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Smooth preferences, symmetries and expansion vector fields

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Abstract Tyson (J Math Econ 49(4): 266–277, 2013) introduces the notion of symmetry vector field for a smooth preference relation, and establishes necessary and sufficient conditions for a vector field on consumption space to be a symmetry vector field. The structure of a such a condition is discussed on both geometric and economic grounds. It is established that symmetry vector fields do commute (i.e. have vanishing Lie bracket) for additive and joint separability. The marginal utility of money is employed as a normalization of the expansion vector field (Mantovi, J Econ 110(1): 83–105, 2013) which results in the fundamental (expansion-) symmetry vector field. Finally, a characterization of symmetry vector fields is given in terms of their action on the distance function, and a pattern of complete response is discussed for additive preferences. Examples of such constructions are explicitly worked out. Potential implications of the results are discussed.

Keywords Utility function · Symmetry · Separability · Vector field · Expansion path · Distance function

JEL Classification D01 · D04 · D11

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1 Introduction

The utility representation of the properties of preferences has driven fundamental advances for both theoretical and empirical analyses of consumption (see Christensen et al. 1975, and Deaton[1] and Muellbauer 1980, for classical contributions). In particular, the functional representation of the several facets of separability (for instance, Houthakker 1960; Pollak 1972; Gorman 1976; Blackorby et al. 1978) resulted in a fundamental line of progress for microeconomic theory in general, and duality theory in particular.

The significance of separability has been sharply pointed out by Gorman (1976): separability has to do with the "natural structure" of a choice problem. As long pointed out, in a number of relevant cases, choice problems can be assumed to consist of a series—perhaps a hierarchy—of subproblems that can be dealt with "in at least partial isolation from each other" (ivi, p. 224), possibly, budgeting problems concerning consumption aggregates. Such separations have been thoroughly addressed in terms of the functional structure of objective functions (direct and indirect utility functions, cost and distance functions: Blackorby et al. 1978). Noticeably, recent geometrical advances in the analysis of preferences (Tyson 2013; Mantovi 2013) point at the symmetries of preferences as a promising direction for deepening the "natural structure" of a choice problem. It is the aim of the present contribution to set forth a number of advances in such respects, meant to exploit the powerful analytical language of vector fields.[2]

On the one hand, Mantovi (2013) introduces, among other things, the *expansion vector field* on consumption space, whose integral curves define a scaling parametrization of income expansion paths, thereby sheding new light on the benchmark relevance of homothetic models, in connection with the commutativity of *finite* expansion and substitution effects. On the other hand, Tyson (2013) introduces *symmetries* of preferences as transformations of choice space which do preserve the preference relation, and then *symmetry vector fields*, whose flows embody 1-parameter semigroups of continuous symmetries of such a space. The Author characterizes such symmetries in terms of partial differential equation (PDE) systems for the utility representation of preferences, which he applies to univariate, multivariate and joint separability.

Building on such premises, it is the aim of the present contribution[3] to provide further insights on the characterization of symmetry vector fields set forth by Tyson (2013), and establish that symmetry vector fields for multivariate and joint separability have vanishing Lie bracket (a strong analytical condition which shall be given due economic interpretation). Then, we shall find that the marginal utility of money defines a "normalization" of the expansion vector field which results in the *fundamen-*

[1] In his Nobel lecture (2015, December 8th) Prof. Deaton recalls the intellectual challenges faced in the search of an "almost ideal" demand system.

[2] A vector field defines a 'velocity' vector at each point of a space, i.e. a first order ordinary differential equation (ODE) system; tangent curves to such vectors (solutions to the ODE system) do not cross each other, and, as such, represent a *flow* on such space. In fact, fluid motion is the typical physical picture meant to foster intuition about the properties of vector fields and flows. See Appendix 1 for a brief introduction to vector fields and dynamical systems.

[3] Part of the following arguments circulated via the working paper "Differential duality", Department of Economics, Parma (Italy).

tal symmetry vector field, whose flow superposes to expansion paths and therefore preserves marginal rates of substitution. Finally, a characterization of symmetry vector fields shall be given with respect to the distance function, thereby complementing Tyson's results with a perspective on the radial structure of preferences, together with an insightful approach to multivariate separabilty. In addition, we shall gain further insights on the benchmark relevance of the scale symmetry which characterizes homothetic models (think for instance of the conditions for the existence of a cost of living index: Samuelson and Swamy 1974; Gorman 1976), which share the same expansion vector field, namely, the generator of scale transformations (formula 1 below).

True, the relevance of preference symmetries does extend beyond the problem of separability. In first instance, on conceptual grounds, recall that the significance of preferences and utility functions is not confined to the microeconomic representation of static consumption problems. For instance, one can consider the macroeconomic models encompassing utility functions over intertemporal consumption aggregates (for instance Barro 2001), or the utility functions representing the tradeoff between consumption and leisure (for instance, King et al. 1988). Such examples enlighten the relevance of preference symmetries in the representation of significant policy problems.

A (public) policy is typically an investment of (public) resources meant to pursue definite goals, as gauged by some welfare index. It is often the case that the properties of the preferences of a representative agent, or of heterogeneous agents, are employed in the very representation of the policy problem. In such respects, the relevance is well known of employing analytically tractable functional forms (the Cobb-Douglas case stands out in such respects); true, the symmetries of the problem at hand may represent the bottom line of the modelling process: one can consider for instance the symmetry requirements discussed by Kimball and Shapiro (2008) for the consumption-leisure tradeoff. In addition, the *reversibility* aspects of a policy framework may be strongly connected with the assumptions on the preferences of agents, as discussed by Mantovi (2013) for the commutativity of finite expansion and substitution effects.[4] All in all, the symmetries of utility functions seem to represent a relevant line of inquiry for both micro- and macro- economic analysis.

The plan of the rest of the paper is as follows. In Sect. 2 we introduce symmetry vector fields and establish a few original results. In Sect. 3 we introduce our expansion vector field, and in Sect. 4 we discuss the fundamental expansion-symmetry vector field. In Sect. 5 we characterize symmetry vector fields via the distance function, and discuss a pair of insightful examples. A final section sketches potential lines of progress of our approach. Appendix 1 provides a short introduction to vector fields and Lie brackets. Appendix 2 addresses a technical result, upon which a fundamental lemma in the main text builds.

2 Symmetries

Following Tyson (2013), let us define a *discrete symmetry* of a preference relation as a bijection of choice (possibly, consumption) space B which does preserve the

[4] Noticeably, such a commutativity is 'isomorphic' to the equivalence of standard and reversed Farrell decompositions of productive efficiency recently established by Bogetoft et al. (2006).

preference relation between the elements of B, and therefore maps indifference subsets onto indifference subsets. As a familiar example, any scale transformation (homothety) of the positive orthant $\mathcal{R}^n_{++} = (0, \infty) \times \cdots \times (0, \infty)$ (n copies) is a discrete symmetry of any homothetic preference relation on such a space.

Then, define a *continuous symmetry* for a preference relation on B as a 1-parameter family $\sigma \colon [0, 1] \times \mathsf{B} \to \mathsf{B}$ of discrete symmetries which contains the identity transformation $\sigma(0, \cdot) : \mathsf{B} \to \mathsf{B}$ and is of differentiable class C^2. Evidently, the class of homotheties on $\mathcal{R}^n_{++}$ is a continuous symmetry of homothetic preferences on $\mathcal{R}^n_{++}$.

Finally, define a *symmetry vector field* of a *smooth* preference relation (i.e. represented by a C^2 utility function which has no stationary point, and induces a monotone continuous complete preorder; see the classical contribution by Debreu (1972)) on B as a vector field on B whose integral curves trace out a continuous symmetry. The vector field

$$\mathbf{Z} = \sum_{k=1}^{n} q^k \frac{\partial}{\partial q^k} \tag{1}$$

is the generator of scale transformations (Mantovi 2013) on $\mathcal{R}^n$, and therefore its action on homogeneous functions of degree 1 results in the identity transformation; in fact, in terms of the vector field (1), Euler's formula (Mas-Colell et al. 1995) can be written $\mathbf{Z}\,(g) = g$. Then, evidently, (1) is a symmetry vector field for any homothetic preference relation on $\mathcal{R}^n_{++}$ (Mantovi 2013; Tyson 2013).

