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ORIGINAL PAPER

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New adiabatic invariants for disturbed non-material volumes

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Abstract This paper investigates Mei symmetry and new adiabatic invariants of the disturbed non-material volumes. A infinitesimal transformation group and the infinitesimal transformations vectors of generators are proposed. The definition of Mei symmetry and the determining equation for the systems are presented. The perturbation to the Mei symmetry and new adiabatic invariants for non-material volumes is employed, two types of Mei adiabatic invariant induced by Mei symmetrical perturbation are obtained. Two theorems on new adiabatic invariants are given, and the corresponding deductions about new exact invariants are derived. An example is given to illustrate the application of the method, the corresponding adiabatic invariants are obtained. The example is verified numerically, and it proofs that the theoretical derivation is correct.

1 Introduction

In the last decade, the field of non-material volumes has become a focal research topic, which possesses profound theoretical significance and important engineering application value [1,2]. Researchers have addressed many fundamental principles of the non-material volumes. Irschik and Holl [3] derived the Lagrange's equation of a non-material volume which instantaneously coincides with some part of a continuous and possibly deformable body. Casetta and Pesce [4] constructed the generalized Hamilton's principle for a non-material volume utilizing Reynolds' transport theorem. They [5] also proposed the inverse problem of Lagrangian mechanics for Meshchersky's equation. Casetta [6] depicted the inverse problem of Lagrangian mechanics for a non-material volume by introducing the method of Darboux, proposed Hamiltonian formalism and a conservation law. Irschik and Holl [7] derived a formulation of Lagrange's equations for non-material volumes, computed local forms and global form of Lagrange's equations in the framework of the Lagrange description of continuum mechanics. Irschik et al. [8] concerned with Lagrange's equations, applied to a deformable body in the presence of rigid body degrees of freedom. Casetta et al. [9] developed the generalization of Noether's theorem for a non-material volume, proposed a Noether conserved quantity and the corresponding Killing equations. Jiang and Xia [10] presented the Lie symmetry of non-material volumes, and obtained four kinds of conserved quantities. Jiang et al. presented the algebraic structure and Poisson brackets [11], Mei symmetry and a new conserved quantity [12], dynamical equation of relative motion [13], conformal invariance [14] and Noether symmetrical perturbation and adiabatic invariants [15] for nonmaterial volumes. However, to the authors' knowledge, the adiabatic invariants induced by Mei symmetrical perturbation of the non-material volumes have not been investigated yet.

Symmetry is a promising approach to seek conserved quantities for dynamical systems, which has profound theoretical significance. In 2000, Mei [16] pioneered a new symmetry, i.e., Mei symmetry which is an invariance

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of dynamical functions under infinitesimal transformations of Lie group. Subsequently, Jiang et al. [17] derived Mei conserved quantities for higher-order nonholonomic systems. Xia and Chen [18,19] promoted the Mei symmetry to different difference systems. Zhang et al. [20] described Mei conserved quantities of generalized Hamilton systems with additional terms. Wang and Xue [21] considered Mei conserved quantities of thin elastic rod. Ding and Fang [22] reported Mei adiabatic invariants caused by Mei symmetry for Nonholonomic controllable dynamical systems. Song and Zhang [23] employed Mei symmetrical perturbation and adiabatic invariant of disturbed El-Nabulsi's fractional Birkhoff system. Luo et al. [24] addressed Mei adiabatic invariants of disturbed fractional generalized Hamiltonian equation. So far, to the authors' best knowledge, there is no Mei symmetrical perturbation analysis [25,26] on non-material volumes. To address the lack of research in this aspect, the present work develops the symmetrical perturbation technique to determine the adiabatic invariant of non-material volumes.

The paper is organized as follows. In Sect. 2, we review the differential equations of the non-material volumes. In Sect. 3, we consider the definition of Mei symmetry for disturbed non-material volumes, obtain the determining equation of Mei symmetry of the non-material volumes. In Sect. 4, we treat Mei symmetrical perturbation and new adiabatic invariants of the non-material volumes. In Sect. 5, we give an example to illustrate the application of the method, obtain two types of new adiabatic invariants of the system under the Mei symmetrical transformations. Conclusions are presented in Sect. 6.

