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Carrera unified formulation (CUF) for the shells of revolution. II. Navier close form solutions

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Abstract Here, we present a Navier close form solution method for some type of the higher-order theories for elastic shells of revolution developed using the CUF approach. The higher-order models of elastic shells of revolution are developed using the variational principle of virtual power for 3-D equations of the linear theory of elasticity and generalized series in the coordinates of the shell thickness. The higher-order cylindrical supported on the edges and axisymmetric shells, as well as the shallow spherical shells with rectangular planform, are considered. Numerical calculations were performed using the computer algebra software Mathematica. The resulting equations can be used for theoretical analysis and calculation of the stress–strain state, as well as for modeling thin-walled structures used in science, engineering, and technology. The numerical results can be used as benchmark examples for finite element analysis of the higher-order elastic shells.

1 Introduction

Classical theories of beams, rods, plates, and shells are widely used for theoretical analysis and for design of thin-walled structures. Thin shells have a remarkable ability to withstand significant loads with a minimum thickness. This property of thin shells makes it possible to create lightweight structures from them with good rigidity and strength characteristics, which contributes to the use of shells as structural elements in mechanical, civil, aerospace, marine, biomedical, automotive, or other engineering applications, where light weight is very important. These theories are based on the well-known Kirchhoff–Love and Timoshenko–Mindlin physical hypotheses which significantly simplify final equations. They are very popular among the engineering community because of their relative simplicity and physical clarity. Numerous books and monographs have been written on this subject. For general references among others, we can recommend the following: Ambartsumyan [1], Calladine [2], Flügge [23], Gould [24], Huang [27], Kraus [32], Niordson [35], Novozhilov [36], Pelekh [37], Reddy [42], Ugural [47], Timoshenko and Woinowsky-Krieger [46], Ventsel and Krauthammer [50], Vlasov [51], Wempner and Talaslidis [53]. But unfortunately, the classical theories have some shortcomings and logical contradictions, such as their proximity and inaccuracy and, as result, in some cases, not good agreement with the results obtained using the 3-D theory of elasticity approach and experiments. Therefore, there is a need to develop new, more accurate theories.

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Another approach to the development of the theory of shells consists of expansion of the stress–strain field components into polynomials series in thickness. This approach was first proposed by Cauchy and Poisson for the statistical equilibrium of plates in the early nineteenth century. As significant development and extension on the theory of shells this approach was made by Kil’chevskii [30] and Vekua [49]. Later, it was generalized and applied to anisotropic shells by Khoma [29] and Pelekh et al. [39], to laminate anisotropic plates and shells by Pelekh and Lazko [38], to contact problems for plates and shells by Pelekh and Sukhorol’skii [40] and Pelekh et al. [39], Zozulya [55, 56], to thermoelasticity of plates and shells by Guliaev et al. [26], Pelekh and Sukhorol’skii [40], Zozulya [57]. We, aslo would like to note recently published papers on higher-order rods, plates and shells by Zozulya [58] and its appliation to analysis of the functionaly graded shells by Czekanski and Zozulya [19] and Zozulya and Zhang [62], as well as to micropolar and couple stress theories of plates and shells by Zozulya [59, 60].

The Carrera Unified Formulation (CUF) approach can be considered a generalization of the polynomial expansion method for beams, plates, and shells, including laminated structures and multi-fields loadings. The CUF approach was first presented in Carrera [3]. Hundreds of articles are available on the CUF theory, and its applications published by Carrera and by many other authors. The most complete information related to the CUF can be found in the books of Carrera et al. [7, 9, 11] and references there. In particular, higher-order theory of beams and comparison with classical theories considered in Carrera et al. [9], multi-field analysis of higher-order plates analysis of plates and shells, their modeling, analysis, and applications presented in Carrera et al. [4–6, 10, 11] and the finite element analysis of the higher-order multilayer anisotropic composite plates and shells studied in Carrera et al. [6, 7, 11]. We also would like to note articles on the higher-order theory of the micropolar theory of beams by Carrera and Zozulya [12], of the micropolar theory of plates by Carrera and Zozulya [13, 14], and of the micropolar theory of by shells by Carrera and Zozulya [15, 16] and Zozulya and Carrera [61], where provided more detailed information on the approach used in this publication.

