Acta Mech 234, 4315–4321 (2023)
https://doi.org/10.1007/s00707-023-03587-2

ORIGINAL PAPER

Yu. A. Chirkunov · E. O. Pikmullina · V. V. Molodin · A. A. Lapidus · D. V. Topchiy

Submodels of the model of dynamic deformation of a transversally isotropic thermoelastic medium for solving the problems of horizontal crack formation at 3D printing

Received: 13 August 2022 / Revised: 22 March 2023 / Accepted: 24 April 2023 / Published online: 9 June 2023

Abstract A three-dimensional dynamic model of a transversally isotropic thermoelastic medium satisfying the Gassman conditions is used to study the problem of the formation of horizontal cracks in a printed array arising during 3D printing during its cooling and maturation. For a system of second-order differential equations defining this model, a group foliation is performed with respect to the pseudogroup admitted by this system. As a result, a system of first-order differential equations equivalent to the equations of the original model is obtained. This system consists of an automorphic system and a resolving system. With a help of a resolving system, a system (R) of first order equations for the components of the displacement vector and temperature is obtained. The system (R) contains fewer additional functions than the union of the automorphic and resolving systems of the performed group stratification. Two submodels are found, which are determined by exact invariant solutions of the system (R). The first submodel describes a wave traveling inside of the layer along one of the coordinate axes. The second submodel describes a plane wave traveling inside of the layer. For these submodels, heating modes are indicated that do not lead to the formation of horizontal cracks in the product, and modes in which horizontal cracks will necessarily appear.

1 Introduction

The most acute problem in 3D printing of enclosing structures of buildings and structures is the formation of a large number of horizontal cracks in the printed array during its cooling and maturation [1–8]. The study of the physical processes of thermoelastic deformation of the material occurring in this case becomes a central issue in the development of 3D printing technology. In this paper, to study this problem, we took a three-dimensional dynamic model of a transversely isotropic thermoelastic medium [9–11], which is used to describe the thermoelastic deformation of materials with anisotropy of elastic properties with the predominant direction of anisotropy. These materials include layered and composite materials used in construction, engineering, aircraft and shipbuilding, and also mountain ranges and glaciers.

For this model, the components of the strain tensor are determined by the relations

$$\varepsilon_{11} = \frac{\partial u^1}{\partial x'}, \varepsilon_{22} = \frac{\partial u^2}{\partial y'}, \varepsilon_{33} = \frac{\partial u^3}{\partial z'}, 2\varepsilon_{12} = \frac{\partial u^1}{\partial y'} + \frac{\partial u^2}{\partial x'}, 2\varepsilon_{13} = \frac{\partial u^1}{\partial z'} + \frac{\partial u^3}{\partial x'},$$

Yu. A. Chirkunov (✉) · E. O. Pikmullina · V. V. Molodin
Novosibirsk State University of Architecture and Civil Engineering (Sibstrin), 113 Leningradskaya St., Novosibirsk, Russia 630008
e-mail: chr101@mail.ru

A. A. Lapidus · D. V. Topchiy
National Research Moscow State University of Civil Engineering (NRU MGSU), 26 Yaroslavskoe Shosse, Moscow, Russia 129337

$$2\varepsilon_{23} = \frac{\partial u^2}{\partial z'} + \frac{\partial u^3}{\partial y'}, \tag{1}$$

where $u^1 = u^1(x', y', z', t'), u^2 = u^2(x', y', z', t'), u^3 = u^3(x', y', z', t')$ are the displacement vector components. Here $\boldsymbol{x'} = (x', y', z')$ are the coordinates of a point in space, t' is a time.

Hooke's law for a transversely isotropic thermoelastic medium has the form

$$\sigma_{11} = (\lambda + 2\mu)\varepsilon_{11} + \lambda\varepsilon_{22} + \lambda'\varepsilon_{33} - \beta\theta', \sigma_{22} = \lambda\varepsilon_{11} + (\lambda + 2\mu)\varepsilon_{22} + \lambda'\varepsilon_{33} - \beta\theta',$$
$$\sigma_{33} = \lambda'(\varepsilon_{11} + \varepsilon_{22}) + (\lambda' + 2\mu')\varepsilon_{33} - \beta'\theta', \sigma_{12} = 2\mu\varepsilon_{12}, \sigma_{13} = 2G'\varepsilon_{13}, \sigma_{23} = 2G'\varepsilon_{23}, \tag{2}$$

where σ_{ij} are the stress tensor components, $\lambda, \mu, \lambda', \mu', G'$ are the coefficients of a transversely isotropic medium, β, β' are the thermal expansion coefficients, and $\theta' = \theta'(x', y', z', t')$ is the temperature.

