J Econ (2017) 120:193–217
DOI 10.1007/s00712-016-0507-3

Two-sided competition with vertical differentiation

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Received: 15 February 2015 / Accepted: 19 July 2016 / Published online: 6 August 2016

Abstract This paper studies duopoly in which two-sided platforms compete in differentiated products in a two-sided market. *Direct* competition on *both* sides leads to results that depart from much of the current literature. Under some conditions the unique equilibrium in pure strategies can be computed. It features discounts on one side and muted differentiation as the cross-market externality intensifies competition. Less standard, that equilibrium fails to exist when the externality is too powerful (that side becomes too lucrative). A mixed-strategy equilibrium always exists and is characterized. These results are robust to variations in the extensive form. The model may find applications in the media, internet trading platforms, search engine competition, social media or even health insurance (HMO/PPO).

Keywords Two-sided market · Vertical differentiation · Industrial organization · Platform competition

JEL Classification C72 · D43 · D62 · L13 · L15

"The only thing advertisers care about is circulation, circulation, circulation."

Edward J. Atorino, analyst Fulcrum Global Partners, New York
June 17, 2004 (The Boston Globe).

1 Introduction

In many markets, firms must satisfy two constituencies: consumers on one side and advertisers on the other in the case of media, policyholders and service providers for

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HMOs and PPOs, search engine users and advertisers, or application developers and users of software platforms. This paper analyzes platform competition when these firms engage in vertical differentiation and set prices. The model herein departs from much of the current literature in that platforms compete *directly* on *both* sides. Doing so qualitatively alters equilibria the understanding of which is important in practice. The insights of this paper are robust to changes in the extensive form and of some modeling details, and so may be applied to multiple markets like newsprint, operating systems or video game consoles, and even healthcare and education (see Bardey and Rochet 2010a and Bardey et al. 2010b). The results also extend where prices are zero on one side, such as broadcasting, search engine competition or social media.

The game considered has three stages: quality setting on one side (B) then price setting on the same side, and price setting on the other side (A). Because of a cross-market externality, the dominant platform on side B is the more attractive one for A-side agents, so vertical differentiation arises endogenously on side A. A unique pure-strategy equilibrium exists only when the A-side is not too lucrative. In this case the optimal quality level of the top firm is lower than the benchmark of maximal differentiation established by Shaked and Sutton (1982, now S&S) and Gabszewicz and Thisse (1979). Usually differentiation is a means of extracting consumer surplus at the cost of surrendering market share to the competition (Hotelling 1929; S&S). But here every B agent allows the platform to extract surplus from side A as well, and so is more valuable. This enhances competition for them. Thus B agents receive a discount commensurate with the profits that can be extracted from side A; then a lesser quality is necessary to attract the marginal B consumer.

The more lucrative is side A, the harder platforms compete for B agents, so the lower are B prices. Beyond a well-defined threshold, the quality-adjusted price of the high-quality firm is so low that it preempts market B, and consequently side A as well. But then the excluded firm possesses a non-local deviation and can monopolize the market too. One must play in mixed strategies. The market may be preempted ex post (in the continuation play), which is a distinct feature of two-sided markets in practice; there is a single eBay, a single Microsoft and a single newspaper in any U.S. city (except for New York City). This pre-emption result is also the only equilibrium when B-side prices are fixed, or bounded, at 0. This applies well to search engine competition or social media: quality is costly and in equilibrium there is a single dominant player (Google, Facebook). Pre-emption and ex post monopolization owe not to a contraction of market B but rather to an *expansion* of the A market, which induces more aggressive competition for B-side consumers.

In a discussion I argue that the results are quite widely applicable. I explain that the introduction of a second externality from A to B does not qualitatively alter any of the results. I also discuss bottlenecks (introduced by Armstrong (2006)) and show that the results are robust to a change in the extensive form.[1]

Capturing the phenomena of quality distortion and pre-emption *requires* there to be direct (price) competition on both side. That is, no platform should be a bottleneck. If a platform is a bottleneck, it is insulated from (price) competition in market A and so it

[1] A "bottleneck" arises when a platform can exercise its market power and thus restrict access to its consumers.

behaves like a monopolist in that market. Then (i) a pure-strategy equilibrium always exists; and (ii) there can be no pre-emption (see details in the Sect. 4). Typically a bottleneck is modeled as allowing a buyer to purchase up to two units, with at most one from each supplier; it can only generate monopoly pricing. The bottleneck assumption severely understates the full extent of the competition between firms, and rules out playing in mixed strategies.

In this paper *direct* competition is re-introduced in the form of a 'single-homing' assumption: both sides have unit demand. Single-homing exacerbates competition.[2] With this, price competition for A-side consumers generates a premium to being the dominant platform on side B. This premium effect is subsided when platforms are bottleneck: they are both local monopolists. When side B is lucrative enough, the premium effect induces payoffs that are not quasiconcave; it is precisely this lack of quasiconcavity that leads to a breakdown of the pure-strategy equilibrium. I also note that single-homing finds empirical support in Kaiser and Wright (2006) in the context of German magazines, in Argentesi and Filistrucchi (2007) in Italian newspapers and in Jin and Rysman (2010), who study sportscard conventions.

The works closest to this paper are Gabszewicz et al. (2001, hereafter GLS) and Dukes and Gal-Or (2003, now DGO), which both study a media duopoly. GLS allow advertisers to place at most one ad on each platform; this is what creates the bottleneck. For a small externality the location equilibrium displays maximal differentiation; if the externality is large enough firms co-locate. In DGO the payoff function is additive over advertisers; this linear separability induces the bottleneck. The equilibrium exhibits minimal differentiation. In the present model there cannot be a pure-strategy equilibrium with minimal differentiation; instead one must play in mixed strategies. Hence we see that the nature of equilibrium varies greatly depending on whether platforms are bottlenecks. Armstrong and Wright (2007) study a model of bottlenecks that shares the essential features of GLS and generates results similar in spirit.

In Ferrando et al. (2008) locations as fixed and prices are set simultaneously on both sides. The equilibria are coordination equilibria in which the market may be preempted by one platform. Gabszewicz et al. (2004) derive three mutually exclusive rational-expectation equilibria: a symmetric, Bertrand equilibrium; a preemption equilibrium and an interior (asymmetric) equilibrium. Here the extensive form calls for subgame perfection, which leads to a unique equilibrium. In the context of health care, Bardey and Rochet (2010a) allow insurance companies to compete for patients (through premia) and service providers (through rebates). Patients are heterogenous in their health risk and thus may value health services differently. This affects health plans' payments to physicians and hospitals, but there is no direct competition. The authors assert that little changes with direct competition on both sides. This suggestion should be weighed with some caution in light of our results. Reisinger (2012) allows for direct competition for homogenous advertisers, while differentiated consumers do not pay for the platforms. Advertisers do not care for the relative quality of a platform, but only for the number of consumers, hence there is no premium effect. This tames competition. Armstrong and Weeds (2007) use a model of horizontal differentiation augmented

[2] What is important for the characteristics of an equilibrium is whether there exists competition on *both* sides, not whether agents single-home or multi-home. Supplement available from the author.

with a quality investment to study the welfare effects of competition between broadcasters. Quality, although not the source of differentiation, may be under-provided in equilibrium. Gonzales-Maestre and Martinez-Sanchez (2015) use a similar model to evaluate the provision of quality and the quantity of advertising shown when a private broadcaster competes for viewers with a welfare-maximizing (public) broadcaster. The presence of the public broadcaster increases the quality of the private provider.

The next Section introduces the model. Section 3 covers the characterization and some implications. Section 4 presents an extensive discussion in which robustness checks are performed. All proofs are sent to the Appendix, as well as some additional technical material.

2 Model

There are two platforms, identified with the subscripts 1 and 2, that market a good (for example, news) to a continuum of B-side consumers of mass 1. Simultaneously it also sells another commodity (such as informative advertising) on the A side. All players have an outside option normalized to 0.

B agents' net utility function is expressed as $u(b, \theta_i, p_i^B) := \theta_i b - p_i^B;\ i = 1, 2$ when facing a price p_i^B. All B agents value quality in the sense of vertical differentiation—there is no ambiguity as to what quality is. The benefit b is uniformly distributed on an interval $[\underline{\beta}, \overline{\beta}]$ and $\theta \in \Theta = [\underline{\theta}, \overline{\theta}]$ denotes the quality parameter of each good. Let $\mathbf{p}^B := (p_1^B, p_2^B), \theta := (\theta_1, \theta_2)$. These consumers buy at most one unit (say, one newspaper). When $\theta_1 > \theta_2$, define the measure $D_1(\mathbf{p}^B, \theta) := Pr(\theta_1 \beta - p_1^B \geq \max\{0, \theta_2 \beta - p_2^B\})$. Hence they will purchase from provider 1 over provider 2 as long as $\beta \geq \hat{\beta} := \frac{p_1^B - p_2^B}{\theta_1 - \theta_2}$.

A agents have gross payoff $eD_i a - p_i^A;\ i = 1, 2$ from consuming one unit of the good, where e is a scaling parameter and a represents the marginal benefit of attracting more B-side agent. The more B agents any A agent can reach, the more they value this service. In the media example this is the marginal benefit of reaching one more consumer. A-side agents are heterogenous in this parameter, which is uniformly distributed on $[\underline{\alpha}, \overline{\alpha}]$ with mass 1. The difference in the platforms' market shares on the B side defines their relative attractiveness on the other side. Given prices $\mathbf{p}^A := (p_1^A, p_2^A)$ and coverage $\mathbf{D} := (D_1, D_2)$, an A-side agent purchases from 1 over 2, only if $eD_1 a - p_1^A \geq \max\{0, eD_2 a - p_2^A\}$. This decision rule generates the measure $Pr(eD_1 a - p_1^A \geq \max\{0, eD_2 a - p_2^A\}) := q_1(\mathbf{p}^A, \mathbf{D})$. Without significant loss there is no externality from the A to the B side (see the Sect. 4). There is no capacity constraint and zero marginal cost.[3]

Assumption 1 $\overline{\beta} - 2\underline{\beta} > 0,\ \overline{\alpha} - 2\underline{\alpha} > 0$ and $\underline{\theta}\underline{\beta} \geq \frac{1}{3}(\overline{\theta} - \underline{\theta})(\overline{\beta} - 2\underline{\beta})$.

This assumption rules out the trivial case in which the low-quality platform necessarily faces zero demand in the price subgames on both sides; it is also sufficient for market coverage on both sides.

[3] A capacity constraint is either trivially exogenous, or endogenous as in Kreps and Scheinkman (1983), which may induce a quantity-setting game instead of the price game.

Quality θ_i is costly and is modeled as an investment with cost $k\theta_i^2$, where we impose.