Shells of revolution present a very important class of thin-walled structures, that are widely used as structural elements in civil, mechanical, and biomedical engineering such as pressure vessels, liquid-filled tubes and tanks, roof domes, etc. A middle surface of shells of revolution is generated by rotating a plane curve through 360 degrees about straight line in plane of the curve. Most of the results in analysis modeling and applications have been obtained for shells of cylindrical, conical, and spherical geometries. The theory of shells of revolution is presented in most of the books on the general theory of shells mentioned above. Among others, we would like to note the books by Kovarik [31], Kraus [32], Novozhilov [36], Reddy [43], Vlasov [51] as well as a book by Chernina [18] specially devoted to the shells of revolution. Brief review of the shells of revolution of various shape is presented in the paper of Carrera and Zozulya [17].

Shell theory equations are complicate, especially in the case of higher-order theories. Therefore, the corresponding boundary value problems are usually solved numerically, for example, using finite element methods, see Carrera et al. [7]. Only in some special cases, for example for supported on the edges rectangular plates and cylindrical shells, there are exact Navier solutions. Corresponding examples can be found in the classical book by Woinowsky-Krieger [46] and other books on shell theory mentioned above. Vlasov’s [51] shows that the method of Navier analytical solution can be extended to some classes of the shallow shells, especially to shallow shells with rectangular planform. More results on shallow shells and their analysis are presented in books by Dikovitch [20], Nazarov [34], Rzhantsyn [45], Vajnberg and Zhdan [48] for statistical analysis, and in books by Jin et al. [28], Leissa and Qatu [33], Qatu [41] for vibration analysis. There are many particles on the static and vibration analysis of the shallow shells with rectangular planform, among others we would like to mention Carrera and Giunta [8], Carrera et al. [9], Carrera and Zozulya [10, 14], D’Ottavio et al. [21], Eisenberger and Godoy [22], Grigorenko and Parkhomenko [25], Yan et al. [54].

Here, on the basis of higher-order models developed using the principle of virtual displacements and based on CUF, the Navier close form solutions are presented for the case of the freely supported cylindrical shell and shells of revolution with rectangular planform. Numerical calculations of displacements and stress tensors were performed using the computer algebra software Mathematica, and the results of calculation are presented in the form of graphs and surfers and density plots. The resulting equations can be used for theoretical analysis and numerical calculation of the stress–strain state, as well as for modeling thin-walled structures that are used in science, engineering, and technology. These numerical results can be used as a benchmark example for finite element analysis of the elastic higher-order shells.

2 Carrera unified formulation for a shell of revolution

We consider here the exact Navier solution for shells of revolution of a higher order. The theoretical consideration and the corresponding differential equations were developed in detail and presented in Carrera and Zozulya [17]. Useful and more detailed information can be found in our papers on the higher-order theory of micropolar shells, in particular, the general higher-order theory of micropolar shells considered in Carrera and Zozulya [15], the completely linear expansion case (CLEC) in Zozulya and Carrera [16], the model based on Timoshenko-Mindlin hypothesis in Zozulya and Carrera [61]. Therefore, here, we present only brief information on CUF adapted to the case of shells of revolution.

Let us now consider an elastic shell of revolution occupying a region $V = \Omega \times [-h, h]$ in a 3-D Euclidean space, where Ω is the middle surface and $2h$ is the thickness of the shell. For convenience, we introduce curvilinear coordinates $\mathbf{x} = (x_1, x_2, x_3)$ associated with the middle surface of the shell. Coordinate x_1 is axis of rotation, coordinate x_2 is an angle of rotation, and coordinate x_3 is perpendicular to the middle surface. The coordinates (x_1, x_2) are connected to the principal curvatures k_1 and k_2 of the middle surface of shell. Then, position of any point occupied by material points of the shell in the domain V may be presented by vector $\mathbf{R}(\mathbf{x})$, as

$$\mathbf{R}(\mathbf{x}) = \mathbf{r}(x_1, x_2) + x_3 \mathbf{n}(x_1, x_2) \quad (1)$$

where $\mathbf{r}(x_1, x_2)$ is the position vector of the points located in the middle surface of the shell, and $\mathbf{n}(x_1, x_2)$ is a unit vector normal to the middle surface of the shell.

Since introduced coordinates are orthogonal, Lamé coefficients and their derivatives can be presented in the form

$$\begin{aligned} H_\alpha &= A_\alpha(1 + k_\alpha x_3) \text{ for } \alpha = 1, 2 \text{ and } H_3 = 1, \\ \frac{\partial H_\beta}{\partial x_\alpha} &= \frac{\partial A_\beta}{\partial x_\alpha}(1 + k_\alpha x_3), \quad \frac{\partial H_\beta}{\partial x_3} = k_\beta A_\beta, \quad \frac{\partial H_3}{\partial x_i} = 0 \end{aligned} \quad (2)$$

where $A_\alpha(x_1, x_2) = \sqrt{\frac{\partial \mathbf{r}(x_1, x_2)}{\partial x_\alpha} \cdot \frac{\partial \mathbf{r}(x_1, x_2)}{\partial x_\alpha}}$ are the coefficients of the first quadratic form of a surface, k_α are the principal curvatures, and $\alpha = 1, 2$.

