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On Input and Output Translation Homotheticity

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This note determines necessary and sufficient conditions for a production technology to exhibit both input translation homotheticity and output translation homotheticity without invoking joint efficiency.

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1 Introduction

Several recent papers have investigated conditions under which technologies can exhibit both input translation homotheticity and output translation homotheticity in the sense of Chambers and Fare (1998) (Briec and Kerstens, 2004; Fukuyama, 2002). Following Färe and Primont (1995), Fukuyama (2002) has referred to this property as inverse translation homotheticity. Inverse translation homothetic (and inverse homothetic) technologies have a number of convenient properties. In particular, as shown by Briec and Kerstens (2004), inverse translation homotheticity is required for the Luenberger-Hicks-Moorsteen and Luen berger dfference-based productivity indicators to coincide. Similarly, Färe, Grosskopf, and Roos (1996) have shown that inverse homotheticity is required for the Hicks-Moorsteen and Malmquist productivity indicators to coincide.

The approach taken in both Briec and Kerstens (2004) and Fukuyama (2002) generalizes an approach developed by Färe and Primont (1995) in their investigation of conditions sufficient for technologies to exhibit both input homotheticity and output homotheticity. The characterizations in all of these papers are restricted to technologies that satisfy a joint efficiency assumption.[1] As it turns out, this joint efficiency assumption rules out very broad classes of well-understood, and frequently used, technologies. In particular, nonparametric data envelopment analysis (DEA) routinely relies on technical representations that may not globally satisfy this joint efficiency assumption. Hence, at both an empirical and theoretical level, a weakening of this restriction seems desirable.

This note determines necessary and sufficient conditions for a production technology to exhibit both input translation homotheticity and output translation homotheticity without invoking joint efficiency. Relaxing this assumption is straightforward. An analogous argument (not pursued in detail here) applied to radial distance functions also allows one to derive necessary and sufficient conditions for a technology to exhibit inverse homotheticity under weaker restrictions than those employed by Färe and Primont (1995). Thus, the results of this note are of interest both for the consideration of inverse translation homotheticity and inverse homotheticity.

2 Notation and Assumptions

Let $x \in \Re_+^N$ denote an input vector and let $y \in \Re_+^M$ be an output vector. We model the technology in terms of the input requirement sets induced by the input correspondence:

$$L(y) = \{x : x \text{ can produce } y\}, \quad y \in \Re_+^M. \tag{1}$$

We assume that $L(y)$ satisfies the standard set of properties discussed by, e.g., Shephard (1970) and Färe (1988): (1) strong disposability of inputs, i.e., $x \geq x' \in L(y) \Rightarrow x \in L(y)$, (2) strong disposability of outputs, i.e., $y \geq y' \Rightarrow L(y) \subseteq L(y')$, and (3) $T = \{(x, y) : x \in L(y)\}$ is closed. The directional input distance function (Luenberger, 1992; Chambers et al., 1996) is defined as

$$\vec{D}_i(y, x; g_x) = \max\{\beta : x - \beta g_x \in L(y)\}, \tag{2}$$

if there exists a β such that $x - \beta g_x \in L(y)$ and $-\infty$ otherwise. Here $g_x \in \Re_+^N, g_x \neq 0$, is the direction in which the input vector x is contracted.

1 The joint efficiency assumptions in Briec and Kerstens (2004) and Fukuyama (2002) are subtly different from those in Fare and Primont (1995). The former impose joint efficiency in terms of directional distance functions while the latter imposes joint efficiency in terms of radial distance functions.

When g_x is the unit vector, the directional input distance function is equivalent to Blackorby and Donaldson's (1980) translation function.

We also have an equivalent, but alternative representation of the technology, in terms of its output set

$$Y(x) = \{y : x \in L(y)\},$$

and the directional output distance function

$$\vec{D}_o(x, y; g_y) = \max\{\beta : y + \beta g_y \in Y(x)\}$$

if there exists a β such that $y + \beta g_y \in Y(x)$ and $-\infty$ otherwise.