Assumption 2 $k > (2\overline{\beta} - \underline{\beta})^2/18\overline{\theta}$.

to obtain an interior solution in the benchmark problem (Shaked and Sutton 1982, now S&S).

Game: Platforms first choose a quality level simultaneously. Given observed qualities, they each set prices to B consumers, who make purchasing decisions. With $\mathbf{D}$ observed, they set prices to A agents in a third stage. This extensive form captures some real-life situations.[4] An alternative timing is discussed in Sect. 4; the results are robust to it. The equilibrium concept is Nash subgame-perfect. The three-stage game is denoted Γ. For any platform $i = 1, 2$, the objective function reads

$$\Pi_i := D_i(\mathbf{p}^B, \theta)p_i^B - k\theta_i^2 + q_i(\mathbf{p}^A, \mathbf{D})p_i^A, \tag{2.1}$$

where we see there is an indirect effect from B-side prices (and thus quality) onto side A through demands D_1, D_2. This model is stark and simple. Yet its analysis is somewhat involved, which reflects the fact that two-sided markets present us with intricate problems.

3 Equilibrium analysis

We proceed in three steps, starting with the A side where the platforms' behavior is not directly affected by B-side quality choices.

3.1 Price subgames

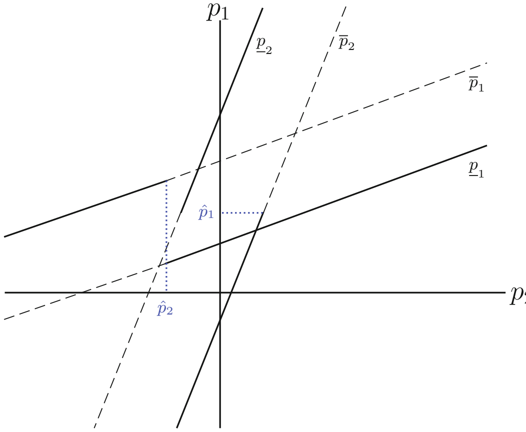
A **-market subgame.** This stage replicates the result of the classical analysis of vertical differentiation. Let $e\Delta D = e\cdot(D_1 - D_2)$ denote the scaled difference in the platforms' quality. Then equilibrium payoffs take a simple form, for which the proof is standard and therefore omitted (see Tirole 1988).

Lemma 1 *Suppose $D_1 \geq D_2$ w.l.o.g. There may be three pure strategy equilibria in the A market. When $D_1 > D_2 > 0$, the profit functions write $\overline{\Pi}_1^A = e\Delta D \cdot (\frac{2\overline{\alpha}-\underline{\alpha}}{3})^2$; $\underline{\Pi}_2^A = e\Delta D \cdot (\frac{\overline{\alpha}-2\underline{\alpha}}{3})^2$. When $D_1 > D_2 = 0$, platform 1 is a monopolist and its profits are $\Pi_1^{AM} = eD_1 \cdot (\frac{\overline{\alpha}}{2})^2$. For $D_1 = D_2$, the Bertrand outcome prevails and platforms have zero profits.*

It is also helpful to recall the equilibrium demand functions on side B : $D_i = \overline{\beta} - \frac{p_i^B - p_j^B}{\Delta\theta}$, $D_j = \frac{p_i^B - p_j^B}{\Delta\theta} - \underline{\beta}$ for $\theta_i > \theta_j$. As usual, denote $\Delta\theta = \theta_i - \theta_j$ and for convenience $\overline{A} = (\frac{2\overline{\alpha}-\underline{\alpha}}{3})^2$ and $\underline{A} = (\frac{\overline{\alpha}-2\underline{\alpha}}{3})^2$.

[4] For example, in the case of media, B-prices (cover prices or subscription rates) are more difficult to change than A-prices (advertising rates), and the media format even more so. Also, readership is often reported to advertisers, so known to them when they purchase.

Fig. 1 Best replies and unique equilibrium



***B*-side price subgame.** From Lemma 1 three distinct configurations may arise on the equilibrium path. In the first case platform 1 dominates the B market, in the second one both share the B market equally and in the last one it is dominated by firm 2. Hence the profit function (2.1) rewrites

$$\Pi_i = p_i^B D_i(\mathbf{p}^B, \theta) - k\theta_i^2 + \begin{cases} \overline{\Pi}_i^A, & \text{if} \quad D_i > D_j; \\ 0, & \text{if} \quad D_i = D_j; \\ \underline{\Pi}_i^A, & \text{if} \quad D_i < D_j. \end{cases} \tag{3.1}$$

This function is continuous with a kink at the profile of prices $\tilde{\mathbf{p}}^B$ such that $D_1 = D_2$.[5] More importantly it is not quasi-concave because of the externality generated by A-side revenue; therefore the best response is discontinuous. It is nonetheless possible to construct a *unique* equilibrium in pure strategies, which always exists. (Note that observing $\theta_1 > \theta_2$ acts like a coordination device; it rules out multiple equilibria.) The demonstration is left to the Appendix, Section C; here we discuss it briefly and focus on its outcome.

First, from (3.1), it is immediate that any price profile $\tilde{\mathbf{p}}^B$ such that $D_1 = D_2$ is dominated. Next we can define 'quasi best responses' $p_i(p_j)$ corresponding to platforms playing as if either $D_1 > D_2$ or $D_1 < D_2$ (for example, $\overline{p}_2$, $\underline{p}_2$ in Fig. 1), from which we can construct the true best replies—discontinuous at the points $\hat{p}_1$, $\hat{p}_2$.[6] Last, a necessary and sufficient condition for existence is verified by construction. In summary,

[5] Continuity implies that the monopoly outcome is also nested.

[6] The discontinuity set is not trivial: mixed strategies cannot restore the second candidate equilibrium—see Fig. 1. Indeed an outcome such that $\theta_1 > \theta_2$ and $D_1 < D_2$ entails playing a weakly dominated strategy for player 2. So the intuitive reasoning whereby the low-quality firm may find it profitable to behave very aggressively in order to access large advertising revenue does not hold true. That is, playing $\theta_i < \theta_2$ but offering a very low price p_i^B so that $D_i > D_j$.

Proposition 1 *Let $\theta_1 > \theta_2$ w.l.o.g. There may be two possible configurations arising in the B-side price subgame. For each, there exists a unique Nash equilibrium in pure strategies*:-

- *For* $\Delta\theta > \frac{2e(\overline{A}+\underline{A})}{\overline{\beta}-2\underline{\beta}}$

$$p_1^{B*} = \frac{1}{3}[\Delta\theta(2\overline{\beta} - \underline{\beta}) + 2e(\underline{A} - 2\overline{A})]$$
$$p_2^{B*} = \frac{1}{3}[\Delta\theta(\overline{\beta} - 2\underline{\beta}) + 2e(2\underline{A} - \overline{A})]$$

- *If* $\Delta\theta \leq \frac{2e(\overline{A}+\underline{A})}{\overline{\beta}-2\underline{\beta}}$

$$p_1^{B*} = \frac{\Delta\theta\overline{\beta}}{2} - e\overline{A}; \quad p_2^{B*} = 0$$

B-side prices resemble the S&S prices but include a discount ($\underline{A}-2\overline{A} < 2\underline{A}-\overline{A} < 0$) that is linear in the A-side profits. Platforms internalize the full value of the B agents, which intensifies competition for their patronage, and pass it on to them in the form of this price reduction. The quality spread $\Delta\theta$, which is fixed in the first stage, may be too narrow to sustain two firms in the price subgame. That is, the high-quality platform may be able to pre-empt the market with its quality choice, thank to the cross-market externality.

In the first stage, platforms face the profit function (3.1) given equilibrium prices, which they each maximize by choice of their quality variable θ_i. In doing so they are subject to the constraint

$$\hat{\beta} := \frac{p_i^{B*} - p_j^{B*}}{\theta_i - \theta_j} \in [\underline{\beta}, \overline{\beta}], \tag{3.2}$$

which is a natural restriction guaranteeing that the endogenous threshold $\hat{\beta}$ remain within the exogenous bounds $[\underline{\beta}, \overline{\beta}]$.[7]

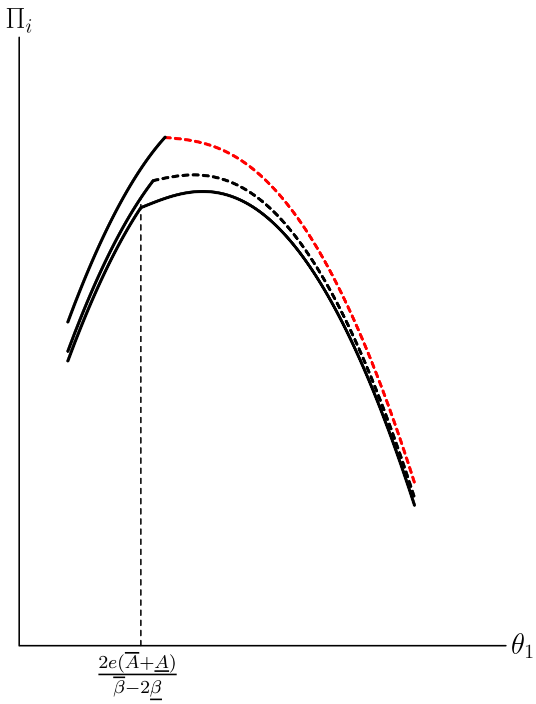
These profit functions are not necessarily well-behaved. Section A of the Appendix studies $\Pi_1(\theta_1, \theta_2)$ in the details necessary to support the results. Next we delineate when the equilibrium features pre-emption (and not).

3.2 Pure-strategy equilibrium

When the externality from side B to side A is not too large, the function $\Pi_1(\cdot, \cdot)$ is well-behaved. It remains increasing (and concave) on the portion *beyond* a well-defined threshold labeled $\tilde{\theta}(e)$ for the high quality firm, where it admits a maximizer.

[7] $\theta_i \to \theta_j \Rightarrow \hat{\beta} \to \infty$. On the equilibrium path Constraint (3.2) can be rearranged as a pair of inequalities: $\Delta\theta(2\overline{\beta} - \underline{\beta}) + 2e(\overline{A} + \underline{A}) \geq 0$ and $\Delta\theta(\overline{\beta} - 2\underline{\beta}) - 2e(\overline{A} + \underline{A}) \geq 0$. Only the second one is constraining.