The system of the equations for dynamic deformation of a transversally isotropic thermoelastic medium, due to (1), (2), is written in the form [9–11]

$$\rho u^1_{t't'} = (\lambda + 2\mu)u^1_{x'x'} + \mu u^1_{y'y'} + G'u^1_{z'z'} + (\lambda + \mu)u^2_{x'y'} + (\lambda' + G')u^3_{x'z'} - \beta\theta'_{x'},$$
$$\rho u^2_{t't'} = (\lambda + \mu)u^1_{x'y'} + \mu u^2_{x'x'} + (\lambda + 2\mu)u^2_{y'y'} + G'u^2_{z'z'} + (\lambda' + G')u^3_{y'z'} - \beta\theta'_{y'},$$
$$\rho u^3_{t't'} = (\lambda' + G')u^1_{x'z'} + (\lambda' + G')u^2_{y'z'} + G'u^3_{x'x'} + G'u^3_{y'y'} + (\lambda' + 2\mu')u^3_{z'z'} - \beta'\theta'_{z'}. \tag{3}$$

This system is closed by the heat equation

$$c_0\theta'_{t'} + \theta'_0\left(\beta u^1_{t'x'} + \beta u^2_{t'y'} + \beta' u^3_{t'z'}\right) = k\theta'_{x'x'} + k\theta'_{y'y'} + k'\theta'_{z'z'}, \tag{4}$$

where k, k' are the thermal conductivity coefficients, c_0 is the heat capacity coefficient and ρ is density, θ_0 is a temperature of a state in which deformations and stresses are absent.

We will consider a thermodynamic process for which $\beta = \beta' = 1, k = k', \theta_0 = c_0$. In this case, with a help of the variables

$$t = \frac{c_0}{k\rho}t', \boldsymbol{x} = \frac{c_0}{k\sqrt{\rho}}\boldsymbol{x'}, \boldsymbol{x} = (x, y, z), \theta = \frac{k\sqrt{\rho}}{c_0}\theta'$$

Equations (3), (4) take the form

$$u^1_{tt} = (\lambda + 2\mu)u^1_{xx} + \mu u^1_{yy} + G'u^1_{zz} + (\lambda + \mu)u^2_{xy} + (\lambda' + G')u^3_{xz} - \theta_x,$$
$$u^2_{tt} = (\lambda + \mu)u^1_{xy} + \mu u^2_{xx} + (\lambda + 2\mu)u^2_{yy} + G'u^2_{xy} + (\lambda' + G')u^3_{yz} - \theta_y,$$
$$u^3_{tt} = (\lambda' + G')u^1_{xz} + (\lambda' + G')u^2_{yz} + G'u^3_{xx} + G'u^3_{yy} + (\lambda' + 2\mu')u^3_{zz} - \theta_z,$$
$$\theta_t + u^1_{tx} + u^2_{ty} + u^3_{tz} = \theta_{xx} + \theta_{yy} + \theta_{zz}. \tag{5}$$

Let $h = h(\boldsymbol{x})$ is any harmonic function. The functions $\theta = const, \boldsymbol{u} = \nabla h(x, y, z)$ are a solution of system (5) only in the following four cases: (1) $\lambda' = \lambda + 2\mu - 2G', \mu' = G'$; (2) $h_{zzz} = 0, \lambda' = \lambda + 2\mu - 2G'$, (3) $h_{zz} = const$; (4) $h_{zz} = f(z), \mu' = G'$.

Only in the first case there are no restrictions on the harmonic function. Restrictions are imposed only on the elastic modules. In this case, Gassmann condition [12, 13], which is widely used in geophysics to study wave propagation in transversely isotropic elastic media, is certainly satisfied.

