

## **A Closed-form Solution of the Uzawa-Lucas Model of Endogenous Growth**

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This paper presents a closed-form solution to the Uzawa-Lucas endogenous growth model with human and physical capital. Our result also applies if an external effect in the use of human capital in goods production occurs. Using the “guess-and-verify” method, we determine the two value functions of the social planner in the centralized economy and of the representative agent in the decentralized case. We show that the introduction of income taxes on wages and of a subsidy on physical capital earnings helps the decentralized economy in reaching the social optimum, while keeping the policy maker’s budget balanced. Finally, we derive the time-series implications of the model’s solution.

*Keywords:* closed-form solution, value function, Uzawa-Lucas endogenous growth model.

*JEL Classifications:* C61, C62, C68.

### **1 Introduction**

In this paper, we present the value function of a stochastic version of the Uzawa (1965) and Lucas (1988) model of endogenous growth in discrete time. The externality of human capital in goods production inherent to the model causes a difference between the social planner’s solution and the market outcome. We pay attention to this fact by treating both cases separately and by presenting both the social planner’s and the representative agent’s value function. Furthermore, we show that the inefficiency of the decentralized economy can be overcome by introducing taxes and

subsidies on factor compensations. Resorting on lump-sum taxation is not required to satisfy the government's balanced budget condition.

The main feature of the Uzawa-Lucas model is the fact that the agents have to allocate their human capital between two production sectors. On the one hand, there is a goods sector where a single good usable for consumption and physical capital investment is produced. This sector exhibits a production technology that uses human and physical capital. On the other hand, there is a schooling sector where agents augment their stock of human capital. Here, human capital is the only input factor. In short, agents have to "learn or to do" (Chamley, 1993). In his seminal paper, Lucas (1988) argues that the economy's average level of human capital contributes to total factor productivity in goods production. In a decentralized economy the individual's accumulation of human capital has no appreciable influence on this average level and agents are only compensated for their respective factor supplies. This incentive structure leads to non-efficient equilibria. Since agents are not able to coordinate their actions, their discounted utility could be higher without making a single agent worse off. As a result, the solution for the centralized economy deviates from that of the decentralized case.

The theoretical model considered here differs from that studied by Lucas (1988) in two ways. First, there is our choice of the utility function. We assume logarithmic preferences which imply that the constant intertemporal elasticity of substitution is equal to one. This assumption reduces the number of parameters by one and simplifies the calculations. Second, we assume discrete time where the two capital stocks depreciate fully at the end of the period. This way the closed-form solution of the stochastic one sector growth model with logarithmic preferences and full depreciation of physical capital (cf. McCallum, 1989) is extended to the case with two capital goods.

The advantages of closed-form solutions are obvious. The user of an economic model does not need to apply sophisticated solution methods but can instead directly turn to the model's implications. This way the solution is suitable for sharpening economic intuition. Moreover, closed-form solutions are exact and, if necessary, easy to implement. Hence they also provide (rare) examples that allow the evaluation of approximation methods. Apart from McCallum's setup, there are only a few other closed-form solutions in business cycle theory. The best-known is the linear-quadratic formulation (cf. Sargent, 1987, sect. IX.8). Another case is given in Saint-Paul (1996, sect. 2.1) where an example with logarithmic

preferences and depreciation is presented.<sup>1</sup> Moreover, there are some varieties of the McCallum setup with closed-form solutions. Bénassy (1995) suggests geometric depreciation of physical capital. In this case, the model's solution can still be stated in terms of an explicit functional form of the value function. Another example for a variation of McCallum (1989) with a closed-form solution is Huffman (2004). He endogenizes investment-specific technological change and solves the model by using the explicit form of the value function.

The paper is organized as follows. Section 2 introduces the model. Section 3 presents the value function in closed form as the solution to the social planner's dynamic optimization problem (DOP). In Section 4, we present the value function of the representative agent. Section 5 shows that the solutions are saddle path stable and determines their time-series implications. Section 6 summarizes our results and concludes.

## 2 The Model

We consider a closed economy populated by an infinite number of homogeneous, infinitely-lived agents. The representative agent enters every period  $t$  with predetermined endowments of human and physical capital,  $h_t$  and  $k_t$ , respectively. Furthermore, there are two sectors in the economy. Firms produce a single homogeneous good and a schooling sector provides educational services.

### 2.1 The Household

The population is assumed to be constant and normalized to one. The representative agent has logarithmic preferences over sequences of consumption:

$$U(c_0, c_1, \dots) = E_0 \left[ \sum_{t=0}^{\infty} \beta^t \ln(c_t) \right], \quad (1)$$

where  $c_t$  is the level of consumption in period  $t \in \mathbb{N}_0$  and  $\beta \in (0, 1)$  is the subjective discount factor. Expectations are formed over the sequence of

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<sup>1</sup> Depreciation does not only affect the predetermined capital stock but also current investments.