Fig. 2 Profit functions for different values of the A-side profits



This is illustrated in Fig. 2 (the higher curve corresponds to the complementary case, when it is not well-behaved).[8] To ensure this is the case we impose

Assumption 3 $e < \bar{e} \equiv \min\left\{1, \left(\frac{(2\overline{\beta}-\underline{\beta})^2}{27k} - \underline{\theta}\right)\frac{\overline{\beta}-2\underline{\beta}}{2(\overline{A}+\underline{A})}\right\}$,[9]

which is tantamount to saying the A market is not too lucrative. Assumption 3 ensures that when $\hat{\theta}_1$ solves the first-order condition, the quality difference $\Delta\theta$ is large enough: $\Delta\theta \geq \frac{2e(\overline{A}+\underline{A})}{\overline{\beta}-2\underline{\beta}}$ so that both platforms operate (Proposition 1). Then,

Proposition 2 *Suppose Assumption 3 holds. The game* Γ *admits a unique equilibrium in pure strategies in which both platforms operate and choose different qualities. It is characterized by the triplet* $(\mathbf{p}^{B*}, \mathbf{p}^{A*}, \theta^*)$ *defined by Proposition 1, Lemma 1, and the optimal actions* $\theta_2^* = \underline{\theta}$ *and* θ_1^*, *where* θ_1^* *uniquely solves*

$$(2\overline{\beta} - \underline{\beta})^2 = 18k\theta_1 + \left(\frac{2e(\overline{A} + \underline{A})}{\Delta\theta}\right)^2 \tag{3.3}$$

[8] The term of interest is $e(\overline{A}+\underline{A})$, which needs to be small enough. Then beyond $\theta_1 = \tilde{\theta}(e) := \underline{\theta} + \frac{2e(\overline{A}+\underline{A})}{\overline{\beta}-2\underline{\beta}}$, the function Π_1 remains concave in θ_1.

[9] This arises from the condition $(\hat{\theta}_1 - \underline{\theta})(\overline{\beta} - 2\underline{\beta}) > (\theta_1^f - \underline{\theta})(\overline{\beta} - 2\underline{\beta}) \geq 2e(\overline{A} + \underline{A})$, where $\theta_1^f = \frac{(2\overline{\beta}-\underline{\beta})^2}{27k}$ is defined in Appendix Section A.

The second term in (3.3) is labeled the 'cross-market effect'; it acts as an incentive to reduce quality. Condition (3.3) trades off the marginal benefit of quality (the left-hand side) with its *total* marginal cost. That total cost includes the marginal loss of A-side profit induced by differentiation: the cross-market effect. The intuition is quite simple. More differentiation leads to higher B-side prices; but higher prices means surrendering B-side market share, thereby foregoing A-side profits. So the cross-market effect increases the cost of differentiation. Comparative statics show that θ_1 is decreasing in e: the more attractive the A-side profit, the more powerful the cross-market effect and the more muted is the Differentiation Principle. We can expand on the insights of Proposition 2, where we take S&S to be the benchmark.

Corollary 1 *In any pure-strategy equilibrium of the game* Γ, *quality is lower than it would be absent the* A*-market externality.*

Differentiation is known to soften price competition, but here the cross-market externality puts emphasis back on market share and forces the platforms to engage in more intense price competition for B consumers. Lower consumer prices relax the need to provide costly quality: the marginal B consumer demands a lesser product. This result owes to the increased value of each B-side consumer, which renders differentiation costlier.

3.3 Mixed strategies

When the externality from B to A is sufficiently large a pure strategy equilibrium fails to exist. The mechanics are quite intuitive. The extent of the discount firm must offer increases in the externality, and the high-quality platform can increase its price dominance by lowering quality. That is, it has an incentive to select θ_1 low enough so that $\Delta\theta$ is too narrow for firm 2 to have positive B-side market share, if firm 2 selects $\theta_2 = \underline{\theta}$. But firm 2 does not have to play $\underline{\theta}$. In fact it can "leap" over firm 1 and become the monopolist at a negligible incremental cost. Then one must play in mixed strategies.[10]

The Appendix (Section B) shows that a mixed-strategy equilibrium always exists. Let $H_i(\theta_i)$ be the probability distribution over i's play and $h_i(.)$ the corresponding density, Θ_i^N the relevant support of H_i and θ_i^c the upper bound of the support. Let also H_i^* be a best response and $R_i(\theta_i, \theta_j) := D_i(\mathbf{p}^B, \theta)p_i^B + q_i(\mathbf{p}^A, \mathbf{D})p_i^A$ denote the revenue accruing to i.[11]

Proposition 3 *The symmetric mixed-strategy equilibrium of the game* Γ *is characterized by the pair of distributions* H_1, H_2 *on* $\Theta_i^N \equiv \{\underline{\theta}\} \cup [\tilde{\theta}(e), \theta^c]$, $i = 1, 2$ *satisfying*

[10] Technically, when Assumption 3 is not satisfied, the necessary first-order condition (3.3) fails to hold entirely. As can be seen on Fig. 2, the high-quality firm would like to pick the point $\tilde{\theta}(e)$, where $\Pi_1(\cdot,\cdot)$ reaches is maximum. But this cannot be an equilibrium; the incremental cost is $k(\theta_1+\varepsilon)^2 - k\theta_1^2$.

[11] The proof adapts the work of Sharkey and Sibley (1993), who characterize mixed strategies in a problem of entry with sunk cost. The major difference is there is no proper entry stage; playing $\theta_i = \underline{\theta}$ cannot be interpreted as a decision to not enter the market because $\Pi_i(\underline{\theta}_i, \theta_j) > 0$ whenever $\Delta\theta > \frac{2e(\overline{A}+\underline{A})}{\overline{\beta}-2\underline{\beta}}$.

$$H_i(\underline{\theta}) \int_{\Theta_j^N} R_i(\underline{\theta}, \theta_j) dH_j^*(\theta_j) + \int_{\theta_i=\theta_j}^{\theta^c} R_i(\theta_i, \theta_j) d(H_i(\theta_i) \times H_j^*(\theta_j))$$
$$= k \int_{\tilde{\theta}(e)}^{\theta_i^c} \theta_i^2 d(H_i(\theta_i) \times H_j^*(\theta_j)) \tag{3.4}$$

with

$$H_i^*(\underline{\theta}) \in (0, 1), \; H(\theta^c) = 1 \quad \textit{and} \quad h_i(\theta_i) = 0, \; \theta_i \in (\underline{\theta}, \tilde{\theta}(e))$$

and θ^c *defined by* $\theta^c = \max\{\theta_i' | \Pi_i(\underline{\theta}, \theta_i') = 0, \Pi_i(\tilde{\theta}(e), \theta_i') = 0\}, i = 1, 2.$[12]

Condition (3.4) balances the expected benefit from adopting the distribution H_i with its expected cost. An interesting feature of the mixed-strategy equilibrium is that the platforms do not mix over all the pure actions that are available to them. To see why, suppose firm 1 picks any action higher than firm 2's (so $\theta_1 > \theta_2$); playing $\theta_1 = \tilde{\theta}(e)$ dominates any other play below $\tilde{\theta}(e)$ because profits are increasing on that range (see Fig. 2). In response, playing anything but $\theta_2 = \underline{\theta}$ is dominated because $\underline{\theta}$ secures 0 while any other play generates a loss. That is, the range $(\underline{\theta}, \tilde{\theta}(e))$ is dominated and no mass should be assigned on it. Even if platform 1 selects a quality beyond the preemption point $\tilde{\theta}(e)$, firm 2's profits are still maximized by playing $\underline{\theta}$ because they decrease in θ_2. Hence $\underline{\theta}$ remains a best response to any quality $\theta_1 \geq \tilde{\theta}(e)$. Therefore there must be an atom at that point. Last, platform 1 must assign some probability mass on the range $(\tilde{\theta}(e), \theta^c]$ otherwise it is necessarily preempted by 2's non-local deviation.

In a mixed-strategy equilibrium the realizations of qualities (θ_1, θ_2) are random variables. Hence these equilibrium distributions do not rule out an outcome such that $\Delta\theta$ is actually too small to sustain two firms; they guarantee that it does not happen with probability one. It is helpful to know under what conditions two platforms may operate in the continuation game after choosing their quality.

Proposition 4 *Suppose $e > \bar{e}$. For two platforms to have positive market share in the price subgame, one of them must select the lowest quality $\underline{\theta}$. Otherwise the market is necessarily monopolized ex post.*

Recall Proposition 1; depending on the choice of θ_1, θ_2, platform 2 may or may not have any market share on the equilibrium path. However the length of the interval $[\tilde{\theta}(e), \theta^c]$ is not sufficient to accommodate two firms.[13] So for both platforms to survive, at least one of them must choose the lowest quality.

Proposition 4 compares favorably to some industries' idiosyncrasies. First, either monopolization or duopoly may be an ex post outcome, which fits some industry patterns. Markets such as print media, internet trading platforms or search engines are known to tip. This suggests an alternative rationale for the observed concentration in

[12] The notation $\theta_i = \theta_j$ in the second integral of (3.4) reflects that for $\tilde{\theta}(e) \leq \theta_i < \theta_j$, firm i collects zero.

[13] $\theta^c - \tilde{\theta}(e) < 2e(\overline{A} + \underline{A})/(\overline{\beta} - 2\underline{\beta})$—although clearly $\theta^c - \underline{\theta} \geq 2e(\overline{A} + \underline{A})/(\overline{\beta} - 2\underline{\beta})$, corresponding to the condition $\Delta\theta \geq 2e(\overline{A} + \underline{A})/(\overline{\overline{\beta}} - 2\underline{\beta})$.

these markets. According to this model, some players may be driven out not because of a market contraction on the B side, but because of an *expansion* on the other one. Second, ex post profits are not monotonically ranked: the action profile $(\underline{\theta}_1, \theta_2^c)$ implies $\Pi_1 > \Pi_2 = 0$ although $\theta_1 < \theta_2$. So too in media for example, where the higher-quality shows (e.g. HBO) or magazines (e.g. The New Yorker) do not necessarily yield higher profits. This implication departs from standard vertical differentiation models (such as S&S), and from the pure strategy equilibrium, where higher quality implies higher profits.

3.4 Zero prices on one side

Many two-sided markets feature zero prices on at least one side. This may be an equilibrium outcome or an exogenous imposition (or both in the sense of binding constraint). Examples include broadcasting, internet search engine or social media usage.

Proposition 5 *Fix $p_1^B = p_2^B = 0$. A pure-strategy equilibrium does not exist. A mixed-strategy equilibrium exists and is characterised as in Proposition 3.*

Proposition 5 tells us we should expect pre-emption in these markets. The examples of Google (users do not pay) or eBay (buyers do not pay fees) lend credence to this claim. These outcomes do not arise in a model without competition on both sides.

4 Discussion

This Discussion is offered largely without proof. These proofs do exist and are available from the author.

4.1 One-sided or two-sided externality

The model ignores any externality the side A exerts on B agents. Media consumers may dislike advertising; game developers seek more gamers to market to, and these likely enjoy games' diversity.

