W_t$  determines the weight of these real options  $V_n(X)$ , which generates the wealth effect. Furthermore, we find that the optimal investment threshold  $\bar{X}$  depends on the entrepreneur's liquid wealth  $W$ . We discuss the economic intuition and model implications in Sect. 5.

Following Miao and Wang (2007), we define the value of the investment option  $G(W, X)$  as:

$$J(W + G(W, X), 0) = J(W, X), \quad (26)$$

which means that the entrepreneur is indifferent between exercising the investment option in the future and a total current wealth level of  $W + G$ . Then, we have

$$G(W, X) = P(W, X) - W. \quad (27)$$

## 5 Quantitative results and discussions

We report the quantitative results in this section. We use the following annualized baseline parameter values as Grenadier and Wang (2007): the risk-free rate  $r = 0.05$ , the expected growth rate  $\mu = 0$ , the volatility of the project  $\sigma = 0.35$ , and the investment cost  $I = 1$ . In addition, we consider the widely used value for the coefficient of relative risk aversion  $\gamma = 2$ . Since the investment decision in our model is affected by the wealth, we choose  $W = 5$  as the baseline parameter value.

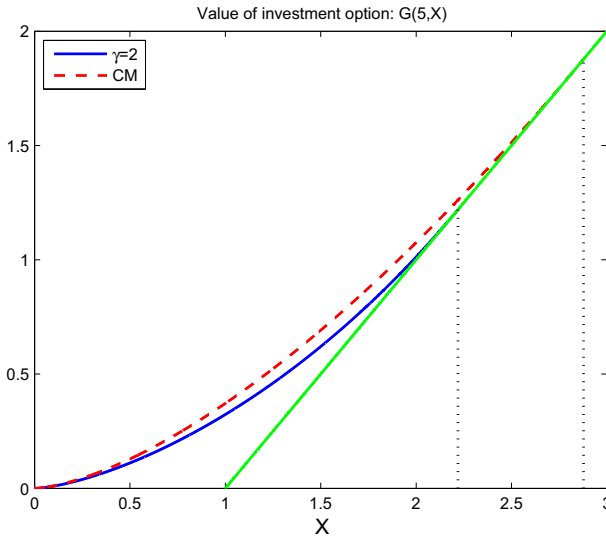


Fig. 1 Value of the investment option for  $W = 5$

### 5.1 Implied option value and wealth effects

Figure 1 shows the value of the investment option as a function of the project value  $X$  when the entrepreneur's wealth  $W = 5$ . First, market incompleteness significantly lowers the option value  $G(5, X)$  from its CM benchmark  $V(X)$ . Quantitatively, when  $X = 1$ , we have  $G(5, 1) = 0.3230$ , which is much lower than the  $V(1) = 0.3716$  under CM. Furthermore, market incompleteness significantly accelerates the exercise of the investment option. In CM's model, the entrepreneur will invest in the project when  $X$  reaches 2.8775. In our model, the entrepreneur invests when  $X$  reaches 2.2195.

The left panel of Fig. 2 shows the value of the investment option as a function of the entrepreneur's wealth  $W$  when the project value  $X = 1$ .<sup>7</sup> In the CM model, the value of the investment option is 0.3716, which is irrelevant to the entrepreneur's wealth  $W$ . In our model, market incompleteness significantly lowers the option value. This reduction is severe when the entrepreneur's wealth is close to zero. The increase in her wealth level substantially enhances the value of the investment option. Quantitatively, when  $W = 20$ , we have  $G(20, 1) = 0.3560$ , which increases by more than 40% relative to its counterpart as  $W = 1$ . The right panel of Fig. 2 shows the marginal option value of wealth as a function of the entrepreneur's wealth  $W$  when the project value  $X = 1$ . In the CM model, it is always zero since the value of the investment option relies solely on the project value  $X$ . When the market is incomplete, the marginal option value of wealth becomes positive and decreases in the wealth level. Quantitatively, when  $W = 1$ , we have  $G_W(1, 1) = 0.0561$ .

<sup>7</sup> In Figs. 2, 3 and 4, the left endpoint of  $W$  is 0.68 and the corresponding investment threshold is 1.6292. Thus the convergence condition (25) is satisfied.