2 The differential equations of the non-material volumes

The Lagrange's equation of non-material volumes is pioneered by Irschik and Holl ([3], p. 243, Eq. (5.6)), which can be given as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_u}{\partial \dot{q}_k}-\frac{\partial T_u}{\partial q_k}-\int\limits_{\partial V_u}\frac{1}{2}\rho\upsilon^2\left(\frac{\partial \boldsymbol{v}}{\partial \dot{q}_k}-\frac{\partial \boldsymbol{u}}{\partial \dot{q}_k}\right)\cdot \boldsymbol{n}d\partial V_u+\int\limits_{\partial V_u}\rho\boldsymbol{v}\frac{\partial \boldsymbol{v}}{\partial \dot{q}_k}(\boldsymbol{v}-\boldsymbol{u})\cdot\boldsymbol{n}d\partial V_u=Q_k+\varepsilon W_k \tag{1}$$

where $T_u{=}T_u\,(\dot{q}_k, q_k, t)$ is the total kinetic energy of the material particles come with non-material volume V_u, q_k represent generalized coordinates, $\boldsymbol{v}$ is the velocity of the material particles, $\boldsymbol{u}$ is the velocity of the fictitious particles, ρ is the volumetric mass density, Q_k is the k-th generalized force apply to the material body, $\boldsymbol{n}$ is the outer normal unit vector at the surface of V_u, ∂V_u depicts the bounding surface of V_u.

Then the motion differential equation of the system (1) can be rewritten as the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_u}{\partial \dot{q}_k}-\frac{\partial T_u}{\partial q_k}=Q_k+Z_k+\varepsilon W_k \tag{2}$$

where

$$Z_k=\int\limits_{\partial V_u}\frac{1}{2}\rho\upsilon^2\left(\frac{\partial \upsilon}{\partial \dot{q}_k}-\frac{\partial u}{\partial \dot{q}_k}\right)\cdot nd\partial V_u-\int\limits_{\partial V_u}\rho\upsilon\frac{\partial \upsilon}{\partial \dot{q}_k}(\upsilon-u)\cdot nd\partial V_u \tag{3}$$

Suppose that the systems (1) are nonsingular, i.e.

$$\Delta=\det\left(\frac{\partial^2 T_u}{\partial \dot{q}_s\partial \dot{q}_k}\right)\neq 0 \tag{4}$$

Expanding Eq. (2), we can obtain all generalized accelerations of the disturbed system as

$$\ddot{q}_s=\alpha_s(t,q,\dot{q})+\varepsilon\frac{\Delta_{sl}}{\Delta}W_l,\,(s,l=1,2,\ldots,n) \tag{5}$$

3 Mei symmetry of the disturbed non-material volumes

Choose the infinitesimal transformations t and q_i of Lie group as the following

$$t^* = t + \Delta t,\ \ q_i^*(t^*) = q_i(t) + \Delta q_i, (i = 1, 2, \ldots, n) \tag{6}$$

and their expanding forms are

$$t^* = t + \varepsilon\tau(t, q_j, \dot{q}_j), q_i^*(t^*) = q_i(t) + \varepsilon\xi_i(t, q_j, \dot{q}_j), \tag{7}$$

where ε is an infinitesimal parameter, and τ and ξ_i represent the infinitesimal generators. Take the infinitesimal generator vector is

$$X^{(0)} = \tau\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} \tag{8}$$

and its first extension vector

$$X^{(1)} = X^{(0)} + (\dot{\xi}_s - \dot{q}_s\dot{\tau})\frac{\partial}{\partial \dot{q}_s} \tag{9}$$

From the definition of the Mei symmetry, we can easily obtain the criterion of the Mei symmetry for the disturbed non-material volumes, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial X^{(1)}(T_u)}{\partial \dot{q}_k} - \frac{\partial X^{(1)}(T_u)}{\partial q_k} = X^{(1)}(Q_k) + X^{(1)}(Z_k) + \varepsilon X^{(1)}(W_k) \tag{10}$$

Definition 1 If the generators of infinitesimal transformations (7) accords with the condition (10), then the invariance is called the Mei symmetry of the disturbed non-material volumes.

Moreover, Eq. (10) is called the determining equation of Mei symmetry of the disturbed non-material volumes.