Introducing a second externality from A to B does not modify the results qualitatively, which implies the results are quite widely applicable. A negative A-to-B externality effectively damages the B-side quality of the platforms. In response they must offer a further discount; the dominant platform can offer a steeper discount than the dominated platform. This feedback thus hardens competition on side B. This narrows the range of parameters on which the pure-strategy equilibrium can be sustained. This is in line with DGO's results, for example, who show that the negative externality associated with adverts leads to minimal differentiation.

To see why, rewrite the B-side utility function as $u_i = \theta_i b - p_i^B - \delta q_i$, where δq_i is a disutility from A-side consumption level. A-side demand is defined as before; suppose $\theta_1 > \theta_2$, B demands are $D_1 = \overline{\beta} - \frac{\Delta p^B + \delta \Delta \tilde{q}}{\Delta \theta}$ and $D_2 = \frac{\Delta p^B + \delta \Delta \tilde{q}}{\Delta \theta} - \underline{\beta}$. The new term is $\delta \Delta \tilde{q}$: the utility impact of the difference in A-side *expected* consumption levels; these can be computed given $(\theta, \mathbf{p})$. It can be shown that $\Delta\tilde{q} = (\overline{\alpha} + \underline{\alpha})/3$: a *constant*. Let $(\overline{\alpha} + \underline{\alpha})/3 \equiv \hat{A}$, eventually the condition for platform 2 to be active turns into $D_2 \geq 0 \Leftrightarrow \Delta\theta(\overline{\beta} - 2\underline{\beta}) \geq 2e(\overline{A} + \underline{A}) + \hat{A}$, which is *more* restrictive than the one of Proposition 1.

4.2 Bottlenecks and preemption

Suppose that A-side agents are able to place at most one ad on *each* platform, as in GLS. Then they are a monopoly on side A with profits $\pi_i^A = \overline{\alpha}^2 e D_i/4$. Equilibrium prices can be computed as

$$p_1^B = \frac{1}{3}\left[\Delta\theta(2\overline{\beta} - \underline{\beta}) - \frac{3e\overline{\alpha}^2}{4}\right]; \quad p_2^B = \frac{1}{3}\left[\Delta\theta(\overline{\beta} - 2\underline{\beta}) - \frac{3e\overline{\alpha}^2}{4}\right]$$

The standard price functions $p_i(\theta)$ are only shifted by $e\overline{\alpha}^2/4$ each—*independently* of what the other platform does. After simple manipulations, the profits functions write

$$\Pi_1 = \Delta\theta\left(\frac{2\overline{\beta} - \underline{\beta}}{3}\right)^2 - k\theta_1^2; \quad \Pi_2 = \Delta\theta\left(\frac{\overline{\beta} - 2\underline{\beta}}{3}\right)^2 - k\theta_2^2$$

exactly as in S&S. So the externality is present and affects prices, but not the quality choices. When platforms are bottlenecks, the pass-through is perfect: consumers (B) receive a discount that exactly exhausts what platforms can extract from the other side (A). Then the incentives at the quality setting stage are standard. There is no incentive to decrease quality nor for endogenous pre-emption through quality. The exact same outcome obtains if introducing a A-to-B externality together with the bottleneck assumption.

4.3 Robustness check: simultaneous moves

The three-stage game suits some industries well (e.g. media), but not necessarily all. For example, Hagiu (2006) studies the problem of game console manufacturers, who must *simultaneously* commit to a price on each side of the platform. The analysis is robust to this change in timing, except for one small variation.[14] Consider the platforms' problem at the price-setting stage given some $\theta_1 > \theta_2$ and expected $\widetilde{D}_1 > \widetilde{D}_2$:-

$$\max_{p_1^A, p_1^B} \Pi_1 = p_1^B\left[\overline{\beta} - \frac{p_1^B - p_2^B}{\Delta\theta}\right] + p_1^A e\left[\overline{\alpha} - \frac{p_1^A - p_2^A}{\Delta\widetilde{D}}\right]$$
$$\max_{p_2^A, p_2^B} \Pi_2 = p_2^B\left[\frac{p_1^B - p_2^B}{\Delta\theta} - \underline{\beta}\right] + p_2^A e\left[\frac{p_1^A - p_2^A}{\Delta\widetilde{D}} - \overline{\alpha}\right]$$

[14] Here we discuss the results; the derivations can be found in a supplement available from the author.

The first-order condition with respect to p_i^A, $i = 1, 2$ remain standard; from this $p_1^A = e\frac{\Delta\widetilde{D}}{3}[2\overline{\alpha} - \underline{\alpha}]$; $p_2^A = \frac{\Delta\widetilde{D}}{3}[\overline{\alpha} - 2\underline{\alpha}]$ as before. The first-order conditions w.r.t. p_i^B simplify to

$$\Delta\theta\overline{\beta} - (2p_1^B - p_2^B) - \frac{2}{9}e[(2\overline{\alpha} - \underline{\alpha})(\overline{\alpha} + \underline{\alpha})] = \Delta\theta\overline{\beta} - (2p_1^B - p_2^B) - 2A_1 = 0$$

$$-\Delta\theta\underline{\beta} + (p_1^B - 2p_2^B) - \frac{2}{9}e[(\overline{\alpha} - 2\underline{\alpha})(\overline{\alpha} + \underline{\alpha})] = -\Delta\theta\underline{\beta} + (p_1^B - 2p_2^B) - 2A_2 = 0$$

These are linear equations in B prices, as in the sequential move model. This readily suggests that little will change from this new timing. This linearity arises because A profits are still linear in $\Delta\widetilde{D}$. The solution concept is Nash equilibrium, the best replies are discontinuous and there is a unique equilibrium in prices, with a condition on $\Delta\theta$. That condition is also less restrictive than in the sequential-move game.

Things do change a little in the first stage. When the pure-strategy equilibrium can be sustained, both first-order conditions *may* bind, thus yielding interior solutions for *both* platforms. This is in contrast to the sequential game. But this behavior is non-monotonic: for naught A-side profits platform 2 benefits from maximal differentiation, for low A profits it seeks less differentiation (smaller $\Delta\theta$), and for large enough A profits, maximal differentiation again. The reason is that under simultaneous moves, the discount offered by the dominant firm in the B market is smaller. So ΔD—the difference in their market share—is also smaller. As a consequence it is also less dominant in the A market and the condition on $\Delta\theta$ is less tight. This creates an incentive for the low-quality firm to capture some market share in B by increasing quality. In the sequential game, the discounts are such that platform 2 never has such an incentive.

This difference in discounts owes to the timing. By way of (imperfect) analogy, one can consider the difference between a Cournot and a Stackelberg game. In the latter, the dominant firm *commits* to a strategy and the follower takes it as given. By the time they move in the A market, platforms are committed to a strategy in the B market. This generates incentives for platforms to behave more aggressively in the B market in the fist place.

5 Conclusion

This paper has developed an analysis of differentiation in a duopoly of two-sided platforms, where competition prevails on *both* sides of the market. This yields markedly different results, as compared to those typically found in the literature. Direct competition on the A side puts a premium on being the better platform (here meaning covering a larger share) on side B. This exacerbates competition in market B, with consequences on the nature of equilibrium. Whether a pure-strategy equilibrium exists depends on the relative attractiveness of A-side profits; that is, we can identify why it may break down. This paper thus complements prior works, in particular GLS and DGO who analyzed cases of bottleneck competition.

When a pure-strategy equilibrium exists, differentiation is hampered because too costly in terms of market share. The more attractive the A side, the narrower is differentiation. It may be insufficient to sustain two active platforms, at which point the equilibrium breaks down. Then platforms play in mixed strategies and one of them may be preempted ex post. These results are robust to a change in timing; all carry over to quantity competition in the B market and the mixed strategy equilibrium remains valid under horizontal differentiation. Hence they are not exclusive to the chosen extensive form and may find applications in a broad array of industries.

Our ability to compute an equilibrium rests on the simple structure chosen, and in particular on two important assumptions: single-homing and independence between A and B-side consumption decisions. Single-homing is not essential but it is convenient. What is essential is that platforms compete directly for consumers on both sides, which single-homing captures. Independence in consumption decisions is important; it implies that the A side only cares for the B-side market share, not its composition. For example, it asserts that the choice of media consumption is not a signal for good consumption. But we do know that media companies strive to segment their markets to suit advertisers. These characteristics are so far left out for future research.

Acknowledgements I am most grateful to Jean-Charles Rochet for helpful conversations. Thank you to Luis Corchon and Bill Schworm for their comments, and to Carlos Pimienta, Suraj Prasad, Francesco Sobbrio and Luis Vasconcelos for patiently reading earlier versions.

Appendix

The Appendix contains some additional material as well the proofs of the propositions.

Appendix A: Analysis of the high-quality firm's profit function

Let $C \equiv [2e(\overline{A} + \underline{A})]^2$. With the reformulation of constraint (3.2), the objective function of platform 1 writes

$$\Pi_1 = \begin{cases} \frac{1}{9}\left(\Delta\theta(2\overline{\beta} - \underline{\beta})^2 + B_1 + \frac{C}{\Delta\theta}\right) - k\theta_1^2, & \text{if } \Delta\theta > \frac{2e(\overline{A}+\underline{A})}{\overline{\beta}-2\underline{\beta}}; \\ \frac{1}{9}(\Delta\theta(2\overline{\beta} - \underline{\beta})^2 + B_1 + \sqrt{C}(\overline{\beta} - 2\underline{\beta})) - k\theta_1^2, & \text{if } \Delta\theta \leq \frac{2e(\overline{A}+\underline{A})}{\overline{\beta}-2\underline{\beta}} \end{cases} \tag{6.1}$$

where $B_1 = (2\overline{\beta} - \underline{\beta})2e(2\underline{A} - \overline{A}) + 3e(\overline{\beta} + \underline{\beta})\overline{A}$ is a constant. From the first line of (6.1) we can see why (3.2) is necessary: depending on C, the platform may seek a large or small $\Delta\theta$. The second line of (6.1) rules out the artificial case of firm 1 facing a demand larger than the whole market.[15] For platform 2, profits are

$$\Pi_2 = \begin{cases} \frac{1}{9}\left(\Delta\theta(\overline{\beta} - 2\underline{\beta})^2 + B_2 + \frac{C}{\Delta\theta}\right) - k\theta_2^2, & \text{if } \Delta\theta(\overline{\beta} - 2\underline{\beta}) > 2e(\overline{A} + \underline{A}); \\ 0, & \Delta\theta(\overline{\beta} - 2\underline{\beta}) \leq 2e(\overline{A} + \underline{A}) \text{ and } \theta_2 = 0; \\ -k\theta_2^2, & \Delta\theta(\overline{\beta} - 2\underline{\beta}) \leq 2e(\overline{A} + \underline{A}) \text{ and } \theta_2 > 0; \end{cases} \tag{6.2}$$

15 It is derived by taking $\frac{C}{\Delta\theta}$ as fixed at its lowest value, that is, where $\Delta\theta = \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}} = \frac{2e(\overline{A}+\underline{A})}{\overline{\beta}-2\underline{\beta}}$.

with $B_2 = (\overline{\beta} - 2\underline{\beta})2e(\underline{A} - 2\overline{A}) + 3e(\overline{\beta} + \underline{\beta})\underline{A}$. In the sequel $\theta_1 > \theta_2$ without loss of generality. For this Section, take Proposition 1 as established. The profit function Π_1 is obviously continuous for $\theta_1 - \underline{\theta} < \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$ or for $\theta_1 - \underline{\theta} > \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$. Further assume $e < \infty$.

