J Econ (2013) 110:83–105
DOI 10.1007/s00712-013-0337-5

On the commutativity of expansion and substitution effects

Andrea Mantovi

Received: 1 October 2012 / Accepted: 22 February 2013 / Published online: 14 March 2013

Abstract The reversibility of sequential economic choices concerning production and consumption is addressed. A geometric approach to substitution effects and output/income effects is set forth in terms of vector fields on bundle space. By means of suitable fixing relations the 0-homogeneity of such problems can be circumvented, so as to define global parametrizations of effects, for which Lie brackets measure the departure from commutativity. A couple of propositions are established, assessing the benchmark relevance of homothetic models. Application to Farrell decompositions, as tailored by Bogetoft et al. (Eur J Oper Res 168:450–462, 2006), results in complete agreement with the results found by such Authors. The theoretical relevance of the approach is thoroughly discussed.

Keywords Partial equilibrium · Homotheticity · Substitution effect · Output effect · Income effect · Technical efficiency · Allocative efficiency

JEL Classification D00 · D11 · D20

> "If all we ever had to consider were pure substitution effects—sliding around indifference curves—life would have been so much easier [. . .] income effects are nearly always a nuisance"
>
> Frank Cowell, *Microeconomics*

Presented at the International Workshop "The Economics of Irreversible Choices", Brescia, 13–14 Settembre 2012.

A. Mantovi (✉)
Dipartimento di Economia, Università degli Studi di Parma, Parma, Italy
e-mail: andrea.mantovi@unipr.it

Initial bundle	EXE →	Expansion modified bundle
SUE ↓		SUE ↓
Substitution modified bundle	EXE →	Final bundle

Fig. 1 A commutative diagram for EXE and SUE (a truism for the infinitesimal effects in Slutsky equations)

1 Introduction

Irreversibility has long been recognized as an inherent feature of relevant economic choices, concerning for instance exploitation of exhaustible resources, commitment to binding agreements (possibly, contracts) and exercise of (partially) irreversible investment opportunities under uncertainty, typically modelled via stochastic processes. Furthermore, bounds of rationality (for instance, endowment effects, framing effects, biases, etc.) may preclude smooth reversions of resource allocations. On top of that, according to Stiglitz (2000, p. 1459), "there are natural irreversibilities associated with the creation of knowledge: history *has* to matter."

Differentiating from such lines of inquiry, we shall address the irreversibility of *sequences* of choices of producers and consumers. It is the aim of the present contribution to argue about the relevance of the *order* with which basic effects drive sequential economic choices, so that reversing such an order may not re-establish initial conditions (choices). Definitely, we shall tailor an analytical framework for the commutativity (Fig. 1) of independent and finite substitution effects ("SUE" henceforth) and output/income effects (*expansion* effects, "EXE" henceforth), so as to address a facet of the irreversibility of microeconomic choices which the extant literature does not seem to encompass. Perhaps, Bogetoft et al. (2006) represent the closest perspective to the one we shall be dealing with, in which the order of effects and the benchmark role of homotheticity are at stake. Weber (2010) argues about the reversibility of the compensated-income function.

The 'nuisances' generated by income effects are well known to be rooted in the inherent entanglement between EXE, price changes and SUE. On the one hand, producers' and consumers' optimization problems are 0-homogeneous in expenditure/income and prices (for instance, doubling both income and prices has no effect on the solution bundle; we shall denote such a property by "0-h"). On the other hand, price changes onset both EXE and SUE according to Slutsky equations: the variation in an optimal bundle can be decomposed as the superposition of a notional infinitesimal EXE and a notional infinitesimal SUE, being the order of such fictitious effects inherently irrelevant (Cowell 2005, p. 87). In fact, applied Slutsky equations typically employ functional forms lacking generality,[1] thereby setting inherent limits to the analysis of EXE and SUE.

[1] For instance, as pointed out by Chambers (1988), "one would never use a Cobb–Douglas function to investigate the magnitude of different elasticities of substitution since it forces all Allen elasticities to equal unity." According to Cornes (1992), "Its extreme simplicity is both a strength and a weakness of the Cobb–Douglas form. Empirically, it is not sensible to impose unit income elasticities and zero uncompensated cross-price responses, because both generally conflict with empirical findings."

Furthermore, comparative statics deals, by definition, with commuting effects: definitely, the first order Taylor expansion of a function $g(x, y)$,

$$g(x+h, y+k) - g(x, y) \approx \frac{\partial g}{\partial x}h + \frac{\partial g}{\partial y}k \tag{1}$$

entails a (commutative) sum of contributions which reverse their signs together with the 'driving effects' h and k; so to say, comparative statics is bound to the commutativity of infinitesimal effects (recall, according to Baumol (1973), "We have become used to comparative statics arguments whose results are remarkable for their banality"). As is well known, once finite effects are at stake, the commutativity represented in (1) may not hold anymore, as manifested in the problem of the commutativity of vector fields in the theory of dynamical systems, which we shall employ in our approach to global (i.e. defined on all of bundle space), independent and finite EXE and SUE, so as to fix the homotheticity benchmark in terms of scale symmetry, and then measure the departure from such a benchmark in terms of the Lie bracket[2] of expansion and substitution flows, thereby tailoring a unified approach to both finite and infinitesimal effects.

We shall be concerned with the microeconomics of both single output producer's and consumer's choices, between which we shall not differentiate unless suggested by the sharpness of the arguments; we shall therefore generally employ the following unifying conventions. By the term *input* we shall refer to both production inputs and consumption goods, and employ x and y to denote both the inputs and the variables representing the quantities of such inputs.[3] By *expansion paths* we shall refer to both output and income expansion paths. We shall denote both isoquants and indifference curves by the expression *level curves*. By *budget constraint* and *marginal rate of substitution* (MRS) we shall denote corresponding notions for producers and consumers. SUE shall be typically parametrized by the input ratio[4] $\kappa = y/x$ or MRS (equivalent to price ratio by FOC). Our manifold shall be the space $\mathcal{B} = (0, \infty) \times (0, \infty)$ (endowed with the natural differentiable structure) of the combinations (bundles) of inputs x and y: we shall disregard problems admitting 'corner' solutions (thereby ruling out perfect substitutability), which would force us to introduce more 'costly' geometry (manifolds with boundary) without correspondingly beneficial insights. For the same reasons we shall disregard multioutput settings. Objective functions shall be smooth and convex, so as to make sense of the following arguments.

Our geometric approach is meant to help escape the straitjacket of comparative statics, which is well suited for addressing *local* conditions (typically FOC) and at the same time 'myopic' with respect to *global* conditions, like the ones represented in the definition of normal and luxury goods. As long established, it is the global picture of

2 For the geometric concepts we shall be dealing with see for instance Abraham and Marsden (1987), Arnold (1992), for which "Poisson bracket" stands for "Lie bracket", and Williams (2008).

3 We shall confine main arguments to a pair of inputs, aligning with standard approaches (for instance, Varian 1992; Silberberg 2008).

4 Chambers (2002) argues about the naturality of such an index: "economists routinely prefer to work in terms of quantities which are unit free" (ivi, p. 756), as represented for instance by elasticity measures. In our perspective, the relevance of input ratios is grounded in their being adapted to the homothetic symmetry.

expansion paths which establishes whether a good is luxury or not; the slope of Engel curves at a single point cannot tell luxury. Needless to say, our framework is not meant to deviate from the mainstream of microeconomic analysis, but rather to take a natural step for embracing Engel curves and expansion paths in a global framework, in which additive and scaling expansion flows on input space tailor EXE and their commutation properties with SUE. Our differential geometric approach compares, to some extent, with the one employed by Williams (2008) in the analysis of communication in mechanism design.