At the elastic modules $\lambda' = \lambda + 2\mu - 2G', \mu' = G'$ system (5) takes the form

$$\boldsymbol{u}_{tt} - (\lambda + 2\mu)\nabla \mathrm{div}\, \boldsymbol{u} + \mathrm{rot}\,(M\ \mathrm{rot}\, \boldsymbol{u}) + \nabla\theta = \boldsymbol{0}, \theta_t + div\, \boldsymbol{u}_t = \Delta\theta, \tag{6}$$

where matrix $M = \mathrm{diag}(G', G', \mu), \nabla = \partial_x, \Delta = \nabla^2$.

Mathematical models of many phenomena of the real world and, of course, models of continuum mechanics are formulated in the form of differential equations. Group analysis of these models is one of the most effective ways to obtain quantitative and qualitative characteristics of the physical processes they describe.

The symmetry (group) analysis of differential equations is based on the theory of Lie groups and Lie algebras. It has shown well in finding classes of exact solutions of the differential equations. The fundamental beginning was made by the Norwegian mathematician Sophus Lie (1842–1899). In Russia, the method developed as a theory of dimensions. The theory of Lie groups for a long time remained aloof from possible applications to differential equations of mathematical physics. However, since the middle of the last century, studies carried out by academician L. V. Ovsiannikov, his students and followers have shown that the methods of Lie group theory are an effective way to study the structure of the solution set of differential equations (for example, see [14–16] and references given there). At present, this mathematical direction is called the group or symmetry analysis of differential equations.

We will study the system (6) by methods of group analysis of differential equations.

2 Group foliation

It is easy to check that among the operators admitted by system (6) there are operators of the form:

$$X = \nabla h(\boldsymbol{x}) \cdot \partial_{\boldsymbol{u}} + \partial_\theta, \tag{7}$$

where $h = h(\boldsymbol{x})$ is any harmonic function. Operators (7) generate a Lie pseudogroup of transformations, which is an infinite subgroup of the main group of system (6). First-order differential invariants for this pseudogroup are determined by the formulas

$$I_1 = t,\ I_2 = \boldsymbol{x},\ I_3 = \operatorname{rot}\boldsymbol{u},\ \ I_4 = \operatorname{div}\boldsymbol{u},\ I_5 = u^1 - h_x\theta,\ I_6 = u^2 - h_y\theta,\ I_7 = u^3 - h_z\theta,$$
$$I_8 = \boldsymbol{u}_t,\ I_9 = \theta_t,\ I_{10} = \theta_x,\ I_{11} = \theta_y,\ I_{12} = \theta_z.$$

The structure of these invariants makes it possible to carry out a group foliation of system (6). That is, to represent this system as an equivalent union of two first-order systems: automorphic and resolving.

An automorphic system has a form

$$\boldsymbol{u}_t = \boldsymbol{v}(t,\boldsymbol{x}),\ \operatorname{rot}\boldsymbol{u} = \boldsymbol{\omega}(t,\boldsymbol{x}),\ \operatorname{div}\boldsymbol{u} = q(t,\boldsymbol{x}),\ \nabla\theta = \boldsymbol{\varphi}(t,\boldsymbol{x}),\ \theta_t = \psi(t,\boldsymbol{x}), \tag{8}$$

where $\boldsymbol{v} = \left(v^1, v^2, v^3\right)$, $\boldsymbol{\omega} = \left(\omega^1, \omega^2, \omega^3\right)$, $\boldsymbol{\varphi} = \left(\varphi^1, \varphi^2, \varphi^3\right)$, q, ψ are additional functions of the variables $t, \boldsymbol{x}$, determined from the resolving system.

A resolving system has a form

$$\boldsymbol{v}_t = (\lambda + 2\mu)\nabla q - \operatorname{rot}(M\boldsymbol{\omega}) - \boldsymbol{\varphi},\ \operatorname{div}(\boldsymbol{\varphi} - \boldsymbol{v}) = \psi,\ \boldsymbol{\omega}_t = \operatorname{rot}\boldsymbol{v},\ q_t = \operatorname{div}\boldsymbol{v},$$
$$\boldsymbol{\varphi}_t = \nabla\psi,\ \operatorname{div}\boldsymbol{\omega} = 0,\ \operatorname{rot}\boldsymbol{\varphi} = \boldsymbol{0}. \tag{9}$$

The pseudogroup generated by operators (7) acts transitively on an automorphic system. That is, if one solution of system (8) is known, then its general solution is obtained as a result of the action of a pseudogroup on this solution. This pseudogroup acts identically on an automorphic system.