Claim 1 *The function* Π_1 *is continuous for* $\Delta\theta = \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$.

Proof For ease of notation, let $\Pi_1(\theta_1, \theta_2) = \Pi_1^L$ for all $\Delta\theta \geq \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$ and $\Pi_1(\theta_1, \theta_2) = \Pi_1^R$ otherwise. To the left platform 1 is a monopolist whose profits Π_1^L are necessarily bounded. The function is defined as $\Pi_1^L : \Theta_1 \times \Theta_2 \subseteq \mathbb{R}^2 \mapsto \mathbb{R}$, therefore Theorem 4.5 in Haaser and Sullivan applies. So $\Pi_1^L(\theta_1, \theta_2)$ is continuous at $\Delta\theta = \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$, and is necessary the left-hand limit of the same function Π_1^L. Now consider a sequence θ_1^n such that $\Delta\theta > \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$ converging to $\frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$ from above for some fixed θ_2. This sequence exists and always converges for $\Theta_1 \subseteq \mathbb{R}$ is complete. As $e < \infty$ and $\overline{A}$ and $\underline{A}$ are necessarily bounded, C is finite so there is some n and some arbitrarily small δ such that $\Pi_1^R(\theta_1^n, \theta_2) - \Pi_1^L(\theta_2 + \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}, \theta_2) < \delta$. That is, $\lim_{\theta_1^n \to \theta_2 + \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}} \Pi_1^R(\theta_1^n) = \Pi_1^L(\theta_2 + \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}, \theta_2)$. Hence Π_1 is continuous for $\Delta\theta = \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$. □

When C becomes large enough, $\Pi_1(\theta_1, \theta_2)$ is no longer well behaved.

Claim 2 *There exists some* $C^f := [\frac{(2\overline{\beta}-\underline{\beta})^2}{27k} - \underline{\theta}]^2(\frac{(2\overline{\beta}-\underline{\beta})^2}{3})$ *such that* Π_1 *admits a binding first-order condition for* $C \leq C^f$ *only. When* $C > C^f$*, its maximum is reached at the kink:* $\theta_1 = \underline{\theta} + \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}} := \tilde{\theta}(e)$.

Proof Seeking first-order conditions of $\Pi_1(.,.)$ with respect to θ_1 yields

$$\frac{\partial \Pi_1}{\partial \theta_1} = \begin{cases} \left(\frac{2\overline{\beta}-\underline{\beta}}{3}\right)^2 - 2k\theta_1 = 0, & \text{for } \Delta\theta \leq \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}; \\ \left(\frac{2\overline{\beta}-\underline{\beta}}{3}\right)^2 - \frac{C}{(3\Delta\theta)^2} - 2k\theta_1 = 0, & \text{for } \Delta\theta > \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}} \text{ and } C \leq C^f; \\ \left(\frac{2\overline{\beta}-\underline{\beta}}{3}\right)^2 - \frac{C}{(3\Delta\theta)^2} - 2k\theta_1 < 0, & \text{for } \Delta\theta > \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}} \text{ and } C > C^f; \end{cases} \tag{6.3}$$

The second line of system (6.3) can be rearranged as $(2\overline{\beta} - \underline{\beta})^2 = \phi(\theta_1)$, with slope $\phi'(\theta_1) = 18k - \frac{2C}{(\Delta\theta)^3}$. This FOC has at most two solutions: one where $\phi'(\theta_1) < 0$ and the other with $\phi'(\theta_1) > 0$. The SOC requires $\phi'(\theta_1) \geq 0$ for the FOC to identify a maximiser, so there exists a unique local maximiser of Π_1, denoted $\hat{\theta}_1$. Let θ_1^0 be the (unique) solution of the first line of system (6.3) with $\hat{\theta}_1 < \theta_1^0$. Therefore if $\theta_1^0 - \theta_2 \leq \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$ it cannot be that θ_1 is a best response to θ_2. That is, firm 1 would not play the first line of (6.1), but the second one. Hence the FOC cannot bind when $\Delta\theta \leq \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$:

$$\frac{\partial \Pi_1}{\partial \theta_1} = \left(\frac{2\overline{\beta}-\underline{\beta}}{3}\right)^2 - 2k\theta_1 > 0; \quad for\ \Delta\theta \leq \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$$

When the FOC does bind (second line of 6.3), the function Π_1 is concave for $C \leq C^f$ and $\hat{\theta}_1$ is a global maximizer. To pin C^f, the binding first-order condition defines a function $C(\theta_1, \theta_2) := (\Delta\theta)^2[(2\overline{\beta}-\underline{\beta})^2 - 18k\theta_1]$, whence $\frac{dC(.)}{d\theta_1} = 0 \Leftrightarrow \theta_1^f = \frac{(2\overline{\beta}-\underline{\beta})^2}{27k}$. Substituting back into $C(\theta_1, \theta_2)$ gives the cut-off value $C^f := [\frac{(2\overline{\beta}-\underline{\beta})^2}{27k} - \theta_2]^2(\frac{(2\overline{\beta}-\underline{\beta})^2}{3})$. When $C > C^f$, the first-order condition (6.3) is everywhere negative, hence

$$\frac{d\Pi_1}{d\theta_1}|_{\theta_1 < \underline{\theta} + \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}} > 0; \quad \frac{d\Pi_1}{d\theta_1}|_{\theta_1 > \underline{\theta} + \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}} < 0$$

and is not differentiable at $\Delta\theta = \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$. By Claim 1 it is continuous, and monotonic on either side of $\Delta\theta = \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$. Therefore, $\hat{\theta}_1$ such that $\Delta\theta = \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$ is the unique maximiser of $\Pi_1(\theta_1, \theta_2)$ given some fixed θ_2. □

Appendix B: Existence of a mixed-strategy equilibrium

Take Proposition 1 as established. Throughout consider suppose $\theta_1 \geq \theta_2$ w.l.o.g.

Proposition 6 *A mixed-strategy equilibrium of the game* Γ *always exists.*

This assertion holds trivially when Assumption 3 holds. When it fails the payoff correspondences are not upper-hemicontinuous and their sum is not necessarily so either.

Denote $\tilde{\theta} := \underline{\theta} + \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$ for some e. Let θ_1^c the threshold such that $\Pi_1(\theta_1^c, \underline{\theta}) = 0$ when $\theta_1 > \theta_2$. This point exists and exceeds $\tilde{\theta}_1$ because $\frac{d\Pi_1}{d\theta_1}|_{\theta_1 > \tilde{\theta}_1} < 0$ and the cost function is convex. Neither platform wants to exceed that threshold, so the set of pure actions over which firms randomize is $[\underline{\theta}, \theta_i^c] \subseteq \Theta_i,\ i = 1, 2$. Any distribution over this set must assign zero mass to any $\theta_i \in (\underline{\theta}, \tilde{\theta})$: any action in this interval is dominated by either $\underline{\theta}$ or $\tilde{\theta}$. Take $\theta_1 > \theta_2 > \underline{\theta}$ and suppose $\Delta\theta > \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$ and $\Pi_1 > \Pi_2 > 0$. Let θ_2 increase, both Π_1 and Π_2 vary smoothly with $\lim_{\theta_2^n \uparrow \theta_1} \Pi_1 = \Pi_1 > 0$, $\lim_{\theta_2^n \downarrow \theta_1} \Pi_1 = -k\theta_1^2$, and similarly for firm 2. Both payoff functions are discontinuous at the point $\theta_1 = \theta_2$. In this case neither the payoffs nor their sum are even upper-hemicontinous. Following Dasgupta and Maskin (1986), it is first necessary to characterize the discontinuity set $\Upsilon_0 := \{(\theta_1, \theta_2) | \theta_1 = \theta_2, \theta_i \in [\tilde{\theta}_i, \theta_i^c]\ \forall i\}$, on which the payoffs are discontinuous. Further define the probability measure $\mu(\theta_1, \theta_2)$ over the set $\Theta^N = \{\underline{\theta}_1\} \cup [\tilde{\theta}_1, \theta_1^c] \times \{\underline{\theta}_2\} \cup [\tilde{\theta}_2, \theta_2^c]$. It is immediate that Υ_0 has Lebesgue measure zero, so that $\Pr((\theta_1, \theta_2) \in \Upsilon_0) = 0$. Next we claim

Lemma 2 *Suppose $\theta_1 = \theta_2 = \underline{\theta}$, an equilibrium in mixed strategies exists in the B-side price subgame.*

Proof Let $\theta_1 = \theta_2 = \underline{\theta}$. The sum of profits $\Pi = \Pi_1 + \Pi_2$ is almost everywhere continuous. Either $\Pi = \Pi_1 > 0 \; \forall p_1^B < p_2^B$, or $\Pi = \Pi_2 > 0 \; \forall p_1^B > p_2^B$, both of which are continuous except at $p_1^B = p_2^B$, where $\Pi = \Pi_1 + \Pi_2 = 0$. But the set $\Psi := \{(p_1^B, p_2^B) | p_1^B = p_2^B, (p_1^B, p_2^B) \in \mathbb{R}^2\}$ has Lebesgue measure zero. Theorem 5 of Dasgupta and Maskin (1986) directly applies and guarantees existence of an equilibrium in mixed strategies. □

Therefore the pair $\theta_1 = \theta_2 = \underline{\theta}$ may be part of an equilibrium of the overall game.