shocks  $\{\varepsilon_t\}_{t=0}^{\infty}$  entering goods production. The logarithmic utility function implies that the intertemporal elasticity of substitution is equal to one. In each period, agents have a fixed endowment of time, which is normalized to one unit. The variable  $u_t$  denotes the fraction of time allocated to goods production in period  $t$ . Furthermore, as agents do not benefit from leisure, the whole time budget is allocated to the two production sectors. The fraction  $1 - u_t$  of time is spent in the schooling sector. Note that in any solution the condition

$$u_t \in [0, 1] \quad (2)$$

has to be fulfilled. The variables  $c_t$  and  $u_t$  are the two control variables of the agent. When maximizing her discounted stream of utility, the agent has to pay attention to the following budget constraint:

$$\tau_r r_t k_t + \tau_w w_t u_t h_t = c_t + k_{t+1}, \quad \forall t \in \mathbb{N}_0, \quad k_0 > 0, \quad (3)$$

where  $k_t$  is the agent's physical capital stock in period  $t$ . The terms  $\tau_r r_t k_t$  and  $\tau_w w_t u_t h_t$  are, respectively, the net returns on physical capital and work effort after taxation. We assume that both parameters  $\tau_r$  and  $\tau_w$  are positive. If the parameter  $\tau_r$  is smaller than 1, we have a tax on physical capital, if it is larger than 1, we have a subsidy. The same is true for the parameter  $\tau_w$ . If  $\tau_w < 1$ , work effort is taxed, if  $\tau_w > 1$ , work effort is subsidized. Hence, the rates of taxation are given by  $\tau_r - 1$  and  $\tau_w - 1$ , respectively. The above constraint implies full depreciation of physical capital. The variables  $r_t$  and  $w_t$  are the real interest rate and the real wages, respectively. Prices and tax rates are endogenous to the model. The former via the market clearing mechanism and the latter via the government's balanced budget condition. Despite this fact, prices and tax rates are taken as given by the representative agent. The left-hand side describes her income derived from physical capital plus the income stemming from effective work, which is determined by the worker's level of human capital  $h_t$  multiplied by the fraction of time spent in the goods sector in period  $t$ , i.e.,  $h_t u_t$ . We assume that the initial values  $k_0$  and  $h_0$  are strictly positive. On the right-hand side, the spending of the agent's earnings appears, which she can either consume or invest. Another constraint the agent has to keep in mind is the evolution of her stock of human capital when allocating  $1 - u_t$  to the schooling sector.

## 2.2 The Schooling Sector

The creation of human capital is determined by a linear technology in human capital only:

$$h_{t+1} = B(1 - u_t)h_t, \quad \forall t \in \mathbb{N}_0, \quad h_0 > 0, \quad (4)$$

where we assume that  $B$  is positive.<sup>2</sup> If we set  $u_t$  in Eq. (4) equal to zero, we get the potential stock of next period's human capital. If we set  $u_t$  equal to one, tomorrow's stock of human capital is equal to zero. The schooling technology implies that the potential marginal and average product of human capital coincide and are equal to  $B$ , whereas the realized marginal and average products are equal to  $B(1 - u_t)$ . Note that the depreciation rate of human capital is 100 percent per period.

## 2.3 The Goods Sector

We assume an infinitely large number of profit-maximizing firms producing a single good. They are using a Cobb-Douglas technology in physical capital  $k_t$  and effective work  $h_t u_t$ . Furthermore, the average skill of workers  $h_{a,t}$  has a positive influence on total factor productivity. Hence, output  $y_t$  is determined by:

$$y_t = A_t k_t^\alpha (u_t h_t)^{1-\alpha} h_{a,t}^\gamma. \quad (5)$$

The parameter  $\alpha$  is the output elasticity of physical capital and we assume  $\alpha \in (0, 1)$ . The parameter  $\gamma$  is nonnegative and measures the degree of the external effect of human capital. If we set  $u_t$  equal to one, we get the potential output in the goods sector. The homogeneity of the agents implies that:

$$h_{a,t} = h_t, \quad \forall t \in \mathbb{N}_0. \quad (6)$$

The state variable  $A_t$  denotes total factor productivity. Throughout this paper, we assume that  $\ln A_t$  follows a first-order autoregressive process, i.e.:

$$\ln A_{t+1} = \rho \ln A_t + \varepsilon_{t+1}, \quad \forall t \in \mathbb{N}_0 \quad \text{and} \quad \varepsilon \sim N(0, \sigma^2). \quad (7)$$

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<sup>2</sup> The case when  $B$  equals 0 corresponds to the neoclassical growth model.

The firm has to rent physical and human capital on perfectly competitive factor markets. In the decentralized economy, the representative firm's profit  $\Pi$  in period  $t$  is given by:

$$\Pi(k_t, h_t; A_t, h_{a,t}) = A_t k_t^\alpha (u_t h_t)^{1-\alpha} h_{a,t}^\gamma - r_t k_t - w_t u_t h_t,$$

where the semicolon indicates that the whole paths of  $h_{a,t}$  and  $A_t$  are treated as exogenous by the representative firm. The first-order necessary conditions for the profit-maximizing factor demands are:

$$r_t \equiv \frac{\partial y_t}{\partial k_t} = \frac{\alpha y_t}{k_t} \quad \text{and} \quad w_t \equiv \frac{\partial y_t}{\partial (u_t h_t)} = \frac{(1-\alpha)y_t}{u_t h_t}. \quad (8)$$

These market-clearing factor prices ensure that the zero-profit condition holds. Inserting the prices into the agent's budget constraint (3) yields:

$$\tau_r \alpha y_t + \tau_w (1-\alpha)y_t = c_t + k_{t+1}, \quad \forall t \in \mathbb{N}_0. \quad (9)$$

#### 2.4 The State Sector in the Decentralized Economy

In each period  $t$ , we require the state's budget to be balanced. Therefore:

$$(\tau_r - 1)r_t k_t = (1 - \tau_w)w_t u_t h_t \quad (10)$$

must hold for all  $t \in \mathbb{N}_0$ . This means that if we consider a tax on physical capital returns, we are subsidizing work effort at the same time and vice versa. This remark ends the presentation of the model. In Section 3, we solve the centralized version of this model.