System (9) contains subsystems that define widely known models of continuum mechanics and mathematical physics. Namely [17]:

(1) For $\boldsymbol{\omega} = \boldsymbol{const}$ system (5) coincides with the system of the equations for irrotational acoustics with thermodynamics:

$$\boldsymbol{v}_t = (\lambda + 2\mu)\nabla q - \boldsymbol{\varphi},\ \boldsymbol{\varphi}_t = \operatorname{grad}\psi,\ q_t = \operatorname{div}\boldsymbol{v},\ \operatorname{div}(\boldsymbol{\varphi} - \boldsymbol{v}) = \psi,\ \operatorname{rot}\ \boldsymbol{v} = \boldsymbol{0},\ \operatorname{rot}\boldsymbol{\varphi} = \boldsymbol{0}.$$

(2) For $q = const$ system (5) coincides with Maxwell's equations in an inhomogeneous medium with thermodynamics:

$$\boldsymbol{v}_t = -\operatorname{rot}(M\boldsymbol{\omega}) - \boldsymbol{\varphi},\ \boldsymbol{\omega}_t = \operatorname{rot}\boldsymbol{v},\ \boldsymbol{\varphi}_t = \nabla\psi,\ \operatorname{div}\boldsymbol{v} = 0,\ \operatorname{div}\boldsymbol{\omega} = 0,\ \operatorname{div}\boldsymbol{\varphi} = \psi,\ \operatorname{rot}\boldsymbol{\varphi} = \boldsymbol{0}.$$

(3) For $\psi = const$ we get a system with a stationary thermodynamics.

3 System (R)

In resolving system (9) we denote a function $\boldsymbol{v}$ by $\boldsymbol{u}$ and function ψ by θ. In this case, system (9) takes the form

$$\boldsymbol{u}_t = (\lambda + 2\mu)\nabla q - \mathrm{rot}(M\boldsymbol{\omega}) - \boldsymbol{\varphi},\ \mathrm{div}\,(\boldsymbol{\varphi} - \boldsymbol{u}) = \psi,\ \boldsymbol{\omega}_t = \mathrm{rot}\,\boldsymbol{u},\ q_t = \mathrm{div}\,\boldsymbol{u},$$
$$\boldsymbol{\varphi}_t = \nabla\theta,\ \mathrm{div}\,\boldsymbol{\omega} = 0,\ \mathrm{rot}\,\boldsymbol{\varphi} = \mathbf{0}. \tag{R}$$

Differentiation with respect to t of the first and second equations of system (R) in view of the remaining equations of this system gives system (6). Consequently, for any solution $(\boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{\varphi}, q, \theta)$ of system (R) a pair of the functions is the solution of system (6).

System (R) contains fewer additional functions than system (8), (9), namely: $\boldsymbol{\omega}$, $\boldsymbol{\varphi}$ and q. Therefore, to describe a dynamic deformation of a transversally isotropic thermoelastic medium we use system (R).

Operator, admitted by system (R), is sought in the form

$$X = \xi^0(t, \boldsymbol{x}, \boldsymbol{p})\partial_t + \boldsymbol{\xi}(t, \boldsymbol{x}, \boldsymbol{p}) \cdot \partial_{\boldsymbol{x}} + \boldsymbol{\eta}(t, \boldsymbol{x}, \boldsymbol{p}) \cdot \partial_{\boldsymbol{p}},$$

where ξ^0, $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ are unknown functions of its variables, and $\boldsymbol{p} = (\boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{\varphi}, q, \theta)$.