In turn, our geometrical perspective deepens the benchmark relevance of homothetic symmetry. In fact, our approach enlightens the relevance of the independence of technical and allocative inefficiencies, with respect to which Bogetoft et al. (2006) set forth a reversed decomposition of overall efficiency which counters the standard approach named after Farrell (1957), thereby raising concerns about the consistency of such measures. "From a *conceptual* point of view, we suggest that the interpretations associated with the notions of technical and allocative efficiency are more ambiguous if the size of the effects depends on the order of decomposition" (Bogetoft et al. 2006, p. 451). Such an instance, as will be seen, can be given a fundamental microeconomic status, accounting for the consistency of decompositions in the benchmark case of homothetic problems; in fact, according to Chambers and Mitchell (2001), "Homotheticity may be the most common functional restriction employed in economics."

The plan of the rest of the paper is as follows. In Sect. 2 we introduce our first proposition on commutativity and homotheticity. In Sect. 3 we introduce vector fields and Lie brackets. In Sect. 4 we set forth our global picture for EXE and SUE and their commutation setting, and tailor our main proposition. In Sect. 5 we apply such a framework to Farrell decompositions. A final section sketches possible avenues for future research. A pair of appendices collect technical results supporting main arguments.

2 Effects under scale symmetry

Recall, EXE are defined, at fixed prices, along a given expansion path, and parametrized by expenditure/income. As is well known, such a definition is not 'natural' once we consider classes of expansion paths: on the one hand, scaling prices is equivalent to an income/expenditure effect; on the other hand, changes in price ratios onset entangled EXE and SUE according to Slutsky equation. It is therefore of fundamental relevance to deepen the benchmark role of homothetic problems, in which EXE can be globally 'disentangled' from SUE along the following lines.

2.1 Slutsky equations

Slutsky equations represent the cornerstone of the canonical approach to income and substitution effects associated with infinitesimal changes in the prices of consumption goods. Let I denote income, let D^i represent Marshallian demand functions, and let H^i represent Hicksian demand functions, with indices ranging from 1 to the number n of goods; then, each of the n^2 Slutsky equations

$$\frac{\partial D^i(\mathbf{p}, I)}{\partial p_j} = \frac{\partial H^i(\mathbf{p}, \upsilon)}{\partial p_j} - x_j^* \frac{\partial D^i(\mathbf{p}, I)}{\partial I} \tag{2}$$

defines an equilibrium disentanglement of changes in each demand function in terms of a notional infinitesimal SUE $\frac{\partial H^i}{\partial p_j} dp_j$ (a comparative statics for the H^i at the same level υ of the utility function) and a notional infinitesimal EXE $x_j^* \frac{\partial D^i}{\partial I} dp_j$ proportional to the j-th component of the initial optimal bundle. Recall, (2) represent total differentials of ordinary demand functions with respect to prices, once the identities (write C for the cost/expenditure function)

$$H^i(\mathbf{p}, \upsilon) = D^i(\mathbf{p}, C(\mathbf{p}, \upsilon)) \tag{3}$$

and Shephard's lemma are properly taken into account. It is not difficult to represent the 'directions' in input space of the effects in (2) by considering the whole matrix of Slutsky equations and multiplying them by the infinitesimal changes in optimal bundles (see for instance Varian 1992, p. 121).

As is well known, Slutsky equations are crucial for establishing a number of results, for instance, the symmetry of SUE and the sign of own price effects. Furthermore, Eq. (2) represent the integrability conditions guaranteeing the existence of a utility function underlying a given set of demands functions (Varian 1992). True, in and by themselves, the decompositions (2) do not convey sharp insights about the directions of change of input ratios, unless definite functional forms are employed. Appendix 1 employs a Cobb–Douglas model in order to pin down the significance of such terms in a benchmark homothetic case as a preliminary to our framework, which builds on the benchmark role of straight expansion paths.

Recall, the classes of expansion paths, one for each price vector, embody the solutions to the optimization problems of producers and consumers; such problems are 0-h, as a consequence of the fundamental constraint for ordinary demand functions (for instance, Cowell 2005, p. 85), which in a two input setting reads

$$p_x D^x + p_y D^y = I. \tag{4}$$

According to (4), scaling prices and income by the same factor does not alter the solution, i.e. the problem is 0-h; as is well known, such an invariance property plays a pivotal role in the analysis of partial equilibrium[5] (needless to say, Slutsky equations do share such an invariance property). Being a constraint, (4) holds both on equilibrium, where the D's are Marshallian demand functions (of prices and income), and off-equilibrium, where the D's may represent the compositions of affordable bundles, possibly specified according to some behavioral response to prices and income. Then, (4) provides a sharp insight concerning straight expansion paths: for fixed prices, if the D's increase proportionally with I, then EXE act along rays; homotheticity is in fact the property (symmetry) guaranteeing that, for fixed input prices, Marshallian demand functions grow proportionally along expansion paths (unit income elasticities).

[5] In the words of Silberberg (2008), "Consumers respond to changes in relative prices, not absolute prices."

In such a setting, a recipe for the commutation of EXE and SUE can be tailored as follows.

2.2 Homotheticity and commutativity

The benchmark relevance of homothetic models has long been assessed in terms of straight expansion paths (rays through the origin of bundle space) and separability of cost functions. Explicit functional forms (in first instance, Cobb–Douglas) are often employed in devising arguments, due to the sharpness of the insights which can be thereby represented. It is the aim of this subsection to deepen the relevance of homotheticity by virtue of the "blow ups" (in the words of Chambers and Mitchell (2001)) which generate homothetic profiles; such "blow-up" symmetry[6] supports the statement of Proposition 1 (below), establishing the commutativity of EXE and SUE in homothetic settings. In a nutshell, homotheticity enables one to 'bypass' 0-h by virtue of the fact that expansion paths are straight lines, rays, which can be cogently parametrized by scale, whereas, in the general problem, expansion paths are not straight and cannot sensibly be parametrized by scale. Evidently, the (duality) mapping from the space of parameters (prices and income) to (primal) input space is not a bijection (due to 0-h) irrespective of homotheticity; the 'surplus' characterizing homothetic models is that scale invariance is *adapted* to 0-h.

Figure 2 pictures a representative homothetic problem, in which the spots may represent optimal choices (bundles) given a definite price ratio and a definite level of income, but may as well represent off-equilibrium states to which we can apply both EXE and SUE (for this reason no budget constraint is represented). It seems to be implicitly accepted in the literature, though not explicitly stated (for instance, no mention is to be found in Varian (1992), nor in Mas-Colell et al. (1995)) that in homothetic problems the sequence of a finite EXE and a finite SUE can be reversed in both orders, thereby re-establishing the original state (Fig. 1). As is well known, underpinning such property is the fact that MRS are constant along rays, so that optimal bundles consist of the same proportions of inputs once only EXE are in place.

The so called "blow up" symmetry of the objective function (which, recall, does not imply constant return to scale), guarantees that MRS, and therefore FOC, are unchanged along rays, in perfect analogy with the similitude of euclidean geometry: once we expand a pattern (for instance, a set of curves), the same proportions hold as in the initial pattern. It is remarkable that on the basis of such a simple observation we are in a position to prove our first proposition, an insightful step in the direction of our full fledged argument.

Consider a consumer[7] endowed with homothetic preferences over two goods, and facing a definite budget constraint, thereby identifying an optimal bundle A (Fig. 2).

[6] Recall, a symmetry is a property of a system that is preserved with respect to the action of a group of transformations of the system. Homotheticity entails properties preserved by "blow-ups"; any class of homothetic level curves is invariant with respect to the action of scale transformations.

[7] The focus on consumer, as already pointed out, is meant to sharpen the argument; an 'isomorphic' argument concerning a single output producer may as well be conceived, as enlightened by application to Farrell decompositions (Sect. 5).

