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Why are firms sometimes unwilling to reduce costs?

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Abstract This paper identifies the environments in which it does not pay for a multiproduct firm to engage in small cost reductions. Specifically, it shows that a multiproduct Bertrand firm's profits will *decrease* in response to a small reduction in one product's marginal cost if and only if the output share of the cost-reducing unit is below a threshold. Because cost reductions by a single-product firm or by a multiproduct Cournot firm always increase the firm's profits, this result is unique to multiproduct Bertrand firms.

Keywords Effect of cost reduction · Multiproduct oligopoly · Price competition

JEL Classification C63 · D43 · L13

1 Introduction

This paper explains why firms sometimes are unwilling to reduce cost by identifying the strategic environments in which it does not pay to engage in small cost reductions or small technological innovations. Specifically, it shows that reducing a multiproduct Bertrand firm's marginal cost will *reduce its profits* if and only if the output share of the cost-reducing unit is below a threshold. Such negative profit effect of small cost reductions is a unique feature of multiproduct Bertrand firms, because in our linear

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model the profit effect of small cost reductions by single-product Bertrand firms and by multiproduct Cournot firms are both positive.

Previous studies have characterized the possible negative welfare effects of small cost reductions, which come in variety of forms, such as adoption of new technology in production or management or an upgrade of equipment or reduction in labor force. Most of such studies involved a single-product oligopoly, with the exception of Lapan and Hennessy (2007) involving a Cournot oligopoly with arbitrary number of multi-products.[1] This study is the first to report and characterize the possible negative profit effects of small cost reductions.[2] It derives two closed-form critical levels for identifying the profit effects of cost reduction: 1) critical output share: a small reduction in one product's marginal cost by the multiproduct Bertrand firm *reduces* its profits if and only if the output share of the cost-reducing unit is below the critical level; 2) critical size of cost reduction: a small reduction in one product's marginal cost by the multiproduct Bertrand firm *reduces* its profits if and only if the size of the cost reduction is below the critical level.

It is important to note three points in understanding the negative profit effects of small cost reductions. First, the counter-intuitive effect is the confluence of two main economic factors: strategic complementarity, and strategic interaction.[3] If the choices are strategic substitutes (such as in linear Cournot oligopolies), or if there are no strategic interactions (such as in a monopoly), the profit effects of cost reductions will always be positive. Given these two factors, a firm's small cost reduction has two opposing effects on its profits: a direct positive effect (lower cost affects a firm's profit positively, as in a monopoly), and a strategic negative effect (the rivals tend to respond with lower prices, which affects the firm's profit negatively). In practice it is often true that the direct effect dominates. Although it is known that the net effect on profits could possibly be negative, these two factors alone are insufficient for a negative profit effect. For example, these two factors exist in linear multiproduct Bertrand oligopolies in which each firm has identical marginal cost for all its products, yet the involved profit effect of small cost reductions is always positive. Besides, the two factors exist in linear single-product Bertrand oligopolies where a firm's small cost reduction always increases its profits.

Second, the negative effect requires the working of two other economic factors: price competition and cost asymmetry within the multiproduct firm. Cost asymmetry within a multiproduct firm allows output reallocation due to the chain effects caused by a small cost reduction. A cost reduction in one unit increases the production in this unit, but decreases productions in all other units inside the firm. Such reallocation raises

[1] For recent works on the welfare effects of cost reduction, see Février and Linnemer (2004), Lapan and Hennessy (2007), Wang and Zhao (2007), and Nakayama (2009).

See Cabral and Villas-Boas (2005) and Lapan and Hennessy (2007) for survey of the small literature on multi-product oligopoly, most of which were completed without using the general equilibrium expressions. Lapan and Hennessy (2006) studied the welfare effects in a special Cournot model in which the number of multi-products is two or three.

[2] A related result is Bulow et al. (1985, Section II), who provided an example of multiproduct Cournot duopoly with nonlinear joint cost (or economies of scope) in which a large cost reduction by the multiproduct firm reduces its profit, but no general characterization of such negative profit effect is known.

[3] A definition of strategic complementarity is given in Sect. 4.

the profit of the cost-reducing unit but lowers the profits of all other units inside the multiproduct firm. When the cost-reducing unit is small, its profit gain is outweighed by profit losses from all other units, leading to a reduction in the multiproduct firm's total profits. Under quantity competition, however, the firm's profit rises following a cost reduction.[4]

Third, for readers who still find the result counter-intuitive, please keep in mind that the comparative statics analysis assumes the separation of price (or quantity) decision and the cost cutting decision. The validity of this assumption is supported by the empirical observation that engineers working in R&D departments often know nothing and have no say about management's choice of price and quantity, and by the theoretical observation that a firm's profit maximization problem can not be defined if the firm is able to simultaneously choose price (or quantity) and choose technology. The paper is not claiming an explanation of why firm's cost is high nor suggesting that firms will increase profits by increasing cost—the authors would argue that firms in real world do not want to take such cost increasing measures because the size of such profit increase is small and they cause damage to the firm's reputation. The paper is just making a modest claim that sometimes it does not pay for a multiproduct firm to engage in small cost reductions. In such situations, it only pays if the magnitude of cost reduction is large enough or above the critical size established in this paper.

These new results are established by calculating and analyzing the general expressions for Bertrand equilibrium in linear multiproduct oligopolies, which are previously unknown. Because deriving the expressions is non-trivial, the authors hope that scholars working in related areas would find the expressions useful in their future research.

The remainder of the paper is organized as follows. Section 2 describes the model and derives the closed-form expressions for multiproduct Bertrand equilibrium in asymmetric linear oligopolies. Section 3 studies the profit effects of a Bertrand firm's cost reduction, Sect. 4 provides analogous results for multiproduct Cournot oligopolies, Sect. 5 provides two numerical examples, Sect. 6 concludes the paper, and the appendix provides all proofs.

2 Description of the model

A linear Bertrand oligopoly with n goods, or the Bertrand-Shubik model, is defined by n demand and n cost functions (see Bertrand 1883; Shubik 1980):

$$q_i(p) = V - p_i - \gamma(p_i - \overline{p}), \quad C_i(q_i) = c_i q_i, \quad i \in N = \{1, \dots, n\}, \tag{1}$$

where $V > 0$ is the common intercept of demand functions, p_i is the price of good i, $p = (p_1, \dots, p_n)^\top$ is the price vector, $\gamma \geq 0$ is the substitutability parameter, $\overline{p} = (\sum_{j=1}^{n} p_j)/n$ is the average price, and c_i is the constant marginal (or average) cost of producing good i. These goods are independent if $\gamma = 0$, and they become closer substitutes as γ increases toward infinity. As done in most comparative

[4] In Sect. 4 we show that small cost reductions are essentially impossible to reduce profits in homogeneous Cournot oligopolies with strong strategic complementarity (nonlinear demand) and linear costs.

statics analysis of oligopoly, we do not consider the case of $\gamma < 0$ or goods that are complements.

A multiproduct oligopoly with k firms ($1 \leq k \leq n$) is given by an arbitrary partition $\Delta = \{S_1, S_2, \ldots, S_k\}$ of N (i.e., $S_i \neq \emptyset$, $S_i \cap S_j = \emptyset$, all $i \neq j$, and $\cup S_j = N$), with each firm j or S_j ($j = 1, \ldots, k$) producing $|S_j| = n_j$ products (i.e., $\sum_{j=1}^{k} n_j = n$). The multiproduct monopoly is the coarsest partition $\Delta_m = \{N\}$, and the single-product oligopoly is the finest partition $\Delta_0 = \{1, 2, \ldots, n\}$. For each firm $S \in \Delta$, let $p_S = \{p_i |\, i \in S\}$ and $q_S = \{q_i |\, i \in S\}$ denote its price and output vectors, and $p_{-S} = \{p_i |\, i \in N \backslash S\}$ denote the vector of other firms' prices. Then, for each $p = (p_S, p_{-S}) = (p_1, \ldots, p_n)^\top$, the profit of a firm $S \in \Delta$ is given by

$$\pi_S(p) = \pi_S(p_S, p_{-S}) = \sum_{i \in S} \pi_i(p) = \sum_{i \in S} q_i(p)(p_i - c_i), \tag{2}$$

and the Bertrand equilibrium (or Nash equilibrium or strategic equilibrium) is a price vector $p^* = \{p_S^* |\, S \in \Delta\} = (p_1^*, \ldots, p_n^*)^\top$ such that for each firm $S \in \Delta$, p_S^* is its best response to p_{-S}^*, or that each p_S^* solves $Max\{\pi_S(p_S, p_{-S}^*) | p_S \geq 0\}$. Throughout the paper we assume that a unique and interior equilibrium always exists.[5]