It is well-known that

$$\vec{D}_i(y, x; g_x) \geq 0 \Leftrightarrow x \in L(y) \Leftrightarrow y \in Y(x) \Leftrightarrow \vec{D}_o(x, y; g_y) \geq 0.$$

The directional input and output distance functions are nondecreasing in inputs and nonincreasing in outputs. Moreover, they are translatable in the direction of g_x and g_y, respectively,

$$\vec{D}_i(y, x + \alpha g_x; g_x) = \vec{D}_i(y, x; g_x) + \alpha, \alpha \in \Re,$$

$$\vec{D}_o(x, y + \alpha g_y; g_y) = \vec{D}_o(x, y; g_y) - \alpha, \alpha \in \Re.$$

Other properties are discussed in Chambers and Färe (1998).

3 Translation Homotheticity

The technology is *input translation homothetic in the direction of* g_x if $L(y)$ can be written as

$$L(y) = H(y; g_x)g_x + L(1_y), y \in \Re^M_+, \tag{3}$$

where $L(1_y) = \{x : x \text{ can produce } 1_y\}$, where 1_y is a reference output vector, and where $H(y; \cdot)$ is non-decreasing in y and satisfies $H(1_y; g_x) = 0$. Translation homotheticity can be characterized in terms of the directional input distance function as (Chambers and Färe, 1998):

$$\vec{D}_i(y, x; g_x) = \vec{D}_i(1_y, x; g_x) - H(y; g_x). \tag{4}$$

Similarly, the technology is *output translation homothetic in the direction of* g_y if

$$Y(x) = M(x; g_y)g_y + Y(1_x),$$

where 1_x is now a reference input vector, and M is non-decreasing in x and $M(1_x; g_y) = 0$ Alternatively, the technology is translation homothetic if and only if

$$\vec{D}_o(x, y; g_y) = M(x, g_y) + \vec{D}_o(1_x, y; g_y).$$

Briec and Kerstens (2004) and Fukuyama (2002) investigate technologies that are simultaneously input translation homothetic and output translation homothetic. (Färe and Primont, 1995, investigate technologies that are both input and output homothetic.) They restrict attention to technologies which satisfy the following joint efficiency criterion

$$\vec{D}_i(y, x; g_x) = 0 \Leftrightarrow \vec{D}_o(x, y; g_y) = 0.$$

To see how limiting this type of restriction might be consider the following single-output, two-input technology described by the Leontief production function:

$$L(y) = \{x : y \leq \min\{\gamma_1 x_1, \gamma_2 x_2\}\}.$$

If $y = \min\{\gamma_1 x_1, \gamma_2 x_2\}$ with $x_2 > \frac{y}{\gamma_2}$ then $\vec{D}_0(x, y; 1) = 0$ but $\vec{D}_i(y, x; (0, 1)) > 0$ so that the joint efficiency criterion cannot be generally met for this or any other Leontief technology. Moreover, it is quite easy to extend this demonstration to show that non-parametric DEA representations of technologies can fail this joint efficiency criterion. Hence, this restriction rules out very broad classes of technologies that are used in empirical analyses.

Fortunately, the joint efficiency criterion is not required.

Proposition: A technology is both input translation homothetic and output translation homothetic if and only if

$$\vec{D}_i(y, x; g_x) = \vec{D}_i(1_y, x; g_x) + H^*\left(-\vec{D}_0(1_x, y; g_y); g_x\right),$$
$$\vec{D}_0(x, y; g_y) = \vec{D}_0(1_x, y; g_y) + F\left(\vec{D}_i(1_y, x; g_x), g_y\right),$$

with F non-decreasing in its first argument and H^* non-increasing in its first argument.