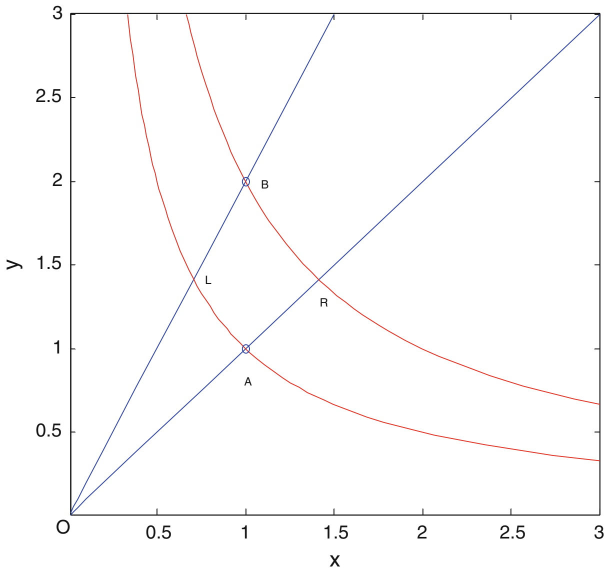

Fig. 2 Sample expansion paths (*blue lines*) and level curves (*red curves*) for a representative homothetic problem. Marginal rates of substitution are constant along expansion paths, as implied by homothetic/scale symmetry (color figure online)

Then, consider a change in the prices of the goods, and possibly a change in income, determining a new optimal bundle B. Capitalizing on homothetic symmetry, we can connect A with B by a pair of paths generalizing Slutsky decompositions towards the definition of independent EXE and SUE, as depicted in Fig. 2. A first path consists of a sub-path, starting at A, generated by an income effect, leading to the point R on the level curve containing B. A second sub-path, generated by a suitable substitution effect, consists of the part of the level curve joining R and B. The income effect acts along an income expansion path (right bound blue line), and connects optimal bundles for the same relative prices (prices to numeraire) for varying income. The substitution effect acts along the final indifference curve (upper red curve).

As depicted in Fig. 2, a second path connecting A with B can be conceived by reversing the order of effects. A first substitution effect pushes A towards the point L on the initial level curve, the one point with the final goods ratio. Then, an income effect pushes L towards the final point B. Indeed, it is hardly surprising that both paths lead to the same final point, since such paths have been constructed for such a purpose. Furthermore, it is somewhat trivial to notice that both substitution effects entail the same initial and final goods ratio, since they connect the same straight expansion paths. Noticeably, the two income effects correspond to *the same proportional increase in income*: in fact, being

$$\frac{\mathrm{AR}}{\mathrm{OR}} = \frac{\mathrm{LB}}{\mathrm{OB}}$$

(equivalently, the triangles ORB and OAL are similar), the segments AR and LB correspond to the same proportional increase in income, despite their different euclidean length,[8] and to the same Shephard's distance from the final utility level.

We have thereby established that for homothetic consumers, any pair of optimal bundles (choices) can be joined by properly defined finite and independent EXE and SUE in both orders. We are thus in a position to state

Proposition 1 *Finite and independent SUE (parametrized by input ratio) and scaling EXE do commute on a region of (possibly the whole of) input space if the objective function is homothetic on that region.*

Proposition 1 accounts for both well known results and quite advanced issues, like the consistency of standard and reversed Farrell decompositions, which we discuss in Sect. 5. Proposition 1, being established on geometrical grounds, paves the way for the introduction of expansion and substitution flows on input space, which will lead us to our full fledged result (Proposition 2).

3 Vector fields and Lie brackets

The standard physical picture motivating the relevance of vector fields poses that a vector field on a space may represent the 'velocity' field of 'particles' moving in such a space, whose trajectories do not cross each other (thereby defining a *flow*) being the velocity at any point uniquely determined. A dynamical parameter represents the 'time' it takes for the particles to travel along trajectories, given an initial position. Such a mechanical picture has been progressively deepened in its mathematical structure during the twentieth century (we refer the reader to the literature on differential geometry for a thorough account of vector fields and flows).

A vector field $\mathbf{X}$ on a manifold generates a flow on such manifold in terms of its integral curves, i.e. curves everywhere tangent to $\mathbf{X}$ and parametrized by the proper 'velocity'. The trajectories which constitute flows are more than one-dimensional sets on the manifold, they are parametrized sets, i.e. they are *functions* from real intervals to the manifold, for which the 'velocity' parameter does matter. So to say, integral curves of a vector field are somewhat like the timetables of trains, for which not only the position of the railway matters, but also the speed at which such railways are travelled. In such a setting, the Lie derivatives $\mathcal{L}_{\mathbf{X}}$ generalize the ordinary directional (partial) derivatives of calculus (linearity and Leibniz rule hold), and are defined on every smooth tensor field on the manifold; our microeconomic perspective enables us to confine attention to Lie derivatives of functions and vector fields.

The Lie derivative of a smooth function g with respect to the vector field $\mathbf{X}$ is the field of directional derivatives of g along the flow of $\mathbf{X}$; write

$$\mathcal{L}_{\mathbf{X}}(g) = \mathbf{X}(g) = \frac{d}{dt} g(t), \tag{5}$$

8 Recall, the Euclidean distance between points in input space bears no economic significance; for instance, Shephard's distance can be defined irrespective of any Riemann structure on the manifold of bundles.

with the smooth function g considered along each trajectory as a function of the flow parameter t. Analogously, the Lie derivative of a vector field $\mathbf{Y}$ with respect to the vector field $\mathbf{X}$ is a vector field which represents the 'derivative' of $\mathbf{Y}$ along the flow of $\mathbf{X}$; write (for any smooth function g)

$$\mathcal{L}_{\mathbf{X}}(\mathbf{Y})(g) \equiv [\mathbf{X}, \mathbf{Y}](g) \equiv \mathbf{X}(\mathbf{Y}(g)) - \mathbf{Y}(\mathbf{X}(g)) \tag{6}$$

and call it *Lie bracket* of $\mathbf{X}$ and $\mathbf{Y}$. Such bracket is skew symmetric (its sign reverses with the order of the entries) and satisfies the Jacobi identity. Remarkably, the Lie bracket characterizes the commutativity of the flows of the two vector fields (therefore two vector field are said to *commute* if their Lie bracket vanishes).

In order to start grasping such a profound issue, notice that coordinate vector fields do commute. The underlying intuition poses that the change in the value of a function $g(x, y)$ in passing from a point to another can always be decomposed as the sum of an increment along the x direction and an increment along the y direction, namely,

$$g(x + h, y + k) - g(x, y) = \Delta_x g + \Delta_y g \tag{7}$$

being the order in which such directions are contemplated evidently irrelevant. Compare (7) with formula (1) which represents the *differential* of the function. In geometric representation, the infinitesimal commutativity encompassed in (7) can be expressed in terms of the Lie brackets of the coordinate vector fields, i.e.,

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] = 0 \Leftrightarrow \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) = 0 \tag{8}$$

for any smooth function g on the manifold. The significance of such commutativity is easily spelled out: to say that coordinate vector fields do commute is tantamount to say that the change in the function f from following a coordinate curve, and then following another coordinate curve, does not depend on the order in which such coordinate paths are sequenced. The commuting mixed derivative $\frac{\partial^2}{\partial x \partial y}$ (Schwartz's theorem of standard calculus) is guest: once coordinate vector fields are at stake, the relative flows do commute, contrary to the general Lie bracket, which can be written (Arnold 1992, p. 207)

$$\left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right] g = \left.\frac{\partial^2 g}{\partial s \partial t}\right|_{s=t=0}. \tag{9}$$

with respect to the parameters s and t of the flows. Such a bracket measures the departure from commutativity of the two flows. One could argue that the commutativity represented in (8) reflects the 'orthogonality'[9] of the vector fields in the brackets; that is not the case, as the following examples will make clear.

[9] Recall, one is not in fact in a position to define orthogonality unless a Riemann structure has been introduced on the manifold; definitely, our arguments do not employ the Euclidean metric on bundle space.