Proof of Proposition 6 We only need showing that the payoff functions Π_i $i = 1, 2$ are lower-hemicontinuous in their own argument θ_i. We know that Π_1 is continuous for any $\theta_1 > \theta_2$ (refer Appendix Section A). It is immediate that Π_2 is continuous for $\theta_1 > \theta_2$. Last, for $i = 1, 2$

$$\Pi_i = \begin{cases} 0, & \text{if } \theta_1 = \theta_2 = \underline{\theta}; \\ -k\theta_i^2, & \text{if } \theta_1 = \theta_2 > \underline{\theta}. \end{cases}$$

that is, $\Pi_i,\ i = 1, 2$ is l.h.c. Since $(\theta_2, \theta_1) s.t\ \theta_2 = \theta_1 \in \Upsilon_0$, Theorem 5 in Dasgupta and Maskin (1986) can be applied, whence an equilibrium in mixed strategies must exist. □

Appendix C: Proofs

Proof of Proposition 1 The proof begins by showing existence of an equilibrium, then characterizes it.[16]

Definition 1 For $i = 1, 2$, the platforms' 'quasi-best responses' are defined as the solution to the problem $\max_{p_i^B} \; \Pi_i(p_i^B, D_i(\mathbf{p}^B, \theta); \Pi_i^A(D_i, D_j))$, where the profit function is defined by (3.1). Therefore, letting $\theta_1 > \theta_2$ w.l.o.g,

$$p_1^B(p_2^B) = \begin{cases} \underline{p}_1^B(p_2^B) = \frac{1}{2}(p_2^B + \Delta\theta\overline{\beta} - 2e\overline{A}), & \text{if } D_1 > D_2; \\ \frac{1}{2}(p_2^B + \Delta\theta\overline{\beta}), & \text{if } D_1 = D_2; \\ \overline{p}_1^B(p_2^B) = \frac{1}{2}(p_2^B + \Delta\theta\overline{\beta} + 2e\underline{A}), & \text{if } D_1 < D_2; \end{cases}$$

and

$$p_2^B(p_1^B) = \begin{cases} \underline{p}_2^B(p_1^B) = \frac{1}{2}(p_1^B - \Delta\theta\underline{\beta} - 2e\overline{A}), & \text{if } D_1 < D_2; \\ \frac{1}{2}(p_1^B - \Delta\theta\underline{\beta}), & \text{if } D_1 = D_2; \\ \overline{p}_2^B(p_1^B) = \frac{1}{2}(p_1^B - \Delta\theta\underline{\beta} + 2e\underline{A}), & \text{if } D_1 > D_2; \end{cases}$$

[16] The conditions of Theorem 2 of Dasgupta and Maskin (1986) are not met, and neither are those of Reny (1999). The sufficient conditions (Proposition 1) of Baye et al. (1993) also fail here, so their existence result cannot be readily applied. A recent contribution by Bich (2008) establishes existence by introducing a measure of the lack of quasi-concavity that resembles ironing. Our construction does remain essential in that we face a potential multiplicity of equilibria and seek a characterization.

While it is always possible to find some point where 'quasi-best responses' intersect, it by no means defines an equilibrium. Doing so assumes that in some sense platforms coordinate on a particular market configuration—for example, such that $D_1 < D_2$. We first need to pin down the firms' true best replies.

Lemma 3 *Let $\theta_1 > \theta_2$ w.l.o.g. There exists a pair of actions $(\hat{p}_1, \hat{p}_2)$ such that the best response correspondences are defined as*

$$p_1^B(p_2^B) = \begin{cases} \underline{p}_1^B(p_2^B), & \textit{for } p_2 \geq \hat{p}_2; \\ \overline{p}_1^B(p_2^B), & \textit{for } p_2 < \hat{p}_2; \end{cases} \tag{8.1}$$

and

$$p_2^B(p_1^B) = \begin{cases} \overline{p}_2^B(p_1^B), & \textit{for } p_1 < \hat{p}_1; \\ \underline{p}_2^B(p_1^B), & \textit{for } p_1 \geq \hat{p}_1; \end{cases} \tag{8.2}$$

Lemma 3 says that platform 1, for example, prefers responding with $\underline{p}_1^B(p_2^B)$ for any prices $p_2 \geq \hat{p}_2$ and switches to $\overline{p}_1^B(p_2^B)$ otherwise. The best reply correspondence is discontinuous at that point where platforms are indifferent between being the dominant platform and not, that is, between the combination of prices $(\underline{p}_i^B(p_j^B), p_i^A(\underline{p}_i^B))$ and $(\overline{p}_i^B(p_j^B), p_i^A(\overline{p}_i^B))$.

Proof Any profile $\tilde{\mathbf{p}}^B$ such that $D_1 = D_2$ can never be an equilibrium. When $D_1 = D_2$ A profits Π_i^A are nil for both platforms. Both players have a deviation strategy $p_i^B + \varepsilon$ in either direction since $\overline{\Pi}_i^A > \underline{\Pi}_i^A > 0,\ i = 1, 2$ as soon as $D_i \neq D_{-i}$. Maximizing the profit function (3.1) leaves us with two 'quasi-reaction correspondences', for each competitor, depending on whether $D_1 > D_2$ or the converse. Depending on firm 2's decision, platform 1's profit is either

$$\Pi_1 = \begin{cases} \Pi_1(\underline{p}_1^B(p_2^B), p_2^B; \Pi_i^A) = \Pi_1\left(\frac{1}{2}\left(p_2^B + \Delta\theta\overline{\beta} - 2e\overline{A}\right), p_2^B; \Pi_i^A\right), & \text{or;} \\ \Pi_1(\overline{p}_1^B(p_2^B), p_2^B; \Pi_i^A) = \Pi_1\left(\frac{1}{2}\left(p_2^B + \Delta\theta\overline{\beta} + 2e\underline{A}\right), p_2^B; \Pi_i^A\right). \end{cases}$$

Define $g_1(p_2^B) \equiv \Pi_1(\overline{p}_1^B(p_2^B), p_2^B; \Pi_i^A) - \Pi_1(\underline{p}_1^B(p_2^B), p_2^B; \Pi_i^A)$. This is the difference in profits generated by firm 1 when it chooses one 'quasi-best response' over the other. For p_2^B sufficiently low, $g_1 > 0$. This function is continuous and a.e differentiable. Using the definitions of equilibrium A-side profits, $\frac{dg_1}{dp_2^B} = \frac{d\Pi_1^A(\overline{p}_1^B, p_2^B)}{dp_2^B} - \frac{d\Pi_1^A(\underline{p}_1^B, p_2^B)}{dp_2^B} < 0$, and $\frac{d^2 g_1}{d(p_2^B)^2} = 0$, whence there exists a point $\hat{p}_2^B$ such that $g_1(\hat{p}_2^B) = 0$. At $\hat{p}_2^B$, $\Pi_1(\underline{p}_1^B(\hat{p}_2^B), \hat{p}_2^B) = \Pi_1(\overline{p}_1^B(\hat{p}_2^B), \hat{p}_2^B)$; platform 1 is indifferent between either best response $\underline{p}_1^B(\hat{p}_2^B)$ or $\overline{p}_1^B(\hat{p}_2^B)$. The same follows for platform 2, which defines $\hat{p}_1^B$. It follows that

$$\begin{aligned} \Pi_1\left(\underline{p}_1^B(p_2^B), p_2^B; \Pi_i^A\right) \geq \Pi_1\left(\overline{p}_1^B(p_2^B), p_2^B; \Pi_i^A\right) \Leftrightarrow p_2^B &\geq \hat{p}_2^B \\ &\equiv -\left(\Delta\theta\underline{\beta} + e(\overline{A} - \underline{A})\right) \end{aligned}$$

and

$$\Pi_2\left(p_1^B, \underline{p}_2^B(p_1^B); \Pi_i^A\right) \geq \Pi_2\left(p_1^B, \overline{p}_2^B(p_1^B); \Pi_i^A\right) \Leftrightarrow p_1^B \geq \hat{p}_1^B \equiv \Delta\theta\overline{\beta} - e(\overline{A} - \underline{A})$$

□

For each firm, its price must be an element of the best reply correspondence and these correspondences must intersect. From the 'quasi-best responses', an equilibrium candidate is a pair of prices such that

$$\left(p_1^{*B}, p_2^{*B}\right) = \begin{cases} \underline{p}_1^B(p_2^B) \cap \overline{p}_2^B(p_1^B), & \text{if } D_1 > D_2 \text{ or;} \\ \overline{p}_1^B(p_2^B) \cap \underline{p}_2^B(p_1^B), & \text{if } D_1 < D_2; \end{cases}$$

An equilibrium exists only if these intersections are non-empty. Together, the definitions of a best-response profile (relations (8.1) and (8.2)) and of an equilibrium candidate sum to

Condition 1 *Either*

$$\hat{p}_1^B \geq p_1^{*B} \;\; \textit{and} \;\; \hat{p}_2^B \leq p_2^{*B} \;\; \textit{or} \;\; \hat{p}_1^B \leq p_1^{*B} \;\; \textit{and} \;\; \hat{p}_2^B \geq p_2^{*B}$$

or both.

Consider an action profile $\mathbf{p}^{*B}$ satisfying this condition; from Lemma 3 each p_i^{*B} is an element of i's best response. For it to be an equilibrium the reaction functions must intersect. This is exactly what Condition 1 requires. For example, the first pair of inequalities tells us that player 1's optimal action has to be low enough and simultaneously that of 2 must be high enough. When they hold, player 2's reaction correspondence is continuous until 1 reaches the maximizer p_1^{*B}, and similarly for firm 1.

Lemma 4 *Condition 1 is necessary and sufficient for at least one equilibrium $\mathbf{p}^{*B} = (p_1^{*B}, p_2^{*B})$ to exist. When both inequalities are satisfied, the game admits two equilibria.*

When Condition 1 holds, the Nash correspondence $p_1^B(p_2^B) \times p_2^B(p_1^B)$ has a closed graph and standard theorems apply; it provides us with a pair of easy-to-verify conditions in terms of prices.