Thus, a symmetry vector field is a dynamical system on choice space whose trajectories fill such a space and sweep indifference surfaces with the proper 'speed', so as to map indifference surfaces onto indifference surfaces. The following proposition establishes a basic property of symmetry vector fields.

Proposition 1 *If* $\mathbf{A}$ *is a symmetry vector field for the preference relation represented by the smooth utility function* U*, then* $\bar{\mathbf{A}} \equiv \frac{1}{\mathbf{A}(U)}\mathbf{A}$ *is a symmetry vector field as well.*

Proof The integral curves of $\bar{\mathbf{A}}$ are smooth reparametrizations of the integral curves of $\mathbf{A}$, and in fact the flow of $\bar{\mathbf{A}}$ can be considered as the flow of $\mathbf{A}$ measured at 'unit speed', being $\bar{\mathbf{A}}(U) \equiv 1$. Then, the conditions for $\bar{\mathbf{A}}$ to be a symmetry vector field are satisfied. □

Tyson (2013) establishes a system of PDEs for the components of symmetry vector fields. Write MRS^j_k for the marginal rate of substitution of good j and good k, i.e. the function

$$\mathrm{MRS}^j_k(q) \equiv \frac{\frac{\partial U}{\partial q^j}}{\frac{\partial U}{\partial q^k}} \tag{2}$$

on consumption space (as is well known, such functions are independent of the utility representation). Then the condition that the flow of the vector field $\mathbf{S}$ on B does preserve such a function can be written

$$\mathbf{S}(\mathrm{MRS}^j_k) \equiv \sum_{l=1}^{n} S^l(q)\frac{\partial \mathrm{MRS}^j_k(q)}{\partial q^l} = 0 \tag{3}$$

Such a condition does in fact identify a symmetry vector field in some cases (for instance, Tyson (2013), formula 3), but the general conditions for the existence of a such a field are identified by Tyson as consequences of the following: the gradient (differential) of a utility representation U at the transformed point $\mathbf{q}_t$ is obtained by multiplying the gradient of U at $\mathbf{q}$ by a function of $\mathbf{q}$ and t. The Author establishes (ivi, Theorem 2.10) necessary and sufficient conditions for $\mathbf{S}$ to be a symmetry vector field. The representation $\mathbf{S} = \sum_{k=1}^{n} S^k(x)\frac{\partial}{\partial x^k}$ of a vector field $\mathbf{S}$ enables us to write such conditions as[5]

$$\begin{aligned}\mathbf{S}(\mathrm{MRS}^j_k) &= \sum_{i=1}^{n} S^i \frac{\partial \mathrm{MRS}^j_k}{\partial q^i}\\ &= \sum_{i=1}^{n} \mathrm{MRS}^i_k\left(\mathrm{MRS}^j_k \frac{\partial S^i}{\partial q^k} + \frac{\partial S^i}{\partial q^j}\right)\end{aligned} \quad 1 \le j < k \le n \tag{4}$$

(compare Tyson 2013, equation 22), already gaining some insight into their significance: the left hand side (LHS) of (4) turns out to be the derivative of MRS^j_k along the flow of $\mathbf{S}$ (Lie derivative of MRS^j_k with respect to $\mathbf{S}$; Spivak 1999). Being MRSs constant along expansion paths, the right hand side (RHS) can be interpreted as a *measure of deviation* of the flow of $\mathbf{S}$ from the expansion flow. Thus, the RHS of condition (4) may admit an insightful interpretation in terms of substitution effects, once a budget constrained optimal consumption problem is at stake.

In order to start grasping the significance of (4), one can exploit the well known analytical tractability of Cobb-Douglas (CD) preferences. Let the strictly positive n-orthant $\mathcal{R}^n_{++}$ be our consumption space B, endowed with the utility representation $U(x) = \sum_{k=1}^{n} a_k \ln x^k$ of CD preferences. (Tyson 2013, Corollary 3.4) shows that the n vector fields $\mathbf{S}_k \equiv x^k \frac{\partial}{\partial x^k}$ are symmetry vector fields for CD preferences. One can in fact easily check conditions (4) with $\mathrm{MRS}^j_k(q) \equiv \frac{a^j x^k}{a^k x^j}$. The RHS of (4) does not vanish, signalling that the flow of $\mathbf{S}_k$ does not superpose to expansion paths. In fact, for each k between 1 and n, the vector field $\mathbf{S}_k$ is parallel to the coordinate vector field $\frac{\partial}{\partial x^k}$, and therefore its integral curves are straight lines parallel to the k axis, and can be written

$$q_j(t) = \begin{cases} q_j(0)e^t & \text{for } j = k \\ q_j(0) & \text{else} \end{cases} \tag{5}$$

Such paths do not superpose to expansion paths for CD preferences, well known to be rays as for any homothetic model. It is not difficult to convince oneself that, for any $t \in (0, \infty)$, any pair of initial conditions $\mathbf{q}(0)$, $\mathbf{r}(0)$ on the same indifference surface are dragged onto the same (different) indifference surface by such vector fields, since

5 Notice that the index in S^k has tensor character, whereas the indices in MRS^j_k have not.

$U(\mathbf{q}(0)) = U(\mathbf{r}(0)) \Leftrightarrow U(\mathbf{q}(t)) = U(\mathbf{r}(t))$, being U any utility representation of the preferences under consideration.

One then finds that the sum of the vector fields $\mathbf{S}_k$ is again a symmetry vector field, and this is a key element for our discussion, in that the vector field (1) is the well known generator of scale transformations (homotheties) on $\mathcal{R}^n$, and its integral curves represent a sound parametrization of expansion effects (Mantovi 2013) for any homothetic preference relation. Such a property represents one more instance of the benchmark relevance of homothetic models. Tyson (2013) does not emphasize such a benchmark role for scale symmetry, which, in the author's view, represents in fact a key motivation for deepening the implications of the symmetries of preferences, for instance in terms of separability properties.

In such respects, a key result established by (Tyson 2013, Proposition 3.3) is that additive separability of a utility function U on $\mathcal{R}^n_{++}$, say $U(q) = g_1(q^1)+\cdots+g_n(q^n)$, is equivalent to the fact that the n vector fields

$$\frac{1}{g_k'(q^k)}\frac{\partial}{\partial q^k} \tag{6}$$

are symmetry vector fields. The strong implications of additive separability have been long established; definitely, "additivity reduces the scope for substitution and complementarity to the barest minimium" (Houthakker 1960, p. 246). In such respects, in the author's vision, the point of Tyson's result is the economic insight conveyed by such a result: each subutility function g_k, upon differentiation, gives the (reciprocal of the) component of a directional symmetry vector field, i.e. the 'speed' at which utility increases along such direction. One is then in a position to sweep the indifference map in any direction with the proper (point dependent) speed so as to map indifference surfaces onto indifference surfaces. Then, being "large aggregates, such as clothing, food, etc." (ivi) the natural canditates for an additively separable representation, one thereby envisions an insightful framework for conceiving separation of budgeting problems, recalling that consumption aggregates can be reasonably partitioned into luxuries and necessities (mainly, food and housing; see for instance Deaton and Muellbauer 1980).