Being the Lie derivative linear, we can represent basic insights by focusing vector fields with only one component. First, it is not difficult to convince oneself that proportional vector fields have vanishing Lie derivative. Then, simple classes of commuting 'orthogonal' vector fields are given by $\varphi(x)\frac{\partial}{\partial x}$ and $\psi(y)\frac{\partial}{\partial y}$, being φ and ψ smooth functions; true, for any smooth function $g(x, y)$,

$$\left[\varphi(x)\frac{\partial}{\partial x}, \psi(y)\frac{\partial}{\partial y}\right] g(x, y) \equiv \varphi(x)\frac{\partial}{\partial x}\left(\psi(y)\frac{\partial g}{\partial y}\right) - \psi(y)\frac{\partial}{\partial y}\left(\varphi(x)\frac{\partial g}{\partial x}\right)$$
$$= \varphi(x)\psi(y)\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] g(x, y) = 0$$

Corresponding to the vanishing of such local measure, we expect the associated flows, for finite parameter intervals, to commute: in fact, the flow of the first vector field is made of x-coordinate lines, and its velocity is independent of y; an analogous property holds for the second vector field. Then, one can follow the flow of one vector field for a finite interval, and then follow the flow of the other vector field for another finite interval, and the final point does not depend on the order in which flows are sequenced.

Such insights can be sharpened by focusing the commutativity of the coordinate vector field $\frac{\partial}{\partial x}$ with the vector fields $y\frac{\partial}{\partial y}$ and $x\frac{\partial}{\partial y}$. The Lie brackets under inquiry result in (as usual, for any smooth function g)

$$\left[\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\right] g(x, y) \equiv \frac{\partial}{\partial x}\left(y\frac{\partial g}{\partial y}\right) - y\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) = 0, \qquad (10)$$

$$\left[\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}\right] g(x, y) \equiv \frac{\partial}{\partial x}\left(x\frac{\partial g}{\partial y}\right) - x\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial}{\partial y}(g). \qquad (11)$$

The vanishing of the Lie bracket (10) on all input space entails the commutation of the flows, as can be intuitively grasped by noticing that such flows consist of coordinate lines along which the velocity does not depend on the fixed coordinate. On the opposite, the nonvanishing Lie bracket (11) (which results in the y coordinate vector field) entails that the flows under inspection *do not* commute: the velocity of the vector field $x\frac{\partial}{\partial y}$ depends on the x coordinate, so that it makes a difference if one follows its flow for different values of x, and hence the order matters with which our flows are sequenced (see Appendix 2). In fact, flows represent global geometric 'pictures' for first order ODEs (dynamical systems), whose analytical structure provides the local coordinate characterization of the system; once a pair of flows commute, the associated dynamical systems define different paths leading to the same point, the trivial case being represented by coordinate vector fields.

4 Expansion and substitution flows

It is the aim of the present section to introduce a geometric representation of EXE and SUE by means of flows on input space generated by suitable vector fields, thereby establishing a proper commutation settings for EXE and SUE so as to guarantee

(i) commutation for homothetic problems (Proposition 1), and (ii) consistency with standard representations of effects. As a preliminary insight, recall that in two dimensions there is a trivial level of analysis for the commutativity of flows: provided level curves do span all values of input ratios (the 'regular' case we are interested in), one can always find a suitable SUE which connects any pair of expansion paths, so that one can connect any two points on bundle space by sequences of portions of expansion paths and level curves. Evidently, our analysis triggers more structure: effects are to be properly parametrized, and their commutativity is expected to connect effects of the *same size*, so as to enter equations with a sound significance.

4.1 Additive expansion flows

As already pointed out, the very possibility of meaningfully representing EXE in terms of flows on input space reflects the 'regularity' property our expansion paths are meant to fit, i.e. we want such paths to fill (foliate) bundle space and never intersect with each other. Equivalently, we expect prices and income to uniquely determine an optimal point in bundle space. True, each expansion curve is naturally parametrized by income for given prices, yet, connecting different curves by income parametrization is not straightforward, due to 0-h. As is well known, we are in a position to define a global parametrization at the expense of fixing a relation between income and prices (the choice of a numeraire is a typical example of such fixing); building on the resulting bijection, we can define expansion flows as follows.

Consider the Marshallian demand functions (of prices and income) solving the optimization problem of a consumer. For given prices, and income ranging from zero to infinity, we define an income expansion curve, parametrized by income. Therefore, by differentiating such demand functions with respect to income we obtain the components of a vector field with respect to coordinate vector fields (which define a basis of the tangent space at each point of the manifold, a global frame), namely,

$$\mathbf{X} = \frac{\partial D^x(\mathbf{p}, I)}{\partial I}\frac{\partial}{\partial x} + \frac{\partial D^y(\mathbf{p}, I)}{\partial I}\frac{\partial}{\partial y}. \tag{12}$$

The flow of (12) consists, by definition, of expansion paths parametrized by income. Notice, the components of (12) are not unfamiliar, being parts of the EXE terms in Slutsky equations. (12) embodies the geometric properties of income effects, which can thereby be represented by portions of the integral curves of (12), as measured by income increases (needless to say, such parameter intervals do not define euclidean lengths). For normal goods such flow is 'expanding'; the bending of such paths towards one axis defines a luxury good. The vector field (12) is the generator of the flow defined by expansion paths; EXE terms is Slutsky equations are proportional to the components in (12). On the other hand, SUE terms represent infinitesimal paths along level curves, built out of Hicksian demand functions, which identify optimal bundles as functions of the level of output or utility and of prices. In essence, Slutsky equations connect points on nearby expansion paths in terms of, on the one hand, SUE measured by changes in compensated demand, and, on the other hand, EXE measured by changes in direct demand. We shall introduce (Sect. 4.3) a vector field generating SUE. Call *additive EXE* those generated by (the flow of) the vector field (12), since they are measured by (additive) increases in income, unlike the proportional increases which characterize homothetic symmetry (Proposition 1), for which a suitable vector field is needed.

Evidently, in and by itself, expression (12) does not represent a vector field on bundle space, since its components are not functions of x and y; yet, one can employ FOC in order to turn (12) into a vector field on input space, provided a relation between income and prices is fixed, so as to break 0-h. See Appendix 1 for the familiar Cobb–Douglas case, by means of which we shall build our main Proposition.

4.2 Generator of homothetic (scale) symmetry

As established by Proposition 1 (by means of similitude arguments), homothetic models admit a natural scaling parametrization of EXE. Thus, let us look for a 'connection' between geometric similitude and analytical mappings in order to characterize the benchmark role of homothetic problems with respect to general problems, and better grasp the sense in which input ratio defines a parametrization of SUE *adapted* to the benchmark homothetic symmetry. First, recall the well known fact that the "blow up" symmetry can be generated by the *scaling* vector field

$$\mathbf{Z} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \tag{13}$$

a radial vector field whose flow consists of scale transformations, solutions to the initial value problems associated with the ODE system $\dot{x} = x$, $\dot{y} = y$, equivalent to (13); the well known solution mapping

$$x(t; x_0, y_0) = x_0 e^t, \qquad y(t; x_0, y_0) = y_0 e^t \tag{14}$$

is an exponential mapping which drags any point in bundle space along the corresponding ray through the origin, so that any scale transformation is accomplished by the homothety e^t : we thereby assess that the vector field (13) generates the "blow ups" with respect to which homotheticity is defined. Recall, furthermore, that Shephard's input distance function is defined in terms of proportional (radial) reductions of inputs. Evidently, for any positive α, the vector field $\alpha\mathbf{Z}$ is as well a generator of "blow ups" (with the proper rescaling of the flow parameter). We call (14) exponential mapping since it shares the defining property of exponentials, namely, $\exp(t\mathrm{Z})\exp(s\mathrm{Z}) = \exp((t+s)\mathrm{Z})$, for any t, s.