### 3 The Centralized Solution of the Model

In the centralized economy, the social planner internalizes the contribution of the economy's average level of human capital to goods production. That is, the planner is able to reach the efficient allocation of resources without the instrument of taxation. Therefore, we assume  $\tau_r = \tau_w = 1$  throughout this section.

The central planner internalizes the social returns of human capital when choosing his optimal controls. This means that he exploits the symmetry condition stated in Eq. (6) and writes his DOP as follows:

$$U = \sup_{\{c_t, u_t\}_{t=0}^{\infty}} E_0 \left[ \sum_{t=0}^{\infty} \beta^t \ln c_t \right],$$

with respect to the state dynamics

$$k_{t+1} = A_t k_t^\alpha u_t^{1-\alpha} h_t^{1-\alpha+\gamma} - c_t, \quad \forall t \in \mathbb{N}_0, \quad (11)$$

$$h_{t+1} = B(1 - u_t)h_t, \quad \forall t \in \mathbb{N}_0, \quad (12)$$

$$\ln A_{t+1} = \rho \ln A_t + \varepsilon_{t+1}, \quad \forall t \in \mathbb{N}_0, \quad (13)$$

$$k_t \geq 0 \quad \text{and} \quad h_t \geq 0, \quad \forall t \in \mathbb{N}_0.$$

Since the social planner uses the symmetry from (6), he has simply dropped the index  $a$ . Furthermore, the initial values  $k_0$ ,  $h_0$ , and  $A_0 > 0$  are assumed to be given and the social planner has to ensure that

$$c_t > 0 \quad \text{and} \quad 0 \leq u_t \leq 1$$

hold for all  $t \in \mathbb{N}_0$ . He defines the value function as the solution to his optimization problem from time  $t$  onwards:

$$V(k_t, h_t, A_t) \equiv \sup_{\{c_s, u_s\}_{s=t}^{\infty}} E_t \left[ \sum_{s=t}^{\infty} \beta^{s-t} \ln c_s \right] \quad \text{s.t. (11), (12), and (13)}.$$

The Bellman equation associated with the planner's DOP is given by:

$$V(k_t, h_t, A_t) = \sup_{c_t, u_t} \{ \ln c_t + \beta E_t [V(k_{t+1}, h_{t+1}, A_{t+1})] \}. \quad (14)$$

The first-order necessary conditions for the optimal consumption choice and the optimal allocation of human capital between the two sectors are given by:

$$c_t \quad : \quad \frac{1}{c_t^*} = \beta E_t \left[ \frac{\partial V_{t+1}}{\partial k_{t+1}} \right], \quad (15)$$

$$u_t \quad : \quad u_t^* = \left( \frac{E_t \left[ \frac{\partial V_{t+1}}{\partial k_{t+1}} \right] (1 - \alpha) A_t}{E_t \left[ \frac{\partial V_{t+1}}{\partial h_{t+1}} \right] B} \right)^{\frac{1}{\alpha}} \frac{k_t}{h_t^{\frac{\alpha-\gamma}{\alpha}}}, \quad (16)$$

where  $V_t^*$  stands for  $V(k_t, h_t, A_t)$  and the asterisk denotes optimality. Equation (15) describes the behavior along the optimal consumption path. When shifting a marginal unit of today's output from consumption to investment, today's marginal change in utility should equal the expected discounted marginal change of wealth with respect to tomorrow's capital stock. Equation (16) states that the weighted expected marginal change of wealth with respect to physical capital equals the weighted expected marginal change of wealth with respect to human capital. The first weight is the marginal product of human capital in goods production, given a certain choice of  $u_t$ . The second weight is the potential marginal product of human capital when the remaining fraction of human capital is allocated to the educational sector.

We now turn to the Euler equations. The envelope property with respect to physical capital is straightforward. Together with the above first-order necessary conditions (15) and (16), it gives rise to the following Euler equation in consumption:

$$\frac{1}{c_t} = \beta E_t \left[ \frac{1}{c_{t+1}} \frac{\alpha y_{t+1}}{k_{t+1}} \right]. \quad (17)$$

This is the Lucas asset pricing equation (cf. Lucas, 1978) with the constant of relative risk aversion being equal to one. Along the optimal consumption path, marginal utility of consumption in time  $t$  must be equal to the discounted expected marginal utility from the return on investment in the next period. The envelope condition for the stock of human capital is given by:

$$\frac{\partial V_t}{\partial h_t} = \beta E_t \left[ \frac{\partial V_{t+1}}{\partial h_{t+1}} \frac{B(1 - \alpha + \gamma u_t)}{1 - \alpha} \right]. \quad (18)$$

The last term on the right-hand side,  $\gamma u_t$ , indicates that the expected marginal social gain of exploiting the external effect in goods production has an impact on the evolution of the shadow price of human capital. To be precise, today's shadow price of human capital is positively influenced by the degree of the external effect of human capital in goods production. This is the mechanism by which the external effect enters the second Euler equation along the optimal allocation of human capital between the two sectors:



$$u_t = \left( \frac{\mathbb{E}_t \left[ \frac{1}{c_{t+1}} \frac{\alpha y_{t+1}}{k_{t+1}} \right] (1 - \alpha) A_t}{\mathbb{E}_t \left[ \frac{(1 - \alpha + \gamma u_{t+1}) y_{t+1}}{c_{t+1} h_{t+1} u_{t+1}} \right] B} \right)^{\frac{1}{\alpha}} \frac{k_t}{h_t^{\frac{\alpha - \gamma}{\alpha}}}. \quad (19)$$