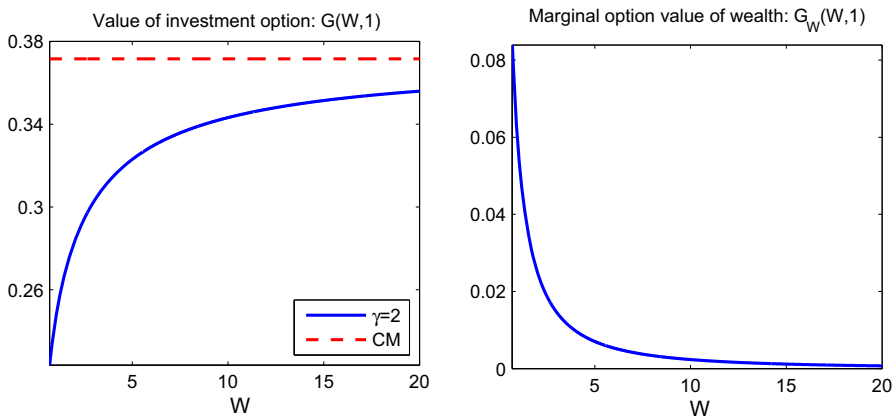


Fig. 2 Value of the investment option and marginal option value of wealth for  $X = 1$

### 5.2 Effective risk aversion

To investigate the influence of market incompleteness on the entrepreneur’s risk attitude, we define the effective risk aversion as

$$\begin{aligned}
 h(W, X) &\equiv -\frac{P(W, X)J_{WW}(W, X)}{J_W(W, X)} \\
 &= \gamma P_W(W, X) - \frac{P(W, X)P_{WW}(W, X)}{P_W(W, X)},
 \end{aligned}
 \tag{28}$$

where the first identity sign gives the definition of  $h(W, X)$  and the second equality follows from (15). When markets are incomplete, the entrepreneur’s effective risk aversion is captured by the curvature of his value function  $J(W, X)$ . We can characterize the entrepreneur’s coefficient of endogenous absolute risk aversion with the standard definition:  $-J_{WW}/J_W$ . In order to link this absolute risk aversion measure to a relative risk aversion measure, we need to multiply absolute risk aversion  $-J_{WW}/J_W$  with an appropriate measure of the entrepreneur’s wealth. In our model, the entrepreneur’s certainty equivalent wealth  $P(W, X)$  is a natural measure. This motivates our definition of  $h(W, X)$  in (28). In the CM model, since the total wealth  $P^*(W, X)$  is simply additive in wealth  $W$  and the project value  $V(X)$ , the effective risk aversion is constant, as  $h(W, X) = \gamma$ . However, in the presence of market incompleteness,  $h$  is dependent on wealth  $W$  owing to the appearance of wealth effects.

Figure 3 plots the effective risk aversion as a function of wealth. First, we find that market incompleteness induces the entrepreneur to be more risk averse. For instance, when  $W = 1$ , the entrepreneur’s effective risk aversion is 2.1898, which is higher than  $\gamma$ . Moreover, the entrepreneur’s effective risk aversion  $h$  decreases with  $W$  as self insurance becomes more effective. As one would expect, in the limit, as  $W \rightarrow \infty$ ,  $h(W, X)$  approaches  $\gamma$ .