Definitely, we are in a position to shed new light on additive separability. Our differential geometric approach to the "natural structure" of the choice problem enables us to state

Proposition 2 *Given the utility representation $U(q) = g_1(q^1)+\cdots+g_n(q^n)$ of additive preferences, the vector fields (6) have vanishing Lie brackets over all consumption space.*

Proof (Spivak 1999, chapter 5) derives the algorithm for computing the Lie bracket $[\mathbf{A}, \mathbf{B}] = \mathsf{L}_{\mathbf{A}}\mathbf{B}$ (the Lie derivative of $\mathbf{B}$ with respect to $\mathbf{A}$) for a coordinate representation of vector fields $\mathbf{A}$, $\mathbf{B}$, namely,

$$\mathsf{L}_{\mathbf{A}}\mathbf{B} = \sum_{k=1}^{n}\left(\sum_{j=1}^{n} A^j(q)\frac{\partial B^k(q)}{\partial q^j} - B^j(q)\frac{\partial A^k(q)}{\partial q^j}\right)\frac{\partial}{\partial q^k} \tag{7}$$

being $\mathbf{A} = \sum_{k=1}^{n} A^k(q)\frac{\partial}{\partial q^k}$ and $\mathbf{B} = \sum_{l=1}^{n} B^l(q)\frac{\partial}{\partial q^l}$. A straightforward application of formula (7) confirms that the vector fields (6) have vanishing Lie bracket with one another over all consumption space. □

As long established, the vanishing of the Lie bracket $[\mathbf{A}, \mathbf{B}]$ entails the commutativity of the flows defined by such vector fields (see Appendix 1). Then, Proposition 2 sheds new light on additive preferences, for which n commuting symmetry vector fields exist, i.e. n independent ways to sweep indifference surfaces so that one can follow the flow of one symmetry vector field for a parameter value a, and then the flow of another vector field for a parameter value b, or the other way round, and obtain the same transformation of choice space (Spivak (1999)).

Additive separability is a strong (in a sense, the 'strongest') form of separability. We are in a position to enlarge the connection between symmetry vector fields of separable preferences and the commutativity of such vector fields to the case of joint separability. Following (Tyson 2013, Proposition 3.5), let us confine to the case $n = 3$.

Proposition 3 *Consider the symmetry vector fields* $\mathbf{H}_1 \equiv \frac{1}{\frac{\partial h}{\partial q^1}}\frac{\partial}{\partial q^1}$, $\mathbf{H}_2 \equiv \frac{1}{\frac{\partial h}{\partial q^2}}\frac{\partial}{\partial q^2}$ *and* $\mathbf{G} \equiv \frac{1}{\frac{\partial g}{\partial q^3}}\frac{\partial}{\partial q^3}$ *for the preferences represented by the utility function* $U(q^1, q^2, q^3) = \lambda h(q^1, q^2) + g(q^3)$ *on* $\mathcal{R}^3_{++}$, *with* $\frac{\partial^2 h}{\partial q^1 \partial q^2}$ *not identically vanishing. Then* $[\mathbf{H}_1, \mathbf{G}] = [\mathbf{H}_2, \mathbf{G}] = 0$.

Proof (Tyson 2013, Proposition 3.5) establishes that the vector fields $\mathbf{H}_1$, $\mathbf{H}_2$ and $\mathbf{G}$ are symmetry vector fields for the utility function $U(q^1, q^2, q^3) = \lambda h(q^1, q^2) + g(q^3)$. Then, a straightforward application of (7) yields the vanishing of the Lie brackets $[\mathbf{H}_1, \mathbf{G}]$ and $[\mathbf{H}_2, \mathbf{G}]$. □

One can compare such a result with Proposition 2: for additive preferences, each subutility function defines a symmetry vector field, and such fields do commute with one another; analogously, for the joint separability addressed in Proposition 3, the vector fields associated with the subutility function h do commute with the vector field associated with the subutility function g. Thus, the Lie bracket of symmetry vector fields for multivariate and joint separability seems to open an intuitive perspective on the problem of separability (compare the role of the Lie bracket for the integrability problem as discussed by Debreu (1972)).

3 Expansion

The basic elements of a normative economic problem are an objective function, to be optimized, and a set of constraints, which restrict the set of feasible solutions, and typically represent the scarcity of resources in the problem under consideration. The analysis of preference symmetries set forth by Tyson (2013) deals with the properties of objective (utility) functions, and not with the consequences of such properties on the solution of consumption problems. It is the aim of this section to move along such direction, and in particular to focus expansion effects.

Recall, an expansion path is a curve in consumption space obtained by fixing prices and letting expenditure range over $(0, \infty)$. In other words, if one considers Marshallian demand at fixed prices, one obtains the optimal consumption path as expenditure varies. In fact, the usefulness has been established of considering Marshallian demand as a function of *normalized* prices (prices to income; Cornes 1992), due to the 0-homogeneity of the budget constraint $\mathbf{Pq} = \mathrm{M}$, which can be written $\mathbf{pq} = 1$ in terms of normalized prices. From a duality standpoint such a choice is somewhat natural, since Marshallian demand is then defined between *spaces of the same dimension*, a property upon which we will soon capitalize.

Write A for the space $\mathcal{R}^n_{++}$ of positive normalized prices. Rays in A generate expansion effects, in that along any such ray prices scale equivalently with income. As one approaches the origin of A, one obtains the infinite income limit; on the opposite, as one moves away from the origin of A, one approaches the small income limit; such a recipe reflects the well established duality connection between the origin of primal space and infinity in dual space, and viceversa. Thus, Marshallian demand maps rays in A onto expansion paths on B so that expanding (contracting) prices are made to correspond with contracting (expanding) consumption bundles. We are in a position to deepen the introduction of the expansion vector field set forth by Mantovi (2013), which is primarily aimed at deepening the geometric representation of homothetic models.

To begin with, evidently, let us follow Tyson (2013) in assuming smooth preferences, and in fact consider

Assumption 1 Let preferences be smooth, strongly monotone and strictly-convex.

Such a standard assumption (compare Cornes 1992, section 2.1, or Blackorby et al. 1978, conditions R-1, R-1') guarantees the existence and uniqueness of the solution of consumption problems so that Marshallian demand $\mathbf{q}(\mathbf{p})$ results in a *diffeomorphism* $\mathsf{A} \rightarrow \mathsf{B}$ (i.e. a bijection of a definite differentiable class), and expansion paths do fill consumption space, and therefore, once properly parametrized, define a flow on B. Such a flow can be naturally defined as follows. Consider the scaling vector field

$$\Xi = \sum_{k=1}^{n} \Xi_j \frac{\partial}{\partial p_j} = \sum_{k=1}^{n} p_j \frac{\partial}{\partial p_j} \tag{8}$$

on the dual space A, i.e. the generator of homotheties (scale transformations) on the space of normalized prices; one thereby defines the dynamical system on A with trajectories $p_j(t) = p_j(0)e^t$. Then, with respect to the Marshallian demand $\mathbf{q}(\mathbf{p})$, define the *expansion* vector field $\mathbf{X}$ on B as the *push-forward* (Abraham and Marsden 1987) of the vector field Ξ on A. Notice that, by Assumption 1, Marshallian demand $\mathbf{q}(\mathbf{p})$ maps bijectively curves on A onto curves on B: thus, each vector tangent at $\mathbf{p}$ to a curve on A is uniquely mapped onto a vector tangent at $\mathbf{q}(\mathbf{p})$ to the image curve on B. Evidently, the *expansion* vector field thus defined is independent of the utility representation of preferences.

Such a differential geometric approach to expansion paths seems to have been conceived by Mantovi (2013) for the first time; perhaps the closest correspondent is given by the use of (Smale 1982) of tangent vectors to expansion paths in a direct sum decomposition of tangent spaces to choice space. Definitely, our geometric construction is the natural translation of the standard definition of expansion effects into the language of vector fields. Evidently, the economic justification for such an approach is to be found in the insights thereby obtained; true, underlying such a differential geometric construction is an attempt to sharpen the geometric insights embodied by duality theory (long praised by Cornes 1992), in first instance, the fact that Marshallian demand maps rays in the (dual) space of normalized prices onto expansion paths on the (primal) space of consumption bundles.

Evidently, the linearization of Marshallian demand in the neighborhood of a point is the standard recipe for obtaining the components X^k of $\mathbf{X}$, which are given by the matrix multiplication $\sum_{l=1}^{n} -J^{kl}\,\Xi_l$, being

$$J = \begin{pmatrix} \frac{\partial q^1}{\partial p_1} & \cdots & \frac{\partial q^1}{\partial p_n} \\ & \cdots & \\ \frac{\partial q^n}{\partial p_1} & \cdots & \frac{\partial q^n}{\partial p_n} \end{pmatrix} \tag{9}$$

the familiar jacobian matrix representation of the tangent component of Marshallian demand $\mathbf{q}(\mathbf{p})$. We thereby obtain a manageable approach to the properties of the push-forward, in terms of standard calculus, which we exploit as follows.