The condition of invariance [14–16] of the manifold, defined by system (9), with respect to this operator, and the splitting by parameter derivatives gives a overdetermined system of the determining equations. After the second continuation, this system is reduced to involution and can be integrated. The solution of this system shows that a main Lie group of transformations of system (9) (factor group by the standard invariant subgroup associated with the linearity and homogeneity of this system) is generated by the operators

$$X_1 = \partial_t,\ X_2 = \partial_x,\ X_3 = \partial_y,\ X_4 = \partial_z.$$

4 Invariant submodels of rank 1

We consider dynamic invariant submodels of rank 1. All these submodels are determined by the invariant solutions of rank 1 and describe the traveling waves.

In the subsequent formulas: $\alpha, \beta, \gamma, c_k (k = 1, 2, \dots, 7)$ are arbitrary real constants.

4.1 Invariant $\langle X_1 + \alpha X_2, X_3, X_4\rangle$- submodel $(\alpha \neq 0)$

This submodel is defined by the solution of the form

$$\boldsymbol{u} = \boldsymbol{U}(\xi) = \left(U^1, U^2, U^3\right), \boldsymbol{\omega} = \boldsymbol{\Omega}(\xi) = \left(\Omega^1, \Omega^2, \Omega^3\right), \boldsymbol{\varphi} = \boldsymbol{\Phi}(\xi) = \left(\Phi^1, \Phi^2, \Phi^3\right),$$
$$\theta = \Theta(\xi), q = Q(\xi), \xi = \alpha t - x. \tag{10}$$

Submodel (10) describes a wave traveling inside of the layer along the Ox axis.

Substitution (10) into system (R) gives the factor system

$$\alpha U^1_\xi + (\lambda + 2\mu)Q_\xi + \Phi^1 = 0, \alpha U^2_\xi + \mu\Omega^3_\xi + \Phi^2 = 0, \alpha U^3_\xi - G'\Omega^2_\xi + \Phi^3 = 0, U^1_\xi = \Phi^1_\xi + \Theta,$$
$$\alpha\Phi^1_\xi = -\Theta_\xi, U^1_\xi = -\alpha Q_\xi, U^2_\xi = -\alpha\Omega^3_\xi, U^3_\xi = -\alpha\Omega^2_\xi, \Omega^1_\xi = 0, \Phi^3_\xi = \Phi^2_\xi = 0. \tag{11}$$

From (11) it follows that the characteristic of this wave is determined from the relation.

$$Q_{\xi\xi} = c_1 \exp\left(\alpha\left(\alpha^2 - 1 - (\lambda + 2\mu)\right)\xi\right).$$

If $\alpha^2 \neq 1 + (\lambda + 2\mu)$, then the components of the displacement vector and the temperature are determined by the formulas

$$U^1 = -\frac{c_1}{\alpha\left(\alpha^2 - 1 - (\lambda + 2\mu)\right)^2} \exp\left(\alpha\left(\alpha^2 - 1 - (\lambda + 2\mu)\right)\xi\right) + c_4\xi + c_5,$$
$$U^2 = -\frac{\alpha_2}{\mu + \alpha^2}\xi + c_6, U^3 = -\frac{\alpha c_3}{G' + \alpha^2}\xi +_7,$$

$$\Theta = -\frac{{}_1\left(1 + \left(\alpha^2 - (\lambda + 2\mu)\right)\left(\alpha^2 - 1 - (\lambda + 2\mu)\right)\right)}{\left(\alpha^2 - 1 - (\lambda + 2\mu)\right)} \exp\left(\alpha\left(\alpha^2 - 1 - (\lambda + 2\mu)\right)\xi\right) - c_4. \tag{12}$$

For $c_2 = c_3 = c_4 = 0, c_1 < 0, \alpha > 0$ and $\left(\alpha^2 - (\lambda + 2\mu)\right) < 0$ displacement U^1 and temperature Θ decrease exponentially with a time. This means that under such conditions, horizontal cracks will not appear in the product. And at $\left(\alpha^2 - 1 - (\lambda + 2\mu)\right) > 0$ displacement U^1 and temperature Θ exponentially increase with a time. This means that under such conditions, horizontal cracks will definitely appear.