Capitalizing on the insights represented in proposition 1, call *scaling EXE* those generated by the vector field (13); the properties of the flow (14) underlie Proposition 2 (below). In Appendix 1 we find that (13) is a simple function of the additive expansion vector field for a Cobb Douglas model, namely, $\mathbf{X}_f = a\frac{\partial}{\partial x} + a\frac{y}{x}\frac{\partial}{\partial y} = a\frac{1}{x}\mathbf{Z}$, provided x is taken as numeraire. It is not difficult to conjecture that the same holds for any homothetic problem. By the properties of Lie brackets, we shall therefore reduce the commutation properties of additive expansion vector fields to those of the scaling vector field $\mathbf{Z}$ (13), with the caveat that the effects generated by $\mathbf{Z}$ are scaling effects, guest of Proposition 1. True, despite its radial nature, the flow (14) does *not* parallel drag along budget constraints, so that it cannot be employed as a generator of 'additive' EXE.

4.3 Substitution flows

Turn now to the representation of SUE. It has not been difficult to connect our additive expansion flows with the income effects in Slutsky equations, since such terms contain the components of (12). Definitely, the substitution terms in Slutsky equations contain derivatives with respect to prices, but evidently prices do not represent viable parametrizations in the primal setting of bundle space. Therefore, we have to transform price derivatives into derivatives with respect to a cogent primal parametrization: it is natural to elect input ratio to natural parametrization in that it is adapted to the homothetic benchmark.

Let us introduce a vector field whose flow is meant to generate the substitution effects along a given class of level curves. We proceed in two steps. A first step identifies a field of vectors parallel to level curves, as explicitly[10] dependent on the objective function f according to

$$\tilde{\mathcal{S}}_f = \frac{\partial f}{\partial y}\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial y} \tag{15}$$

(evidently, equivalent utility functions f give rise to parallel vector fields according to such formula). By construction, the objective function f is constant along the flow of the vector field (15): $\tilde{\mathcal{S}}(f) \equiv \mathcal{L}_{\tilde{\mathcal{S}}}(f) = \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}\right) - \frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0$; such an algebraic structure resembles the euclidean scalar products between the vector field (15) and the gradient of f, and one may be tempted to conclude that such vector fields are orthogonal; that is not the case (see footnote 9). For our workhorse Cobb Douglas model $f(x, y) = x^a y^{1-a}$ such a vector field reads

$$\tilde{\mathcal{S}}_f = (1-a)\left(\frac{x}{y}\right)^a \frac{\partial}{\partial x} - a\left(\frac{y}{x}\right)^{1-a} \frac{\partial}{\partial y}.$$

Our task has been partially performed: we have constructed a vector field with the proper direction at each point in bundle space, which is uniquely determined by the objective function under inquiry. In order to complete out task, we have to check that our substitution vector field is parametrized by input ratio, so as to comply with the commutation setting tailored by Proposition 1. Thus, we normalize the vector field (15) by its action on the input ratio, thereby obtaining a flow whose parameter grows exactly like the input ratio, namely,

$$\mathcal{S}_f \equiv \frac{1}{\tilde{\mathcal{S}}_f(\frac{y}{x})}\left(\frac{\partial f}{\partial y}\frac{\partial}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial}{\partial y}\right) \tag{16}$$

[10] Once a vector field is defined, one of two possible verses for the flow is fixed; evidently, for our purposes, the opposite vector field (opposite components) does the job as well.

For the Cobb Douglas case above we find $\tilde{\mathcal{S}}_f\left(\frac{y}{x}\right) = -x^{a-2}y^{1-a}$, so that the substitution vector field results in

$$\mathcal{S}_f = -(1-a)x^2y^{-1}\frac{\partial}{\partial x} + ax\frac{\partial}{\partial y}. \tag{17}$$

with $\mathcal{S}_f\left(\frac{y}{x}\right) = 1$.

To sum up, we have succeeded in defining flows generating SUE like the ones entering Slutsky equations. It is not difficult to convince oneself that the terms $\frac{\partial H^i}{\partial p_j}$ in Slutsky equations are parallel to the vector field (16) (see Varian 1992, p. 121).

4.4 Flows and commutation

Finally, we are in a position to employ our flows on bundle space in order to characterize locally the degree of departure from the homothetic benchmark. Proposition 1 establishes that commutativity holds for homothetic models once scaling EXE are considered; we thus expect such commutativity not to hold for additive expansion effects; furthermore, we expect Lie brackets to provide useful insights in such respects.

For our Cobb–Douglas models $f(x, y) = x^a y^{1-a}$ the relevant Lie bracket reads

$$\begin{aligned}\left[\mathcal{S}_f, \mathrm{X}_f\right] = &\left(-(1-a)x^2y^{-1}\frac{\partial}{\partial x} + ax\frac{\partial}{\partial y}\right) a\left(\frac{\partial}{\partial x} + \frac{y}{x}\frac{\partial}{\partial y}\right)\\ &-a\left(\frac{\partial}{\partial x} + \frac{y}{x}\frac{\partial}{\partial y}\right)\left(-(1-a)x^2y^{-1}\frac{\partial}{\partial x} + ax\frac{\partial}{\partial y}\right)\end{aligned}$$

(being x the numeraire). By the linearity of the Lie bracket, we can expand the algebra according to the general structure (Arnold 1992 p. 208; Williams 2008 p. 39)

$$\begin{aligned}\left[\mathcal{S}_f, \mathrm{X}_f\right]_x &= (\mathcal{S}_f)_x\frac{\partial(X_f)_x}{\partial x} + (\mathcal{S}_f)_y\frac{\partial(\mathrm{X}_f)_x}{\partial y} - (\mathrm{X}_f)_x\frac{\partial(\mathcal{S}_f)_x}{\partial x} - (\mathrm{X}_f)_y\frac{\partial(\mathcal{S}_f)_x}{\partial y}\\ \left[\mathcal{S}_f, \mathrm{X}_f\right]_y &= (\mathcal{S}_f)_x\frac{\partial(\mathrm{X}_f)_y}{\partial x} + (\mathcal{S}_f)_y\frac{\partial(\mathrm{X}_f)_y}{\partial y} - (\mathrm{X}_f)_x\frac{\partial(\mathcal{S}_f)_y}{\partial x} - (\mathrm{X}_f)_y\frac{\partial(\mathcal{S}_f)_y}{\partial y}\end{aligned} \tag{18}$$

Instead of performing such algebra, compute first the Lie bracket $[\mathcal{S}_f, \mathbf{Z}]$, which we expect to vanish on account of Proposition 1: being SUE measured by input ratio, we expect the substitution vector field to commute with the scaling vector field (13). In fact, performing the proper algebra we get

$$\begin{aligned}\left[\mathcal{S}_f, Z\right]_x &= (\mathcal{S}_f)_x\frac{\partial Z_x}{\partial x} + (\mathcal{S}_f)_y\frac{\partial Z_x}{\partial y} - Z_x\frac{\partial(\mathcal{S}_f)_x}{\partial x} - Z_y\frac{\partial(\mathcal{S}_f)_x}{\partial y}\\ &= -(1-a)x^2y^{-1}\cdot 1 + ax\cdot 0 - x(a-1)2xy^{-1} - y(1-a)x^2y^{-2} = 0\end{aligned} \tag{19}$$

$$\begin{aligned}\left[\mathcal{S}_f, Z\right]_y &= (\mathcal{S}_f)_x \frac{\partial Z_y}{\partial x} + (\mathcal{S}_f)_y \frac{\partial Z_y}{\partial y} - Z_x \frac{\partial (\mathcal{S}_f)_y}{\partial x} - Z_y \frac{\partial (\mathcal{S}_f)_y}{\partial y} \\ &= -(1-a)x^2y^{-1} \cdot 0 + ax \cdot 1 - xa - y \cdot 0 = 0\end{aligned} \tag{20}$$