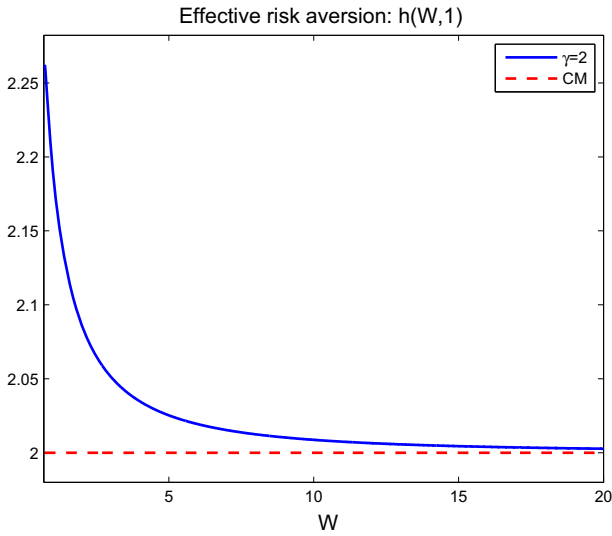


Fig. 3 Effective risk aversion for  $X = 1$

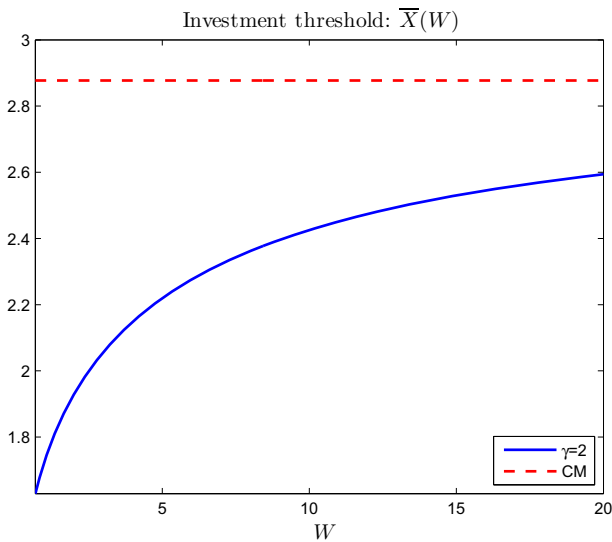


Fig. 4 Optimal investment threshold

Cressy (2000) documents that higher wealth makes the risk-averse entrepreneur more prone to take risks, i.e., a higher wealth level makes the entrepreneur less risk averse. Moreover, Alpanda and Woglom (2007) state that the prediction that wealthier people are more risk-averse implied by CARA utility is counterintuitive and is contradicted by empirical studies. In our model, the effective risk aversion is

decreasing in the wealth  $W$ , indicating that the wealthier the entrepreneur, the less risk-averse she is. This is in line with the aforementioned fact.

### 5.3 Investment threshold and wealth effects

In Fig. 4, we plot the optimal investment threshold as a function of the entrepreneur's wealth. Quantitatively, when  $W = 1$ , the investment threshold in our model is 1.7275, which is significantly lower than the 2.8775 in CM. Therefore, market incompleteness results in over-investment. With the increase in the entrepreneur's wealth, the optimal investment threshold increases and converges to its counterpart in CM. For instance, when  $W = 20$ , the investment threshold in our model is 2.5941.

### 5.4 Idiosyncratic risk premium

Based on (16), we obtain the PDE for the value of the investment option  $G(W, X)$  as

$$\begin{aligned} & \left[ r + \frac{\gamma\sigma^2 X^2 G_X(W, X)^2}{2G(W, X)(W + G(W, X))} \right] G(W, X) \\ & = rWG_W(W, X) + \mu XG_X(W, X) + \frac{1}{2}\sigma^2 X^2 G_{XX}(W, X). \end{aligned} \quad (29)$$

Then we define the idiosyncratic risk premium as

$$\pi(W, X) = \frac{\gamma\sigma^2 X^2 G_X(W, X)^2}{2G(W, X)(W + G(W, X))}. \quad (30)$$

Due to the existence of the wealth effect, we can discuss the influence of the entrepreneur's wealth  $W$  on the idiosyncratic risk premium  $\pi$ . In the extreme case  $W \rightarrow \infty$ , we have  $\pi \rightarrow 0$ .

Figure 5 plots the idiosyncratic risk premium as a function of the project value. First, we find that the idiosyncratic risk premium increases in the project value. Furthermore, the increase in the entrepreneur's wealth significantly lowers the idiosyncratic risk premium. For instance, when  $X = 2$ , we have  $\pi(20, 2) = 0.0154$ , which decreases by more than 70% relative to its counterpart when  $W = 5$ . Intuitively, the increase in wealth implies more effective self-insurance, which lowers the idiosyncratic risk premium.