Proof: Suppose the technology is input translation homothetic. Then its output directional distance function can be written as

$$\begin{aligned}\vec{D}_0(x, y; g_y) &= \max\{\beta : \vec{D}_i(1_y, x; g_x) \geq H(y + \beta g_y; g_x)\}\\ &= \vec{D}_0^H\left(\vec{D}_i(1_y, x; g_x), y; g_y\right),\end{aligned}$$

where

$$\vec{D}_0^H(h, y; g_y) = \max\{\beta : h \geq H(y + \beta g_y; g_x)\}$$

is the directional output distance function for the function H. Because H is non-decreasing in y, $\vec{D}_0^H$ is non-increasing in y and non-decreasing in h. Therefore, the technology is both input translation homothetic and output translation homothetic only if for all (x, y)

$$M(x, g_y) + \vec{D}_0(1_x, y; g_y) = \vec{D}_0^H\left(\vec{D}_i(1_y, x; g_x), y; g_y\right),$$

whence

$$\begin{aligned}M(x, g_y) &= \vec{D}_0^H\left(\vec{D}_i(1_y, x; g_x), y; g_y\right) - \vec{D}_0(1_x, y; g_y)\\ &\equiv F\left(\vec{D}_i(1_y, x; g_x), g_y\right),\end{aligned}$$

after renormalization with F non-decreasing in $\vec{D}_i$. This establishes necessity. To go the other way, suppose that

$$\vec{D}_0(x, y; g_y) = F\left(\vec{D}_i(1_y, x; g_x), g_y\right) + \vec{D}_0(1_x, y; g_y).$$

This technology is obviously output translation homothetic. But it also implies that

$$\begin{aligned}\vec{D}_i(y, x; g_x) &= \max\{\beta : F\left(\vec{D}_i(1_y, x - \beta g_x; g_x), g_y\right) + \vec{D}_0(1_x, y; g_y) \geq 0\}\\ &= \max\{\beta : F\left(\vec{D}_i(1_y, x; g_x) - \beta, g_y\right) + \vec{D}_0(1_x, y; g_y) \geq 0\}\\ &= D^F\left(-\vec{D}_0(1_x, y; g_y), \vec{D}_i(1_y, x; g_x)\right)\\ &= D^F\left(-\vec{D}_0(1_x, y; g_y), 0\right) + \vec{D}_i(1_y, x; g_x),\end{aligned}$$

where

$$D^F(d, z) = \max\{\gamma \in \Re : F(z - \gamma, g_y) \geq d\}$$

is the directional distance function for the function F, the second equality follows by translatability of directional input distance functions, and the last equality follows by the fact that directional distance functions are always translatable. Because $F(z, g_y)$ is non-decreasing in z, $D^F(d, z)$ is non-increasing in d. This form is input translation homothetic and satisfies the form in the statement of the proof for the input distance function. □

4 Conclusion

This note has developed necessary and sufficient conditions for a technology to exhibit input and output translation homotheticity simultaneously using weaker regularity conditions than existing demonstrations. The demonstration here only requires free disposability of inputs and outputs as well as closedness restrictions. Fukuyama (2002), and Briec and Kerstens (2004) both impose the restriction that

$$\vec{D}_0(x, y; g_y) = 0 \Leftrightarrow \vec{D}_i(y, x; g_x) = 0,$$

and then follow a method of proof originally developed by Färe and Primont (1995) in the context of radial input and output distance functions to deduce representations parallel to the forms identified above.[2] Hence, if an output bundle, y, is judged to be efficient in the direction of g_y then the input bundle must also be judged to be efficient in the direction of g_x. Well understood technologies failing this restriction include technologies with either "flat" isoquants or "flat" product transformation curves. Thus, the definition of inverse translation homotheticity essentially due to Fukuyama (2002), and Briec and Kerstens (2004), as does the original result on inverse homotheticity due to Färe and Primont (1995), convolutes efficiency restrictions with homotheticity-type restrictions. The efficiency conditions can be relaxed.

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2 Because stronger regularity conditions are involved in their analysis, they obtain stronger regularity conditions for the resulting directional distance functions than the weaker properties resulting from our weaker assumptions.

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