In first instance, it is not difficult to convince oneself that for all homothetic models one has $\mathbf{X} = \mathbf{Z}$ (see Mantovi 2013): in fact, by the scale symmetry of homothetic preferences, Marshallian demand, as a function of normalized prices $\mathbf{p}$, is homogeneous of degree—1, i.e. $\mathbf{q}(\lambda\mathbf{p}) = \mathbf{q}(\mathbf{p})/\lambda$ for any positive λ, so that $X^k = -\frac{\partial q^k(\lambda p)}{\partial \lambda}|_{\lambda=1} = q^k(\mathbf{p})$. Such a result goes beyond the well known fact that expansion paths are rays for any homothetic model. The expansion vector field $\mathbf{Z}$ fixes the speed at which consumption increases with expenditure: it turns out that such a speed is the same for all homothetic models once a scaling parametrization of expenditure is employed.

Then, we can exploit the analytical tractability of the following classical example, in order to appreciate the naturality of the introduction of the expansion vector field. Consider the quasilinear preferences represented by the additive direct utility function

$$U(x, y) = \ln x + y \tag{10}$$

on consumption space $(0, \infty) \times (0, \infty)$, which do satisfy Assumption 1 (and have been employed by Silberberg (1972) in his celebrated analysis of consumer's surpluses). The functional form (10) displays a continuous symmetry, namely, translation along y: as represented in Fig. 1, indifference curves can be obtained from one another by vertical translation; such a simple symmetry fixes a transparent setting for discussing the link with expansion effects.

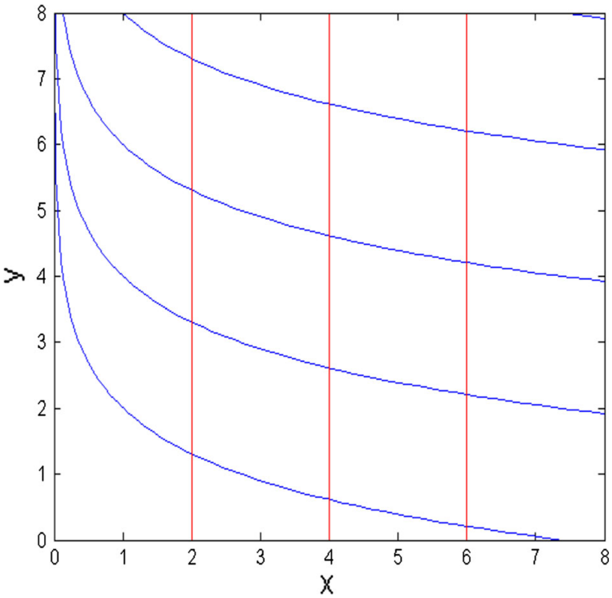

Fig. 1 Sample indifference curves (*blue curves*) and expansion paths (*red lines*) for preferences (10) (color figure online)

Optimal consumption is uniquely determined by the FOC

$$\frac{1}{x\mathbf{Z}(U)} = p_x$$
$$\frac{1}{\mathbf{Z}(U)} = p_y \tag{11}$$

with $\mathbf{Z}(U) = 1 + y$ playing the role of "marginal utility of money" as a function on B (see next section). The resulting Cartesian equation $x = \frac{p_y}{p_x}$ for expansion paths represents straight lines parallel to the y axis, along which the MRS is constant. Therefore, we expect the expansion vector field to be of the form $f(x, y)\frac{\partial}{\partial y}$, and thus parallel to the symmetry vector field $\frac{\partial}{\partial y}$ for the preferences (10): being $x(s) = x(0)$; $y(s) = y(0) + s$ the integral curves of $\frac{\partial}{\partial y}$ for any initial condition and $s \in (-y(0), \infty)$, indifference curves are mapped onto indifference curves by the flow of $\frac{\partial}{\partial y}$.

In fact, following the lines previously discussed, one can compute the expansion vector field by employing the Marshallian demand

$$x(p_x, p_y) = \frac{p_y}{p_x}, \quad y(p_x, p_y) = \frac{1}{p_y} - 1 \tag{12}$$

and its inverse

$$p_x = \frac{1}{x(1+y)}, \quad p_y = \frac{1}{1+y} \tag{13}$$

(the general structure of Marshallian demand for additive preferences is of course well known, see for instance Pollak 1972) in order to compute the jacobian matrix (9), which results in

$$\begin{pmatrix} -\frac{p_y}{p_x^2} & \frac{1}{p_x} \\ 0 & -\frac{1}{p_y^2} \end{pmatrix} \tag{14}$$

One thereby obtains, employing (13), the explicit expression

$$\mathbf{X}(x, y) = (1+y)\frac{\partial}{\partial y} \tag{15}$$

of the expansion vector field for preferences (10). The integral curves of (15) can be written

$$\begin{aligned} x(s) &= x(0) \\ y(s) &= y(0)e^s - 1 \end{aligned} \tag{16}$$

for any initial condition $x(0)$, $y(0)$, with $s \in (–\ln y(0), \infty)$. Thus, the flow of $\mathbf{X}$ does not map indifference curves onto indifference curves, and therefore $\mathbf{X}$ is not a symmetry vector field. Still, one can normalize $\mathbf{X}$ by means of $\mathbf{Z}(U) = 1 + y$ and define the vector field $\mathbf{Y} \equiv \frac{1}{1+y}\mathbf{X} = \frac{\partial}{\partial y}$, which thus results in the generator of y-translations. Notice that $\mathbf{X}(U) = \mathbf{Z}(U)$, a result whose generality is the subject of the following section.

As we learned in the previous section, by the additive separability of (10) we should expect the vector field $\frac{1}{(\ln x)'}\frac{\partial}{\partial x} = x\frac{\partial}{\partial x}$ to be a symmetry vector field as well. In fact, it is not difficult to convince oneself that the integral curves $x(t) = x(0)e^t$, $y(t) = y(0)$ of such a vector field map indifference surfaces onto indifference surfaces. Such a result is less intuitive than the symmetry related to vertical translation, and thus is worth consideration; perhaps even less intuitive is the result that any linear combination of $\frac{\partial}{\partial y}$ and $x\frac{\partial}{\partial x}$ results in a symmetry vector field.

Evidently, an elementary instance of separability is embodied by the previous problem: being Marshallian demand for good y independent of the price of good x, the budgeting problem can be considered as concerned first with the choice of how much y to consume, and only in a second round with the allocation of the remaining budget to good x. The superposition of expansion paths with the integral curves of the symmetry vector field $\frac{\partial}{\partial y}$ provides an insightful analytical representation of such an elementary instance of separability, which seems to witness the soundness of our approach.

4 Expansion and symmetry

The previous example has shown that, in some cases, our expansion vector field may be normalized so as to obtain a symmetry vector field; one may argue about the generality of such an occurrence. It is in fact the aim of this section to show that a definite normalization scheme for the expansion vector field results in the "fundamental" symmetry vector field. The basic ingredient of the following arguments is given by

Lemma 1 *For any utility representation U of preferences which satisfy Assumption 1,* $\mathbf{X}(U) = \mathbf{Z}(U)$.