The vanishing of the Lie bracket $[\mathcal{S}_f, \mathbf{Z}]$, assessed componentwise by (19) and (20), parallels the content of Proposition 1 and introduces a differential perspective in which the parametrization of the substitution vector field (input ratio) is adapted to the homothetic symmetry, which can be generated by $\mathbf{Z}$. As already pointed out, we then expect the Lie bracket $[\mathcal{S}_f, \mathbf{X}_f]$ not to vanish, as can be easily checked as follows: for any smooth function g,

$$\left[\mathcal{S}_f, \mathrm{X}_f\right](g) = \left[\mathcal{S}_f, \frac{1}{x}\mathrm{Z}\right](g) = \mathcal{S}_f\left(\frac{1}{x}\mathrm{Z}(g)\right) - \frac{1}{x}\mathrm{Z}(\mathcal{S}_f(g)) = \mathcal{S}_f\left(\frac{1}{x}\right)\mathrm{Z}(g) \tag{21}$$

The following interpretation of such a formula is insightful. The nonvanishing Lie bracket (21) signals that sequences of paths built out of finite effects (a finite EXE followed by a finite SUE and the other way round) do not end up at the same final point. True, being SUE parametrized by input ratio, they connect rays, so that the 'missing paths' must be along rays: in fact, (21) identifies a radial vector field, which generates the (finite or infinitesimal) 'missing paths' which determine the non commutativity of flows. In much the same way, had we measured EXE in the standard additive guise, we would have missed the commutativity of effects in Proposition 1.

To sum up, the differential geometric perspective represented in such brackets complements the euclidean geometric perspective represented in Proposition 1. We thereby re-establish, on differential geometric grounds, the benchmark relevance of homothetic models, at the same time paving the way for the Lie bracket to assess a fundamental measure of the (non)commutativity of EXE and SUE. The following proposition distils the heart of our analysis.

Proposition 2 *SUE (measured by input ratio) do commute with scaling EXE if and only if the problem is homothetic. The Lie bracket between the scaling and substitution vector fields provides a meaningful local measure of the departure from commutativity which characterizes non homothetic models, in that it defines a flow on (and therefore a direction at each point of) input space.*

Then, building on the insights generated by formula (21), we can state the following

Conjecture *In homothetic problems, the Lie bracket between the expansion vector field* (12) *and the substitution vector field* (16) *is a vector field parallel to* (12).

Such a conjecture points at the general problem of setting free, to some extent, from the straitjacket of functional forms, an issue which our geometric approach is meant to trigger.

Finally, we are in a position to apply our theoretical approach to applied production analysis, so as to attach 'flesh and blood' to our framework in terms of measures of productive inefficiency.

5 Standard and reversed decompositions of overall productive efficiency

As already anticipated, our Propositions account for the benchmark relevance of homothetic models established by Bogetoft et al. (2006, "BFO" henceforth) with respect to the definition of productive efficiency. Recall, Farrell (1957) set forth a decomposition of productive overall (in)efficiency (OE) as the product of technical (in)efficiency (TE) and allocative (in)efficiency (AE), thereby setting the stage for a basic strand of applied production analysis. Farrell (1957) did posit that TE is to be computed first, by projecting the actual inefficient input bundle onto its efficient counterpart along the ray. The ratio between technically efficient inputs and initial inputs is called TE; evidently, TE is maximized (TE = 1) if the initial bundle is already technically efficient. Then, along the efficient isoquant defining TE, one determines the optimal input bundle as function of input prices; the ratio between the cost of the optimal bundle and the cost of the TE bundle is called AE, which is maximized (AE = 1) if the TE bundle is the one of least cost. Then, OE = TE·AE; despite the commutative property of products of real numbers, the order matters, on both theoretical and empirical grounds, with which such *standard* effects are computed: "One can say that allocative efficiency is treated as the residual when evaluating the overall performance" (BFO, p. 450).

In fact, BFO set forth a *reversed* decomposition of OE meant to enlighten the relevance of the order with which TE and AE are computed. Evidently, such a reversed approach does fit our commutation framework, provided one interprets, on the one hand, TE as a scaling EXE, and, on the other hand, AE as a SUE measured by bundle cost. The naturality of such a correspondence should not be difficult do grasp on intuitive grounds by comparing our Fig. 2 with Fig. 1 in BFO. True, BFO consider multioutput problems, yet TE and AE are measured on input space,[11] upon which homotheticity plays a pivotal role.

Propositions 1 and 2 in BFO establish the benchmark role of (input- and ray-) homothetic models, in which standard and reversed OE decompositions do coincide. The proofs of such Propositions define an insightful algebraic perspective which complements our geometric arguments for Propositions 1 and 2. The vanishing of the Lie bracket represented in (19) and (20) assesses the benchmark role of homothetic models, with respect to which the Lie bracket may define a natural local differential measure of the second order allocative (in)efficiency AAE defined in BFO.

Beyond the theoretical relevance of the issue, which our previous analysis was meant to enlighten, the managerial and organizational relevance of the hierarchies between choices comes to the fore: the analysis of hierarchies between strategies of improving TE and AE represents a beautiful setting (at least, in the author's view) for putting the theory at work. Following BFO, one may envisage TE as improving how to do things right, whereas AE can be considered as ameliorating the way to do the right things. Being organizations in charge of doing the right things right, and being organizations *complex* structures which cannot be reconfigured straightforwardly, the order matters with which such reconfigurations are studied and implemented: "it may

[11] It is not difficult to convince oneself that, in single output settings, the hypotheses (A1), (A2) and (A3) in BFO do collapse onto the hypothesis that we can identify the (efficient) isoquant corresponding to the initial (inefficient) input combination, and then define SUE along such a curve.

be easier to reallocate resources within a hierarchy or via markets, than to actually change the production procedures (including the culture, power configuration, incentive structure etc.) used in the individual production units" (BFO, p. 451). Needless to say, our framework is not meant to provide algorithms for solving such complex problems, but rather to enlarge the *language* by means of which such problem can be represented. After all, managing complexity entails, in first instance, providing a sound representation of the underlying degrees of freedom. In such respects, our geometric approach may contribute handles for dealing with recent advances in microeconomics.

For instance, Chambers and Färe (1998) introduce the notion of *translation homotheticity* ("TH") as a generalization of homotheticity, whose rationale can be grasped as follows. Let L denote input requirements sets for our technology, and consider translations of the reference input requirements set $L(1)$; our technology is TH if, "as one moves out from any point on L(1) in the direction of g, that movement will cut isoquants or indifference curves at points having the same marginal rate of substitution as the point L(1)." (ivi, p. 632). Noticeably, such Authors represent the explicit mapping between a TH technology and a homothetic technology for single output. Since translating vector fields is a well defined notion w.r.t. a given coordinate system (inputs), and such translated fields display "blow up" symmetry w.r.t a new "origin", TH may be enlightened by our geometric approach to flows on input space. Correspondingly, more advanced notions (such as *input and output* TH, Chambers 2005, and references therein) may benefit from the geometric toolkit defined by flows and Lie brackets.

6 Perspectives

It was the aim of the present contribution to argue about the relevance of enlarging the language of microeconomics towards the commutativity of EXE and SUE as an insightful perspective on partial equilibrium. We have gone through a differential geometric approach to the parametrizations of EXE and SUE meant to embrace well known instances (like expansion paths and Slutsky equations) and sparkle promising advances (like the commutativity of EXE and SUE). It is quite natural to conjecture the relevance of applying such a framework to the analysis of welfare, general equilibrium and market failures (in first instance, nonconvexities); in fact, before that, a number of promising avenues of inquiry can be envisioned.