### 5.5 Comparative analysis of risk

Miao and Wang (2007) find that an increase in volatility  $\sigma$  may encourage the agent to exercise her option sooner, which is contrary to the standard real options result. To investigate the relationship between project volatility and investment timing, we plot the investment threshold as a function of inverse volatility in the left panel of Fig. 6. As shown in Miao and Wang (2007), this negative connection is more pronounced when the risk aversion coefficient is sufficiently high. However, the

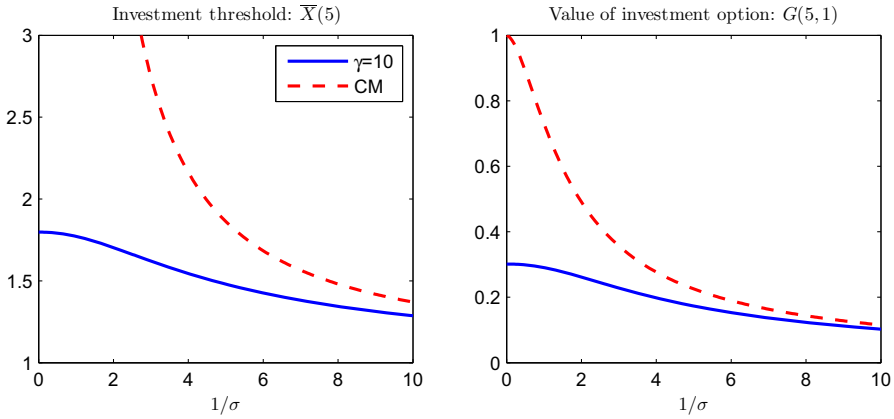


Fig. 5 Idiosyncratic risk premium

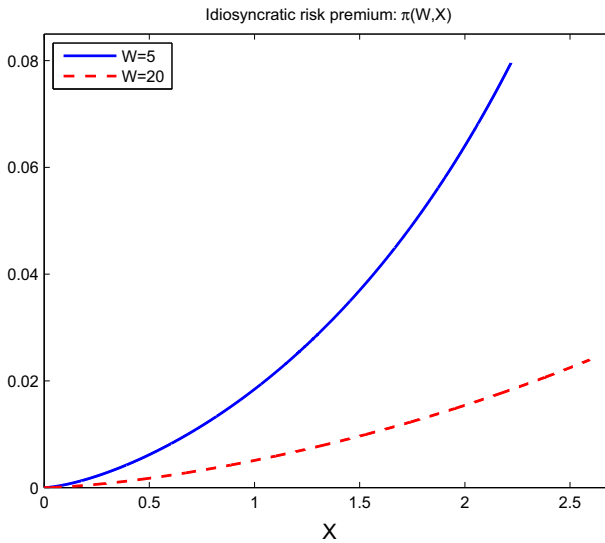


Fig. 6 The optimal investment threshold and the value of the investment option for  $W = 5$  and  $X = 1$

asset pricing literature (e.g., Mehra and Prescott 1985) documents that the relative risk aversion coefficient should be no more than 10. As a result, we set  $\gamma = 10$  to highlight the negative effect of volatility on investment timing.

We find that market incompleteness weakens the positive effect of volatility on investment threshold. For instance, when volatility  $\sigma$  increases from 0.3 to 0.4, the investment threshold in the CM model increases by 31.9%. However, in our model, the investment threshold increases by only 4.2%. Unfortunately, given reasonable parameter values, an increase in project volatility always leads to a higher investment threshold in our model. In the extreme case in which  $\sigma \rightarrow \infty$  ( $1/\sigma \rightarrow 0$ ),

we have  $\beta_n = 1$  for  $n = 1, 2, \dots, \infty$ . Then, the optimal investment threshold  $\bar{X}(W)$  is the solution of the following equation:

$$\frac{(W + \bar{X} - I)^\gamma}{1 - \gamma} \left[ (W + \bar{X} - I)^{1-\gamma} - W^{1-\gamma} \right] = \bar{X}. \quad (31)$$

Quantitatively, when project volatility goes to infinity, the investment thresholds in our model converge to 1.7980. In addition, we plot the value of the investment option as a function of inverse volatility in the right panel of Fig. 6. Similar to the investment threshold, market incompleteness also weakens the positive effect of volatility on the value of the investment option. In the extreme case  $\sigma \rightarrow \infty$  ( $1/\sigma \rightarrow 0$ ), we have

$$G(W, X) = \left( \frac{X}{\bar{X}} \right)^{\frac{1}{1-\gamma}} (W + \bar{X} - I) - W. \quad (32)$$

Notably, we do not intend to conclude that a negative relationship between project volatility and the investment threshold is impossible. What we want to show is that it is difficult to produce such a negative relationship for reasonable parameter values.