Proof Consider the Hotelling-Wold identities

$$p_j(\mathbf{q}) \sum_{k=1}^{n} q^k \frac{\partial U}{\partial q^k}(\mathbf{q}) = \frac{\partial U}{\partial q^j}(\mathbf{q}) \tag{17}$$

(Cornes 1992) for the inverse Marshallian demand functions $p_j(\mathbf{q})$. Such identities can be considered as the component representation of the identity

$$dU = \mathbf{Z}(U)\mathbf{p} \tag{18}$$

between a pair of 1-forms on consumption space B. The 1-form dU on the LHS is the differential (gradient) of the utility function U, which Tyson (2013) writes ∇U. The 1-form on the RHS is the product of the function $\mathbf{Z}(U)$ and the 1-form $\mathbf{p}$ whose components are the inverse Marshallian demand functions $p_j(\mathbf{q})$[6]. It follows that

$$dU(\mathbf{X}) \equiv \mathbf{X}(U) = Z(U)\mathbf{p}\mathbf{X} \tag{19}$$

Thus, the pairing $\mathbf{pX}$ equals the ratio between $\mathbf{X}(U)$ and $\mathbf{Z}(U)$. Then, since the identity

$$1 = \sum_{k=1}^{n} p_k X^k \tag{20}$$

holds for the marginal propensities to consume (Cornes 1992, p. 47), we obtain $\mathbf{X}(U) = \mathbf{Z}(U)$. □

Such a lemma paves the way to the proof of our fundamental result.

Proposition 4 *Given the utility representation U of preferences, and the associated expansion vector field* $\mathbf{X}$*, the vector field*

$$\mathbf{Y} \equiv \frac{1}{\mathbf{Z}(U)}\mathbf{X} \tag{21}$$

is a symmetry vector field.

6 An element of the dual space A can be considered a 1-form on the primal space B; see Appendix 2.

Proof Being $\mathbf{X}(U) = \mathbf{Z}(U)$, one has $\mathbf{Y}(U) = 1$ identically on B. Let $\mathbf{q}$ and $\mathbf{r}$ lie on the same indifference surface, and consider the integral curves $\mathbf{q}_t$, $\mathbf{r}_s$ of $\mathbf{Y}$ emanating from such points. Since $\mathbf{Y}(U)$ =1, such curves sweep indifference surfaces at the same speed; therefore, for any positive t, $\mathbf{q}_t$ and $\mathbf{r}_t$ lie on the same indifference surface, that is, the flow of $\mathbf{Y}$ maps indifference surfaces onto indifference surfaces. $\mathbf{Y}$ is therefore a symmetry vector field. □

The simplicity of the above proof seems to witness the soundness of our approach to the connection between expansion effects and symmetries. In first instance, let us consider the case of homothetic models, which fix the benchmark perspective on our result. Following Mantovi (2013), we can unite all homothetic models into a single benchmark case for Proposition 4, in that for any homothetic model one has $\mathbf{X} = \mathbf{Z}$. Then, whatever the representation of utility (whether 1-homogeneous or not), one has $\mathbf{Y} \equiv \frac{1}{\mathbf{Z}(U)}\mathbf{Z}$, such that $\mathbf{Y}(U)$ =1, as a symmetry vector field (recall Proposition 1). The benchmark relevance of homothetic models emerges once again: for such models, the expansion vector field is already a symmetry vector field, and, as expected, our procedure (Proposition 4) yields another symmetry vector field.

Thus, the expansion vector field yields a symmetry vector field via the "normalization" $\mathbf{Z}(U)$, that can be interpreted as the "marginal utility of money" (Afriat 1987; or "marginal utility of wealth", Mas-Colell et al. 1995) once we work with normalized prices p_k, with respect to which Hotelling-Wald (HW) identities can be written as (17). The standard definition of marginal utility of money as the multiplier λ is represented by $\mathbf{Z}(U) = \lambda M$(being M the expenditure). $\mathbf{Z}(U)$ is evidently representation dependent: for $\tilde{U} = \phi(U)$ one has $\mathbf{Z}(\tilde{U}) = \phi'\mathbf{Z}(U)$, and the HW identity is satisfied for $\tilde{U}$ as well. The solution to the constrained optimization problem fixes a value of the multiplier, which is typically considered a function of prices and expenditure; by Proposition 4, we can consider it as the function $\mathbf{Z}(U)$ on B. Recall, the marginal utility of money has long been recognized as a 'pivotal' element for demand systems; consider for instance the assumption of a constant or "quasi constant" marginal utility of money (Georgescu Roegen 1968), or the role of the expression $\mathbf{Z}(U) = \sum_{k=1}^{n} q^k \frac{\partial U}{\partial q^k}$ in the characterization of additive preferences as a special case of generalized separability (Pollak 1972, in particular formula 3C.8 and subsequent considerations). Perhaps our perspective on the marginal utility of money may provide some 'unifying' insights on its multifaceted relevance.

As a methodological consideration, notice that, much like Marshallian demand, the expansion vector field is independent of the utility representation, whereas $\mathbf{Z}(U)$ does depend on the representation, and therefore our expansion-symmetry vector field (21) as well. In fact, we can easily establish

Proposition 5 *Whatever the utility representation, (21) is a (representation dependent) symmetry vector field.*

Proof Straightforward consequence of Proposition 1. □

Thus, we have succeeded in fixing the basic properties of the *fundamental* symmetry vector fields, namely, the ones whose flow superpose to expansion paths, and whose 'velocity', so to say, represents the utility parametrization of the preference relation.

Our results imply that a symmetry vector field exists under standard assumptions (Sect. 3); then, being the set of symmetry vector fieds a convex cone (Tyson 2013, Propostion 2.12), if another symmetry vector exists, our expansion-symmetry vector field can be employed in order to generate one more symmetry vector field.

5 Symmetry and distance

In the previous sections we have succeeded in deepening the notion of symmetry vector field, and providing an explicit construction of the fundamental symmetry vector field. It is the aim of this section to set forth a further characterization of symmetry vector fields in terms of a radial perspective on preferences.

It has been long established that the distance function $D(\mathbf{q}, u)$ provides a complete primal representation of preferences (Deaton 1979, and references therein). Thus, in principle, it should be possible to characterize the symmetries of preferences in terms of D. It is in fact the aim of this section to devise such a characterization, and argue that it can be considered an insightful complement to the one set forth by (Tyson 2013, theorem 2.10). We shall thereby shed further light on the nature of our expansion-symmetry vector field, and more generally on the 'response' of D to symmetry vector fields.

The distance function $D(\mathbf{q}, u)$ measures the radial contraction which drags the bundle $\mathbf{q}$ onto the u-indifference surface, i.e.

$$U\left(\frac{\mathbf{q}}{D(\mathbf{q}, u)}\right) = u \tag{22}$$

If $\mathbf{q}$ is strictly (not) preferred to the bundles on the u-indifference surface, then D is larger (smaller) than 1. Thus, the distance function gauges the *radial structure* of a preference relation; it is homogeneous of degree 1 in $\mathbf{q}$, and therefore satisfies Euler's formula $\mathbf{Z}(D) = D$. Still, "the distance function is completely *ordinal*; it is defined with reference to an indifference surface, and not with respect to any cardinalization of preferences" (Deaton 1979). Shephard's theorem

$$D(\mathbf{q}, u) = \mathbf{Z}(D(\mathbf{q}, u)) = \sum_{j=1}^{n} q^j \frac{\partial D}{\partial q^j}(\mathbf{q}, u) = \sum_{j=1}^{n} q^j \xi_j(\mathbf{q}, u) \tag{23}$$

(Blackorby et al. 1978, and references therein) enlightens the significance of the Euler's formula: the components $\frac{\partial D}{\partial q^j}(\mathbf{q}, u)$ of the $\mathbf{q}$-differential of D represent the inverse conditional demand functions $\xi_j(\mathbf{q}, u)$, i.e. the marginal willingness to pay for each good j as a function of utility and quantity supplied (Deaton 1979); as such, the $\xi_j(\mathbf{q}, u)$ are homogeneous of degree 0 in $\mathbf{q}$, consistently with the structure of formula (23).

A characterization of symmetry vector fields in terms of the distance function D amounts to the representation of the response of D to symmetry vector fields (analogously, Tyson represents the response of MRSs to symmetry vector fields). In order to compute such responses, notice that, by (22), the condition $U(\mathbf{q}) = u$ can be written $D(\mathbf{q}, u) = 1$; let us employ such a finite condition in order to obtain a differential correspondent, meant to characterize the action of a symmetry vector field $\mathbf{S}$ on D (the response of D to $\mathbf{S}$).