The transversality conditions with respect to the two capital stocks that establish the sufficiency of the two Euler equations (17) and (19) are given by:

$$\lim_{T \rightarrow \infty} \beta^T \mathbb{E}_0 \left[ \frac{\alpha y_T}{c_T k_T} k_T \right] = 0 \quad \text{and} \quad \lim_{T \rightarrow \infty} \beta^T \mathbb{E}_0 \left[ \frac{(1 - \alpha + \gamma u_T) y_T}{c_T u_T h_T} h_T \right] = 0. \quad (20)$$

The conditions (20) assert that the intertemporal budget constraints are met by the planner's decisions. Since the social planner exploits the external effect of human capital, the derivative of the production function with respect to human capital looks different from that in a decentralized economy below. This derivative is the sum of the private marginal return from  $u_t h_t$  and the marginal social gain of the average stock of human capital  $h_t$ .

Using the guess and verify method, one can show that the value function has the following form:

$$V(A_t, k_t, h_t) = \theta + \theta_B \ln B + \theta_A \ln A_t + \theta_k \ln k_t + \theta_h \ln h_t, \quad (21)$$

where the  $\theta_i$ 's, with  $i \in \{k, h, B, A\}$ , are defined as follows:

$$\begin{aligned} \theta_B &:= \frac{(1 - \alpha + \gamma)\beta}{(1 - \beta)^2(1 - \alpha\beta)}, & \theta_A &:= \frac{1}{(1 - \rho\beta)(1 - \alpha\beta)}, \\ \theta_k &:= \frac{\alpha}{1 - \alpha\beta}, & \theta_h &:= \frac{1 - \alpha + \gamma}{(1 - \alpha\beta)(1 - \beta)}. \end{aligned}$$

Moreover, the constant term is given by:

$$\begin{aligned} \theta &:= \frac{\ln[1 - \alpha\beta]}{1 - \beta} + \frac{(1 - \alpha) \ln[1 - \beta]}{(1 - \beta)(1 - \alpha\beta)} + \frac{\alpha\beta \ln \alpha}{(1 - \beta)(1 - \alpha\beta)} \\ &+ \frac{(1 - \alpha\beta + \gamma)\beta \ln \beta}{(1 - \beta)^2(1 - \alpha\beta)} + \frac{(1 - \alpha) \ln[1 - \alpha]}{(1 - \alpha\beta)(1 - \beta)} \\ &+ \frac{\beta(1 - \alpha + \gamma) \ln[1 - \alpha + \gamma]}{(1 - \alpha\beta)(1 - \beta)^2} - \frac{(1 - \alpha + \beta\gamma) \ln[1 - \alpha + \beta\gamma]}{(1 - \alpha\beta)(1 - \beta)^2}. \end{aligned}$$

The function  $V$  implies the following controls along the welfare-maximizing consumption and human capital allocation paths:

$$c_t = (1 - \alpha\beta)y_t \quad \text{and} \quad u_t = \frac{(1 - \alpha)(1 - \beta)}{1 - \alpha + \beta\gamma} := u. \quad (22)$$

Note that  $0 \leq u \leq 1$  is satisfied even in the strict sense. Furthermore, the allocation of human capital is constant regardless of the respective endowments of human and physical capital. The central planner simply devotes a constant share of goods production to consumption. Surely, findings (22) must also hold in period  $T$  such that it is easy to see that the Euler equations (17) and (19) and the transversality conditions (20) are satisfied. This remark closes the discussion of the centralized case. In the next section, we turn to the decentralized economy.

#### 4 The Decentralized Solution of the Model

In the decentralized case, we assume a representative agent with rational behavior. The agent knows that her stock of human capital equals the average level of human capital in the economy. Furthermore, she knows that the external effects of human capital in goods production, captured by the term  $h_{a,t}^\gamma$ , may increase her and all the other agents' wealth. But here, in the decentralized case, the market mechanism prevents a coordination of agents' actions. This can be understood as a Nash game producing the prisoner's dilemma. For this reason, we introduce a government that taxes and subsidizes the respective factor compensations. In the first subsection we write down the representative agent's optimization problem. Then, the second subsection characterizes the agent's optimal behavior. Finally, the third subsection determines the government's optimal taxation policy.

##### 4.1 The Representative Agent's Optimization Problem

Although the external effect of the economy's average human capital stock in period  $t$  may be not exploited, the whole path of  $h_{a,t}$  is predictable and is therefore treated as given by the agents. The representative agent's DOP is given by:

$$U = \sup_{\{c_t, u_t\}_{t=0}^{\infty}} E_0 \left[ \sum_{t=0}^{\infty} \beta^t \ln c_t \right],$$

with respect to the state dynamics

$$k_{t+1} = \tau_r r_t k_t + \tau_w w_t u_t h_t - c_t, \quad \forall t \in \mathbb{N}_0, \quad (23)$$

$$h_{t+1} = B(1 - u_t)h_t, \quad \forall t \in \mathbb{N}_0, \quad (24)$$

$$h_{a,t+1} = B(1 - u_{a,t})h_{a,t}, \quad \forall t \in \mathbb{N}_0, \quad (25)$$

$$\ln A_{t+1} = \rho \ln A_t + \varepsilon_{t+1}, \quad \forall t \in \mathbb{N}_0, \quad (26)$$

$$k_t \geq 0 \quad \text{and} \quad h_t \geq 0, \quad \forall t \in \mathbb{N}_0.$$

The variable  $u_a$  stands for the average human capital allocation in the decentralized economy, the value of which cannot be influenced by the representative agent.