## 5.6 Implications for real estate markets

The result of our model provides an important implication for the real estate overbuilding. Miao and Wang (2007) predicts that individual entrepreneurs are likelier to invest earlier than institutional investors since market incompleteness lowers individual developers' value of waiting. But the wealth does not have an impact on the investment timing since there is no wealth effect. If the wealth effect is embedded, this model can provide an alternative but related explanation for the phenomenon of real estate overbuilding. To be specific, consider the real estate development example suggested by Miao and Wang (2007). Suppose, in addition, that we have a sample containing an individual developer and an institutional investor such as public traded real-estate-investment-trusts (REITs), both of which face market incompleteness. As reported in Ciochetti et al. (2000), institutions owned \$72.7 billion or 52.6% of REIT stocks in 1998. Then it is natural to believe that institutional investors should be wealthier than individual developers. Under this circumstance, the investor's wealth should affect the investment decision.<sup>8</sup> As shown in Fig. 4, the investment threshold is increasing in the wealth level. If the investor owns relatively less wealth, it will choose to hasten investment (implying more investment), indicating that the individual entrepreneurs are more likely to develop earlier than the public traded REITs. Such early development could be a contributor to the tendency for developers to overbuild. In fact, the noninstitutional investors like merchant builders are often blamed for causing overbuilding in US

<sup>8</sup> Of course, other parameters like risk aversion and discount rate also affect the investment decision. However, we only focus on the effect of wealth in describing our implications since it is the focus of this paper.

office markets. For example, in an April 4, 2001 article in Barron's, merchant builders were accused of contributing to oversupply in suburban office markets (Gose 2001). Since there is no wealth effect in Miao and Wang (2007), they assume that institutional investors face a CM, which is equivalent to the extreme case  $W \rightarrow \infty$  in our model.

## 6 Robustness

Our main results are based on the assumptions of geometric Brownian motion payoff processes. We now relax this assumption by using arithmetic Brownian motion process. Then the dynamic process of  $X_t$  is replaced by

$$dX_t = \mu dt + \sigma dZ_t, \quad (33)$$

where  $Z$  is a standard Brownian motion. Following the similar procedure, we derive the solutions and summarize them in the following proposition. We provide the details in the appendix.

**Proposition 3** *The entrepreneur's certainty equivalent wealth  $P(W, X)$  is given by (21), where*

$$V_n(X) = e^{\theta_n(X-\bar{X})} (\bar{X} - I)^n, \quad (34)$$

$$\theta_n = \frac{\sqrt{\mu^2 + 2nr\sigma^2} - \mu}{\sigma^2}. \quad (35)$$

The optimal investment threshold  $\bar{X}(W)$  is the solution of the following equation:

$$\frac{(W + \bar{X} - I)^\gamma}{1 - \gamma} \sum_{n=1}^{\infty} \binom{1 - \gamma}{n} \theta_n W^{1-\gamma-n} (\bar{X} - I)^n = 1. \quad (36)$$

To guarantee convergence, we need (25).

In Fig. 7, we plot the optimal investment threshold as a function of the entrepreneur's wealth. We demonstrate that our main results on the investment timing are robust to the arithmetic Brownian motion specification of the payoff process. When  $W = 1$ , the investment threshold in our model is 1.782, which is significantly lower than the 2.1068 in CM. In addition, the optimal investment threshold is increasing in the entrepreneur's wealth and converges to its counterpart in CM.