Start with any u_0-indifference surface as "initial condition". For any $\mathbf{q}_0$ belonging to such a surface, such that $D(\mathbf{q}_0, u_0) = 1$, one can consider the integral curve $\mathbf{q}_t$ of a symmetry vector field $\mathbf{S}$ starting at $\mathbf{q}_0$, such that $D(\mathbf{q}_t, u_t) = 1$ for any positive t.[7] A standard Taylor expansion about the initial condition $\mathbf{q}_0$ enables us to write

$$D(\mathbf{q}_\varepsilon, u_\varepsilon) = 1 = D(\mathbf{q}_0, u_0) + \sum_{j=1}^{n} \left(\frac{\partial D}{\partial q^j} + \frac{\partial D}{\partial u} \frac{\partial U}{\partial q^j} \right) |_{\mathbf{q}_0} \varepsilon S^j + o(\varepsilon) \tag{24}$$

which implies

$$0 = \sum_{j=1}^{n} S^j \left(\frac{\partial D}{\partial q^j} + \frac{\partial D}{\partial u} \frac{\partial U}{\partial q^j} \right) |_{D=1} = \sum_{j=1}^{n} S^j \left(\xi_j + \frac{\partial D}{\partial u} \frac{\partial U}{\partial q^j} \right) |_{D=1} \tag{25}$$

for any utility level. Condition (25) establishes the *response* of D to the action of the symmetry vector field $\mathbf{S}$: in order to preserve the condition $D(\mathbf{q}_t, u_t) = 1$, $\mathbf{S}$ must belong to the kernel of the 1-forms $\left(dD + \frac{\partial D}{\partial u} dU\right) |_{D=1}$ on consumption space. Such a characterization does not seem to represent a 'substitute' for the one set forth by Tyson (2013), which focuses the action of a symmetry vector field on MRSs; rather, it may be considered a 'complement', in that it focuses the action of the symmetry vector field on the radial structure of the preference relation, as represented by D. Evidently, the analytical structure of condition (25) is independent of the utility representation.

Condition (25) can be specialized to our expansion symmetry vector field $\mathbf{Y}$, such that $\mathbf{Y}(U) = 1$, as

$$\sum_{j=1}^{n} Y^j \xi_j |_{D=1} = -\frac{\partial D}{\partial u} |_{D=1} \tag{26}$$

which can be written $dD(\mathbf{Y})|_{D=1} = \mathbf{Y}(D)|_{D=1} = -\frac{\partial D}{\partial u}|_{D=1}$. In fact, (26) holds for any vector field $\mathbf{W}$ such that $\mathbf{W}(U) = 1$.

Let us check the soundness of our characterization of symmetry vector fields in terms of the examples discussed in the previous sections. First, consider the benchmark case of homothetic models, in which D is well known to admit a factorized representation (Cornes 1992): given any utility representation U which is homogeneous of degree 1, one has $D(\mathbf{q}, u) = \frac{U(\mathbf{q})}{u}$, and then $\frac{\partial D}{\partial u} = -\frac{D}{u}$. Thus, with $\mathbf{X} = \mathbf{Z}$, $\mathbf{Z}(D) = D$ and $\mathbf{Z}(U) = U$, one has $\mathbf{Y}(D)|_{D=1} = \frac{\mathbf{Z}(D)}{\mathbf{Z}(U)}|_{D=1} = \frac{D}{u}|_{D=1} = -\frac{\partial D}{\partial u}|_{D=1}$

[7] Evidently, the condition $D(\mathbf{q}_t, U(\mathbf{q}_t)) = 1$ holds identically along any integral curve of *any* vector field on consumption space, but it is only for symmetry vector fields, which map indifference surfaces onto indifference surfaces, that the path $U(\mathbf{q}_t) \equiv u_t$ is the same for all integral curves originating from the initial indifference surface.

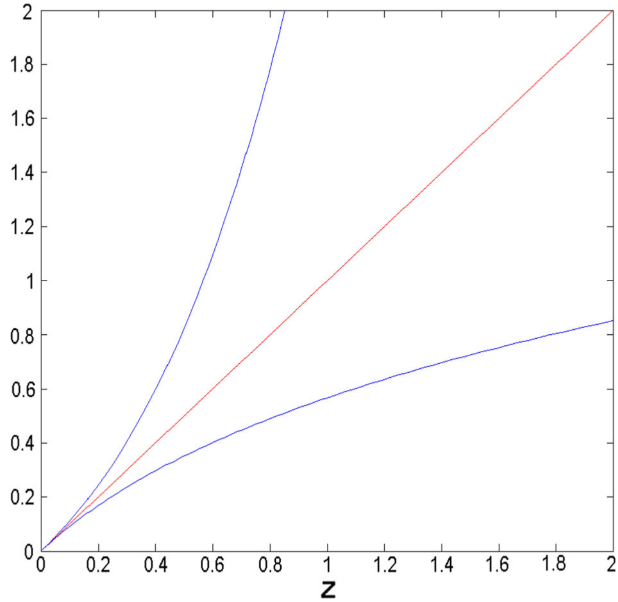

Fig. 2 Plot of the function $\varphi(z) = ze^z$ (convex curve), and its inverse (concave curve)

for any homothetic model. Furthermore, concerning the specific homothetic models discussed in Sect. 2, recall that the n vector fields $\mathbf{S}_j = q^j \frac{\partial}{\partial q^j}$ are symmetry vector fieds for CD preferences; one can easily check (25) in such respects.

As a second example, consider the distance function associated with preferences (10), which display a symmetry with respect to y-translations. One can thus expect the variable y to set a preferred analytical standpoint for representing D. In fact, one can devise an explicit representation of D for the preference relation (10) by writing condition (22) as

$$\frac{x}{y} \frac{y}{D(x, y; u)} \exp \frac{y}{D(x, y; u)} = \exp u \tag{27}$$

so that, in terms of the function $\varphi(z) = ze^z$(as a holomorphic bijection of the positive real axis; Fig. 2), one can write

$$\frac{x}{y} \varphi \left(\frac{y}{D(x, y; u)} \right) = e^u \tag{28}$$

We thereby factorize the 0-homogeneous term $\frac{x}{y}$, and let the variable y, a 1-homogeneous term, gauge the structure of the distance function in terms of the function

φ. Write ψ for its inverse, and obtain[8]

$$D(x, y; u) = \frac{y}{\psi\left(e^{u}\frac{y}{x}\right)} \tag{29}$$

As expected, (29) is homogeneous of degree 1 in (x, y), it is decreasing in u, and attains unit value for $xe^{y} = e^{u}$, which is equivalent to (10), since $\psi(ye^{y}) \equiv y$ (notice though, that the function $\psi\left(e^{u}\frac{y}{x}\right)$ is homogeneous of degree 0 in (x, y)). We are thereby in a position to check our characterizations (25) and (26) in terms of the distance function (29) for the symmetry vector fields of the preferences under consideration.

Consider first the symmetry vector field $\frac{\partial}{\partial y}$. Differentiation of (29) with respect to y, in conjuction with $xe^{y} = e^{u}$ and $\psi'(ye^{y}) = \frac{1}{\varphi'(y)}$, results in

$$\frac{\partial D}{\partial y}|_{D=1} = \frac{D}{y}|_{D=1} - \frac{D}{\psi}|_{D=1}\psi'(ye^{y})\frac{e^{u}}{x} = \frac{1}{y} - \frac{1}{y}\psi'(ye^{y})\frac{e^{u}}{x} = \frac{1}{1+y} \tag{30}$$

which does indeed coincide with

$$-\frac{\partial D}{\partial u}|_{D=1} = \frac{D}{\psi}|_{D=1}\psi'(ye^{y})\frac{ye^{u}}{x} = \frac{1}{y}\psi'(ye^{y})\frac{ye^{u}}{x} = \frac{1}{1+y} \tag{31}$$

Notice that the expression $\frac{1}{1+y}$ is not homogeneous of degree 0 in (x, y), whereas the inverse conditional demand function $\xi_y = \frac{\partial D}{\partial y} = \frac{1}{\psi} - \frac{y}{\psi^2}\frac{\partial \psi}{\partial y}$ is indeed homogeneous of degree 0, on account of the homogeneity of degree 0 of $\psi\left(e^{u}\frac{y}{x}\right)$. Then, for the continuous symmetry generated by $\frac{\partial}{\partial y}$, our recipe enables us to recover the intuitive relation $\frac{\partial D}{\partial y}|_{D=1} = -\frac{\partial D}{\partial u}|_{D=1}$, that one may have conjectured simply by looking at Fig. 1 (another clue of the soundness of our framework).