4 Mei symmetrical perturbation and new adiabatic invariants of non-material volumes

Under the action of small forces of perturbation, the primary symmetries and invariants of the system may vary. Assume that the disturbed generators $\tau(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ and $\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ of infinitesimal transformations are small perturbation on the basis of the generators of symmetrical transformations of an undisturbed system, then we have

$$\tau = \tau^0 + \varepsilon\tau^1 + \varepsilon^2\tau^2 + \cdots,\quad \xi_s = \xi_s^0 + \varepsilon\xi_s^1 + \varepsilon^2\xi_s^2 + \cdots \tag{11}$$

By virtue of Eqs. (9), and (5) can be rewritten as

$$X^{(1)} = \varepsilon^m X_m^{(1)}, (m = 0, 1, 2, \cdots, z) \tag{12}$$

where

$$X_m^{(1)} = \tau^m\frac{\partial}{\partial t} + \xi_s^m\frac{\partial}{\partial q_s} + (\dot{\xi}_s^m - \dot{q}_s\dot{\tau}^m)\frac{\partial}{\partial \dot{q}_s}.$$

Analogously, the gauge function can be expressed as the

$$G_{\mathrm{M}} = G_{\mathrm{M}}^0 + \varepsilon G_{\mathrm{M}}^1 + \varepsilon^2 G_{\mathrm{M}}^2 + \cdots \tag{13}$$

Substituting Eqs. (11) and (12) into Eq. (10), and comparing the coefficients of ε^m on the left side of equality with those on the right side, yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial X_m^{(1)}(T_u)}{\partial \dot{q}_k} - \frac{\partial X_m^{(1)}(T_u)}{\partial q_k} + \frac{\Delta_{sl}}{\Delta}W_l\frac{\partial^2 X_{m-1}^{(1)}(T_u)}{\partial \dot{q}_k\partial \dot{q}_s} = X_m^{(1)}(Q_k) + X_m^{(1)}(Z_k) + X_{m-1}^{(1)}(W_k) \tag{14}$$

Definition 2 If the generators of infinitesimal transformations (7) accords with the condition (14), then the invariance is called the Mei symmetrical perturbation of the disturbed non-material volumes.

In addition, Eq. (14) is called the determining equation of Mei symmetrical perturbation of the disturbed non-material volumes.

For the non-material volumes, the Mei symmetrical perturbation can directly lead to Mei adiabatic invariants. The following theorem gives the condition of the existence of adiabatic invariants directly induced by Mei symmetrical perturbation.

Theorem 1 *For the disturbed non-material volumes* (1)*, if the generators of the infinitesimal transformations* (7) *satisfy the Mei criterion* (14) *and exists a gauge function satisfying the following condition*

$$\sum_{\alpha=0}^{m}\left\{X_{\alpha}^{(1)}\left(T_{u}\right)\left(\dot{\tau}^{m-\alpha}+\frac{\Delta_{kl}}{\Delta}W_{l}\frac{\partial\tau^{m-\alpha-1}}{\partial\dot{q}_{k}}\right)+X_{m-\alpha}^{(1)}\left[X_{\alpha}^{(1)}\left(T_{u}\right)\right]+\frac{\Delta_{kl}}{\Delta}W_{l}\frac{\partial G_{M}^{m-1}}{\partial\dot{q}_{k}}\right.$$
$$\left.+X_{\alpha}^{(1)}\left(Q_{k}+Z_{k}\right)\left(\xi_{s}^{m-\alpha}-\dot{q}_{s}\tau^{m-\alpha}\right)+X_{\alpha-1}^{(1)}\left(W_{k}\right)\left(\xi_{s}^{m-\alpha}-\dot{q}_{s}\tau^{m-\alpha}\right)+\dot{G}_{M}^{m}=0\right. \tag{15}$$

then the system (1) *furnishes the following Mei adiabatic invariants,*

$$I_{Mz}=\sum_{m=0}^{z}\varepsilon^{m}\left\{\sum_{\alpha=0}^{m}\left[X_{\alpha}^{(1)}\left(T_{u}\right)\tau^{m-\alpha}+\frac{\partial X_{\alpha}^{(1)}\left(T_{u}\right)}{\partial\dot{q}_{s}}\left(\xi_{s}^{m-\alpha}-\dot{q}_{s}\tau^{m-\alpha}\right)\right]+G_{M}^{m}\right\} \tag{16}$$