We start the analysis of the decentralized economy with the definition of the value function as the solution to the representative agent's problem:

$$V(k_t, h_t; A_t, h_{a,t}) \equiv \sup_{\{c_s, u_s\}_{s=t}^{\infty}} E_t \left[ \sum_{s=t}^{\infty} \beta^{s-t} \ln c_s \right] \quad \text{s.t. (2)–(8)}.$$

Note that the value function of the representative agent is restricted to a given path of  $h_{a,t}$ . The corresponding Bellman equation is given by:

$$V(k_t, h_t; A_t, h_{a,t}) \equiv \sup_{c_t, u_t} \{ \ln c_t + \beta E_t [V(k_{t+1}, h_{t+1}; A_{t+1}, h_{a,t+1})] \}. \quad (27)$$

Taking the derivatives with respect to the two controls and inserting the market-clearing factor prices (8) gives us the following first-order necessary conditions:

$$c_t \quad : \quad \frac{1}{c_t^*} = \beta E_t \left[ \frac{\partial V_{t+1}}{\partial k_{t+1}} \right], \quad (28)$$

$$u_t \quad : \quad u_t^* = \left( \frac{E_t \left[ \frac{\partial V_{t+1}}{\partial k_{t+1}} \right] (1 - \alpha) \tau_w A_t}{E_t \left[ \frac{\partial V_{t+1}}{\partial h_{t+1}} \right] B} \right)^{\frac{1}{2}} \frac{k_t h_{a,t}^2}{h_t}, \quad (29)$$

where  $V_{t+1}$  is a shortcut for  $V(k_{t+1}, h_{t+1}; A_{t+1}, h_{a,t+1})$ . Equation (28) is very standard and characterizes the effect of shifting one unit of today's output from consumption to investment. Today's marginal change in utility should equal the expected discounted marginal change in tomorrow's wealth with respect to tomorrow's capital stock. Equation (29) considers the shifting of a marginal unit of human capital from the goods production sector to the schooling sector, or vice versa. The condition states that the marginal change in goods production due to this shifting, weighted by the expected shadow price of physical capital, should equal the marginal change in the schooling sector weighted by the expected shadow price of human capital. Using the envelope property of the optimal decision rules:

$$c_t^* = c(k_t, h_t; A_t, h_{a,t}) \quad \text{and} \quad u_t^* = u(k_t, h_t; A_t, h_{a,t}), \quad (30)$$

leads us to the following envelope conditions:

$$\frac{\partial V_t}{\partial k_t} = \beta E_t \left[ \frac{\partial V_{t+1}}{\partial k_{t+1}} \right] \frac{\alpha \tau_r y_t}{k_t}, \quad \text{and} \quad \frac{\partial V_t}{\partial h_t} = \beta E_t \left[ \frac{\partial V_{t+1}}{\partial h_{t+1}} \right] B.$$

These conditions together with the above first-order necessary conditions along the optimal consumption path (28) and for the optimal allocation of human capital (29) imply the following Euler equations:

$$\frac{1}{c_t} = \beta E_t \left[ \frac{1}{c_{t+1}} \frac{\tau_r \alpha y_{t+1}}{k_{t+1}} \right], \quad (31)$$

$$u_t = \left( \frac{E_t \left[ \frac{1}{c_{t+1}} \frac{\alpha \tau_r y_{t+1}}{k_{t+1}} \right] A_t}{E_t \left[ \frac{1}{c_{t+1}} \frac{y_{t+1}}{u_{t+1} h_{t+1}} \right] B} \right)^{\frac{1}{\alpha}} \frac{k_t h_{a,t}^{\frac{\alpha}{\alpha-1}}}{h_t}. \quad (32)$$

The two Euler equations (31) and (32) are necessary for a policy to attain the optimum. Together with the following transversality conditions they are also sufficient:

$$\lim_{T \rightarrow \infty} \beta^T E_0 \left[ \frac{1}{c_T} \frac{\tau_r \alpha y_T}{k_T} k_T \right] = 0 \quad \text{and} \quad \lim_{T \rightarrow \infty} \beta^T E_0 \left[ \frac{1}{c_T} \frac{\tau_w (1 - \alpha) y_T}{u_T h_T} h_T \right] = 0. \quad (33)$$

Note that the first fraction in both conditions is the derivative of the utility function and the second fraction is the derivative of the goods sector production function with respect to the inputs of physical and human capital. To be more precise, the last derivative is taken with respect to the fraction of human capital that is allocated to the goods sector, i.e.,  $u_t h_t$ . These derivatives are multiplied by the respective state variable. The transversality conditions tell us that the expected discounted marginal utility of an additional unit of the capital stocks in the “last period” is equal to zero. These requirements rule out that the agent plays Ponzi games.