For our purpose of analyzing the profit effects of a multiproduct firm's cost reductions, it suffices to focus on oligopolies with a single multiproduct firm given by $\Delta = \{S, m+1, \ldots, n\} = \{\{1, \ldots, m\}, m+1, \ldots, n\}$. The expressions for equilibria with arbitrary partitions are given in the appendix. Under the uniqueness assumption, the Bertrand equilibrium for $\Delta = \{S, m+1, \ldots, n\}$ solves the following first-order conditions:

$$\frac{\partial \sum_{k \in S} \pi_k}{\partial p_i} = 0, \quad \text{all } i \in S; \quad \text{and} \quad \frac{\partial \pi_j(p)}{\partial p_j} = 0, \text{ all } j \notin S. \tag{3}$$

Direct calculation (e.g., using the inverse of A in (26)) shows that such Bertrand equilibrium is equal to

$$p_i^* = \frac{n(2n(1+\gamma)-\gamma)V}{\omega_1} + \frac{\gamma^2 m\,(n-m)\,\bar{c}_S}{2\omega_1} + \frac{\gamma(n(1+\gamma)-\gamma)(n-m)\bar{c}_{-S}}{\omega_1} + \frac{c_i}{2}, \tag{4}$$

$$p_j^* = \frac{n(2n(1+\gamma)-m\gamma)V}{\omega_1} + \frac{\gamma m\,(n(1+\gamma)-m\gamma)\,\bar{c}_S}{\omega_1} + \frac{\gamma\,(n(1+\gamma)-\gamma)\,(2n(1+\gamma)-m\gamma)\,(n-m)\,\bar{c}_{-S}}{(2n(1+\gamma)-\gamma)\omega_1} + \frac{(n(1+\gamma)-\gamma)c_j}{2n(1+\gamma)-\gamma}, \tag{5}$$

for each unit $i \in S$ and each single-product firm $j \notin S$, where $\bar{c}_S = \sum_{i \in S} c_i / m$ and $\bar{c}_{-S} = \sum_{i \notin S} c_i/(n-m)$ are the multiproduct firm's and the outsiders' average

[5] This assumption is equivalent to the assumption that each firm has a positive market share at all equilibria. In linear cases, the conditions are similar to those for a single-product Cournot oligopoly in Zhao (2001), which have been adopted in Pham and Folmer (2003).

marginal cost, respectively, and $\omega_1 = \omega_1(n, m, \gamma) > 0$ is given by

$$\omega_1(n, m, \gamma) = \gamma^2 (n - m)(m + 2n - 2) + 2n\gamma (3n - m - 1) + 4n^2. \tag{6}$$

It follows from (4–5) (or by rearranging (3) without solving it) that the equilibrium markups satisfy the following properties:

$$\frac{p_i^* - c_i}{q_i^*} = \frac{1}{1+\gamma} + \frac{m\gamma(\overline{p}_S^* - \overline{c}_S)}{n(1+\gamma)q_i^*}, \quad \text{all } i \in S; \quad \text{and} \tag{7}$$

$$\frac{\overline{p}_S^* - \overline{c}_S}{\overline{q}_S^*} = \frac{n}{n(1+\gamma) - m\gamma} > \frac{p_j^* - c_j}{q_j^*} = \frac{n}{n(1+\gamma) - \gamma}, \quad \text{all } j \notin S, \tag{8}$$

where $\overline{p}_S^* = \sum_{i \in S} p_i^*/m$ and $\overline{q}_S^* = \sum_{i \in S} q_i^*/m$ are the multiproduct firm's average price and supply, respectively. Hence, different units in the multiproduct firm may have different markup/supply ratios, but single-product firms have an identical markup/supply ratio, which is smaller than the average-markup/average-supply ratio of the multiproduct firm.

It is useful to note that closed-form expressions for multiproduct Bertrand and Cournot equilibria with arbitrary partitions (i.e., p^* in (28) and q^* in (39) in the appendix) have not been reported in the existing literature.[6] We hope that other scholars working in related areas will find them useful in their future studies.

In the next section, we conduct the comparative static analysis of the above Bertrand equilibrium.

3 Cost reductions in price competition

Plugging the Bertrand equilibrium in (4–5) into the demand system (1) and simplifying, one obtains the equilibrium outputs as below: for each $i \in S$ and $j \notin S$,

$$\begin{aligned} q_i^* &= \frac{(2n(1+\gamma)-\gamma)(n(1+\gamma)-m\gamma)V}{\omega_1} + \frac{\gamma(n(1+\gamma)-\gamma)(n(1+\gamma)-m\gamma)(n-m)\overline{c}_{-S}}{n\omega_1} \\ &\quad + \frac{\gamma[\gamma^2(3n-2)(n-m)+\gamma n(7n-3m-2)+4n^2]m\overline{c}_S}{2n\omega_1} - \frac{(1+\gamma)c_i}{2}, \\ q_j^* &= \frac{(2n(1+\gamma)-m\gamma)(n(1+\gamma)-\gamma)V}{\omega_1} + \frac{\gamma m(n(1+\gamma)-m\gamma)(n(1+\gamma)-\gamma)\overline{c}_S}{n\omega_1} \\ &\quad + \frac{\gamma(n(1+\gamma)-\gamma)^2(2n(1+\gamma)-m\gamma)(n-m)\overline{c}_{-S}}{n(2n(1+\gamma)-\gamma)\omega_1} - \frac{(1+\gamma)(n(1+\gamma)-\gamma)c_j}{2n(1+\gamma)-\gamma}, \end{aligned} \tag{9}$$

[6] The only exception is the Bertrand equilibrium (4–5) for $\Delta = \{S, m+1, \ldots, n\}$, which is identical to the postmerger equilibrium in Zhao and Howe (2004). When $c_j = 0$ for all j, (4–5) are identical to the postmerger equilibrium with zero costs in Deneckere and Davidson (1985).

which lead to the following equilibrium prof its:

$$\pi_S^* = \sum_{i\in S} \left(p_i^* - c_i\right) q_i^* = \frac{m\left(n(1+\gamma) - m\gamma\right)(\overline{p}_S^* - \overline{c}_S)^2}{n} + \frac{(1+\gamma)\sum_{i=1}^m (\overline{c}_S - c_i)^2}{4},$$
$$\pi_j^* = \frac{[n(1+\gamma)-\gamma](p_j^* - c_j)^2}{n}, \quad \text{for each } j \notin S, \tag{10}$$

where the equilibrium markups are: for each $j \notin S$ and $i \in S$,

$$\begin{aligned} p_j^* - c_j &= \frac{nq_j^*}{n(1+\gamma)-\gamma}, \\ p_i^* - c_i &= \frac{n(2n(1+\gamma)-\gamma)V}{\omega_1} + \frac{\gamma\left(n(1+\gamma)-\gamma\right)(n-m)\overline{c}_{-S}}{\omega_1} \\ &\quad + \frac{\gamma^2 m\,(n-m)\,\overline{c}_S}{2\omega_1} - \frac{c_i}{2}, \\ \overline{p}_S^* - \overline{c}_S &= \frac{n(2n(1+\gamma)-\gamma)V}{\omega_1} + \frac{\gamma(n(1+\gamma)-\gamma)(n-m)\overline{c}_{-S}}{\omega_1} - \frac{\omega_2\overline{c}_S}{\omega_1}, \end{aligned} \tag{11}$$

where ω_1 is given in (6), q_j^* is given in (9), and $\omega_2 > 0$ is given by

$$\omega_2(n, m, \gamma) = \gamma^2\,(n-1)\,(n-m) + \gamma n\,(3n-m-1) + 2n^2. \tag{12}$$

The proposition below reports the effects of small cost reductions by a unit of the multi-product firm in Bertrand oligopolies, whose closed-form expressions are given in (29–33) in the appendix.[7]

Proposition 1 *Consider the Bertrand oligopoly* (1) *with a single multiproduct firm given by* $\Delta = \{S, m+1, \ldots, n\}$. *A small cost reduction in each unit* $i (\in S)$ *of the multiproduct firm increases output* i, *decreases all other outputs and all single-product firms' profits; it decreases the multiproduct firm's profits if and only if* i*'s output share within the firm is below a critical level, or precisely,* $\partial\pi_S^*/\partial c_i > 0 \Longleftrightarrow t_i^S < \widehat{t}^S = \gamma^2\,(n-m)\,/\omega_1$, *where* $t_i^S = q_i^* / \sum_{j=1}^m q_j^*$ *and* ω_1 *is given in* (6).