Proof Differentiating Eq. (16) with respect to t, utilizing Eqs. (14) and (15), it yields

$$\begin{aligned}
\frac{\mathrm{d}I_{N\mathrm{z}}}{\mathrm{d}t}&=\sum_{m=0}^{z}\varepsilon^{m}\left\{\sum_{\alpha=0}^{m}\frac{\mathrm{d}}{\mathrm{d}t}\left[X_{\alpha}^{(1)}\left(T_{u}\right)\tau^{m-\alpha}+\frac{\partial X_{\alpha}^{(1)}\left(T_{u}\right)}{\partial\dot{q}_{s}}\left(\xi_{s}^{m-\alpha}-\dot{q}_{s}\tau^{m-\alpha}\right)\right]+\frac{\mathrm{d}}{\mathrm{d}t}G_{\mathrm{M}}^{m}\right\}\\
&=\sum_{m=0}^{z}\varepsilon^{m}\left\{\sum_{\alpha=0}^{m}\left[\frac{\mathrm{d}}{\mathrm{d}t}X_{\alpha}^{(1)}\left(T_{u}\right)\tau^{m-\alpha}+X_{\alpha}^{(1)}\left(T_{u}\right)\frac{d}{dt}\tau^{m-\alpha}+\varepsilon\frac{\Delta_{kl}}{\Delta}W_{l}\frac{\partial X_{\alpha}^{(1)}\left(T_{u}\right)}{\partial\dot{q}_{k}}\tau^{m-\alpha}\right.\right.\\
&\quad+\varepsilon X_{\alpha}^{(1)}\left(T_{u}\right)\frac{\Delta_{kl}}{\Delta}W_{l}\frac{\partial\tau^{m-\alpha}}{\partial\dot{q}_{k}}+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial X_{\alpha}^{(1)}\left(T_{u}\right)}{\partial\dot{q}_{s}}\left(\xi_{s}^{m-\alpha}-\dot{q}_{s}\tau^{m-\alpha}\right)\\
&\quad+\frac{\partial X_{\alpha}^{(1)}\left(T_{u}\right)}{\partial\dot{q}_{s}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\xi_{s}^{m-\alpha}-\dot{q}_{s}\tau^{m-\alpha}\right)+\varepsilon\frac{\Delta_{kl}}{\Delta}W_{l}\frac{\partial^{2}X_{\alpha}^{(1)}\left(T_{u}\right)}{\partial\dot{q}_{k}\partial\dot{q}_{s}}\left(\xi_{s}^{m-\alpha}-\dot{q}_{s}\tau^{m-\alpha}\right)\\
&\quad+\varepsilon\frac{\partial X_{\alpha}^{(1)}\left(T_{u}\right)}{\partial\dot{q}_{s}}\frac{\Delta_{kl}}{\Delta}W_{l}\frac{\partial\left(\xi_{s}^{m-\alpha}-\dot{q}_{s}\tau^{m-\alpha}\right)}{\partial\dot{q}_{k}}+\frac{\mathrm{d}}{\mathrm{d}t}G_{\mathrm{M}}^{m}+\varepsilon\frac{\Delta_{kl}}{\Delta}W_{l}\frac{\partial G_{\mathrm{M}}^{m}}{\partial\dot{q}_{k}}\\