## 7 Conclusion

This paper studies the influence of market incompleteness on investment decisions. The existing literature in this field assumes investment results in a proportional increase in the entrepreneur's wealth to avoid the two dimensional free-boundary



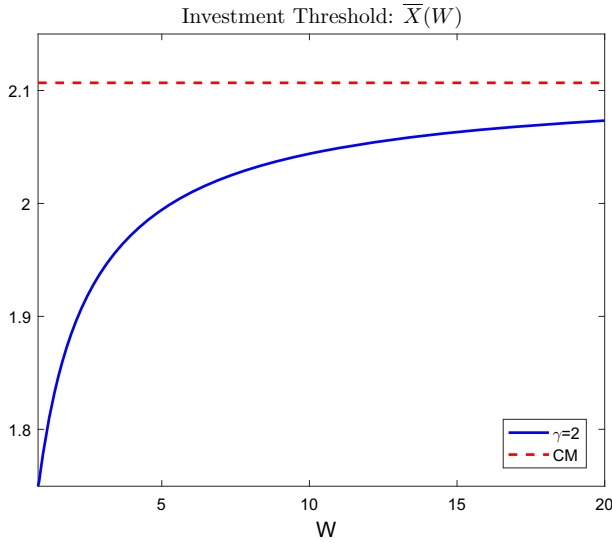


Fig. 7 Optimal investment threshold. The parameter values are the same as those in Fig. 4

problem. To consider the wealth effect, our model uses the additive structure. Consistent with previous studies, we find that market incompleteness significantly reduces the value of the investment option and accelerates investment in a project. Due to the existence of the wealth effect, an increase in wealth enhances the value of the investment option, alleviates the over-investment problem and reduces the idiosyncratic risk premium. Finally, we find that market incompleteness attenuates the positive effect of risk on the option value and investment timing, which is also consistent with previous literature.

### Appendix A: Proof of Proposition 1

First, we find that it is difficult to directly derive the solution of  $P(W, X)$  from the PDE (16). However, we can rewrite (5) as

$$J(W_t, X_t) = \max_{\tau \geq t} \mathbb{E}_t \left\{ \frac{\sum_{n=0}^{\infty} \binom{1-\gamma}{n} W_t^{1-\gamma-n} e^{-nr(\tau-t)} (X_\tau - I)^n}{1-\gamma} \right\}. \tag{A.1}$$

Here, we use the binomial theorem. Since  $W_t$  is known at time  $t$ , we have

$$J(W_t, X_t) = \max_{\tau \geq t} \frac{\sum_{n=0}^{\infty} \binom{1-\gamma}{n} W_t^{1-\gamma-n} \mathbb{E}_t \{ e^{-nr(\tau-t)} (X_\tau - I)^n \}}{1-\gamma}. \tag{A.2}$$

Then, we treat  $V_{n,t} = \mathbb{E}_t \{ e^{-nr(\tau-t)} (X_\tau - I)^n \}$  as the value of a conjectured real option

with discount rate  $nr$  and lump-sum payoff  $(X_\tau - I)^n$ . Based on the standard arguments,  $V_n$  satisfies the following ODE:

$$nrV_n(X) = \mu XV'_n(X) + \frac{1}{2}\sigma^2 X^2 V''_n(X). \tag{A.3}$$

Combining this with boundary condition  $V_n(\bar{X}) = (\bar{X} - I)^n$ , we have

$$V_n(X) = \left(\frac{X}{\bar{X}}\right)^{\beta_n} (\bar{X} - I)^n, \tag{A.4}$$

where

$$\beta_n = \frac{1}{\sigma^2} \left[ -\left(\mu - \frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2nr\sigma^2} \right]. \tag{A.5}$$

Therefore, we obtain the closed-form solution for  $P(W, X)$  as

$$P(W, X) = [(1 - \gamma)J(W, X)]^{\frac{1}{1-\gamma}} = W \left[ \sum_{n=0}^{\infty} \binom{1-\gamma}{n} W^{-n} V_n(X) \right]^{\frac{1}{1-\gamma}}. \tag{A.6}$$

First, we can easily verify that the above solution satisfies boundary conditions (17), (18) and (20). From boundary condition (19), we obtain the optimal investment threshold as

$$\frac{(W + \bar{X} - I)^\gamma}{1 - \gamma} \sum_{n=1}^{\infty} \binom{1-\gamma}{n} \beta_n W^{1-\gamma-n} (\bar{X} - I)^n = \bar{X}. \tag{A.7}$$

To guarantee convergence for the left-hand side of the above equation, we need  $W > \bar{X} - I$ .

For the special cases of  $\gamma = 1$  and  $\gamma = 2$ , the solutions for the entrepreneur’s certainty equivalent wealth and the investment threshold can be further simplified.