An examination of the supply in (9) and the markups in (11) shows that a reduction in c_i increases unit i's production and markup and decreases the production and markups in all other units (see (30) in the appendix for a proof), so a reduction in c_i increases unit i's profits, but at the same time decreases the profits of all other units. The balance of these two opposite effects explains *why a multiproduct firm might be unwilling to reduce its cost*: small cost reductions in one unit will reduce the multiproduct firm's profits if and only if the output share of the cost-reducing unit is sufficiently small,[8]

[7] It is straightforward to verify that a small cost reduction by each single-product firm j $(\in N\backslash S)$ increases its output and profit, and decreases all other outputs and all other firms' profits.

[8] Obviously, if a small reduction in unit i's marginal cost c_i decreases the firm's profits, then a small reduction in the marginal cost of any other unit whose marginal cost is greater than c_i will also reduce the firm's profits.

or equivalently, if and only if the cost-reducing unit is sufficiently inefficient (see Corollary 2 below). The firm's overall profits could decrease following a small cost reduction by an inefficient unit because production will be transferred from efficient units to the inefficient cost-reducing unit, and this is the same reason why welfare in a single-product Cournot oligopoly could decrease in response to a small cost reduction by an inefficient firm.

It is easy to show that $t_i^S > \widehat{t}^S$ always holds if i is an efficient unit (i.e., $\overline{c}_S - c_i > 0$) or if S is a monopoly (i.e., $S = N$), which leads to the following corollary:[9]

Corollary 1 (i) *For each $i \in S$, $\partial\pi_S^*/\partial c_i < 0$ if $c_i < \overline{c}_S$.*
(ii) *Let $m = n$ (i.e., $S = N$) and $\pi_S^* = \pi_N^*$ be the monopoly profit. Then, for all $i \in N$, $\partial\pi_N^*/\partial c_i < 0$.*

The negative profit effects of a small cost reduction also can be characterized by critical levels of marginal costs for each cost-reducing unit, which is given below:[10]

Corollary 2 *Given $\Delta = \{S, m+1, \ldots, n\}$, let π_S^* be the multiproduct firm's profits. Then, for each $i \in S$, $\partial\pi_S^*/\partial c_i > 0 \Leftrightarrow c_i > \widehat{c}_i^S$, where $\widehat{c}_i^S$ is the critical level of unit i's marginal cost given in* (34) *in the appendix.*

The above counter-intuitive negative relationship between small cost reductions and profits is caused by the combined strength of *five* factors in oligopolies with linear costs. The first factor is *strategic interaction* between the cost-reducing firm and other firms. Similar to Zhao and Howe (2004), the above firms' reaction functions (in terms of average prices, $\overline{p}_S = \sum_{i\in S}\overline{p}_i/m$ and $\overline{p}_{-S} = \sum_{j\notin S}\overline{p}_j/(n-m)$) are:

$$\overline{p}_S = h(\overline{p}_{-S}) = \frac{nV + (n + (n-m)\gamma)\overline{c}_S + \gamma(n-m)\overline{p}_{-S}}{2n + (2n - 2m)\gamma}, \quad \text{and} \tag{13}$$

$$\overline{p}_{-S} = g(\overline{p}_S) = \frac{nV + (n + (n-1)\gamma)\overline{c}_{-S} + \gamma m\overline{p}_S}{2n + (m + n - 1)\gamma}. \tag{14}$$

By (13), the multiproduct firm's cost reduction (i.e., a decrease in $\overline{c}_S$) directly causes a reduction in its average price. Such a reduction in $\overline{p}_S$, by (14), leads to a decrease in $\overline{p}_{-S}$, which causes a second-round reduction in $\overline{p}_S$ through the reaction curve (13). If there are no such strategic interactions such as in a monopoly, the profit effects of cost reductions will always be positive.

The second factor is *strategic complementarity*. If the choices are strategic substitutes (such as in linear Cournot oligopolies), the profit effects of cost reductions will always be positive. The above two factors work together to cause a negative effect (the rivals tend to respond with lower prices, which affects the firm's profit negatively),

[9] This corollary can also be obtained by analyzing the effect on cost variance in light of (10). Cost variance has been used to study the welfare effects of cost reduction in Lapan and Hennessy (2007).

[10] The next corollary can be understood geometrically by observing that $\pi_S^* = \pi_S^*(c_i)$ in (10) is convex and quadratic in c_i, with $\widehat{c}_i^S$ as its minimum defined by $\partial\pi_S^*/\partial c_i = 0$. Because π_S^* is symmetric in c_i around $c_i = \widehat{c}_i^S$, small reductions in c_i reduce π_S^* if and only if c_i is on the right half of the profit curve where π_S^* is increasing in c_i (i.e., $c_i > \widehat{c}_i^S$).

which could possibly outweigh the positive effect of cost reductions. However, these two factors alone are insufficient for a negative overall profit effect. For example, these two factors exist in linear single-product Bertrand oligopolies and in multiproduct Bertrand oligopolies in which each firm has identical marginal cost for all its products, yet the overall profit effects in both models are always positive.

The third factor is *cost asymmetry within the multiproduct firm*. As already discussed, the profits of a multiproduct firm with asymmetric costs could decrease following small cost reductions in an inefficient unit because production will be transferred from efficient units to the inefficient cost-reducing unit.

The fourth factor is *smallness of the cost-reducing unit*. As shown in Proposition 1, a small cost reduction will increase the multiproduct firm's profit if the cost-reducing unit is not sufficiently small (i.e., $t_i^S > \widehat{t}^S$).

The final and fifth factor is *the magnitude of cost reductions*. The negative profit effects are caused by small cost reductions in an inefficient unit. As shown in Proposition 2 below, if the magnitude of cost reduction is sufficiently large, the profit effects will be positive.[11]

Proposition 2 *Given* $\Delta = \{S, m+1, \ldots, n\}$ *in the Bertrand oligopoly* (1)*, let unit* 1 *be the most efficient unit of* S (*i.e.,* $c_1 = \min\{c_i | \ 1 \le i \le m\}$). *Consider each unit* $i \in S$ *with* $c_i > \widehat{c}_i^S$*, where* $\widehat{c}_i^S$ *is given in* (34).

- (i) *A large reduction in* c_i *increases the multiproduct firm's profits if and only if the reduction is larger than twice the difference between* c_i *and* $\widehat{c}_i^S$.
- (ii) *The multiproduct firm's profits will increase if* c_i *is reduced to the most efficient level* c_1.

Parts (i) and (ii) together imply that the magnitude of reducing c_i to c_1 is larger than twice the difference between c_i and $\widehat{c}_i^S$ (i.e., $c_i - c_1 > 2(c_i - \widehat{c}_i^S)$). In particular, it implies that a technology spillover within S that reduces all units' marginal costs to c_1 will increase the multiproduct firm's profits. As shown in the proof, by the time unit i becomes an efficient unit (i.e., its output share reaches $1/m$, or equivalently, its marginal cost is reduced to the firm's average marginal cost), the multiproduct firm's profits would have risen above the initial level. When unit i eventually becomes the most efficient unit (i.e., its marginal cost reaches c_1), the multiproduct firm's profits will rise further.

At this point we remark that, although we have focused on Bertrand oligopolies with a single multiproduct firm, the results reported in Propositions 1 and 2 are applicable when there are multiple multiproduct firms. This is due to the fact that the intuition (the five factors) presented above apply equally well when there are multiple multiproduct firms. In particular, the conditions (7) and (8) that equilibrium markups satisfy continue to hold with a corresponding equation like (7) for each multiproduct firm and with m replaced by the corresponding number of products in this firm. Moreover, the firms' reaction functions in average prices given by (13) and (14) continue to hold.

[11] Note that this critical magnitude depends on the assumption that all other firms are single-product firms. If some of the other firms are multi-product firms, a different critical magnitude will arise, which can be obtained using the equilibrium (28) for arbitrary partitions.

There is now a corresponding equation like (13) for each multiproduct firm. These reaction functions capture the strategic interactions between the cost-reducing multiproduct firm and other multi- and single-product firms. Any cost reduction leads to a direct effect within the firm captured by its own price reaction function and the indirect effects through other multi- and single-product firms' price reaction functions.

Naturally, the algebra involved to study an oligopoly with multiple multiproduct firms is much more involved than it already is with a single multiproduct firm as presented in this paper. In Sect. 5, we shall present a numerical example with two two-product firms to illustrate the above remark.