&=\sum_{m=0}^{z}\varepsilon^{m}\left\{\sum_{\alpha=0}^{m}X_{\alpha}^{(1)}\left(T_{u}\right)\frac{\Delta_{kl}}{\Delta}W_{l}\left(\varepsilon\frac{\partial\tau^{m-\alpha}}{\partial\dot{q}_{k}}-\frac{\partial\tau^{m-\alpha-1}}{\partial\dot{q}_{k}}\right)\right.\\
&\quad+\frac{\Delta_{kl}}{\Delta}W_{l}\left(\varepsilon\frac{\partial G_{\mathrm{M}}^{m}}{\partial\dot{q}_{k}}-\frac{\partial G_{\mathrm{M}}^{m-1}}{\partial\dot{q}_{k}}\right)+\sum_{\alpha=0}^{m}\frac{\partial X_{\alpha}^{(1)}\left(T_{u}\right)}{\partial\dot{q}_{s}}\frac{\Delta_{kl}}{\Delta}W_{l}\left(\varepsilon\frac{\partial\xi_{s}^{m-\alpha}}{\partial\dot{q}_{k}}-\frac{\partial\xi_{s}^{m-\alpha-1}}{\partial\dot{q}_{k}}\right)\\
&\quad-\sum_{\alpha=0}^{m}\dot{q}_{s}\frac{\partial X_{\alpha}^{(1)}\left(T_{u}\right)}{\partial\dot{q}_{s}}\frac{\Delta_{kl}}{\Delta}W_{l}\left(\varepsilon\frac{\partial\tau^{m-\alpha}}{\partial\dot{q}_{k}}-\frac{\partial\tau^{m-\alpha-1}}{\partial\dot{q}_{k}}\right)\\
&\quad\left.+\sum_{\alpha=0}^{m}\frac{\Delta_{kl}}{\Delta}W_{l}\left(\xi_{s}^{m-\alpha}-\dot{q}_{s}\tau^{m-\alpha}\right)\left(\varepsilon\frac{\partial^{2}X_{\alpha}^{(1)}\left(T_{u}\right)}{\partial\dot{q}_{k}\partial\dot{q}_{s}}-\frac{\partial^{2}X_{\alpha-1}^{(1)}\left(T_{u}\right)}{\partial\dot{q}_{k}\partial\dot{q}_{s}}\right)\right\}\\
&=\varepsilon^{z+1}\left\{\sum_{\alpha=0}^{z}\left[X_{\alpha}^{(1)}\left(T_{u}\right)\frac{\Delta_{kl}}{\Delta}W_{l}\frac{\partial\tau^{z-\alpha}}{\partial\dot{q}_{k}}+\frac{\Delta_{kl}}{\Delta}W_{l}\left(\xi_{s}^{z-\alpha}-\dot{q}_{s}\tau^{z-\alpha}\right)\frac{\partial^{2}X_{\alpha}^{(1)}\left(T_{u}\right)}{\partial\dot{q}_{k}\partial\dot{q}_{s}}\right.\right.\\
&\quad\left.\left.+\frac{\partial X_{\alpha}^{(1)}\left(T_{u}\right)}{\partial\dot{q}_{s}}\frac{\Delta_{kl}}{\Delta}W_{l}\frac{\partial\xi_{s}^{z-\alpha}}{\partial\dot{q}_{k}}-\dot{q}_{s}X_{\alpha}^{(1)}\left(T_{u}\right)\frac{\Delta_{kl}}{\Delta}W_{l}\frac{\partial\tau^{z-\alpha}}{\partial\dot{q}_{k}}\right]+\frac{\Delta_{kl}}{\Delta}W_{l}\frac{\partial G_{\mathrm{M}}^{z}}{\partial\dot{q}_{k}}\right\}
\end{aligned} \tag{17}$$