#### 4.2 The Representative Agent's Optimal Decisions

Again, the guess-and-verify method can be applied to show that the following function satisfies the Bellman equation (27) and the first-order necessary conditions (28) and (29) simultaneously:

$$V(k_t, h_t; h_{a,t}) = \varphi + \varphi_B \ln B + \varphi_A \ln A_t + \varphi_k \ln k_t + \varphi_h \ln h_t + \varphi_{h_a} \ln h_{a,t}, \quad (34)$$

where the  $\varphi_i$ 's, with  $i \in \{k, h, h_a, B, A\}$ , are defined as follows:

$$\begin{aligned} \varphi_k &:= \frac{\alpha}{1 - \alpha\beta}, & \varphi_h &:= \frac{1 - \alpha}{(1 - \beta)(1 - \alpha\beta)}, \\ \varphi_{h_a} &:= \frac{\gamma}{(1 - \beta)(1 - \alpha\beta)}, & \varphi_B &:= \frac{(1 - \alpha + \gamma)\beta}{(1 - \beta)^2(1 - \alpha\beta)}, \\ \varphi_A &:= \frac{1}{(1 - \rho\beta)(1 - \alpha\beta)}. \end{aligned}$$

Here, the constant is given by:

$$\begin{aligned} \varphi &:= \frac{\ln[1 - \alpha\beta]}{1 - \beta} + \frac{(1 - \alpha) \ln[1 - \beta]}{(1 - \beta)(1 - \alpha\beta)} + \frac{\alpha\beta \ln \alpha}{(1 - \beta)(1 - \alpha\beta)} \\ &+ \frac{(1 - \alpha\beta + \gamma)\beta \ln \beta}{(1 - \beta)^2(1 - \alpha\beta)} + \frac{(1 - \alpha) \ln[\tau_w]}{(1 - \alpha\beta)(1 - \beta)} \\ &- \frac{(1 - \alpha + \beta\gamma) \ln[\beta + \tau_w(1 - \beta)]}{(1 - \alpha\beta)(1 - \beta)^2}. \end{aligned}$$

The optimal controls implied by  $V$  are the following:

$$c_t = (1 - \alpha\beta)y_t \quad \text{and} \quad u_t = \frac{\tau_w(1 - \beta)}{\beta + \tau_w(1 - \beta)}. \quad (35)$$

If the government sets  $\tau_w$  and  $\tau_r$  equal to 1, these results correspond exactly to the non-cooperative equilibrium.  $V$  implies a constant allocation of human capital between the two production sectors, i.e., the evolution of the average stock of human capital  $h_a$  does not enter the first-order necessary condition for  $u_t$  in (35). Hence, there is no linkage between the representative agent's decision and the economy-wide average decision. Therefore the solution strategy of determining the evolution of the agent's stock of capital and then exploiting the symmetry condition (6) is equivalent to the strategy of finding a fixed point where the representative agent's policy rules coincide with the economy-wide average decisions. Hence, the equation:

$$h_{a,t+1} = B \frac{\beta}{\beta + \tau_w(1 - \beta)} h_{a,t}$$

determines the path of the economy-wide average level of human capital in the decentralized economy. Together with the agent's optimal controls, this result implies that the Euler equations (31) and (32) and the transversality conditions (33) are met.

Before we turn to the detailed examination of the government's policy we have a brief note on the uniqueness of the two solutions to the Bellman equations (14) and (27). Logarithmic preferences imply that we deal with unbounded returns. Stokey and Lucas (1989) show that in this case, solving the Bellman equations is necessary but not sufficient for the functions to be the value functions (cf. chap. 9). However, they also present an iterative way to determine the value functions. The basic idea is the following: (1) Find an upper bound for the value function. (2) Apply Theorem 9.12 from their textbook. Since (2) is straightforward, I will briefly discuss (1). For simplicity suppose  $A_t = A$  and  $\gamma = 0$ . Given  $k_0$  and  $h_0$ , consumption in period 0 is limited by  $Ak_0^\alpha h_0^{1-\alpha}$ , in period 1 consumption is limited by  $A(Ak_0^\alpha h_0^{1-\alpha})^\alpha (B\beta h_0)^{1-\alpha}$ , etc. Note that the first limit requires  $u_0 = 1$  while the second assumes  $u_0 = 0$ . However, this is no problem since we are only looking for an upper limit of life-time utility. After some algebra it is indeed possible to show that  $\sum_{t=0}^{\infty} \beta^t \ln c_t$

converges for the consumption stream above even if we assume  $\gamma > 0$ . Allowing for stochastic disturbances means to use the expectations operator throughout. Since the resulting value functions (21) and (34) differ from the well-known closed-form solution in McCallum (1989) only by the logarithmic terms in human capital, I omit the presentation of the complete proofs in the paper at hand. The interested reader is referred to Appendix B in Bethmann (2006).

### 4.3 The Government's Optimal Policy

The government wants to reach the social planner's solution by taxing, respectively subsidizing the agent's factor compensations. Note that the absence of  $\tau_r$  in the first-order conditions (35) implies that the planner's solution can be reached by simply requiring  $u_t$  to be socially optimal. On the other hand, Assumption (10) requires that the state has to ensure that its budget is balanced in each period. These two requirements lead us to the following two conditions:

$$\frac{(1 - \alpha)(1 - \beta)}{1 - \alpha + \beta\gamma} = \frac{\tau_w(1 - \beta)}{\beta + \tau_w(1 - \beta)} \quad \text{and} \quad (\tau_r - 1)\alpha = (1 - \tau_w)(1 - \alpha). \quad (36)$$

This implies the following optimal values of  $\tau_w$  and  $\tau_r$ :

$$\tau_w = \frac{1 - \alpha}{1 - \alpha + \gamma} \quad \text{and} \quad \tau_r = \frac{\alpha - \alpha^2 + \gamma}{(1 - \alpha + \gamma)\alpha}. \quad (37)$$

Hence, the compensation of work effort is reduced by the ratio of the output elasticities of human capital in the decentralized and centralized economy, i.e., by the ratios of private and social marginal returns of human capital in goods production. These tax revenues are then distributed to the owners of the physical capital stock. This result is very intuitive and leads to an increased goods production in the decentralized economy.<sup>3</sup> Note that the planner's and the representative agent's value functions are identical if we apply the above condition on  $\tau_w$  and use (6).