Proof Each platform's action set $p_i^B \subseteq \mathbb{R}$ is compact and convex, and so can be partitioned into two subsets $\underline{P}_i^B = [p_i^{R,min}, \hat{p}_i^B]$ and $\overline{P}_i^B = [\hat{p}_i^B, p_i^{R,max}]$, on which the best-response correspondences defined by (8.1) and (8.2) are continuous for each platform i. Consider any equilibrium candidate (p_1^{*B}, p_2^{*B}). When Condition 1 holds, following the definitions given by Eqs. (8.1) and (8.2), either $p_1^{*B} \in \underline{p}_1^B(p_2^B)$ and $p_2^{*B} \in \overline{p}_2^B(p_1^B)$, or $p_1^{*B} \in \overline{p}_1^B(p_2^B)$ and $p_2^{*B} \in \underline{p}_2^B(p_1^B)$ (or both, if two equilibria exist). Thus at the point (p_1^{*B}, p_2^{*B}) the reaction correspondences necessarily intersect at least once, whence the Nash correspondence has a closed graph and the Kakutani fixed-point theorem applies. To show necessity, suppose a pair (p_1^{*B}, p_2^{*B}) is a Nash equilibrium. By definition, $p_2^B(p_1^B) \cap p_1^B(p_2^B) \neq \varnothing$, and by Lemma 3, either $(p_1^{*B}, p_2^{*B}) = \underline{p}_1^B(p_2^B) \cap \overline{p}_2^B(p_1^B)$ or $(p_1^{*B}, p_2^{*B}) = \overline{p}_1^B(p_2^B) \cap \underline{p}_2^B(p_1^B)$, or both if two equilibria exist. For the first equality to hold, the first line of Condition 1 must hold, and for the second one, the second line of Condition 1 must be satisfied. □

Lemma 5 *An equilibrium in pure strategies of the B-side price subgame always exists.*

Proof First construct a candidate equilibrium as follows. Suppose that platforms maximise $\Pi_1^H = p_1^B D_1(\mathbf{p^B}, \theta) - k\theta_1^2 + \overline{\Pi}_1^A$ and $\Pi_2^H = p_2^B D_2(\mathbf{p^B}, \theta) - k\theta_2^2 + \underline{\Pi}_2^A$, respectively. Solving for the first-order conditions laid out in Definition 1 yields

$$p_1^{*B} = \tfrac{1}{3}[\Delta\theta(2\overline{\beta} - \underline{\beta}) + 2e(\underline{A} - 2\overline{A})]$$
$$p_2^{*B} = \tfrac{1}{3}[\Delta\theta(\overline{\beta} - 2\underline{\beta}) + 2e(2\underline{A} - \overline{A})]$$

Simple algebra yields $D_1 > 0$ and $D_2 > 0$ provided $\Delta\theta > \frac{\sqrt{C}}{\overline{\beta}-2\underline{\beta}}$; otherwise

$$p_1^{*B} = \frac{\Delta\theta\overline{\beta}}{2} - e\overline{A}; \quad p_2^{*B} = 0,$$

both of which verify the first line of Condition 1. So (p_1^{*B}, p_2^{*B}) constitutes an equilibrium by Lemma 4. This equilibrium *always* exists because $\hat{p}_1^B \geq p_1^{*B}$ and $\hat{p}_2^B \leq p_2^{*B}$ are always satisfied. Indeed, either both hold when both platforms are active, or $p_2^{*B} = 0 > \hat{p}_2^B$ when only firm 1 is active. Another candidate equilibrium (p_1^{**B}, p_2^{**B}) can be constructed by letting platform 1 play as if $\Pi_1^L = p_1^B D_1(\mathbf{p^B}, \theta) - k\theta_1^2 + \underline{\Pi}_1^A$ and platform 2 as if $\Pi_2^L = p_2^B D_2(\mathbf{p^B}, \theta) - k\theta_2^2 + \overline{\Pi}_2^A$, whence

$$p_1^{**B} = \frac{1}{3}[\Delta\theta(2\overline{\beta} - \underline{\beta}) + 2e(2\underline{A} - \overline{A})]$$
$$p_2^{**B} = \frac{1}{3}[\Delta\theta(\overline{\beta} - 2\underline{\beta}) + 2e(\underline{A} - 2\overline{A})]$$

An equilibrium such that $p_1^{*B} = 0$; $p_2^{*B} = -\frac{\Delta\theta\underline{\beta}}{2} - e\overline{A}$ cannot exist, for these prices are not best response to each other. For $\theta_1 > \theta_2$ there always exists some price $p_1^B \geq p_2^B$ such that consumers prefer purchasing from platform 1. When both firms are active Condition 1 holds as long as $\Delta\theta(\overline{\beta} + \underline{\beta}) - e(\overline{A} + \underline{A}) \leq 0$. Given that $\Delta\theta \geq \frac{\sqrt{C}}{2\overline{\beta}-\underline{\beta}}$, take the lower bound and substitute into the second line of Condition 1:

$$e(\overline{A} + \underline{A})\left(\frac{2(\overline{\beta} + \underline{\beta})}{\overline{\beta} - 2\underline{\beta}} - 1\right) > 0, \quad \forall\underline{\beta} \geq 0$$

which violates the second pair of inequalities of Condition 1. So the second candidate can never be an equilibrium. For completeness, Condition 1 is also sufficient to rule out deviations from the pairs (p_1^{*B}, p_2^{*B}) and (p_1^{**B}, p_2^{**B}). The SOC of the profit function (3.1) is satisfied at prices p_i^{*B} and p_i^{**B} $\forall i, \forall p_{-i}^B$, there cannot be any local deviation. Consider now deviations involving inconsistent actions, that is, such that both platforms maximise either $p_i^B D_i(p^B, \theta) - k\theta_i^2 + \overline{\Pi}_i^A$ or $p_i^B D_i(p^B, \theta) - k\theta_i^2 + \underline{\Pi}_i^A$. Since (p_1^{*B}, p_2^{*B}) always exists, the first line of Condition 1 always holds. It immediately follows from (8.1) and (8.2) that $\overline{p}_1^B(p_2^B) \cap \overline{p}_2^B(p_1^B) = \varnothing$ and $\underline{p}_1^B(p_2^B) \cap \underline{p}_2^B(p_1^B) = \varnothing$ as well. □

Together these Lemmata conclude the proof. □

Proof of Proposition 2 We begin by characterising the first-stage actions.

Lemma 6 *Let $\theta_1 > \theta_2$ w.l.o.g. and Assumption 3 hold. Optimal actions consist of $\theta_2^* = \underline{\theta}$ and $\theta_1^* = \hat{\theta}_1$, where $\hat{\theta}_1$ uniquely solves*

$$(2\overline{\beta} - \underline{\beta})^2 = 18k\theta_1 + \frac{C}{(\Delta\theta)^2} \tag{C.3}$$

Both platforms operate.

Proof First notice that in any pure-strategy Nash equilibrium (θ_1^*, θ_2^*) such that $\theta_1^* > \theta_2^*$, $\theta_2^* = \underline{\theta}$ necessarily. To see that, assume the FOC (6.3) binds so that $\theta_1^* = \hat{\theta}_1$. Computing the slope of the profit function Π_2 yields

$$\frac{d\Pi_2}{d\theta_2} = \begin{cases} -(\overline{\beta} - 2\underline{\beta})^2 + \frac{C}{(\Delta\theta)^2} - 2k\theta_2 < -2k\theta_2, & \text{if } \Delta\theta(\overline{\beta} - 2\underline{\beta}) > \sqrt{C}; \\ -2k\theta_2, & \text{if } \Delta\theta(\overline{\beta} - 2\underline{\beta}) \leq \sqrt{C}. \end{cases}$$

whence it is immediate that $\frac{d\Pi_2}{d\theta_2}|_{\theta_2 > \underline{\theta}} < \frac{d\Pi_2}{d\theta_2}|_{\underline{\theta}} < 0$. This simplifies the analysis and lets us focus on platform 1's problem. Its first-order condition reads $(2\overline{\beta} - \underline{\beta})^2 - \frac{C}{(\Delta\theta)^2} - 18k\theta_1 = 0$ and admits a unique maximizer $\hat{\theta}_1$. Suppose firm 1 plays $\hat{\theta}_1$, platform 2 cannot increase its quality to any $\theta_2 \in (\underline{\theta}, \hat{\theta}_1)$ (it must play $\theta_2^* = \underline{\theta}$). So the pair $(\hat{\theta}_1, \underline{\theta})$ is an equilibrium as long as firm 2 cannot 'jump' over firm 1 and become the high-quality firm. To guarantee firm 2 operates we need $(\hat{\theta}_1 - \underline{\theta})(\overline{\beta} - 2\underline{\beta}) > \sqrt{C}$ (Assumption 3). The smallest 'leap' firm 2 can undertake is such that $\tilde{\theta}_2 \geq \hat{\theta}_1 + \varepsilon$. Hence the no-deviation condition is $\Pi_2(\hat{\theta}_1, \underline{\theta}) \geq \Pi_2(\hat{\theta}_1, \hat{\theta}_1 + \varepsilon)$, or

$$(\hat{\theta}_1 - \underline{\theta})(\overline{\beta} - 2\underline{\beta})^2 + B_2 + \frac{C}{(\hat{\theta}_1 - \underline{\theta})} \geq B_1 + \sqrt{C}(\overline{\beta} - 2\underline{\beta}) - 9k(\hat{\theta}_1 + \varepsilon)^2$$

$$(\hat{\theta}_1 - \underline{\theta})\left[(\overline{\beta} - 2\underline{\beta})^2 + (2\overline{\beta} - \underline{\beta})^2\right] - 9k\hat{\theta}_1^2 + B_2 \geq B_1 + \sqrt{C}(\overline{\beta} - 2\underline{\beta})$$

using the FOC $(2\overline{\beta} - \underline{\beta})^2 - 18k\hat{\theta}_1 - \frac{C}{(\hat{\theta}_1 - \underline{\theta})^2} = 0$ and the fact that $k\hat{\theta}_1\underline{\theta} = k\underline{\theta}^2 = 0$ (by assumption). When $\hat{\theta}_1 - \underline{\theta} > \frac{\sqrt{C}}{\overline{\beta} - 2\underline{\beta}}$, this condition is always satisfied. □

The optimality of $\theta_2^* = \underline{\theta}$ and $\theta_1^* = \hat{\theta}_1$ is established by Lemma 6. The rest of the claim follows immediately under Assumption 3. □

Proof of Corollary 1 In the first stage of the Shaked and Sutton (1982) model, firms solve

$$\max_{\theta_i \in \Theta_i} \; p_i^* D_i(\mathbf{p}^*, \theta_i, \theta_j^*) - k\theta_i^2$$

for $i = 1, 2$ and with demand $D_1 = \frac{1}{3}(2\overline{\beta} - \underline{\beta})$, $D_2 = \frac{1}{3}(\overline{\beta} - 2\underline{\beta})$ and prices $p_1 = \frac{\Delta\theta}{3}(2\overline{\beta} - \underline{\beta})$, $p_2 = \frac{\Delta\theta}{3}(\overline{\beta} - 2\underline{\beta})$, respectively. This problem is concave and given equilibrium prices p_i^*, has obvious maximizers $\theta_2^0 = \underline{\theta}$ and $\theta_1^0 = \frac{1}{2k}(\frac{2\overline{\beta}-\underline{\beta}}{3})^2$ with $\theta_1^0 < \overline{\theta}$ thanks to $k > \frac{(2\overline{\beta}-\underline{\beta})^2}{18\overline{\theta}}$. These individually optimal maximizers also form a Nash equilibrium as long as there is no profitable deviation. Consider one such deviation: $\tilde{\theta}_2 = \theta_1^0 + \epsilon$. Firm 2 profit from this deviation is $\Pi_2(\theta_1^0, \tilde{\theta}_2) = \epsilon(\frac{2\overline{\beta}-\underline{\beta}}{3})^2 - k\tilde{\theta}_2^2 < 0$ and the marginal profit $(\frac{2\overline{\beta}-\underline{\beta}}{3})^2 - 2k(\theta_1^0 + \epsilon) < 0$. To complete the proof observe that firm 1's first-order condition in the benchmark problem reads $(\frac{2\overline{\beta}-\underline{\beta}}{3})^2 - 2k\theta_1^0 = 0$ and compare it to Eq. (C.3). □