This shows that $\mathrm{d}I_{\mathrm{M}z}/\mathrm{d}t$ is in direct proportion to ε^{z+1}, so I_z is a z-th-order adiabatic invariant of the disturbed non-material volumes (1). □

Theorem 2 *For the disturbed non-material volumes* (1)*, if the generators of the infinitesimal transformations* (7) *satisfy the Mei identities* (14) *and exists a new gauge function satisfying the following condition*

$$\frac{\partial X_m^{(1)}(T_u)}{\partial t} - X_m^{(1)}(Q_k + Z_k)\dot{q}_s - X_{m-1}^{(1)}(W_k)\dot{q}_s + \dot{G}_F^m + \frac{\Delta_{kl}}{\Delta}W_l\frac{\partial G_F^{m-1}}{\partial \dot{q}_k} = 0 \tag{18}$$

then the system (1) *furnishes the following new adiabatic invariants,*

$$I_{Fz} = \sum_{m=0}^{z}\varepsilon^m\left[X_m^{(1)}(T_u) - \frac{\partial X_m^{(1)}(T_u)}{\partial \dot{q}_s}\dot{q}_s + G_F^m\right] \tag{19}$$

Proof Differentiating Eq. (19) with respect to t, utilizing Eqs. (14) and (18), it yields

$$\begin{aligned}
\frac{\mathrm{d}I_{\mathrm{F}z}}{\mathrm{d}t} &= \sum_{m=0}^{z}\varepsilon^m\Bigg\{\frac{\mathrm{d}}{\mathrm{d}t}X_m^{(1)}(T_u) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial X_m^{(1)}(T_u)}{\partial \dot{q}_s}\dot{q}_s - \frac{\partial X_m^{(1)}(T_u)}{\partial \dot{q}_s}\alpha_s \\
&\quad - \varepsilon\dot{q}_s\frac{\Delta_{kl}}{\Delta}W_l\frac{\partial^2 X_m^{(1)}(T_u)}{\partial \dot{q}_k\partial \dot{q}_s} + \frac{d}{dt}G_{\mathrm{F}}^m + \varepsilon\frac{\Delta_{kl}}{\Delta}W_l\frac{\partial G_{\mathrm{F}}^m}{\partial \dot{q}_k}\Bigg\} \\
&= \sum_{m=0}^{z}\varepsilon^m\Bigg[\frac{\partial X_m^{(1)}(T_u)}{\partial t} + \frac{\partial X_m^{(1)}(T_u)}{\partial q_s}\dot{q}_s + \frac{\partial X_m^{(1)}(T_u)}{\partial \dot{q}_s}\alpha_s \\
&\quad + \varepsilon\frac{\Delta_{kl}}{\Delta}W_l\frac{\partial X_m^{(1)}(T_u)}{\partial \dot{q}_s} - \frac{d}{dt}\frac{\partial X_m^{(1)}(T_u)}{\partial \dot{q}_s}\dot{q}_s - \frac{\partial X_m^{(1)}(T_u)}{\partial \dot{q}_s}\alpha_s \\
&\quad - \varepsilon\dot{q}_s\frac{\Delta_{kl}}{\Delta}W_l\frac{\partial^2 X_m^{(1)}(T_u)}{\partial \dot{q}_k\partial \dot{q}_s} - \varepsilon\frac{\Delta_{kl}}{\Delta}W_l\frac{\partial X_m^{(1)}(T_u)}{\partial \dot{q}_s} + X_{m-1}^{(1)}(W_k)\dot{q}_s \\
&\quad + \varepsilon\frac{\Delta_{kl}}{\Delta}W_l\frac{\partial G_{\mathrm{F}}^m}{\partial \dot{q}_k} - \frac{\partial X_m^{(1)}(T_u)}{\partial t} + X_m^{(1)}(Q_k + Z_k)\dot{q}_s - \frac{\Delta_{kl}}{\Delta}W_l\frac{\partial G_{\mathrm{F}}^{m-1}}{\partial \dot{q}_k}\Bigg] \\
&= \sum_{m=0}^{z}\varepsilon^m\Bigg\{\left(\varepsilon\frac{\Delta_{kl}}{\Delta}W_l\frac{\partial G_{\mathrm{F}}^m}{\partial \dot{q}_k} - \frac{\Delta_{kl}}{\Delta}W_l\frac{\partial G_{\mathrm{F}}^{m-1}}{\partial \dot{q}_k}\right) - \dot{q}_s\frac{\Delta_{kl}}{\Delta}W_l\left[\varepsilon\frac{\partial^2 X_m^{(1)}(T_u)}{\partial \dot{q}_k\partial \dot{q}_s} - \frac{\partial^2 X_{m-1}^{(1)}(T_u)}{\partial \dot{q}_k\partial \dot{q}_s}\right]\Bigg\} \\
&= \varepsilon^{z+1}\left[\frac{\Delta_{kl}}{\Delta}W_l\frac{\partial G_{\mathrm{F}}^z}{\partial \dot{q}_k} - \dot{q}_s\frac{\Delta_{kl}}{\Delta}W_l\frac{\partial^2 X_z^{(1)}(T_u)}{\partial \dot{q}_k\partial \dot{q}_s}\right]
\end{aligned} \tag{20}$$

For the disturbed non-material volumes (3) without perturbations, then theorems 1 and 2 give the Mei symmetrical exact invariants of the undisturbed non-material volumes. Moreover, the Mei criterion (14) is reformatted as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial X^{(1)}(T_u)}{\partial \dot{q}_k} - \frac{\partial X^{(1)}(T_u)}{\partial q_k} = X^{(1)}(Q_k) + X^{(1)}(Z_k) \tag{21}$$

□

Deduction 1 For the undisturbed non-material volumes, if the generators of the infinitesimal transformations (7) satisfy the degraded criterion (21), and exists a gauge function satisfying the degraded condition

$$X_0^{(1)}(T_u)\dot{\tau} + X_0^{(1)}\left[X_0^{(1)}(T_u)\right] + X_0^{(1)}(Q_k + Z_k)\left(\xi_s^0 - \dot{q}_s\tau^0\right) + \dot{G} = 0 \tag{22}$$

then the undisturbed system furnishes the exact invariant,

$$I_{\mathrm{M}0} = X_0^{(1)}(T_u)\tau^0 + \frac{\partial X_0^{(1)}(T_u)}{\partial \dot{q}_s}\left(\xi_s^0 - \dot{q}_s\tau^0\right) + G_{\mathrm{M}}^0 = \text{const.} \tag{23}$$

It's worth noting that the exact invariant (23) has been reported and proved by Jiang et al. [12].