**Special case I:**  $\gamma = 1$

When the entrepreneur has log utility, her certainty equivalent wealth  $P(W, X)$  becomes

$$P(W, X) = W \exp \left[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} W^{-n} V_n(X) \right], \tag{A.8}$$

where  $V_n(X)$  is given by (22). The optimal investment threshold  $\bar{X}(W)$  becomes the solution of the following equation

$$(W + \bar{X} - I) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \beta_n W^{-n} (\bar{X} - I)^n = \bar{X}. \tag{A.9}$$

**Special case II:**  $\gamma = 2$

When the entrepreneur’s risk aversion coefficient equals 2, her certainty equivalent wealth  $P(W, X)$  becomes

$$P(W, X) = \frac{W}{\sum_{n=0}^{\infty} (-1)^n W^{-n} V_n(X)}, \tag{A.10}$$

and the optimal investment threshold  $\bar{X}(W)$  becomes the solution of the following equation:

$$(W + \bar{X} - I)^2 \sum_{n=1}^{\infty} (-1)^{n+1} \beta_n W^{-1-n} (\bar{X} - I)^n = \bar{X}. \tag{A.11}$$

### B Proof of Proposition 2

Following Wang, Wang and Yang (2016), we complete markets by introducing a tradable asset that is perfectly correlated with the project value. Since the project risk is idiosyncratic, it can be diversified away at no premium. Hence, the dynamics of this new financial asset are given by

$$\frac{dS_t}{S_t} = rdt + \sigma_S dZ_t, \tag{B.1}$$

where  $\sigma_S$  is the volatility parameter and  $Z$  is the same Brownian motion driving the dynamics of project value. When the market is incomplete, there is no such a tradable asset that is perfectly correlated with the project value. Hence this variable  $S$  disappears in the incomplete market case. Denote  $\phi_t$  as the fraction of the agent’s wealth in this asset; then, the entrepreneur’s wealth  $W$  evolves as follows:

$$dW_t = rW_t dt + \sigma_S \phi_t W_t dZ_t. \tag{B.2}$$

Using the standard principle of optimality, the value function satisfies the following PDE:

$$\begin{aligned} r(1 - \gamma)J(W, X) = & \max_{\phi} rWJ_W(W, X) + \mu XJ_X(W, X) + \frac{1}{2} \phi^2 \sigma_S^2 W^2 J_{WW}(W, X) \\ & + \phi \sigma_S \sigma W X J_{WX}(W, X) + \frac{1}{2} \sigma^2 X^2 J_{XX}(W, X). \end{aligned} \tag{B.3}$$

Compared with (12), there exist two additional terms on the right-hand side of the PDE:  $\frac{1}{2} \phi^2 \sigma_S^2 W^2 J_{WW}$  and  $\phi_t \sigma_S \sigma W X J_{WX}$ . We have  $J_{WW}$  and  $J_{WX}$  due to the stochastic returns for the new risky asset and dynamic hedging, respectively.

Similarly, we write the value function as

$$J(W, X) = \frac{[P(W, X)]^{1-\gamma}}{1-\gamma}. \quad (\text{B.4})$$

Then, we have

$$\begin{aligned} rP(W, X) = & \max_{\phi} rWP_W(W, X) + \mu XP_X(W, X) + \frac{1}{2} \phi^2 \sigma_S^2 W^2 P_{WW}(W, X) \\ & + \phi \sigma_S \sigma W X P_{WX}(W, X) + \frac{1}{2} \sigma^2 X^2 P_{XX}(W, X) - \frac{1}{2} \gamma \phi^2 \sigma_S^2 W^2 \frac{P_W(W, X)^2}{P(W, X)} \\ & - \gamma \phi \sigma_S \sigma W X \frac{P_W(W, X) P_X(W, X)}{P(W, X)} - \frac{1}{2} \gamma \sigma^2 X^2 \frac{P_X(W, X)^2}{P(W, X)}. \end{aligned} \quad (\text{B.5})$$

The FOC for hedging demand  $\phi$  implies

$$\phi = - \frac{\sigma X P_{WX}(W, X) - \gamma \sigma X \frac{P_W(W, X) P_X(W, X)}{P(W, X)}}{\sigma_S W P_{WW}(W, X) - \gamma \sigma_S W \frac{P_W(W, X)^2}{P(W, X)}}. \quad (\text{B.6})$$