In first instance, the comparison between standard and reversed Farrell decompositions may represent an insightful line of progress in the relation between theoretical and applied microeconomics, in which the benchmark role of homothetic models is pivotal. In fact, being technical efficiency measures expressed as ratios (scaled by input distance), the scaling properties represented by our scaling vector field may play a major role in tailoring technical as well as conceptual advances in applied production analysis. Evidently, the hypothesis of single output can be relaxed: in multioutput settings our geometric approach may shed new light on the relevance of non-radial changes (Chambers and Mitchell 2001) and translation homotheticity (Chambers and Färe 1998; Chambers 2005).

An intriguing line of research may focus the argument set forth by Baumol (1973), according to which the ambiguity of the sign of the income effect is crucial to the relevance of the Linder theorem. In such respects, our representation of EXE and SUE may provide 'handles' for addressing such an ambiguity, much like the theory of optimal control provides handles (in first instance the maximum principle) to address intertemporal controlled optimization, which cannot be reduced to a sequence of instantaneous optimizations. Possibly, the introduction of flows on bundle space may pave the way for intertemporal optimizations or evolution processes.[12] Then, the nuisances raised by income effects may turn to pregnant microeconomic insights.

Stiglitz (2000) points out that major advances in twentieth century economics pertain to the economics of information, which, for instance, enables one to tailor the relevance of information imperfection and information asymmetries in the departures from the perfection benchmark represented by general equilibrium. With an 'opposite' attitude, the present contribution aims at deepening the foundations of microeconomics in a deterministic setting with perfect information. True, in the author's vision, the relevance of the present analysis is grounded in the sharp logic defined by the commutativity of EXE and SUE, which deepens the benchmark role of homothetic symmetry as adapted to the 0-h property of partial equilibrium in production and consumption, the 'skeleton' of microeconomics. We have been arguing about the relevance of global parametrizations of basic effects, irrespective of the functional form of objective functions, thereby setting a general framework for posing new problems (like our Conjecture, Sect. 4) and improving the representation of duality, possibly enlarging the methods of empirical analysis.

Definitely, a natural step forward seems to be represented by a geometric analysis of luxury in terms of expansion flows. Luxury goods, recall, represent a general trait of consumption, namely, the case for income effects to shift monotonically the relative composition of optimal bundles. As such, the analysis of luxury may enlighten the generality of the present approach.

Acknowledgments The paper has been conceived as a contribution to the Workshop "The Economics of Irreversible Choices" (Brescia, 2012, September 13th–14th); the author wishes to express sincere appreciation to the organizing committee for the excellent conference setup. Deep gratitude to the conference participants for insightful comments and discussions, in particular Giacomo Corneo, Sergio Vergalli, Luca Di Corato, Peter Kort, Nir Becker, Daniel Heyen, Nicole Glanemann, Andrianos Tsekrekos, Davide Sala and Justus Wesseler. Furthermore, the author gratefully acknowledges an anonymous referee, together with the colleagues at the Department of Economics in Parma, in particular Marco Magnani, for a number of stimulating discussions which contributed shaping the ideas represented in the paper.

Appendix 1. Cobb–Douglas homotheticity

Consider the class of constant returns to scale (CRS) Cobb–Douglas objective functions $f(x, y) = x^a y^{1-a}$, with $0 < a < 1$, for which MRS are constant along rays through the origin: it is not difficult to fix the explicit dependence of MRS on the point

[12] For instance, replicator dynamics is defined by means of vector fields on the simplexes representing the space of mixed strategies (see for instance Weibull 1995; Gintis 2009). For an advanced perspective on the theory of optimal control see Grass et al. (2008).

in bundle space, since along a level curve with level υ one is faced with the functional dependence $y_\upsilon(x) = \upsilon^{\frac{1}{1-a}} x^{\frac{-a}{1-a}}$, out of which one can establish the dependence of MRS on the input ratio $\kappa = y/x$ by first differentiating y_υ with respect to x, and then plugging in the form of f, so as to obtain (Varian 1992, p. 12)

$$-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{\partial y_\upsilon}{\partial x}(\kappa) \equiv \mathrm{MRS}(\kappa) = \frac{-a}{1-a}\kappa, \quad \forall \upsilon \in \mathrm{range}(f). \tag{22}$$

Such a property of homothetic models is thus particularly manageable in Cobb–Douglas models, which we employ for a review of optimization problems preliminary to further developments.

Consider the CRS Cobb–Douglas objective function $f(x, y) = x^{\frac{1}{3}} y^{\frac{2}{3}}$ and the constraint $I = p_x x + p_y y$; as is well known, by virtue of the convexity of f, such an optimization is simply performed in terms of the FOC

$$\frac{\partial f}{\partial x} = \frac{1}{3}\left(\frac{y}{x}\right)^{\frac{2}{3}} = \lambda p_x, \qquad \frac{\partial f}{\partial y} = \frac{2}{3}\left(\frac{x}{y}\right)^{\frac{1}{3}} = \lambda p_y, \tag{23}$$

according to which the gradient of the objective function must be proportional to the price vector (the familiar "price ratio = MRS" rule), the constant of proportionality being the multiplier λ. Being MRS constant along rays, one immediately obtains that the components of optimal bundles, as functions of prices and income, read

$$x^* = \frac{I}{3p_x} = D^x(I, p_x), \qquad y^* = \frac{2I}{3p_y} = D^y(I, p_y), \tag{24}$$

and that the optimal multiplier $\lambda^*$is such that $f = I\lambda^*$. For fixed prices, (24) define the familiar straight Engel curves of Cobb–Douglas models, representing normal goods whose consumption is proportional to income, and weighted by the exponents (1/3 and 2/3 in the case at hand) in the objective function f. As expected, ordinary demand functions are proportional to the ratio of income to own prices, and uncompensated cross-price responses vanish (Cornes 1992, p. 50), a manifestation of the general instance that homothetic problems entail no primal (nor dual) level effects. For fixed prices, the functions (24) define curves in bundle space parametrized by income, the well known straight expansion paths of homothetic models (the ratio of ordinary demand functions is constant), along which MRS is constant. Income effects are easy to compute in this case, and can be disentangled from substitution effects (Proposition 1 in the main text).

For varying price vector, the curves (24) define a flow in bundle space, in that they do not cross each other, and each curve inherits a natural parametrization by income. It is tempting then to define an *expansion* vector field (uniquely determined by the objective function) generating such an expansion flow as

$$\mathbf{X}_f = \frac{1}{3p_x}\frac{\partial}{\partial x} + \frac{2}{3p_y}\frac{\partial}{\partial y} \tag{25}$$

with 'constant' components along each ray, being the ray identified by the price ratio (notice that a flow may consist of straight lines even if the components of the vector field are not constant, they only need be proportional). True, once we try and express the components of (25) in terms of x and y we stumble on 0-h in income and prices, and we need a *fixing* relation

$$\Gamma(I, p_x, p_y) = 0 \tag{26}$$

with respect to which to define a vector field on input space, at the cost of breaking 0-h. A noticeable example of such a fixing procedure is given by the condition of unit cost of the reference bundle in the definition of benefit functions (Luenberger 1996). A natural fixing relation is the choice of a numeraire (compare Williams 2008, p. 67): we can choose for instance $p_x = 1$, thereby measuring both I and p_y in units of p_x; write

$$J \equiv \frac{I}{p_x}, \qquad p \equiv \frac{p_y}{p_x}. \tag{27}$$

Then, express the demand functions (24) in terms of the independent parameters (27) and differentiate with respect to J to obtain the components of the expansion vector field.

$$\mathbf{X}_f(x \text{ numeraire}) = a\frac{\partial}{\partial x} + a\frac{y}{x}\frac{\partial}{\partial y} = a\frac{1}{x}\mathbf{Z}. \tag{28}$$

As expected, such vector field is radial, and its flow is naturally parametrized by 'normalized' income J in (27). (28) is a simple function of the scaling vector field, so that its Lie brackets with relevant vector fields can be computed in terms of $\mathbf{Z}$ (Sect. 4). The income to p_x effects measured by J, evidently, need be connected to ordinary income effects by a proper recipe. A useful fixing relation would be one such that $\mathbf{X}_f = y\frac{\partial}{\partial x} + \frac{y^2}{x}\frac{\partial}{\partial y} = \frac{y}{x}\mathbf{Z}$ (it is evidently relevant to assess the economic significance of such a fixing). Notice that fixing a numeraire has a geometrical drawback in the representation of input space: 0-h does not hold anymore, so that we cannot represent the effects of scaling prices for fixed income, a transformation of parameters which is no more available.