If $\alpha^2 = 1 + (\lambda + 2\mu)$, then the components of the displacement vector and the temperature are determined by the formulas

$$\begin{aligned} &U^1 = -\frac{\alpha_1}{2}\xi^2 + c_4\xi + c_5, U^2 = -\frac{\alpha_2}{\mu + \alpha^2}\xi + c_6, U^3 = -\frac{\alpha c_3}{G' + \alpha^2}\xi + c_7, \\ &\Theta = -\alpha c_1 \xi - \frac{2c_1}{\alpha} + c_4. \end{aligned} \tag{13}$$

Foe $c_2 = c_3 = c_4 = 0, \alpha c_1 < 0$, displacement U^1 increases according to the quadratic law with a time. A temperature Θ increases according linearly with a time. This means that under such conditions, horizontal cracks will definitely appear.

4.2 Invariant $\langle X_1 + \alpha X_3, X_2 + \beta X_3, X_4\rangle$- submodel ($\alpha\beta \neq 0$)

This submodel is defined by the solution of the form

$$\begin{aligned} &\boldsymbol{u} = \boldsymbol{U}(\xi) = \left(U^1, U^2, U^3\right), \boldsymbol{\omega} = \boldsymbol{\Omega}(\xi) = \left(\Omega^1, \Omega^2, \Omega^3\right), \boldsymbol{\varphi} = \boldsymbol{\Phi}(\xi) = \left(\Phi^1, \Phi^2, \Phi^3\right), \\ &\theta = \Theta(\xi), q = Q(\xi), \xi = \alpha t + \beta x - y. \end{aligned} \tag{14}$$

Submodel (14) describes a plane wave traveling inside of the layer.

Substitution (14) into system (R) gives the factor system

$$\begin{aligned} &\alpha U^1_\xi = \beta(\lambda + 2\mu)Q_\xi + \mu\Omega^3_\xi - \Phi^1 = 0, \alpha U^2_\xi = -(\lambda + 2\mu)Q_\xi + \mu\beta\Omega^3_\xi - \Phi^2, \\ &\alpha U^3_\xi = -G'\left(\beta\Omega^2_\xi + \Omega^1_\xi\right) - \Phi^3, \beta\left(\Phi^1_\xi - U^1_\xi\right) - \left(\Phi^2_\xi - U^2_\xi\right), \alpha\Omega^1_\xi = -U^3_\xi, \alpha\Omega^2_\xi = -\beta U^3_\xi, \\ &\alpha\Omega^3_\xi = \beta U^2_\xi + U^1_\xi, \alpha Q_\xi = \beta U^1_\xi - U^2_\xi, \alpha\Phi^1_\xi = \beta\Theta_\xi, \alpha\Phi^2_\xi = -\Theta_\xi, \alpha\Phi^3_\xi = 0_\xi, \beta\Omega^1_\xi - \Omega^2_\xi, \\ &\Phi^3_\xi = 0, \beta\Phi^2_\xi + \Phi^1_\xi = 0. \end{aligned} \tag{15}$$

From (15) it follows that the components of the displacement vector and temperature are the solution of the following system of the equations

$$\begin{aligned} &\left(\beta^2(\lambda + 2\mu) + \mu - \alpha^2\right)U^1_\xi - \beta(\lambda + \mu)U^2_\xi = \frac{\alpha\beta}{\beta^2 + 1}\left(\beta U^1 - U^2\right) + c_1, \\ &-\beta(\lambda + \mu)U^1_\xi + \left((\lambda + 2\mu) + \mu\beta^2 - \alpha^2\right)U^2_\xi = \frac{\alpha}{\beta^2 + 1}\left(-\beta U^1 + U^2\right) + c_2, \\ &\left(G'\left(\beta^2 + 1\right) - \alpha^2\right)U^3_\xi = c_3, \Theta = -\frac{\alpha}{\beta^2 + 1}\left(-\beta U^1 + U^2\right) + c_4. \end{aligned} \tag{16}$$

If $\left(G'\left(\beta^2 + 1\right) - \alpha^2\right) = 0$, then it follows from the third equation of this system that $c_3 = 0$, and the third component of the displacement vector is an arbitrary function. Therefore, submodel (14) has a physical meaning only under the condition

$$\left(G'\left(\beta^2 + 1\right) - \alpha^2\right) \neq 0. \tag{17}$$

In this case a third component of the displacement vector is defined by the formula

$$U^3 = \frac{c_3\xi}{\left(G'\left(\beta^2 + 1\right) - \alpha^2\right)} + c_4. \tag{18}$$