Substituting the optimal hedging demand into the PDE (B.5), we obtain

$$\begin{aligned} rP(W, X) = & rWP_W(W, X) + \mu XP_X(W, X) + \frac{1}{2} \sigma^2 X^2 P_{XX}(W, X) - \frac{1}{2} \gamma \sigma^2 X^2 \frac{P_X(W, X)^2}{P(W, X)} \\ & - \frac{\sigma^2 X^2 \left( P_{WX}(W, X) - \gamma \frac{P_W(W, X) P_X(W, X)}{P(W, X)} \right)^2}{2 \left( P_{WW}(W, X) - \gamma \frac{P_W(W, X)^2}{P(W, X)} \right)}. \end{aligned} \quad (\text{B.7})$$

By using the insights for the CM case, we conjecture

$$P(W, X) = W + V(X). \quad (\text{B.8})$$

Substituting the above-conjectured solution into (B.7) and the four boundary conditions (17) to (20), we have

$$rV(X) = \mu X V'(X) + \frac{1}{2} \sigma^2 X^2 V''(X), \quad (\text{B.9})$$

$$V(X^*) = X^* - I, \quad (\text{B.10})$$

$$V'(X^*) = 1. \quad (\text{B.11})$$

Therefore, (B.8) is indeed the solution for the CM model. Substituting the solution into (B.6) yields the following optimal hedging portfolio:

$$\phi = -\frac{\sigma XV'(X)}{\sigma_S W}. \tag{B.12}$$

Then, we obtain the wealth dynamics  $W_t$ :

$$dW_t = rW_t dt - \sigma X_t V'(X_t) dZ_t. \tag{B.13}$$

Thus, the entrepreneur’s wealth is negatively correlated with the project value. For the total wealth  $P^*(W, X)$ , we have

$$\begin{aligned} dP_t^* &= dW_t + \left( \mu X_t V'(X_t) + \frac{1}{2} \sigma^2 X_t^2 V''(X_t) \right) dt + \sigma X_t V'(X_t) dZ_t \\ &= dW_t + rV(X_t) dt + \sigma X_t V'(X_t) dZ_t = rP_t^* dt, \end{aligned} \tag{B.14}$$

which means that the total wealth is deterministic and increases at rate  $r$ .

### Appendix C: Proof of Proposition 3

When the project has value  $X_t$  follows a arithmetic Brownian motion, the ODE for  $V_n$  is replaced by

$$nrV_n(X) = \mu V'_n(X) + \frac{1}{2} \sigma^2 V''_n(X). \tag{C.1}$$

Combining this with boundary condition  $V_n(\bar{X}) = (\bar{X} - I)^n$ , we have

$$V_n(X) = e^{\theta_n(X-\bar{X})} (\bar{X} - I)^n, \tag{C.2}$$

where

$$\theta_n = \frac{\sqrt{\mu^2 + 2nr\sigma^2} - \mu}{\sigma^2}. \tag{C.3}$$

Therefore, we obtain the closed-form solution for  $P(W, X)$  as

$$P(W, X) = [(1 - \gamma)J(W, X)]^{\frac{1}{1-\gamma}} = W \left[ \sum_{n=0}^{\infty} \binom{1-\gamma}{n} W^{-n} V_n(X) \right]^{\frac{1}{1-\gamma}}. \tag{C.4}$$

Then we obtain the optimal investment threshold as

$$\frac{(W + \bar{X} - I)^\gamma}{1 - \gamma} \sum_{n=1}^{\infty} \binom{1-\gamma}{n} \theta_n W^{1-\gamma-n} (\bar{X} - I)^n = 1. \tag{C.5}$$

To guarantee convergence for the left-hand side of the above equation, we need  $W > \bar{X} - I$  as well.

When the market is complete, the optimal investment threshold  $X^*$  is standard as

$$X^* = I + \frac{1}{\theta}, \quad (\text{C.6})$$

where

$$\theta = \frac{\sqrt{\mu^2 + 2r\sigma^2} - \mu}{\sigma^2}. \quad (\text{C.7})$$

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