4 Cost reductions in quantity competition

The inverse demands of the Bertrand-Shubik demand system given in (1) are:

$$p_i(q) = p_i(q_1, \ldots, q_n) = V - q_i + \frac{\gamma}{1+\gamma}(q_i - \overline{q}), \tag{15}$$

where $\overline{q} = (\sum_{j=1}^{n} q_j)/n$ is the industry's average output. For each multiproduct firm $S \in \Delta$, its profit function is $\pi_S(q) = \pi_S(q_S, q_{-S}) = \sum_{i\in S} \pi_i(q) = \sum_{i\in S}(p_i(q) - c_i)q_i$, where $\pi_i(q) = (p_i(q) - c_i)q_i$. Then, the Cournot equilibrium for an arbitrary partition $\Delta = \{S_1, S_2, \ldots, S_k\}$ is an output vector $q^{C*} = \{q_S^{C*} |\ S \in \Delta\} = (q_1^{C*}, \ldots, q_n^{C*})^\top$ such that for each $S \in \Delta$, q_S^{C*} is its best response to q_{-S}^{C*}, or that each q_S^{C*} solves $Max\{\pi_S(q_S, q_{-S}^{C*}) | q_S \geq 0\}$. Similar to the earlier analysis of Bertrand model, we focus on oligopolies with a single multiproduct firm and provide Cournot equilibria for arbitrary partitions in the appendix. Under the uniqueness assumption, the Cournot equilibrium for $\Delta = \{S, m+1, \ldots, n\}$ solves the first-order conditions given by $\partial\pi_S(q_S, q_{-S})/\partial q_i = 0$, all $i \in S$, and $\partial\pi_j(q_j, q_{-j})/\partial q_j = 0$, all $j \notin S$, which lead to

$$\begin{aligned} q_i^{C*} &= \frac{n(1+\gamma)(2n+\gamma)V}{\omega_3} + \frac{(1+\gamma)(4n+(n-m+2)\,\gamma)m\gamma\overline{c}_S}{2\omega_3} \\ &\quad + \frac{n(1+\gamma)(n-m)\gamma\overline{c}_{-S}}{\omega_3} - \frac{(1+\gamma)c_i}{2}; \quad \text{and} \end{aligned} \tag{16}$$

$$\begin{aligned} q_j^{C*} &= \frac{n(1+\gamma)(2n+m\gamma)V}{\omega_3} + \frac{n(1+\gamma)m\gamma\overline{c}_S}{\omega_3} \\ &\quad + \frac{n\,(n-m)\,(1+\gamma)(2n+m\gamma)\gamma\overline{c}_{-S}}{(2n+\gamma)\omega_3} - \frac{n(1+\gamma)c_j}{2n+\gamma}, \end{aligned} \tag{17}$$

for each unit $i \in S$, and each single-product firm $j \notin S$, where $\omega_3 > 0$ is given by

$$\omega_3 = \omega_3(n, m, \gamma) = m(n-m+2)\gamma^2 + 2n\,(n+m+1)\,\gamma + 4n^2. \tag{18}$$

Substituting (16–17) into (15) gives the following equilibrium prices:

$$p_i^{C*} = \frac{(2n+\gamma)(n+m\gamma)V}{\omega_3} - \frac{m(n-m)\gamma^2\overline{c}_S}{2\omega_3} + \frac{(n-m)(n+m\gamma)\gamma\overline{c}_{-S}}{\omega_3} + \frac{c_i}{2}; \tag{19}$$

$$p_j^{C*} = \frac{(n+\gamma)q_j^{C*}}{n(1+\gamma)} + c_j, \tag{20}$$

for each $i \in S$ and $j \notin S$, which yield the following equilibrium profits:

$$\pi_S^{C*} = \frac{mn(1+\gamma)(\overline{p}_S^{C*} - \overline{c}_S)^2}{n+m\gamma} + \frac{(1+\gamma)\sum_{i=1}^m(\overline{c}_S - c_i)^2}{4}; \quad \text{and}$$

$$\pi_j^{C*} = \frac{n(1+\gamma)(\overline{p}_j^{C*} - c_j)^2}{(n+\gamma)}, \quad \text{all } j \notin S, \tag{21}$$

where the multiproduct firm's average price $\overline{p}_S^{C*} = (\sum_{i\in S} p_i^{C*})/m$ is given in (40), and the equilibrium mark-ups are given in (41).

As shown in the next proposition, a Cournot firm's profits always increase after its cost reductions, the closed-form expressions for such effects are given in (42–44) in the appendix.

Proposition 3 *Consider the Cournot oligopoly* (15) *with a single multiproduct firm.*

(i) *A small reduction in a single product firm's marginal cost increases its product and profit, and it reduces all other firms' products and profits.*
(ii) *A small reduction in the multiproduct firm's marginal cost c_i increases its profit and product i, and it reduces all other products and all single-product firms' profits.*

It is important to discuss two remarks about the difference between multiproduct and single-product firms' profit effects in quantity competition. First, the above result on profit effects for a multiproduct firm is not as obvious as the positive profit effects for single-product firms. In most single-product cases, a reduction in firm i's marginal cost increases both its output and markup so its profits always increase. In the multiproduct case, a reduction in c_i increases output in unit i, increases the markups of all units but decreases the output in all other units within the multiproduct firm (see (43) in the proof), so unit i's profit increases while profits in the multiproduct firm's other units might increase or decrease. This reasoning leads to an ambiguous profit effect for the multiproduct firm. Proposition 3 clarifies such ambiguity and shows that the overall effects on the multiproduct Cournot firm's profits are always positive.

Second, contrary to the argument that with strong strategic complementarity a single-product Cournot firm's small cost reductions will reduce its profits, we show below that the argument is false under the usual conditions. Consider a homogeneous Cournot oligopoly with nonlinear demands and linear costs. Let the profit functions be given by $\pi_i(x) = (p(X) - c_i)x_i$, all i, where $X = \Sigma x_j$. Then, firm i's and j's outputs

$(j \neq i)$ are strategic substitutes if

$$\alpha_i = \partial^2 \pi_i / \partial x_i \partial x_j = p'(X) + x_i p''(X) \leq 0,$$

and strategic complements if $\alpha_i > 0$. Let $E = Xp''(X)/p'(X)$ be the elasticity of the slope of inverse demand, $s_i = x_i/X$ be firm i's market share. Dixit (1986) showed that the stability of the system (i.e., conditions for comparative statistics) requires[12] $\Delta = 1 + \Sigma(\alpha_i/p'(X)) = n + 1 + E > 0$ or

$$E > -(n+1). \tag{22}$$

It is useful to note that strategic complementarity is equivalent to

$$E < -1/s_i, \tag{23}$$

which follows from $\alpha_i > 0 \Leftrightarrow -\alpha_i/p'(X) = -(1 + s_i E) > 0$.

Claim 1 Assume the stability condition (22) holds. Then $\partial \pi_i^* / \partial c_i > 0$ could possibly hold only if $\alpha_i > 0$ and $E \in (-n-1, -n-1/2]$.

Because one half of a unit interval over a half space (i.e., $(-n-1, -n-1/2]$ over $(-n-1, \infty)$) represents an infinitesimally small possibility, this claim implies that it is extremely unlikely that strong strategic complementarity can lead to a single-product Cournot firm's small cost reduction to reduce its profits (i.e., $\partial \pi_i^* / \partial c_i > 0$).

5 Examples

In this section, we provide two numerical examples. The first example illustrates the main results of the paper (Propositions 1 and 2). The second example shows that the result that a multiproduct Bertrand firm's profits can decrease in response to a small reduction in one product's marginal cost is not limited to oligopolies with a single multiproduct firm (which is the focus in our presentation), and can hold for an oligopoly with multiple multiproduct firms.

Example 1 Let $n = 3, V = 9, \gamma = 2, c_1 = 5.9, c_2 = 7.23, c_3 = 4$, and $S = \{1, 2\}$. One gets: $p_1^* \approx 6.8459$, $p_2^* \approx 7.5109$, $p_3^* \approx 5.9795$; $q_1^* \approx 2.0198$, $q_2^* \approx 0.0248$, $q_3^* \approx 4.6189$; and $\omega_1 = 132$. Consider $i = 2$. By Proposition 1 and $t_2^S = q_2^*/(q_1^* + q_2^*) \approx 0.0122 < \widehat{t}^S \approx 0.0303$, or by Corollary 2 and $c_2 = 7.23 > \widehat{c}_2^S \approx 7.1968$, $\partial \pi_S^* / \partial c_2 > 0$ holds. Indeed, one can check that $\partial \pi_S^* / \partial c_2 \approx 0.0371 > 0$. For a small reduction in c_2 from 7.23 to 7.2, the multiproduct firm's profits will decrease from $\pi_S^* = \pi_1^* + \pi_2^* \approx 1.9176$ to $\widetilde{\pi}_S \approx 1.9170$. However, a large reduction in c_2 from 7.23 to 7.15 will, by Proposition 2 and $\Delta c_2 = 0.08 > 2(c_i - \widehat{c}_i^S) \approx 0.0664$, increase the profits from 1.9176 to $\widetilde{\pi}_S \approx 1.9182$.