The system consisting of the first and second equations of system (16) at

$$\left(\beta^2+1\right)\mu(\lambda+2\mu)-\alpha^2(\lambda+3\mu)\neq 0. \tag{19}$$

is reduced to the normal form

$$\frac{d}{d\xi}\begin{pmatrix}U^1\\U^2\end{pmatrix}=\frac{\alpha\beta}{\sigma}\begin{pmatrix}\beta a & -a\\ -\beta b & -b\end{pmatrix}\begin{pmatrix}U^1\\U^2\end{pmatrix}+\begin{pmatrix}c_1\\c_2\end{pmatrix}, \tag{20}$$

where:

$$\begin{aligned}&\sigma=\left(\beta^2+1\right)^2\left(\left(\beta^2+1\right)\mu(\lambda+2\mu)-\alpha^2(\lambda+3\mu)\right),\\ &a=(\lambda+2\mu)+\mu\beta^2-\alpha^2-\beta(\lambda+\mu),\, b=\beta(\lambda+\mu)-\beta^2(\lambda+2\mu)-\mu+\alpha^2.\end{aligned} \tag{21}$$

If $\beta a-b=(\beta+1)\left(\mu\left(\beta^2+1\right)-\alpha^2\right)\neq 0$, then it follows from (20) that components U^1, U^2 of the displacement vector and temperature Θ are determined by the formulas

$$\begin{aligned}U^1&=c_5 a\exp\left(\frac{\alpha\beta(\beta a-b)}{\sigma}\xi\right)+\frac{a(c_2-\alpha_1)}{\beta a-b}\xi+\frac{a\sigma(c_2-\alpha c_1)}{\alpha\beta(\beta a-b)^2}+c_6,\\ U^2&=c_5 b\exp\left(\frac{\alpha\beta(\beta a-b)}{\sigma}\xi\right)+\frac{\beta a(c_2-\alpha c_1)}{\beta a-b}\xi+\frac{b\sigma(c_2-\alpha c_1)}{\alpha\beta(\beta a-b)^2}+\beta c_6,\\ \Theta&=\frac{\alpha}{\beta^2+1}\left(c_5(\beta a-b)\exp\left(\frac{\alpha\beta(\beta a-b)}{\sigma}\xi\right)+\frac{\sigma(c_2-\alpha c_1)}{\alpha\beta(\beta a-b)}\right)+c_7.\end{aligned} \tag{22}$$

Let $c_1=c_2=c_3=0$. At $\frac{\beta(\beta a-b)}{\sigma}<0$ components U^1, U^2 of the displacement vector U^1, U^2 and temperature Θ decrease exponentially with a time. This means that under such conditions, horizontal cracks will not appear in the product. And at $\frac{\beta(\beta a-b)}{\sigma}>0$ components U^1, U^2 of the displacement vector U^1, U^2 and temperature Θ exponentially increase with a time. This means that under such conditions, horizontal cracks will definitely appear.

If $\beta a-b=(\beta+1)\left(\mu\left(\beta^2+1\right)-\alpha^2\right)=0$, then it follows from (20) that components U^1, U^2 of the displacement vector and temperature Θ are determined by the formulas

$$\begin{aligned}&U^1=\left(c_1+\frac{\alpha\beta a(\beta c_5-c_6)}{\sigma}\right)\xi+c_5,\, U^2=\left(c_2+\frac{\alpha\beta a(\beta c_5-c_6)}{\sigma}\right)\xi+c_6,\\ &\Theta=\frac{\alpha}{\beta^2+1}((\beta c_1-c_2)\xi+(\beta c_5-c_6))+c_7.\end{aligned} \tag{23}$$

Components U^1, U^2 of the displacement vector and temperature Θ increase according linearly with a time. This means that under such conditions, horizontal cracks will definitely appear.

5 Conclusion and discussion

The most acute problem in 3D printing of enclosing structures of buildings and structures is the formation of a large number of horizontal cracks in the printed array during its cooling and maturation. The study of the physical processes of thermoelastic deformation of the material occurring in this case becomes a central issue in the development of 3D printing technology. In this paper, to study this problem, we took a three-dimensional dynamic model of a transversely isotropic thermoelastic medium, which is used to describe the thermoelastic deformation of materials with anisotropy of elastic properties with the predominant direction of anisotropy.