Given that we have considered relatively thin shells, we can make the following assumptions:

$$1 + k_\alpha x_3 \approx 1 \rightarrow H_\alpha \approx A_\alpha, \quad \frac{\partial H_\beta}{\partial x_\alpha} = \frac{\partial A_\beta}{\partial x_\alpha}, \quad \frac{\partial H_\alpha}{\partial x_3} = k_\alpha A_\alpha, \quad \alpha, \beta = 1, 2. \quad (3)$$

Following Carrera and Zozulya [17], we introduce here the vector notation and present the symmetric stress $\boldsymbol{\sigma}(x_1, x_2, x_3)$ and strain $\boldsymbol{\varepsilon}(x_1, x_2, x_3)$ tensors, as well as the displacements $\mathbf{u}(x_1, x_2, x_3)$, body $\mathbf{b}(x_1, x_2, x_3)$ and surface $\mathbf{p}(x_1, x_2, x_3)$ forces vectors in the form

$$\begin{aligned} \boldsymbol{\sigma} &= |\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{13}|^T, \quad \boldsymbol{\varepsilon} = |\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{23}, \varepsilon_{13}|^T, \\ \mathbf{u} &= |u_1, u_2, u_3|^T, \quad \mathbf{b} = |b_{x_1}, b_{x_2}, b_{x_3}|^T, \quad \mathbf{p} = |p_{x_1}, p_{u_2}, p_{u_3}|^T. \end{aligned} \quad (4)$$

Kinematic relations in the theory of elasticity relate the displacement vectors to the strain vector introduced in (4) by the following equations:

$$\boldsymbol{\varepsilon} = \mathbf{D} \cdot \mathbf{u} \quad (5)$$

where \mathbf{D} is a matrix differential operator of the form

$$\mathbf{D}^T = \begin{vmatrix} \frac{1}{A_1} \frac{\partial}{\partial x_1} & \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial x_1} & 0 & \frac{1}{A_2} \frac{\partial}{\partial x_2} - \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} - k_1 * 6pt \\ \frac{1}{A_1 A_2} \frac{\partial A_1}{\partial x_2} & \frac{1}{A_2} \frac{\partial}{\partial x_2} & 0 & \frac{1}{A_1} \frac{\partial}{\partial x_1} - \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial x_1} & \frac{\partial}{\partial x_3} - k_2 & 0 * 6pt \\ k_1 & k_2 & \frac{\partial}{\partial x_3} & 0 & \frac{1}{A_2} \frac{\partial}{\partial x_2} - k_2 & \frac{1}{A_1} \frac{\partial}{\partial x_1} - k_1 \end{vmatrix}, \quad (6)$$

To establish constitutive relations, following Carrera and Zozulya [17], we introduce the potential energy density function. In the case of orthotropic linear elastic media, it can be represented in the matrix form

$$W(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon}^T \cdot \mathbf{C} \cdot \boldsymbol{\varepsilon} \quad (7)$$

where \mathbf{C} is the 6×6 matrix of elasticity moduli of the form

$$\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix}. \quad (8)$$

In the case of isotropic material, the corresponding classical moduli of elasticity presented in (8) have the form

$$C_{11} = C_{22} = C_{33} = \lambda + 2\mu, \quad C_{12} = C_{13} = C_{23} = \lambda, \quad C_{44} = C_{55} = C_{66} = \mu \quad (9)$$

where λ and μ are Lamé constants of classical elasticity.

Taking the derivative of the potential energy density function with respect to the strain $\boldsymbol{\varepsilon}$ tensor and substituting the kinematic relations (5) into the obtained result, the classical stress vector can be presented in the following form:

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}} = \mathbf{C} \cdot \boldsymbol{\varepsilon} = \mathbf{C} \cdot \mathbf{D} \cdot \mathbf{u}. \quad (10)$$

Substituting equations for the matrix of material constants (8), operator (6), and displacement vector into (10), we obtain equations for the classical stress vector expressed as a function of the displacement vector components.

Using the CUF approach, the displacement \mathbf{u} vector, which are functions of curvilinear coordinates (x , φ , r), is represented as series of functions of the coordinate x_3 directed orthogonally to the middle surface of the shell, in the form

$$\mathbf{u}(x_1, x_2, x_3) = \mathbf{F}_{\mathbf{u},\tau}(x_3) \cdot \mathbf{u}_\tau(x_1, x_2), \quad \tau = 1, 2, \dots, M \quad (11)$$

where according to the Einstein notation, the repeated subscript τ indicates summation from 0 to M .