Along similar lines, one can check (25) with respect to the symmetry vector field $x\frac{\partial}{\partial x}$, such that $x\frac{\partial U}{\partial x} = 1$, thereby reducing again to condition (26); one then obtains the (perhaps not that intuitive) relation $x\frac{\partial D}{\partial x}|_{D=1} = -\frac{\partial D}{\partial u}|_{D=1}$. In addition, recall from Proposition 2 that the symmetry vector fields $\frac{\partial}{\partial y}$ and $x\frac{\partial}{\partial x}$ do commute (have vanishing Lie bracket), with the implications discussed in Sect. 2.

To sum up, for the simple additive model (10), we have established the response of the distance function to a "complete set" of commuting symmetry vector fields (one for each dimension, as established in Sect. 2). On account of Tyson (2013) Proposition 3.3, such a "complete response" pattern can be obtained for *any* additive preference relation, i.e. one can find n independent symmetry vector fields $\mathbf{S}_1, \ldots, \mathbf{S}_n$ such that

$$\frac{\mathbf{S}_1(D)}{\mathbf{S}_1(U)}|_{D=1} = \cdots = \frac{\mathbf{S}_n(D)}{\mathbf{S}_n(U)}|_{D=1} = -\frac{\partial D}{\partial u}|_{D=1} \tag{32}$$

and it is natural to conjecture that such "complete set" of conditions uniquely determines the distance function D. One thereby envisions a general scheme for the

[8] To the author's knowledge, this is an original result.

representation of multivariate separability, and fixes a sharp connection between additivity and the differential properties of the distance function, which the extant literature does not seem to encompass.

The mainstream approach to the analysis of separability properties is the functional representation problem (Blackorby et al. 1978), whose aim is to fix separability concepts in terms of precise functional forms of the objective—utility, cost, distance (transformation)—function. Such functional representations for the distance function do not seem to play a key role; a major result is the functional representation of homothetic separability via the distance function as given by (Blackorby et al. 1978, Theorems 4.4, 4.12). In such respects, the "differential representation" approach developed in this section seems to open a promising route to the analysis of separability: to the extent that a separability notion can be connected with the action of symmetry vector fields, a distance function representation may provide a pregnant line of analysis. Noticeably, the benchmark relevance of homothetic models provides us with a powerful weapon, namely, the symmetry vector field (1), which enables us to strengthen the analytical grip on the radial structure of preferences.

6 Conclusions

As Gorman (1976) put it, "separability has to do with the 'natural structure' of the problem" defined by an ordinal utility function. Noticeably, as discussed in the previous sections, it turns out that the symmetry properties of preferences as well have to do with the structure of the problem, in first instance, Marshallian demand: we have in fact succeeded in fixing a normalization of the expansion vector field which results in the fundamental expansion-symmetry vector field, along whose flow marginal rates of substitution are preserved. Such a result adds a pregnant element to the relevance of introducing an expansion vector field, and sheds new light on the nature of the marginal utility of money as a function on consumption space, as discussed in Sect. 4.

Furthermore, we have succeeded in complementing Tyson's (2013) characterization of symmetry vector fields in terms of their action on the radial structure of preferences, as embodied by the distance function. The examples discussed in Sect. 5 enlighten the soundness of such a characterization, in terms of the sharp insights represented by the responses of the distance function to infinitesimal symmetry transformations; we have envisioned a complete pattern of responses of a distance function representing additive preferences, which may add an insightful approach to the long established relevance of such preferences (Houthakker 1960); one can further recall the application of additive preferences to the normative analysis of welfare policies in a context of longevity variations (Pestieau and Ponthière 2015).

In the author's view, a relevant line of progress for the previous arguments is represented by the interpretation of condition (4) in terms of expansion and substitution effects. Our expansion-symmetry vector field satisfies such a condition with vanishing RHS, thereby signaling that the RHS is connected with the generation of substitution effects by the flow of a symmetry vector field. One may conceive of (4) as a superposition of expansion and substitution effects, and thereby possibly deepen the relevance of the connection we have been establishing between the symmetries of preferences and the *constrained* optimization problem of the consumer (for which expansion paths embody the solution).

Furthermore, by the well known isomorphism between the microeconomic theories of consumers and producers, our analysis can be applied as well to production problems. Tyson's definition of symmetry vector fields in terms of MRSs can be translated directly, as well as our characterization in terms of the distance function. In a production context, the flow of a symmetry vector field maps isoquants onto isoquants, and results analogous to the previous ones follow for the connection between symmetry and separability. Recall that separability has been long established as a major instance of production analysis, and in fact the functional representation of the properties of production technologies is a key element of the theory of production (see for instance Blackorby et al. 1978; Fuss et al. 1978; Chambers 1988). Furthermore, for production functions such that expansion paths do fill smoothly input space, the expansion vector field can be defined along the same lines followed in Sect. 3; true, the cardinal nature of a production function, as distinct from the ordinal nature of a utility function, entails a somewhat different perspective on the expansion-symmetry vector field.

To conclude, let us point out that a challenging line of application of preference symmetries is discussed by Kimball and Shapiro (2008). The Authors address the properties of utility functions meant to represent the choice between consumption and leisure, and argue about the relevance of postulating scale symmetry in consumption, in order to discuss the well established empirical regularity that "large, permanent differences in the real wage induce at most modest differences in the labor supplied by a household." The present approach to symmetries may provide useful handles in such respects.

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Appendix 1

The present appendix is meant to sketch an essential introduction to vector fields and dynamical systems. We refer to Abraham and Marsden (1987), Spivak (1999) and Taylor (1996) for authoritative references; see also Mantovi (2013).

One can define tangent vectors as a generalization of directional derivatives. Let $\gamma : (-1, 1) \to \mathcal{R}^n$ be a C^1 curve in $\mathcal{R}^n$, and let $f : \mathcal{R}^n \to \mathcal{R}$ be a C^1 function. Define the (Lie) derivative of f at $p = \gamma(0)$ as $\sum_{k=1}^{n} \frac{\partial \gamma^k}{\partial t}|_{t=0} \frac{\partial f}{\partial x^k}|_p$. Such a derivative, evidently, is linear and satisfies Leibniz rule; call it a *tangent vector* at p; call $\frac{\partial \gamma^k}{\partial t}|_{t=0}$ the *components* of the vector with respect to the natural coordinates of $\mathcal{R}^n$. A tangent vector at p is identified by an equivalence class of curves tangent at p (Abraham and Marsden 1987, p. 43). Evidently, any linear combination of tangent vectors is again a tangent vector, and one is in a position to define a *tangent space* at each point of $\mathcal{R}^n$.

A *vector field* on (an open subset of) $\mathcal{R}^n$ is a function which assigns a tangent vector to each point of the space.

On account of the previous considerations, the (local) coordinate representation of a vector field $\mathbf{A}$ reads $\mathbf{A} = \sum_{k=1}^{n} A^k(x)\frac{\partial}{\partial x^k}$, being $x^1, \ldots, x^n$ (local) coordinates, A^k the components of $\mathbf{A}$ with respect to such coordinates, and $\frac{\partial}{\partial x^k}$ the coordinate vector fields in that chart, which define a basis of the tangent spaces at each point. Then, the action of the vector field $\mathbf{A}$ on the function f, which measures the variation of f along the flow of $\mathbf{A}$, can be written $\mathbf{A}(f) = \sum_{k=1}^{n} A^k \frac{\partial f}{\partial x^k}$. Correspondingly, the coordinate representation of a 1-form $\mathbf{w}$ reads $\mathbf{w} = \sum_{k=1}^{n} w_k(x)dx^k$, so that the *pairing* between a 1-form and a vector field can be written $\mathbf{w}(\mathbf{A}) \equiv \mathbf{wA} = \sum_{k=1}^{n} w_k(x)A^k(x)$. Evidently, the differential of the function f is the 1-form $df = \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(x)dx^k$, so that one can write $\mathbf{A}(f) = df(\mathbf{A})$.