Deduction 2 For the undisturbed non-material volumes, if there exists a new gauge function satisfying the following condition

$$\frac{\partial X_0^{(1)}(T_u)}{\partial t} - X_0^{(1)}(Q_k + Z_k)\dot{q}_s + \dot{G}_{\mathrm{F}}^0 = 0 \tag{24}$$

then the undisturbed system furnishes another new exact invariant,

$$I_{\mathrm{F0}} = X_0^{(1)}(T_u) - \frac{\partial X_0^{(1)}(T_u)}{\partial \dot{q}_s}\dot{q}_s + G_{\mathrm{F}}^0 = \mathrm{const.} \tag{25}$$

Proof Differentiating Eq. (25) with respect to t, utilizing Eqs. (21) and (24), it yields

$$\begin{aligned} \frac{\mathrm{d}I_{\mathrm{F0}}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}X_0^{(1)}(T_u) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial X_0^{(1)}(T_u)}{\partial \dot{q}_s}\dot{q}_s - \frac{\partial X_0^{(1)}(T_u)}{\partial \dot{q}_s}\alpha_s + \frac{\mathrm{d}}{\mathrm{d}t}G_{\mathrm{F}}^0 \\ &= \frac{\partial X_0^{(1)}(T_u)}{\partial t} + \frac{\partial X_0^{(1)}(T_u)}{\partial q_s}\dot{q}_s + \frac{\partial X_0^{(1)}(T_u)}{\partial \dot{q}_s}\alpha_s - \frac{d}{dt}\frac{\partial X_0^{(1)}(T_u)}{\partial \dot{q}_s}\dot{q}_s \\ &\quad - \frac{\partial X_0^{(1)}(T_u)}{\partial \dot{q}_s}\alpha_s - \frac{\partial X_0^{(1)}(T_u)}{\partial t} + X_0^{(1)}(Q_k + Z_k)\dot{q}_s \\ &= -\left[\frac{d}{dt}\frac{\partial X_0^{(1)}(T_u)}{\partial \dot{q}_s} - \frac{\partial X_0^{(1)}(T_u)}{\partial q_s} - X_0^{(1)}(Q_k + Z_k)\right]\dot{q}_s \\ &= 0 \end{aligned} \tag{26}$$

Consequently, I_{F0} is a conserved quantity. □

5 Application

In order to express the applicability of the proposed scheme, the rocket motion example will be investigated in the following.

Casetta ([6], p. 11, Eq. (80)) considered the Lagrangian equations of rocket motion which can be expressed as

$$m_u(t)\ddot{q} + \dot{m}_u(t)v_{\mathrm{rel}}(t) - Q(t) = 0 \tag{27}$$

where q represents the displacement of the rocket, $m_u(t)$ is the total mass, $v_{\mathrm{rel}}(t)$ shows the relative velocity of the rocket with respect to the expelled propellant, $Q(t)$ is the external force, and the total of kinetic energy is

$$T_u = \frac{1}{2}m_u(t)\dot{q}^2 \tag{28}$$

and the force of the flow of mass through the control surface is

$$\begin{aligned} Z_k &= \int\limits_{\partial V_u}\frac{1}{2}\rho\upsilon^2\left(\frac{\partial\upsilon}{\partial\dot{q}_k} - \frac{\partial u}{\partial\dot{q}_k}\right)\cdot n d\partial V_u - \int\limits_{\partial V_u}\rho\upsilon\frac{\partial\upsilon}{\partial\dot{q}_k}(\upsilon - u)\cdot n d\partial V_u \\ &= -(\dot{q} - v_{\mathrm{rel}}(t))\dot{m}_u(t) \end{aligned} \tag{29}$$

Suppose the system is disturbed by small quantities $\varepsilon W_k(t, q, \dot{q}) = \varepsilon m_u(t)q$, then the governing equation of motion of the disturbed system is

$$m_u(t)\ddot{q} + \dot{m}_u(t)v_{\mathrm{rel}}(t) - Q(t) = \varepsilon m_u(t)q \tag{30}$$