Now, turn to the significance of Hicksian demand functions. The form is well known of the cost function $C(p_x, p_y, \upsilon) = \upsilon p_x^{\frac{1}{3}} p_y^{\frac{2}{3}}$ for Cobb–Douglas models, so that the Hicksian demand functions

$$H^{x,y}(p_x, p_y, \upsilon) \equiv D^{x,y}(p_x, p_y, C(p_x, p_y, \upsilon))$$

result in

$$H^x(p_x, p_y, \upsilon) = \frac{1}{3}\upsilon\left(\frac{p_y}{p_x}\right)^{\frac{2}{3}}, \qquad H^y(p_x, p_y, \upsilon) = \frac{2}{3}\upsilon\left(\frac{p_x}{p_y}\right)^{\frac{1}{3}}. \tag{29}$$

Such functions represent optimal bundles along level curves: given convexity, along any level curve there is a unique point at which such level curve is tangent to the budget constraint. Such a point can thus be uniquely characterized by the level of the objective function and by the price vector. The functional forms (29) represent a factorized dependence on level and price ratio, as a consequence of (22). We are then in a position to write explicitly Slutsky equations for the problem at hand (Varian 1992, p. 122), for instance

$$D^x_y(p, I) = H^x_y(p, \upsilon) - D^x_I(p, I)H^y(p, \upsilon)$$
$$0 = \frac{1}{3} \cdot \frac{2}{3}\frac{1}{p_y}\upsilon\left(\frac{p_y}{p_x}\right)^{\frac{2}{3}} - \frac{1}{3p_x}\frac{2}{3}\upsilon\left(\frac{p_x}{p_y}\right)^{\frac{1}{3}}$$

The vanishing of the RHS can be considered as a compatibility condition with respect to Cobb–Douglas models, for which the vanishing of the LHS is a consequence of vanishing cross-price elasticities.

Appendix 2. Elementary flows and Lie brackets

Consider the vector fields $\mathbf{X}_1 = \frac{\partial}{\partial x}$, $\mathbf{X}_2 = y\frac{\partial}{\partial y}$, whose Lie bracket vanishes, as established in the main text (Sect. 3). By definition, the flow of the vector field $\mathbf{X}_1$ for a parameter interval t drags the point (x_0, y_0) to the point

$$(x(t), y(t)) = (x_0 + t, y_0);$$

correspondingly, the flow of the vector field $\mathbf{X}_2$ for a parameter interval s drags the point (x_0, y_0) to the point

$$(x(s), y(s)) = (x_0, y_0e^s).$$

Then, consider a path starting at (x_0, y_0), following the flow of $\mathbf{X}_1$ for an interval parameter t, and then following the flow of $\mathbf{X}_2$ for an interval parameter s: such path ends up at the point

$$(x(t, s), y(t, s)) = (x_0 + t, y_0e^s);$$

evidently, a path following the same flows in reversed order, for the same parameter intervals, ends up at the same point: we thereby confirm that the vanishing Lie derivative of the vector fields (a local condition) is associated with the commutativity of the flows (a non local condition).

Consider then the vector field $\mathbf{X}_3 = x\frac{\partial}{\partial y}$, whose Lie bracket with $\mathbf{X}_1 = \frac{\partial}{\partial x}$ is nonvanishing (main text); we expect the associated flows not to commute, as we confirm by the following explicit computation. The flow of $\mathbf{X}_3$ for an interval parameter s drags the point (x_0, y_0) to the point

$$(x(s), y(s)) = (x_0, y_0 + x_0s).$$

Thus, consider a path starting at (x_0, y_0), following the flow of $\mathbf{X}_1$ for an interval parameter t, and then the flow of $\mathbf{X}_3$ for an interval parameter s. Such a path ends at the point

$$(x(t,s), y(t,s)) = (x_0 + t, y_0 + (x_0 + t)s). \tag{30}$$

Conversely, consider a path starting at (x_0, y_0), following the flow of $\mathbf{X}_3$ for an interval parameter s, and then the flow of $\mathbf{X}_1$ for an interval parameter t. Such a path ends at the point

$$(x(s,t), y(s,t)) = (x_0 + t, y_0 + x_0 s). \tag{31}$$

In a nutshell, the x coordinate influences the speed in the y direction, so that it matters for which value of x we follow the flow of $\mathbf{X}_3$, and the two paths (30, 31) do not commute. Notice, in the main text we have established that the Lie bracket of the two vector fields results in $\frac{\partial}{\partial y}$; compare such a *differential* measure of noncommutativity (a tangent vector at each point of the manifold) with the *finite* measure represented by the segment

$$(x_0 + t, y_0 + x_0 s) - (x_0 + t, y_0 + (x_0 + t)s) \tag{32}$$

joining the final points of the two possible paths: the Lie bracket provides a differential measure of noncommutativity of infinitesimal paths, which, upon integration, provides a finite measure of the noncommutativity of finite paths. For the case at hand such an integration is trivial, since the Lie bracket is a coordinate vector field, and its integration can be represented by means of the explicit formula (32).

References

Abraham R, Marsden J (1987) Foundations of mechanics, 2nd edn. Addison-Wesley, UK

Arnold V (1992) Metodi matematici della meccanica classica. Editori Riuniti

Baumol W (1973) Income and substitution effects in the Linder theorem. Q J Econ 87:629–633

Bogetoft P, Färe R, Obel B (2006) Allocative efficiency of technically inefficient production units. Eur J Oper Res 168:450–462

Chambers R (1988) Applied production analysis. Cambridge University Press, Cambridge

Chambers R, Färe R (1998) Translation homotheticity. Econ Theory 11:629–641

Chambers R, Mitchell T (2001) Homotheticity and non-radial changes. J Prod Anal 15:31–39

Chambers R (2002) Exact nonradial input, output, and productivity measurement. Econ Theory 20:751–765

Chambers R (2005) On input and output translation homotheticity. J Econ 85:197–203

Cornes R (1992) Duality and modern economics. Cambridge University Press, Cambridge

Cowell F (2005) Microeconomics, principles and analysis. Oxford University Press, Oxford

Farrell M (1957) The measurement of productive efficiency. J R Stat Soc Ser A Gen 120(3):253–281

Gintis H (2009) Game theory evolving. Princeton University Press, Princeton

Grass D, Caulkins J, Feichtinger G, Tragler G, Behrens D (2008) Optimal control of nonlinear processes. Springer, Berlin

Luenberger D (1996) Welfare from a benefit viewpoint. Econ Theory 7:445–462

Mas-Colell A, Whinston M, Green J (1995) Microeconomic theory. Oxford University Press, Oxford

Silberberg E (2008) Hicksian and Marshallian demands. The New Palgrave Dictionary of Economics

Stiglitz J (2000) The contributions of the economics of information to twentieth century economics. Q J Econ 115:1441–1478

Varian H (1992) Microeconomic analysis, 3rd edn. Norton
Weber T (2010) Hicksian welfare measures and the normative endowment effect. Am Econ J Microecon 2:171–194
Weibull J (1995) Evolutionary game theory. The MIT Press, Cambridge
Williams S (2008) Communication in mechanism design. A differential approach. Cambridge University Press, Cambridge