The *Lie derivative* of the vector field $\mathbf{B}$ with respect to the vector field $\mathbf{A}$ is the vector field $\mathsf{L}_{\mathbf{A}}\mathbf{B}$ which represents, so to say, the derivative of $\mathbf{B}$ along the flow of $\mathbf{A}$. Spivak (1999) provides a rigorous account of such a mechanism, as well as the proof that the action of $\mathsf{L}_{\mathbf{A}}\mathbf{B}$ on functions f results in the *commutator* $\mathbf{A}(\mathbf{B}(f)) - \mathbf{B}(\mathbf{A}(f)) = \mathsf{L}_{\mathbf{A}}\mathbf{B}(f)$. In words, the first term on the LHS is obtained by applying $\mathbf{B}$ to a function f and then applying $\mathbf{A}$ to the function $\mathbf{B}(f)$; the second term is obtained by commuting such operations. Call *Lie bracket* the mapping $(\mathbf{A}, \mathbf{B}) \to \mathsf{L}_{\mathbf{A}}\mathbf{B}$, which is, evidently, bilinear and skew-symmetric. (Spivak 1999, chapter 5) derives the algorithm (formula 7) yielding the components of $\mathsf{L}_{\mathbf{A}}\mathbf{B}$ given the components of $\mathbf{A}$ and $\mathbf{B}$.

The conceptual relevance of the Lie bracket has been long established: the Lie bracket represents the geometric (synthetic) definition of the analytical 'nucleus' which rules the integrability problem, as tailored by Frobenius theorem (Taylor 1996; Spivak 1999). For our purposes, it is enough to appreciate the condition $\mathsf{L}_{\mathbf{A}}\mathbf{B}$ as guaranteeing that the flows of the vector fields $\mathbf{A}$ and $\mathbf{B}$ do *commute*: given any initial point, one can follow the flow of $\mathbf{A}$ for a parameter value a and then the flow of $\mathbf{B}$ for a parameter value b and then reverse the order of such operations and find himself at the same final point (see Spivak 1999, p. 159). Mantovi (2013), Appendix 2 provides a pair of simple examples for such a pattern; one can easily check that coordinate vector fields have vanishing Lie brackets, consistently with the commutation of partial derivatives $\frac{\partial^2 f}{\partial x^j \partial x^k} = \frac{\partial^2 f}{\partial x^k \partial x^j}$ established by a well known theorem (typically named after Schwarz) of elementary calculus.

The vector field $\mathbf{A} = \sum_{k=1}^{n} A^k(x)\frac{\partial}{\partial x^k}$ is equivalent to definition of the first order ordinary differential equation (ODE) system $\dot{x}^k = A^k(x), k = 1, \ldots n$; the integral curves of the vector field are the solutions to the system. Thus, a vector field on some space defines a *dynamical system* in continuous time.

The theory of dynamical systems has long been recognized as a preferred terrain for the cross fertilization of different scientific disciplines. One can think for instance to the physical insights which led Henry Poincaré to envision the geometric approach to dynamical systems, in which it is the properties of flows which drive the development of the theory, and not the analytical form of the differential equations (see the Introduction in Abraham and Marsden 1987). Along such a line of progress, coordinate transformations have become a pivotal element in the study of dynamical systems, for instance in connection with symmetry properties and canonical dynamics. The celebrated address to the 1954 International Congress of Mathematicians set forth by A.N. Kolmogorov ("The General Theory of Dynamical Systems and Classical Mechanics") is a classical reference for the cross fertilizing effects concerning "the complex question of integrating the systems of differential equations of classical mechanics [...] interwoven with problems of the calculus of variations, many dimensional differential geometry, the theory of analytic functions, and the theory of continuous groups."

Noticeably, the development of evolutionary game theory in the last few decades can be considered as one more instance of such historical cross fertilization process: the *replicator* dynamics (Weibull 1995), meant to embody the selection of fitter strategies, is represented naturally in terms of first order ODEs (vector fields) on the space of strategy profiles. Perhaps the most significant economic application of vector fields pertains to the analysis of phase diagrams of growth models (see for instance Barro and Sala-i-Martin 2004).

Our introduction of the expansion vector fields on primal space, meant to represent the class of expansion paths as a *flow*, aims at capitalizing on the aforementioned cross fertilization potentialities associated with the adoption of vector fields. In fact, the introduction of symmetry vector fields by Tyson (2013) represents a clear example of such general pattern: symmetry vector fields on choice space turn out to provide quite effective a setting for deepening the analysis of separability. Vector fields on consumption space can in principle be employed in order to model any family of smooth curves which fill the space, and which embody significant economic effects (expansion effects, scale effects, etc). Evidently, limits to such approaches can be placed in terms of 'cost-benefit' analysis of the analytical description, as gauged by the balance between, on the one hand, the effectiveness of the formalism in establishing sharp conclusions, and on the other hand, the cost of enlarging the analytical toolkit of theoretical microeconomics. True, in the author's view, employing vector fields in the microeconomics of consumption does *not* mean enlarging the toolkit, but simply acknowledging that the tangent vectors to such paths may be employed effectively to gauge economic effects.

Appendix 2

The n-dimensional consumption space $\mathsf{B} = \mathcal{R}^n_{++} = (0, \infty) \times \cdots \times (0, \infty)$($n$ copies) admits the dual space A of normalized price vectors $\mathbf{p}$, with respect to which the budget constraint can be written $\mathbf{pq}$ =1. Despite the fact that B is not a linear space, elements of A are dual to elements of B in the standard algebraic sense, they are linear functionals on B, namely, such that

$$\mathbf{p}(a\mathbf{q}_1 + b\mathbf{q}_2) = a\mathbf{pq}_1 + b\mathbf{pq}_2$$

(provided, evidently, $a\mathbf{q}_1 + b\mathbf{q}_2$ belongs to B).

Then, let us recall the following well known result (see for instance Spivak 1999, chapter 3). A basis of a finite dimensional real linear space uniquely determines global coordinates on such space, i.e. the components of vectors with respect to such a basis.

Then the linear space can be endowed with the structure of real smooth differentiable manifold, and the global chart sets *natural* isomorphisms between the manifold and each of its tangent spaces. A sketch of the proof goes as follows.

Given any $\mathbf{q} \in \mathsf{B}$, the bundles $\mathbf{q}+t\ \mathbf{v}$ belong to B for any $\mathbf{v} \in \mathsf{B}$, for small enough $|\ t\ |$. Then, the vector $\mathbf{V} \in T_q\ \mathsf{B}$ tangent to such line at $\mathbf{q}$ is uniquely determined by $\mathbf{v} \in \mathsf{B}$, and such a mapping is evidently linear and injective. Therefore, the class of isomorphisms $\mathsf{B} \rightarrow T_q\ \mathsf{B}$ is uniquely determined for any $\mathbf{q} \in \mathsf{B}$.

Such a technical result does play a role in our geometric analysis. Tangent vectors to B represent the rate at which economic effects takes place, for instance an expansion effect. A 1-form on B is a linear functional on such vectors: by the previous result, we can consider an element of A (a vector of normalized prices) as a 1-form on B. Given such an interpretation of normalized price vectors as 1-forms, one can consider the Hotelling-Wold identity as defining a "normalization" of the differential dU (a 1-form) of the utility function U, which results in the inverse Marshallian demand (a 1-form), the normalization being performed by the function $\mathbf{Z}(U)$. Such geometric insights enable us to deepen the significance of our expansion-symmetry vector field.

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