We studied this model, using the methods of group analysis of differential equations, which is one of the most powerful and effective means of obtaining exact solutions. The group foliation of the system of second-order differential equations that defines this model was carried out over the pseudogroup allowed by this system under the Gassmann conditions widely used in geophysics. The result was a system of differential equations of the first order, equivalent to the equations of the original model. This system consists of an automorphic system and a resolving system. The unknown functions in an automorphic system are physical variables (displacement vector components and temperature) and additional functions. These additional functions are determined from the resolving system. Using the change of variables in the resolving system, system (R) was obtained in which the unknown functions are the components of displacement vector, temperature, and additional functions. System (R) contains a smaller number of additional functions than the union of automorphic and resolving systems.

Two submodels are found, which are determined by exact invariant solutions of system (R). The first submodel is given by Eqs. (12), (13). It describes a wave traveling inside of the layer along one of the coordinate axes. The second submodel is given by Eqs. (16), (18), (19), (21)–(23). It describes a plane wave traveling inside of the layer. For these submodels, heating modes are indicated that do not lead to the formation of horizontal cracks in the product, and modes in which horizontal cracks will certainly appear.

Acknowledgements The study was carried out with the financial support of Industry Consortium “Construction and architecture”.

References

1. Belgrano, F.S., Diegel, O., Pereira, N., Hatti-Kaul, R.: Cell immobilization on 3D-printed matrices: a model study on propionic acid fermentation. Biores. Technol. **249**, 777–782 (2018)
2. Kelly, B.E., Bhattacharya, I., Heidari, H., Shusteff, M., Spadaccini, C.M., Taylor, H.K.: Volumetric additive manufacturing via tomographic reconstruction. Science **363**(6431), 1075–1079 (2019)
3. Han, P.: Additive design and manufacturing of jet engine parts. Engineering **3**(5), 648–652 (2017)
4. Satyanarayana, B., Prakash, K.J.: Component replication using 3D printing technology. Procedia Materials Sci **10**, 263–269 (2015)
5. Wang, X., Jiang, M., Zhou, Z., Gou, J., Hui, D.: 3D printing of polymer matrix composites: a review and prospective. Compos. Part B Eng. **110**, 442–458 (2017)
6. Galante, R., Figueiredo-Pina, C., Serro, A.P.: Additive manufacturing of ceramics for dental applications. Dent. Mater. **35**(6), 825–846 (2019)
7. Afsana, Jain V., Haider N., Jain K.: 3D printing in personalized drug delivery. Current pharmaceutical design. **24**(42), 5062–5071 (2019)
8. Frank K, Yoichiro K, Akira. 3D printing of continuous carbon fibre reinforced thermo-plastic (CFRTP) tensile test specimens. Open J. Compos. Mater. **6,** 18–27 (2016).
9. Novatskiy, V.: Theory of Elasticity. Mir, Moscow (1975)
10. Chernykh, K.F.: Introduction to Anisotropic Elasticity. Nauka, Moscow (1988)
11. Ilyushin, A.A.: Mechanics of Continuous Medium. Publishing House Mosc. University, Moscow (1990)
12. Gassmann, F.: Introdaction to seismic travel time metods in anisotropic media. Pure. Appl. Geophisics. **58**, 1–224 (1964)
13. Carrier, G.F.: Propagation of waves in orthotropic media. Quarter Appl. Math. **2**(2), 160–165 (1946)
14. Ovsiannikov, L.V.: Group Analysis of Differential Equations. Academic Press, New York (1982)
15. Chirkunov, Yu.A., Khabirov, S.V.: Elements of symmetry analysis of differential equations of continuum mechanics. NSTU, Novosibirsk (2012)
16. Chirkunov, Yu.A.: Group analysis of linear and quasi–linear differential equations. NSUEM, Novosibirsk (2007)
17. Young, H.D., Freedman, R.A.: University Physics with Modern Physics with Mastering Physics. (International Edition). Publisher: Pearson.11th edition (2000).