Proof of Proposition 3 Let θ_i^c denote the upper bound of the support of the distribution of the pure action space, a precise definition of which will soon be provided. Let $H_i(\theta_i)$ be the distribution over i's pure actions $\theta_i \in \{\underline{\theta}\} \cup [\tilde{\theta}, \theta^c]$. For any equilibrium mixing probability $H_2^*(\theta_2)$,

$$\begin{aligned}
\mathbb{E}_{\theta_2}[\Pi_1] &= \int \Pi_1(\underline{\theta}_1, \theta_2) d(H_1 \times H_2^*) + \int_{\tilde{\theta}_1}^{\theta_1'=\theta_2} \Pi_1(\theta_1, \theta_2) d(H_1 \times H_2^*) \\
&\quad + \int_{\theta_1'=\theta_2}^{\theta_1^c} \Pi_1(\theta_1, \theta_2) d(H_1 \times H_2^*) \\
&= H_1(\underline{\theta}_1) \int \Pi_1(\underline{\theta}_1, \theta_2) d(H_2^*) + \int_{\tilde{\theta}_1}^{\theta_1'=\theta_2} \Pi_1(\theta_1, \theta_2) d(H_1 \times H_2^*) \\
&\quad + \int_{\theta_1'=\theta_2}^{\theta_1^c} \Pi_1(\theta_1, \theta_2) d(H_1 \times H_2^*)
\end{aligned}$$

with possibly an atom at $\underline{\theta}_1$. With probability $\int_{\tilde{\theta}_1}^{\theta_1'=\theta_2} d(H_1 \times H_2^*)$ it plays $\theta_1 > \underline{\theta}$ such that 2 is the dominant firm ($\theta_2 \geq \theta_1$); in this case, $\Pi_1(\theta_1, \theta_2) = -k\theta_1^2 < 0$. With probability $\int_{\theta_1'=\theta_2}^{\theta_1^c} d(H_1 \times H_2^*)$ it is the dominant firm (the second integral). First note there is a mass point at $\underline{\theta}_i$, that is, for each agent i, $H_i(\underline{\theta}_i) \in (0, 1)$. To see that, suppose $H_1(\underline{\theta}_1) = 1$, then $\arg\max \mathbb{E}_{\theta_1}[\Pi_2(\underline{\theta}_1, \theta_2)] = \tilde{\theta}_2$, so $H_2(\underline{\theta}_2) = 0$ and $H_2(\theta_2)$ assigns full mass at $\tilde{\theta}_2 : h_2(\tilde{\theta}_2) = 1$. But then firm 1 should play some $\theta_1 > \tilde{\theta}_2$ and become the monopolist for sure. If $H_1(\underline{\theta}_1) = 0$, then 1 necessarily plays on $[\tilde{\theta}, \theta^c]$ and playing $\tilde{\theta}_2$ is a dominated strategy for firm 2. It therefore assigns no mass at this point. But then $\forall\ \theta_2 \in (\tilde{\theta}_2, \theta_2^c]$, $\Pi_1(\underline{\theta}_1, \theta_2) > 0$ and platform 1 should shift some mass to $\underline{\theta}_1$. Then the equilibrium conditions write $\forall \theta_i \in \Theta_i^N$,

$$\begin{aligned} &\mathbb{E}_{\theta_j}\left[\Pi_i(\theta_i, \theta_j)\right] = \Pi_i(\underline{\theta}_i, \tilde{\theta}_j) \\ &\Pi_i(\underline{\theta}_i, \tilde{\theta}_j) = 0 \end{aligned} \tag{C.4}$$

The first line asserts that i's expected payoff cannot be worse than if not investing for sure, in which case j's best response is $\tilde{\theta}_j$. The second one sates that if not investing for sure, a platform can only expect zero profits: expected profits in the mixed-strategy equilibrium must be zero. Next we determine the upper bound θ_i^c of the support of $H_i(\theta_i)$ for each platform $i = 1, 2$. The upper bound θ_i^c solves either

$$\Pi_i(\underline{\theta}_j, \theta_i^c) = 0 \quad \text{or } \Pi_i(\tilde{\theta}_j, \theta_i^c) = 0, \quad \text{hence } \theta_i^c = \max\{\theta_i' | \Pi_i(\underline{\theta}_j, \theta_i') = 0, \Pi_i(\tilde{\theta}_j, \theta_i') = 0\}$$

Rewriting the equilibrium condition (C.4), $\forall\ \theta_i \in \Theta_i^N$,

$$\begin{aligned} &H_i(\underline{\theta}_i) \int_{\Theta_j^N} R_i(\underline{\theta}_i, \theta_j) dH_j^*(\theta_j) + \int_{\theta_i'=\theta_j}^{\theta_i^c} R_i(\theta_i, \theta_j) d(H_i(\theta_i) \times H_j^*(\theta_j)) \\ &= k \int_{\tilde{\theta}_i}^{\theta_i^c} \theta_i^2 d(H_i(\theta_i) \times H_j^*(\theta_j)) \end{aligned}$$

where $R_i(\theta_i, \theta_j)$ stands for platform i's revenue (gross of costs). For any play θ_j, total revenue $R_i(\theta_i, \theta_j)$ is decreasing in $\theta_i \in \Theta_i^N \backslash \underline{\theta}_i$ – refer Conditions (6.1) and (6.2). Thus for any distribution $H_i(\theta_i) \times H_j^*(\theta_j)$ the LHS is bounded as well, and decreasing in θ_i. □

Proof of Proposition 4 When e is large enough platform 1 (the high-quality firm) prefers playing such that $\Delta\theta = \frac{2e(\overline{A}+\underline{A})}{\overline{\beta}-2\underline{\beta}} := z(e)$ for any θ_2 (and θ_1 not so large as to induce negative profits). Its payoffs when $\Delta\theta \le z(e)$ are given by the second line of (6.1), where $B_1(e) = 2e(2\overline{\beta} - \underline{\beta})(2\underline{A} - \overline{A})$. This can be re-arranged as

$$\pi_1(e, \theta) = \frac{1}{9}\left[\Delta\theta(2\overline{\beta} - \underline{\beta})^2 + 2e[\underline{A}(5\overline{\beta} - 4\underline{\beta}) - \overline{A}(\overline{\beta} + \underline{\beta})]\right] - k\theta_1^2$$

for $\Delta\theta \le z(e)$ and

$$\pi_1(e, \theta) = \frac{1}{9}\left[\Delta\theta(2\overline{\beta} - \underline{\beta})^2 + B_1(e) + \frac{[2e(\overline{A} + \underline{A})]^2}{\Delta\theta}\right] - k\theta_1^2$$

if $\Delta\theta > z(e)$. Let $\overline{\pi}_1(e, \theta) = \max \pi_1(e, \theta)$ for any pair $\theta_1 > \theta_2$ such that $\Delta\theta = z(e)$. This is an upper bound on firm 1's profits for any play by firm 2. We know $\pi_1(e, \theta)$ is maximized for $\theta_2 = \underline{\theta}$. Recall that we denote the corresponding value of θ_1 by $\tilde{\theta}_1$. For any e and θ_2, $\frac{\partial \pi_1(e,\theta)}{\partial \theta_1} > 0$ when $\Delta\theta < z(e)$ and $\frac{\partial \overline{\pi}_1(e,\theta)}{\partial \theta_1} < 0$ when $\Delta\theta = z(e)$ and

$\theta_2 > \underline{\theta}$. Therefore $\overline{\pi}_1(e, \theta)$ reaches zero for some value $\theta_1' \leq \theta_1^c$. Thus no firm will play out of these bounds. More precisely,

$$\frac{\partial \pi_1(e,\theta)}{\partial \theta_1} = \frac{2\overline{\beta} - \underline{\beta}}{9} - 2k\theta_1 > 0, \quad \text{when } \Delta\theta < z(e) \quad \text{and}$$
$$\frac{\partial \overline{\pi}_1(e,\theta)}{\partial \theta_1} = \frac{2\overline{\beta} - \underline{\beta}}{9} - 2k\theta_1 < 0, \quad \text{for } \Delta\theta = z(e),\ \theta_2 > \underline{\theta}.$$

with max $\frac{\partial \pi_1(e,\theta)}{\partial \theta_1}$ reached for $\theta_2 = \underline{\theta}$. Since arg max $\Pi_1(\theta_1, \theta_2) > \tilde{\theta}_1$ when $\theta_2 > \underline{\theta}$, it follows that

$$\frac{\partial \pi_1(e,\theta)}{\partial \theta_1} < \left| \frac{\partial \overline{\pi}_1(e,\theta)}{\partial \theta_1} \right|$$

and therefore $| \tilde{\theta}_1 - \theta_1^c | < z(e)$. □

Proof of Proposition 5 First note that when consumer prices are identical a pure strategy equilibrium cannot exist. Suppose $p_1^B = p_2^B$, B demand is given by

$$D_i = \begin{cases} 1, & \text{if } \theta_i > \theta_j; \\ \frac{1}{2}, & \text{if } \theta_i = \theta_j; \\ 0, & \text{if } \theta_i < \theta_j. \end{cases} \quad \text{and,}$$

so platform i faces payoffs

$$\Pi_i = \begin{cases} eD_i \left(\frac{\overline{\alpha}}{2}\right)^2 - k\theta_i^2 \geq 0, & \text{if } \theta_i > \theta_j \geq \underline{\theta}; \\ -k\theta_i^2 \leq 0, & \text{if } \underline{\theta} \leq \theta_i \leq \theta_j; \end{cases}$$

Now take a profile $\theta_1 = \theta_2$, then $D_1 = D_2$ and platforms are Bertrand competitors in the A market with payoffs $-k\theta_i^2 \leq 0$. When $-k\theta_i^2 < 0$, firm i possesses a unilateral deviation: set $\theta_i = \underline{\theta}$. When $-k\theta_i^2 = 0$, it also possesses a unilateral deviation: set $\theta_i > \underline{\theta}$. To complete, set $p_1^B = p_2^B = 0$ exogenously and apply Proposition 6, and the characterization in Proposition 3. □

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