12 Note that most previous works on comparative statics have assumed a much stronger condition, $E > -1$. See Shapiro (1989) for a survey.

It is striking to see that *a small increase in* c_2 *will raise* π_S^*. For example, let c_2 be increased from 7.23 to $c_2^* = 7.2518$, the profits will be raised to $\widetilde{\pi}_S^{**} \approx 1.9187 > \pi_S^* \approx 1.9176$, and $\widetilde{\pi}_3^{**} \approx 9.1587 > \pi_3^* \approx 9.1431$, with the new equilibrium outputs as: $\widetilde{q}_1^{**} \approx 2.0277$, $\widetilde{q}_2^{**} \approx 0.0000$, $\widetilde{q}_3^{**} \approx 4.6228$. It is useful to note that c_2 has been raised to its upper bound $c_2^* = 7.2518$, at which demand for the second product is zero.[13]

The next example confirms that the result presented in Proposition 1 and Example 1 extends to Bertrand oligopolies with multiple multiproduct firms.

Example 2 There are five products ($n = 5$) produced by three firms. Firm 1 produces products 1 and 2 ($S_1 = \{1, 2\}$), firm 2 produces products 3 and 4 ($S_2 = \{3, 4\}$), firm 3 produces product 5. Let $V = 9, \gamma = 2, c_1 = 5.9, c_2 = 7.0, c_3 = 4.0, c_4 = 4.0$, and $c_5 = 4.0$. Applying the Bertrand solution for a general partition of products provided in the appendix, one gets: $p_1^* \approx 6.5673$, $p_2^* \approx 7.1273$, $p_3^* \approx 5.8066$, $p_4^* \approx 5.8066$, $p_5^* \approx 5.6775$; $q_1^* \approx 1.6921$, $q_2^* \approx 0.0121$, $q_3^* \approx 3.9744$, $q_4^* \approx 3.9744$, $q_5^* \approx 4.3616$. Equilibrium profits from the five products are: $\pi_1^* = 1.1256$, $\pi_2^* = 0.0044$, $\pi_3^* = 7.1717$, $\pi_4^* = 7.1717$, $\pi_5^* = 7.3081$. One can check that $\partial \pi_{S_1}^* / \partial c_2 \approx 0.0354 > 0$. It follows that a small increase in the unit cost of product 2 raises the total profits of multiproduct firm 1. Indeed, if c_2 rises from 7.0 to 7.02 firm 1's total profits will increase from $\pi_{S_1}^* = \pi_1^* + \pi_2^* \approx 1.1300$ to 1.1305.

Note that in the last example, multiproduct firm 2's two products have the same production costs as the single-product firm 3 producing product 5. However, this firm's equilibrium choices for its two goods are different from those of the single-product firm, reflecting its internal coordination.[14]

6 Conclusion and discussion

We have provided a new understanding about a multiproduct firm's behavior: reducing a multiproduct firm's cost will reduce its profits in price competition if the cost-reducing unit is sufficiently small. More specifically, we have characterized the critical level of output share below which a small reduction in the marginal cost of a small unit reduces the multiproduct firm's profits, as well as the critical size of cost reduction above which a large reduction in the marginal cost of a small unit increases the multiproduct firm's profit.

[13] The discussion here implies several interesting topics for future study of multiproduct firms' behavior. We present two of them here. First, we believe that our analysis can be modified so that future studies may explain why some firms' costs are high, by analyzing, for example, a two-stage cost-setting model in which firms first choose costs and then engage in price competition. Second, the above situation with a zero demand for an inefficient product provides a new approach for studying the incentives to terminate a product or close a factory. Recall that a single-product firm whose sales are zero is indifferent between exiting and staying. We conjecture that there is sometimes an incentive for a multiproduct firm to keep an inefficient unit idle rather than closing it down.

[14] That firm 3 is more profitable than firm 2 from each of its product is a confirmation of the well established result that non-merged firms can benefit more from merger than the merged firms (since we can regard firm 2 as the result of a merger of two separate firms producing goods 3 and 4, respectively).

Our new results are obviously beyond the boundaries of single-product oligopoly studies, and they indicate that much more remains to be explored in understanding the behavior of multiproduct oligopolies. We hope readers will be encouraged to apply our expressions for a general multiproduct equilibrium in extending oligopoly studies from single- to multiproduct. Such extensions, we believe, are not only a significant step closer to reality, but also will be as rewarding as the extension of calculus from single to multi-variable. In addition to the two topics discussed in the first footnote of Sect. 5, we briefly discuss three other topics. First, observe that the post-merger equilibria[15] in linear single product or multiproduct oligopoly are identical to the equilibrium in multiproduct oligopoly with an arbitrary partition. Hence, our general expressions for equilibrium will be useful in future merger studies involving multiproduct. Second, by introducing asymmetric substitution parameters in the Bertrand-Shubik demand system, one could obtain more general post-merger equilibrium beyond the multi-markets model of Kao and Menezes (2009), which also opens up a large body of future studies. Finally, it will be useful to conduct theoretical and empirical studies regarding the welfare effects of discontinuing a product in multiproduct oligopolies.

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Appendix: Proofs

Multiproduct Bertrand equilibrium with an arbitrary partition

Let $\Delta = \{S_1, S_2, \ldots, S_k\}$ be an arbitrary partition of N, and let $\delta, a, b, c > 0$ and $d = \{d_S |\ S \in \Delta\} = (d_1, \ldots, d_n)^\top$ be defined as

$$\begin{aligned} \delta &= n(1+\gamma) - \gamma; \\ a &= 2\delta, \quad b = 2\gamma, \quad c = \gamma; \text{ and for each } S \in \Delta, \\ d_S &= \{d_i \,|\, i \in S\}, \quad \text{where } d_i = nV + \delta c_i - \gamma \Sigma_{j \in S \setminus i} c_j, \quad \text{all } i \in S. \end{aligned} \tag{24}$$

Then, the first-order conditions for Bertrand equilibrium with $\Delta = \{S_1, S_2, \ldots, S_k\}$ are: $\partial \pi_S(p_S, p_{-S})/\partial p_i = 0$, for each $S \in \Delta$ and all $i \in S$, where $\pi_S(p_S, p_{-S})$ is given in (2). Such equations can be rearranged as

[15] The most general previous postmerger Bertrand equilibrium (Deneckere and Davidson 1985) is equivalent to the class of multiproduct Bertrand equilibria with a single multiproduct firm and with identical marginal costs. Their two-decade old assessment that "*it is no longer possible to write analytical expressions for equilibrium payoffs*" (from an arbitrary coalition structure) (Deneckere and Davidson 1985, p. 481) was still an accurate account of today's literature prior to this study.

$$Ap = d, \quad \text{where } A = A_{n\times n} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{pmatrix} \tag{25}$$

is an $n \times n$ matrix with k^2 blocks whose entries are: 1) for $j = 1, \ldots, k$, A_{jj} is an $n_j \times n_j$ symmetric matrix such that all its main diagonal entries are a, and all its off-diagonal entries are $-b$, where $n_j = |S_j|$ is the number of goods produced by firm $S_j \in \Delta$; 2) for all $i \neq j$, A_{ij} is an $n_i \times n_j$ matrix whose entries are all $-c$; and 3) $\sum_{j=1}^{k} n_j = n$. As shown in Zhao and Howe (2004), the inverse of A has the same block structure of A in (25) and is equal to $A^{-1} = U_{n\times n} =$

$$\{U_{ij}\} = \frac{1}{a+b} I_n + \frac{1}{c} \begin{pmatrix} \frac{b\beta_1(1+\theta_1)}{(a+b)} E_{n_1\times n_1} & \frac{\beta_1\beta_2}{(1-\alpha)} E_{n_1\times n_2} & \cdots & \frac{\beta_1\beta_k}{(1-\alpha)} E_{n_1\times n_k} \\ \frac{\beta_2\beta_1}{(1-\alpha)} E_{n_2\times n_1} & \frac{b\beta_2(1+\theta_2)}{(a+b)} E_{n_2\times n_2} & \cdots & \frac{\beta_2\beta_k}{(1-\alpha)} E_{n_2\times n_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_k\beta_1}{(1-\alpha)} E_{n_k\times n_1} & \frac{\beta_k\beta_2}{(1-\alpha)} E_{n_k\times n_2} & \cdots & \frac{b\beta_k(1+\theta_k)}{(a+b)} E_{n_k\times n_k} \end{pmatrix}, \tag{26}$$