In (11), the basic functions of thickness coordinates $\mathbf{F}_{\mathbf{u},\tau}(x_3)$ and vector of displacement \mathbf{u}_τ have the form

$$\mathbf{F}_{\mathbf{u},\tau}(x_3) = \begin{pmatrix} F_{u_{1,\tau}}(x_3) & 0 & 0 \\ 0 & F_{u_{2,\tau}}(x_3) & 0 \\ 0 & 0 & F_{u_{3,\tau}}(x_3) \end{pmatrix}, \quad \mathbf{u}_\tau(x_1, x_2) = \begin{pmatrix} u_{1,\tau}(x_1, x_2) \\ u_{2,\tau}(x_1, x_2) \\ u_{3,\tau}(x_1, x_2) \end{pmatrix}. \quad (12)$$

In (11), according to the Einstein notation, the repeated subscript τ indicates summation. In general case, the choice of the number M and functions $\mathbf{F}_{\mathbf{u},\tau}(x_3)$ is arbitrary, i.e., for modeling the kinematic field of the shells along their thickness different base functions of any order can be considered. The final equation becomes simple if functions $\mathbf{F}_{\mathbf{u},\tau}$ are polynomials, especially orthogonal polynomials. The expansions coefficients $\mathbf{u}_\tau(x_1, x_2)$ as functions of the coordinates x_1 and x_2 coincided with the middle surface of the shell. The first subscript in basic functions $\mathbf{F}_{\mathbf{u},\tau}$ indicates the component of the displacement vector, and the second index indicates the number of the function in the serial expansion.

Applying matrix differential operators (6) to the displacement vector represented by Eq. (11), one can obtain the strain vector in the form

$$\boldsymbol{\varepsilon} = \mathbf{D}_{\mathbf{u},\tau} \cdot \mathbf{u}_\tau \quad (13)$$

where $\mathbf{D}_{\mathbf{u},\tau}$ is a matrix operator of the form

$$\mathbf{D}_{\mathbf{u},\tau}^T = \begin{pmatrix} \frac{F_{u_{1,\tau}}}{A_1} \frac{\partial}{\partial x_1} & \frac{F_{u_{2,\tau}}}{A_1 A_2} \frac{\partial A_2}{\partial x_1} & 0 & \frac{F_{u_{1,\tau}}}{A_2} \frac{\partial}{\partial x_2} - \frac{F_{u_{2,\tau}}}{A_1 A_2} \frac{\partial A_1}{\partial x_2} & 0 & \frac{\partial F_{u_{1,\tau}}}{\partial x_3} - F_{u_{3,\tau}} k_1 * 6pt \\ \frac{F_{u_{1,\tau}}}{A_1 A_2} \frac{\partial A_1}{\partial x_2} & \frac{F_{u_{2,\tau}}}{A_2} \frac{\partial}{\partial x_2} & 0 & \frac{F_{u_{2,\tau}}}{A_1} \frac{\partial}{\partial x_1} - \frac{F_{u_{1,\tau}}}{A_1 A_2} \frac{\partial A_2}{\partial x_1} & \frac{\partial F_{u_{2,\tau}}}{\partial x_3} - F_{u_{3,\tau}} k_2 & 0 * 6pt \\ F_{u_{1,\tau}} k_1 & F_{u_{2,\tau}} k_2 & \frac{\partial F_{u_{3,\tau}}}{\partial x_3} & 0 & \frac{F_{u_{3,\tau}}}{A_2} \frac{\partial}{\partial x_2} & \frac{F_{u_{1,\tau}}}{A_1} \frac{\partial}{\partial x_1} \end{pmatrix}. \quad (14)$$

Substituting kinematic relations (13) into the generalized Hooke's law (10), the classical force stress vector can be presented as

$$\boldsymbol{\sigma} = \mathbf{C} \cdot \mathbf{D}_{\mathbf{u},\tau} \cdot \mathbf{u}_\tau. \quad (15)$$