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3 Uhlig and Yanagawa (1996) present an opposite result. They study a two period OLG model with endogenous growth where lower labor income taxes correspond to higher capital income taxes. Thereby the young generation is able to generate higher savings which in turn lead to higher growth.

Among others, Garcia-Castrillo and Sanso (2000) and Gomez (2003) also determine optimal policies that replicate the central planner's solution of the Uzawa-Lucas model. Both studies focus on a deterministic setup in continuous time. The former shows that the social optimum can be reached by subsidizing the time spend with schooling, i.e.,  $1 - u_t$ . Furthermore, it is shown that these subsidies can be financed by a continuum of combinations of flat-rate taxes on labor income and lump-sum taxes. Gomez (2003) proves that the government can solely rely on the flat-rate labor income tax and does not need to introduce lump-sum taxation if it subsidizes foregone labor income, i.e.,  $w_t(1 - u_t)h_t$ , instead of subsidizing time. Both studies find that income stemming from returns on physical capital should be untaxed. As pointed out by an anonymous referee the differences between these results and our finding in (37) rests on the assumption that the capital stocks fully depreciate in each period. In this case the government is able to set socially optimal incentives for schooling while meeting simultaneously the balanced budget condition (10). With incomplete or no depreciation the saving/consumption decision is influenced by the tax on physical capital income and the idea of Gomez (2003) to tax human capital income in order to subsidize foregone human capital income is a way to deal with that problem.

In the last two sections, we have studied both the centralized as well as the decentralized version of the Uzawa-Lucas Model of Endogenous Growth. We have found the two value functions and shown that the implied controls satisfy the Euler equations and the transversality conditions. In the next section, we show that the solutions are saddle path stable and determine their time-series implications.

## **5 Stability Properties and Time Series Implications of the Solutions**

In this section, the aim is twofold. First, we want to determine the stability properties of the two solutions. Second, we want to characterize the time-series properties. Lucas (1988) points out that the growth rate of human capital along the balanced growth path is given by  $B(1 - u_{bgp})$ . Furthermore, he shows that the growth rates of physical capital, output, and consumption are  $\frac{1-\alpha+\gamma}{1-\alpha}$  times the growth rate of human capital. Mulligan and Sala-i-Martin (1993) use this property in order to introduce transformed state-like and control-like variables. These new variables remain constant along the balanced growth path. This stationarity together with



the fact that the number of state variables is reduced by one makes the analysis of growth models much simpler. Benhabib and Perli (1994) follow this strategy and define the state-like variable  $x_t$  and the control-like variable  $q_t$ . In principle, we apply the same strategy and argue that the DOP is homogeneous in the initial conditions  $h_0 = h_{a,0}$  and  $k_0$ . However, as in Bethmann and Reiß (2003), our consideration leads us to a different definition of the control-like variable  $q_t$ .<sup>4</sup>

Because of the homogeneity in the initial conditions of the central planner's DOP, we define the state-like variable  $x_t$  and the control-like variable  $q_t$  as follows:

$$x_t := \frac{k_t}{h_t^{1-\alpha+\gamma}} \quad \text{and} \quad q_t := \frac{c_t}{h_t^{1-\alpha+\gamma}}.$$

Similarly, the representative agent's DOP is homogeneous in its initial conditions. The only difference is that we must distinguish between the representative agent's stock of human capital  $h$  and the economy-wide average stock of human capital  $h_a$ . Therefore we redefine the state-like variable  $x_t$  and the control-like variable  $q_t$  as:

$$x_t := \frac{k_t}{h_t h_{a,t}^{\frac{\gamma}{1-\alpha}}} \quad \text{and} \quad q_t := \frac{c_t}{h_t h_{a,t}^{\frac{\gamma}{1-\alpha}}}.$$

The state-like variable can be interpreted as a weighted ratio of the two capital stocks. In the deterministic model, the state-like variable  $x_t$  remains constant along the balanced growth path. Here, we consider a stochastic model such that  $x_t$  may sometimes be above or below its balanced growth path where the dynamics stem from the physical capital stock since our solutions imply that  $h$  evolves deterministically both in the centralized as well as in the decentralized case. In Section 4, we have shown that the government is able to force agents to make socially optimal decisions, i.e., to internalize the external effects stemming from the economy-wide average stock of human capital. Therefore this section focuses on the decentralized case. The representative agent's solution is fully described by the policy rules (35) together with the laws of motion

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<sup>4</sup> In Bethmann and Reiß (2003), we study a continuous time version of the model without depreciation and define  $q$  as we do here. On the other hand, Benhabib and Perli (1994) use  $q = c/k$ .