where I_n is the identity matrix, $E_{n_i\times n_j}$ is an $n_i \times n_j$ matrix of 1s $(i, j = 1, \ldots, k)$,

$$\begin{aligned} U_{ii} &= \frac{1}{a+b} I_{n_i\times n_i} + \frac{b\beta_i(1+\theta_i)}{c(a+b)} E_{n_i\times n_i}, \\ U_{ij} &= \frac{\beta_i(c+b\theta_j)}{c(a+b)} E_{n_i\times n_j}, \quad \text{all } j \neq i; \\ \beta_i &= \left(n_i + \frac{a+(1-n_i)b}{c}\right)^{-1} = \frac{c}{a+b+(c-b)n_i}, \\ \alpha &= \sum_{i=1}^{k} \beta_i n_i = c \sum_{i=1}^{k} \frac{n_i}{a+b+(c-b)n_i}, \\ \theta_i &= \frac{1}{1-\alpha} \left(\beta_i n_i + \frac{c}{b} \sum_{\substack{j=1 \\ j\neq i}}^{k} \beta_j n_j \right) = \frac{1}{1-\alpha} \left(\beta_i n_i + \frac{c}{b}(\alpha - \beta_i n_i)\right), \end{aligned} \tag{27}$$

and it is assumed that $\alpha \neq 1$ and $a + b + (c-b)n_i \neq 0$, all i. Then, the Bertrand equilibrium is equal to

$$p^* = \{p_S^* |\ S \in \Delta\} = (p_1^*, \ldots, p_n^*)^\top = A^{-1} d. \tag{28}$$

Proof of Proposition 1 For each $i \in S$, the effects of its cost reduction on $j \notin S$ are straightforward, so we only need to show the effects on each $j \in S$. Differentiating (9) and (11) with respect to c_i leads to

$$\text{i)} \quad \frac{\partial q_j^*}{\partial c_i} = \begin{cases} \frac{1+\gamma}{2} - \frac{\omega_4}{2n\omega_1} > 0 & \text{if } j \neq i, \\ -\frac{\omega_4}{2n\omega_1} < 0 & \text{if } j = i, \end{cases} \tag{29}$$

$$\text{ii)} \quad \frac{\partial (p_j^* - c_j)}{\partial c_i} = \begin{cases} \frac{\gamma^2(n-m)}{2\omega_1} > 0 & \text{if } j \neq i, \\ \frac{\gamma^2(n-m)}{2\omega_1} - \frac{1}{2} < 0 & \text{if } j = i; \end{cases} \tag{30}$$

where ω_1 is given by (6), and $\omega_4 > 0$ is given by

$$\omega_4(n, m, \gamma) = \gamma^3 (n-m) [2(n-1)^2 + n(m-1)] + \gamma^2 n[(2n-m)(2n+m-5) + n(4n-3m-1) + 2] + 2\gamma n^2 (5n-m-3) + 4n^3. \tag{31}$$

For $j \neq i$, the positive sign of $\partial q_j^* / \partial c_i$ follows from

$$\frac{\partial q_j^*}{\partial c_i} = \frac{1+\gamma}{2} - \frac{\omega_4}{2n\omega_1} = \frac{[\gamma^2 (3n-2)(n-m) + \gamma n (7n-3m-2) + 4n^2]\gamma}{2n\omega_1} > 0$$

and the negative sign of $\partial(p_i^* - c_i)/\partial c_i$ follows from $\partial(p_i^* - c_i)/\partial c_i =$

$$\frac{\gamma^2 (n-m)}{2\omega_1} - \frac{1}{2} = -\frac{4n^2 + 2n\gamma (3n-m-1) + \gamma^2 (n-m)(2n+m-3)}{2\omega_1} < 0.$$

By (29) and (30), the effects on unit i's and j's output and markup satisfy the following properties: for $i \neq j \in S$,

$$\frac{\partial q_i^*}{\partial c_i} = \frac{\partial q_j^*}{\partial c_i} - \frac{1+\gamma}{2}, \quad \text{and} \quad \frac{\partial \left(p_i^* - c_i\right)}{\partial c_i} = \frac{\partial (p_j^* - c_j)}{\partial c_i} - \frac{1}{2}. \tag{32}$$

Using (29–30) and (32), one has

$$\begin{aligned} \frac{\partial \pi_S^*}{\partial c_i} &= \frac{\partial \sum_{j=1}^m \pi_j^*}{\partial c_i} = \frac{\partial[(p_i^* - c_i)q_i^*]}{\partial c_i} + \sum_{j \in S \setminus i} \frac{\partial[(p_j^* - c_j)q_j^*]}{\partial c_i} \\ &= -\frac{q_i^* + (1+\gamma)(p_i^* - c_i)}{2} + \frac{\partial (p_k^* - c_k)}{\partial c_i} \sum_{j \in S} q_j^* \\ &\quad + \frac{\partial q_k^*}{\partial c_i} \sum_{j \in S} (p_j^* - c_j), \quad \text{for any } k \neq i \in S. \end{aligned}$$

Applying (7–8) and (29–30) to the above expression, one has

$$\frac{\partial \pi_S^*}{\partial c_i} = -\frac{q_i^* + (1+\gamma)(p_i^* - c_i)}{2} + \frac{\gamma^2 (n-m) \sum_{j=1}^m q_j^*}{2\omega_1} + \frac{[\gamma^2 (3n-2)(n-m) + \gamma n (7n-3m-2) + 4n^2]\gamma \sum_{j=1}^m (p_j^* - c_j)}{2n\omega_1}$$
$$= -\frac{2nq_i^* + m\gamma(\overline{p}_S^* - \overline{c}_S)}{2n} + \frac{\gamma^2 (n-m) \sum_{j=1}^m q_j^*}{2\omega_1} + \frac{[\gamma^2 (3n-2)(n-m) + \gamma n (7n-3m-2) + 4n^2]\gamma \sum_{j=1}^m q_j^*}{2(n(1+\gamma) - m\gamma)\omega_1}$$
$$= -q_i^* - \frac{n\gamma \sum_{j=1}^m q_j^*}{2n(n(1+\gamma) - m\gamma)} + \frac{\gamma^2 (n-m) \sum_{j=1}^m q_j^*}{2\omega_1} + \frac{[\gamma^2 (3n-2)(n-m) + \gamma n (7n-3m-2) + 4n^2]\gamma \sum_{j=1}^m q_j^*}{2(n(1+\gamma) - m\gamma)\omega_1}$$
$$= -q_i^* + \frac{\gamma^2 (n-m) \sum_{j=1}^m q_j^*}{\omega_1},$$

which leads to

$$\frac{\partial \pi_S^*}{\partial c_i} > 0 \Leftrightarrow t_i^S < \widehat{t}^S = \frac{\gamma^2 (n-m)}{\omega_1}. \tag{33}$$

□

Proof of Corollary 1 (i) By (4), (9) and (11),

$$(p_i^* - c_i) - (\overline{p}_S^* - \overline{c}_S) = \frac{\gamma^2 m (n-m) + 2\omega_2}{2\omega_1}\overline{c}_S - \frac{c_i}{2} = \frac{\overline{c}_S - c_i}{2},$$

which leads to $q_i^* - \overline{q}_S^* =$

$$\frac{[\gamma^2 (3n-2)(n-m) + \gamma n (7n-3m-2) + 4n^2]m\gamma\overline{c}_S}{2n\omega_1} - \frac{(1+\gamma)c_i}{2} + \frac{(n(1+\gamma) - m\gamma)\,\omega_2\overline{c}_S}{n\omega_2} = (1+\gamma)(\overline{c}_S - c_i)/2.$$

Substituting the above two expressions into the expression for $\partial \pi_S^*/\partial c_i$ in the proof of Proposition 1, one has:

$$\frac{\partial \pi_S^*}{\partial c_i} = \frac{m\gamma^2 (n-m)\overline{q}_S^*}{\omega_1} - \overline{q}_S^* - \frac{(1+\gamma)(\overline{c}_S - c_i)}{2} = \frac{-2\omega_2\overline{q}_S^*}{\omega_1} - \frac{(1+\gamma)(\overline{c}_S - c_i)}{2}.$$

By $\omega_1 > 0$ and $\omega_2 > 0, \overline{c}_S - c_i > 0$ implies $\partial \pi_S^*/\partial c_i < 0$, so part (i) holds.