In order to find a conservation law, we attempt to solve particular cases of system parameters, let us suppose that $v_{\mathrm{rel}}(t)$=0 and $Q(t) = -m_u(t)q$, then a simplify equation can be achieved as

$$\ddot{q} + q = \varepsilon q \tag{31}$$

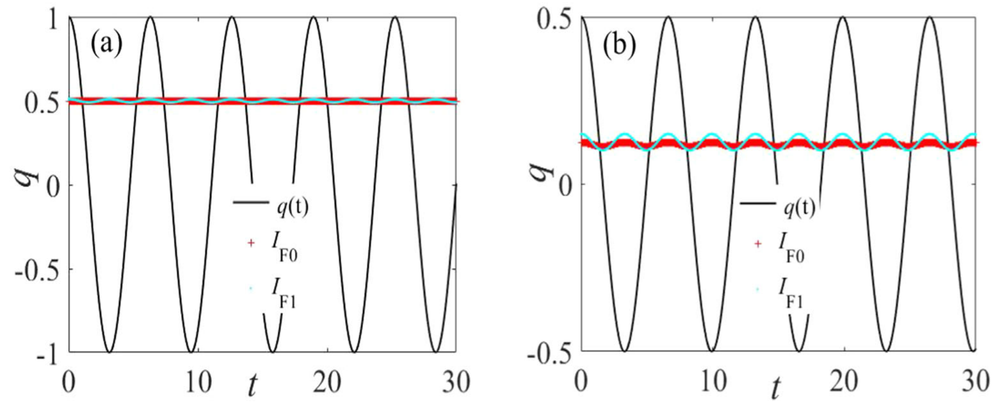

Fig. 1 The numerical results of the system (31) for different system parameter conditions **a** $q(0) = 1$, $\varepsilon = 0.01$, **b** $q(0) = 0.5$, $\varepsilon = 0.1$

Choose infinitesimal generators as

$$\tau^0 = 0.5t, {\xi_1}^0 = q, \tau^1 = 0.5t, {\xi_1}^1 = q \tag{32}$$

It is easy to check that the generators (32) satisfy the determining Eq. (10) of Mei symmetrical perturbation. Moreover, by using structure Eqs. (18) and (24), one has gauge functions

$$G_F^0 = -\frac{1}{2}q^2, G_F^1 = q^2 \tag{33}$$

By introducing Deduction 2, we can get a Mei exact invariant of the undisturbed system

$$I_F = \frac{1}{2}\dot{q}^2 + \frac{1}{2}q^2 \tag{34}$$

Taking account of Theorem 2, we can get a first-order Mei adiabatic invariant of the disturbed system

$$I_{F1} = \frac{1}{2}\dot{q}^2 + \frac{1}{2}q^2 + \varepsilon\left(-\frac{1}{2}\dot{q}^2 + q^2\right) \tag{35}$$

In order to deeply demonstrate the validity of the above results, Fig. 1 plots the time history of displacement, Mei exact invariant of the undisturbed system and Mei adiabatic invariant of the disturbed system under different system parameter conditions, where the black line represents the displacement response, the red plus denotes the Mei exact invariant, the cyan dot defines the Mei adiabatic invariant. From Fig. 1, we see that Mei exact invariant (34) is a conserved quantity, the first-order Mei adiabatic invariant (35) is a small perturbed quantity based on conserved quantity. We also observe from Fig. 1 that a good degree of correlation is obtained both in numerical calculation and theoretical derivation. In addition, as the disturbance parameter ε increases, the disturbance of adiabatic invariant becomes more obvious.

6 Conclusion

This paper has addressed the problem of Mei symmetrical perturbation and new adiabatic invariants for non-material volumes formulated within the framework of Ritzs method. By virtue of the infinitesimal transformation of Lie group, the definition of Mei symmetrical perturbation and determining equation for the system are presented. The corresponding structure equation is derived, two types of Mei adiabatic invariant induced by Mei symmetrical perturbation are obtained. To test the proposed formulation, we have addressed an example to illustrate the application of the method, and the corresponding adiabatic invariant is obtained under the Mei symmetrical perturbation transformations. The example is verified numerically, and it proofs that the theoretical derivation is correct.

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