for  $k_t$ ,  $h_t$ ,  $h_{a,t}$ , and  $A_t$ . Using our results, the dynamics of total factor productivity, of the state-like variable, and of the control-like variable are described by the following equations:

$$\begin{aligned}\ln A_{t+1} &= \rho \ln A_t + \varepsilon_{t+1}, \\ x_{t+1} &= \frac{\alpha \beta u^{1-\alpha}}{B^{\frac{1-\alpha+\gamma}{1-\alpha}} (1-u)^{\frac{1-\alpha+\gamma}{1-\alpha}}} x_t^\alpha A_t, \\ q_t &= \frac{(1-\alpha\beta)u^{1-\alpha}}{B^{\frac{1-\alpha+\gamma}{1-\alpha}} (1-u)^{\frac{1-\alpha+\gamma}{1-\alpha}}} x_t^\alpha A_t.\end{aligned}$$

Taking logarithms and using small letters with a hat in order to indicate this transformation, we arrive at:

$$\begin{aligned}\hat{a}_{t+1} &= \rho \hat{a}_t + \varepsilon_{t+1}, \\ \hat{x}_{t+1} &= \ln \left[ \frac{\alpha \beta u^{1-\alpha}}{B^{\frac{1-\alpha+\gamma}{1-\alpha}} (1-u)^{\frac{1-\alpha+\gamma}{1-\alpha}}} \right] + \alpha \hat{x}_t + \hat{a}_t, \\ \hat{q}_t &= \ln \left[ \frac{(1-\alpha\beta)u^{1-\alpha}}{B^{\frac{1-\alpha+\gamma}{1-\alpha}} (1-u)^{\frac{1-\alpha+\gamma}{1-\alpha}}} \right] + \alpha \hat{x}_t + \hat{a}_t.\end{aligned}$$

The law of motion of total factor productivity is a first-order autoregressive process with stable root  $\rho$ :

$$\hat{a}_t = \frac{\varepsilon_t}{1 - \rho L}.$$

The evolution of the logged state-like variable  $\hat{x}$  is described by a stochastic first-order difference equation with stable root  $\alpha$  and stochastic disturbance  $\hat{a}$ . Hence the logged state-like variable  $\hat{x}$  follows an AR(2) process:

$$\hat{x}_{t+1} = \frac{1}{1-\alpha} \ln \left[ \frac{\alpha \beta u^{1-\alpha}}{B^{\frac{1-\alpha+\gamma}{1-\alpha}} (1-u)^{\frac{1-\alpha+\gamma}{1-\alpha}}} \right] + \frac{\varepsilon_t}{(1-\rho L)(1-\alpha L)},$$

where the constant term on the right-hand side is the unconditional mean of the log state-like variable  $\hat{x}$ . Since the control-like variable  $q_t$  is non-ambiguously determined by  $A_t$  and  $x_t$ , we conclude that the whole system is saddle-path stable. Furthermore, the control-like variable  $q_t$  follows an AR(2) process:

$$\hat{q}_t = \ln[1 - \alpha\beta] + \ln u + \frac{\alpha}{1 - \alpha} \ln \left[ \frac{\alpha\beta}{B^{\frac{1-\alpha+\gamma}{1-\alpha}} (1-u)^{\frac{1-\alpha+\gamma}{1-\alpha}}} \right] + \frac{\varepsilon_{t-1}}{(1 - \rho L)(1 - \alpha L)}.$$

We conclude that the detrended output  $\hat{s}_t := \hat{y}_t - \hat{h}_t - \frac{\gamma}{1-\alpha} \hat{h}_{a,t}$  is also AR(2). Note that  $B(1 - u_{bgp})$  in the decentralized case is equal to  $B \frac{\beta}{\beta + \tau_w(1-\beta)}$ , such that optimal taxation induces a human capital growth rate of  $B\beta \frac{1-\alpha+\gamma}{1-\alpha+\beta\gamma}$ , whereas a laissez-faire policy implies  $B\beta$ , such that the growth rates in the centralized case or in the decentralized case with optimal taxation are indeed higher than in the decentralized economy with suboptimal or no taxation. This concludes the discussion of the time-series implications of our solutions.

## 6 Concluding Remarks

We have proven that the functions (21) and (34) are the value functions of the social planner and of the representative agent, respectively. We can use these functions and the first order necessary conditions along the optimal consumption paths in order to find the optimal level of consumption. The result is the typical consumption rule for the standard *AK* model with logarithmic preferences, Cobb-Douglas technology, and full depreciation of physical capital. It is easy to check that this result does fit the Euler equation in consumption (31). Similarly, we can use (34) and the first order necessary condition for the optimal human capital allocation (29). We find that the optimal way to shift human capital between the two production sectors is to hold  $u_t$  constant, once we have found the optimal allocation. Similar to the consumption rule, it can be shown that this policy rule fulfills the Euler equation (32). Furthermore, the restriction  $u_t \in [0, 1]$  holds. The transversality conditions in (33) ensure that the policy rules (35) of the representative agent are necessary and sufficient for a utility maximizing path. In the centralized case, the optimal stock of human capital employed in the goods sector  $u_t$  is a little bit smaller than in the decentralized case without taxation, although  $u_t \in [0, 1]$  still holds. Hence, the path of human capital in the centralized economy lies above the human capital path in the decentralized economy given the same initial stocks of capital.

Finally, we have shown that the time series properties of the model are similar to those of the standard neoclassical growth model when looking

at the detrended time series. This is due to the fact that the optimal human capital allocation is a constant and thus unaffected by the state variables. As a consequence, the growth rate of human capital is always equal to  $B(1 - u_{b\dot{g}p})$ . Hence, the introduction of the schooling sector does not change the dynamics of the model.

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