(ii) When $m = n, \widehat{t}^S = 0$. By (33), $\partial\pi_S^*/\partial c_i > 0$ is impossible, so $\partial\pi_S^*/\partial c_i < 0$ holds for all i. □

Proof of Corollary 2 By (10), $\pi_S^* = \pi_S^*(c_i)$ is convex and quadratic in c_i, and its minimum point $\widehat{c}_i^S$, or the solution of $\partial\pi_S^*/\partial c_i = 0$, is given by

$$\widehat{c}_i^S = \frac{\omega_5}{4(n(1+\gamma) - m\gamma)(\omega_2)^2 + n(m-1)(1+\gamma)(\omega_1)^2}, \quad (34)$$

where ω_1 and ω_2 are given in (6) and (12), and $\omega_5 > 0$ is given by

$$\omega_5(n, m, \gamma) = 4m(n(1+\gamma)-m\gamma)\omega_2[n(2n(1+\gamma)-\gamma)V+\gamma(n(1+\gamma)-\gamma)(n-m)\overline{c}_{-S}] \\ +[n(1+\gamma)(\omega_1)^2-4(n(1+\gamma)-m\gamma)(\omega_2)^2]\Sigma_{j\in S\setminus i}c_j.$$

Since π_S^* is symmetric at $\widehat{c}_i^S$, a decrease in c_i reduces $\pi_S^* \iff c_i$ is on the right half of the profit curve where π_S^* is increasing in c_i. □

Proof of Proposition 2 (i) Let $\delta_i = c_i - \widehat{c}_i^S > 0$. One has $\pi_S^*(c_i - \delta_i) = \pi_S^*(\widehat{c}_i^S) = Min\left\{\pi_S^*(c_i)|c_i \geq 0\right\}$. By the symmetry of $\pi_S^*(c_i)$ around $\widehat{c}_i^S, \pi_S^*(c_i - 2\delta_i) = \pi_S^*(c_i)$. Therefore, $\Delta c_i > 2\delta_i$ implies $\pi_S^*(c_i - \Delta c_i) > \pi_S^*(c_i - 2\delta_i) = \pi_S^*(c_i)$, which leads to (35). The reverse also holds obviously. Suppose the multiproduct firm's most efficient product is good 1 (i.e., $c_1 = \min\{c_i|\ 1 \leq i \leq m\}$). For $i \in S$ with $c_i > \widehat{c}_i^S$, where $\widehat{c}_i^S$ is given in (34), let $\Delta c_i > 0$ be the reduction in c_i. Let $\pi_S^*(c_i)$ denote the multiproduct firm's profits given in (10) when firm i's marginal cost is c_i. Then, the following two claims hold:

$$\begin{aligned} &\text{(i)} \quad \pi_S^*(c_i - \Delta c_i) > \pi_S^*(c_i) \Leftrightarrow \Delta c_i > 2(c_i - \widehat{c}_i^S); \\ &\text{(ii)} \quad c_i - c_1 > 2(c_i - \widehat{c}_i^S). \end{aligned} \quad (35)$$

(ii) By

$$\frac{\gamma^2(n-m)}{\omega_1} - \frac{1}{2m+2} = \frac{\gamma^2(n-m)(2m+2) - \omega_1}{(2m+2)\omega_1} \\ = -\frac{\gamma^2(n-m)(2n-m-4) + 2n\gamma(3n-m-1) + 4n^2}{(2m+2)\omega_1} < 0,$$

the critical output share $\widehat{t}^S$ in (33) satisfies

$$\widehat{t}^S < \frac{1}{2m+2}. \quad (36)$$

As insider i keeps reducing its marginal cost from $c_i > \widehat{c}_i^S$ to $\widehat{c}_i^S$ and eventually to c_1, its output share will increase from below $\widehat{t}^S$ to above $\widehat{t}^S$, further to $1/m > \widehat{t}^S$, and eventually to above $1/m$, because firm 1 is the most efficient insider. When its marginal cost falls below $\widehat{c}_i^S$, the multiproduct firm's profits will start to increase.

By (36) and by $t_i^S < \widehat{t}^S$,

$$\widehat{t}^S - t_i^S < \frac{1}{2m+2} < \frac{1}{2m}.$$

However, (36) also implies

$$\frac{1}{m} - \widehat{t}^S > \frac{1}{m} - \frac{1}{2m+2} = \frac{m+2}{(2m+2)m} > \frac{1}{2m}.$$

Because insiders' outputs in (9) are linear in marginal costs, the above two inequalities imply that the reduction in c_i equivalent to a share increase from $\widehat{t}^S$ to $1/m$ is much larger than $\delta_i = c_i - \widehat{c}_i^S > 0$, which is the reduction in c_i equivalent to a share increase from t_i^S to $\widehat{t}^S$. Because more reductions are needed for c_i to eventually reach c_1 (i.e., for its output share to increase from $1/m$ to firm 1s output share in S), one must have $c_i - c_1 > 2\delta_i$, which completes the proof of part (ii). □

Multiproduct Cournot equilibrium with an arbitrary partition

Given an arbitrary partition $\Delta = \{S_1, S_2, \ldots, S_k\}$, its Cournot equilibrium solves the first order conditions $\partial \pi_S(q_S, q_{-S})/\partial q_i = 0$, for all $i \in S$ and for each $S \in \Delta$, which can be rearranged as

$$Bq = \overline{d}, \quad \text{where } B = B_{n\times n} = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1k} \\ B_{21} & B_{22} & \cdots & B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ B_{k1} & B_{k2} & \cdots & B_{kk} \end{pmatrix} \tag{37}$$

is identical to A in (25) except that the constants a, b, c, and d in (24) are replaced by $\overline{a}$, $\overline{b}$, $\overline{c}$, and $\overline{d}$ given below:

$$\begin{aligned} \overline{a} &= 2(n+\gamma), \quad \overline{b} = -2\gamma, \quad \overline{c} = -\gamma, \\ \overline{d}_i &= n(1+\gamma)(V-c_i), \quad \text{all } i \in N. \end{aligned} \tag{38}$$

Hence, the inverse B^{-1} is the same as A^{-1} in (26), with its constants a, b, c, and d replaced by the above $\overline{a}$, $\overline{b}$, $\overline{c}$, and $\overline{d}$, and the Cournot equilibrium can be given as

$$q^{C*} = \{q_S^{C*} \mid S \in \Delta\} = (q_1^{C*}, \ldots, q_n^{C*})^\top = B^{-1}\overline{d}. \tag{39}$$

Now, for the single multiproduct firm given by $\Delta = \{S, m+1, \ldots, n\}$, the involved matrix B and its inverse are:

$$B = \begin{bmatrix} [2(n+\gamma)-2\gamma]I_m + 2\gamma E_{m\times m} & \gamma E_{m\times(n-m)} \\ \gamma E_{(n-m)\times m} & [2(n+\gamma)-\gamma]I_{n-m} + \gamma E_{(n-m)\times(n-m)} \end{bmatrix}, \text{ and}$$

$$B^{-1} = \begin{bmatrix} \frac{1}{2n} I_m & 0 \\ 0 & \frac{1}{2n+\gamma} I_{n-m} \end{bmatrix} - \frac{\gamma}{\omega_3} \begin{bmatrix} \frac{4n+(n-m+2)\gamma}{2n} E_{m\times m} & E_{m\times(n-m)} \\ E_{(n-m)\times m} & \frac{2n+m\gamma}{2n+\gamma} E_{(n-m)\times(n-m)} \end{bmatrix},$$

where $\omega_3 > 0$ is given by (18), I_k is the $k \times k$ identity matrix and $E_{k\times j}$ is the $k \times j$ matrix of all 1s. The equilibrium $B^{-1}\overline{d}$ gives the products, prices and profits at the equilibrium in (16–17), (19–20) and (21), where the multiproduct firm's average price and the involved mark-ups are:

$$\begin{aligned} \overline{p}_S^{C*} &= \left(\sum_{i\in S} p_i^{C*}\right) \Big/ m \\ &= \frac{(2n+\gamma)(n+m\gamma)V}{\omega_3} + \frac{(2n^2+n(n+m+1)\gamma+m\gamma^2)\overline{c}_S}{\omega_3} \\ &\quad + \frac{(n-m)(n+m\gamma)\gamma\overline{c}_{-S}}{\omega_3}, \end{aligned} \tag{40}$$

$$\begin{aligned} p_j^{C*} - c_j &= \frac{(n+\gamma)q_j^{C*}}{n(1+\gamma)}, \quad j \notin S; \quad \text{and for each } i \in S, \\ p_i^{C*} - c_i &= \frac{(2n+\gamma)(n+m\gamma)V}{\omega_3} - \frac{m(n-m)\gamma^2\overline{c}_S}{2\omega_3} \\ &\quad + \frac{(n-m)(n+m\gamma)\gamma\overline{c}_{-S}}{\omega_3} - \frac{c_i}{2}, \end{aligned} \tag{41}$$

Proof of Proposition 3 Part (i) For each $i \notin S$, differentiating (16–17) with respect to c_i leads to

$$\frac{\partial q_j^{C*}}{\partial c_i} = \begin{cases} \frac{n(1+\gamma)\gamma}{\omega_3} > 0 & \text{if } j \in S; \\ \frac{n(1+\gamma)(2n+m\gamma)\gamma}{(2n+\gamma)\omega_3} > 0 & \text{if } j \notin S, j \neq i; \\ \frac{-n(1+\gamma)\left[m(n+1-m)\gamma^2+2n(n+m)\gamma+4n^2\right]}{(2n+\gamma)\omega_3} < 0 & \text{if } j \notin S, j = i. \end{cases}$$

The profit effects follow from (21) and (41) and the above product effects.

Part (ii) For each $i \in S$, the effects of its cost reduction on a single-product firm $j \notin S$ are straightforward, so we only need to show the effects on each $j \in S$. Differentiating (16) and $\left(p_i^{C*} - c_i\right)$ in (41) with respect to c_i leads to

$$\frac{\partial q_j^{C*}}{\partial c_i} = \begin{cases} \frac{\gamma(1+\gamma)(4n+(n-m+2)\gamma)}{2\omega_3} > 0 & \text{if } j \neq i, \\ \frac{\gamma(1+\gamma)(4n+(n-m+2)\gamma)}{2\omega_3} - \frac{1+\gamma}{2} < 0 & \text{if } j = i; \end{cases} \tag{42}$$

$$\frac{\partial \left(p_j^{C*} - c_j\right)}{\partial c_i} = \begin{cases} -\frac{(n-m)\gamma^2}{2\omega_3} < 0 & \text{if } j \neq i, \\ -\frac{(n-m)\gamma^2}{2\omega_3} - \frac{1}{2} < 0 & \text{if } j = i. \end{cases} \tag{43}$$

The negative sign of $\partial q_i/\partial c_i$ follows from

$$\frac{\partial q_i^{C*}}{\partial c_i} = \frac{\gamma(1+\gamma)(4n+(n-m+2)\gamma)}{2\omega_3} - \frac{1+\gamma}{2}$$
$$= -\frac{(1+\gamma)[4n^2+2n(n+m-1)\gamma+(m-1)(n-m+2)\gamma^2]}{2\omega_3} < 0.$$

Now, consider the profit effects. For any $i \neq j \in S$, (42–43) lead to

$$\frac{\partial q_i^{C*}}{\partial c_i} = \frac{\partial q_j^{C*}}{\partial c_i} - \frac{1+\gamma}{2}, \quad \text{and}$$
$$\frac{\partial \left(p_i^{C*} - c_i\right)}{\partial c_i} = \frac{\partial \left(p_j^{C*} - c_j\right)}{\partial c_i} - \frac{1}{2}. \tag{44}$$

Using (42–44), one has

$$\frac{\partial \pi_S^{C*}}{\partial c_i} = \frac{\partial \sum_{j=1}^m \pi_j^{C*}}{\partial c_i} = \frac{\partial \sum_{j=1}^m (p_j^{C*} - c_j) q_j^{C*}}{\partial c_i} = \frac{q_i^{C*} \partial (p_i^{C*} - c_i)}{\partial c_i}$$
$$+ \frac{(p_i^{C*} - c_i) \partial q_i^{C*}}{\partial c_i} + \sum_{j \in S \setminus i} \left[\frac{q_j^{C*} \partial (p_j^{C*} - c_j)}{\partial c_i} + \frac{(p_j^{C*} - c_j) \partial q_j^{C*}}{\partial c_j} \right]$$
$$= -\frac{q_i^{C*} + (1+\gamma)\left(p_i^{C*} - c_i\right)}{2}$$
$$+ \sum_{j \in S} \left[-\frac{q_j^{C*} \gamma^2 (n-m)}{2\omega_3} + \frac{(p_j^{C*} - c_j)\gamma(1+\gamma)(4n+(n-m+2)\gamma)}{2\omega_3} \right]$$
$$= -\frac{q_i^{C*} + (1+\gamma)\left(p_i^{C*} - c_i\right)}{2} - \frac{m(n-m)\gamma^2 \overline{q}_S^{C*}}{2\omega_3}$$
$$+ \frac{m\gamma(1+\gamma)[4n+(n-m+2)\gamma](\overline{p}_S^{C*} - \overline{c}_S)}{2\omega_3}.$$

Rearranging the multiproduct firm's markups as $(p_i - c_i) = \left[q_i + m\gamma\overline{q}_S/n\right]/(1+\gamma)$, all $i \in S$, and $\left(\overline{p}_S - \overline{c}_S\right) =, (n+m\gamma)\,\overline{q}_S/\left[n(1+\gamma)\right]$, and substituting into $\partial \pi_S^{C*}/\partial c_i$, one has

$$\frac{\partial \pi_S^{C*}}{\partial c_i} = -\frac{q_i^{C*} + q_i^{C*} + \frac{m\gamma \overline{q}_S^{C*}}{n}}{2} - \frac{m\gamma^2 (n-m)\,\overline{q}_S^{C*}}{2\omega_3}$$
$$+ \frac{m\gamma(1+\gamma)(4n+(n-m+2)\gamma)}{2\omega_3} \frac{(n+m\gamma)\,\overline{q}_S^{C*}}{n(1+\gamma)}$$

$$= -q_i^{C*} - \left[\frac{m\gamma}{2n} + \frac{m\gamma^2\,(n-m)}{2\omega_3} - \frac{m\gamma\,(4n+(n-m+2)\gamma)\,(n+m\gamma)}{2n\omega_3}\right]\overline{q}_S^{C*}$$

$$= -q_i^{C*} - \frac{m\,(n-m)\,\gamma^2}{\omega_3}\overline{q}_S^{C*} < 0$$

□

Proof of Claim 1 The output effects of a small cost reduction are known as given below:

$$\frac{\partial x_i}{\partial c_i} = \frac{(n+1+E)p'(X) - \left[p'(X) + x_i p''(X)\right]}{(n+1+E)\,(p'(X))^2},$$

$$\frac{\partial x_k}{\partial c_i} = \frac{-\left(p'(X) + x_k p''(X)\right)}{(n+1+E)\,(p'(X))^2}, \quad \text{all } k \neq i.$$

Using the envelope theorem and the above expressions, one obtains:

$$\frac{\partial \pi_i^*}{\partial c_i} = x_i p'(X)\Sigma_{j\neq i}\frac{\partial x_j}{\partial c_i} - x_i = x_i\Sigma_{j\neq i}\frac{p'(X) + x_j p''(X)}{(n+1+E)\,(-p'(X))} - x_i \quad (45)$$

$$= -x_i\left[\Sigma_{j\neq i}\frac{1+s_j E}{(n+1+E)} + 1\right] = \frac{-x_i\,[2n+(2-s_i)E]}{(n+1+E)}. \quad (46)$$

Claim 1 follows immediately from the following two conclusions. First, with strategic substitutes (i.e., $\alpha_i \le 0$), (22) and (45) imply that $\partial\pi_i^*/\partial c_i < 0$. Second, with strategic complements (i.e., $\alpha_i > 0$), $\partial\pi_i^*/\partial c_i < 0$ holds if $E > -(n+1/2)$. To see this, substituting (23) into $[2n+(2-s_i)E]$ in (46), one has $[2n+(2-s_i)E] > [2n+2E+1] = 2(E+n+1/2)$. Hence, the condition $E > -(n+1/2)$ (which implies (22)) implies

$$\frac{[2n+(2-s_i)E]}{(n+1+E)} > 0.$$

Applying this to (46) leads to $\partial\pi_i^*/\partial c_i < 0$.[16] □

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[16] For readers who want to find a counter-example with large α_i for $\partial\pi_i^*/\partial c_i > 0$, it is warned that such infinitesimal possibility is further hindered by the second-order condition for π-max. By $\partial^2\pi_i/\partial^2 x_i = 2p'(X) + x_i p''(X) < 0 \Leftrightarrow \alpha_i = p'(X) + x_i p''(X) < -p'(X)$, the second-order condition for π-max places an upper bound on the size of $\alpha_i > 0$, which reduces the possibility of $\partial\pi_i^*/\partial c_i > 0$ in (45).

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