Based on the principle of virtual displacements, see Washizu [52] and by implementing mathematical transformations described in detail in the paper of Carrera and Zozulya [17], we obtain differential equations for high-order elastic shells in the form of displacements, which can be represented in matrix form

$$\mathbf{L}_M^G \cdot \mathbf{u}_M^G = \mathbf{b}_M^G \quad (16)$$

where the global matrix operator \mathbf{L}_n^G , the vectors of unknown functions \mathbf{u}_M^G , and the right hand \mathbf{b}_M^G side have the form

$$\mathbf{L}_M^G = \begin{bmatrix} \mathbf{L}_{1,1}^{loc} & \cdots & \mathbf{L}_{1,M}^{loc} \\ \vdots & \ddots & \vdots \\ \mathbf{L}_{M,1}^{loc} & \cdots & \mathbf{L}_{M,M}^{loc} \end{bmatrix}, \quad \mathbf{u}_M^G = \begin{bmatrix} \mathbf{u}_1^{loc} \\ \vdots \\ \mathbf{u}_M^{loc} \end{bmatrix}, \quad \mathbf{b}_M^G = \begin{bmatrix} \mathbf{b}_1^{loc} \\ \vdots \\ \mathbf{b}_M^{loc} \end{bmatrix}. \quad (17)$$

Matrices $\mathbf{L}_{\tau,s}^{loc}$ are the fundamental nucleus of the differential equations of equilibrium of elastic shells of higher orders. They, as well as the vectors of local unknown functions \mathbf{u}_s^{loc} and local expressions for external body and surface loads \mathbf{b}_s^{loc} , have the form

$$\mathbf{L}_{\tau,s}^{loc} = \begin{bmatrix} L_{u_1,u_1}^{\tau,s} & L_{u_1,u_2}^{\tau,s} & L_{u_1,u_3}^{\tau,s} \\ L_{u_2,u_1}^{\tau,s} & L_{u_2,u_2}^{\tau,s} & L_{u_2,u_3}^{\tau,s} \\ L_{u_3,u_1}^{\tau,s} & L_{u_3,u_2}^{\tau,s} & L_{u_3,u_3}^{\tau,s} \end{bmatrix}, \quad \mathbf{u}_s^{loc} = \begin{bmatrix} u_{1,s} \\ u_{2,s} \\ u_{3,s} \end{bmatrix}, \quad \mathbf{b}_s^{loc} = \begin{bmatrix} \tilde{b}_{u_1,\tau} \\ \tilde{b}_{u_2,\tau} \\ \tilde{b}_{u_3,\tau} \end{bmatrix}. \quad (18)$$

The components of the vector external body and surface loads \mathbf{b}_s^{loc} have the form

$$\tilde{b}_{u_x,\tau} = J_{\tau,s}^{u_x,u_x} b_{u_x,s} + J_{\tau}^{u_x} p_{u_x}, \quad \tilde{b}_{u_y,\tau} = J_{\tau,s}^{u_y,u_y} b_{u_y,s} + J_{\tau}^{u_y} p_{u_y}, \quad \tilde{b}_{u_z,\tau} = J_{\tau,s}^{u_z,u_z} b_{u_z,s} + J_{\tau}^{u_z} p_{u_z}, \quad (19)$$

where

$$J_{\tau,\alpha,\beta}^{u_i,u_j} = \int_{-h}^h \frac{dF_{u_i,\tau}(x_3)}{dx_3} \frac{dF_{u_j,s}(x_3)}{dx_3} dx_3, \quad (i, j) \rightarrow (x_1, x_2, x_3). \quad (20)$$

In the same way performing mathematical transformations described as in the paper of Carrera and Zozulya [17], the natural boundary conditions for elastic shells of higher order can be represented in the matrix form

$$\mathbf{B}_M^{N,G} \cdot \mathbf{u}_M^G = \mathbf{p}_M^G \quad (21)$$

where the global matrix operator $\mathbf{B}_M^{N,G}$, the vectors of unknown functions \mathbf{u}_M^G , and the right-hand side \mathbf{p}_M^G have the form

$$\mathbf{B}_M^{N,G} = \begin{bmatrix} \mathbf{B}_{1,1}^{loc} & \cdots & \mathbf{B}_{1,M}^{loc} \\ \vdots & \ddots & \vdots \\ \mathbf{B}_{M,1}^{loc} & \cdots & \mathbf{B}_{M,M}^{loc} \end{bmatrix}, \quad \mathbf{p}_M^G = \begin{bmatrix} \mathbf{p}_1^{loc} \\ \vdots \\ \mathbf{p}_M^{loc} \end{bmatrix}. \quad (22)$$

The matrices $\mathbf{B}_{\tau,s}^{loc}$ are the fundamental nucleus for the natural boundary for higher-order elastic shells, and \mathbf{p}_s^{loc} are the vectors of local expression for the external load applied at the ends of the shells. They can be presented as

$$\mathbf{B}_{\tau,s}^{loc} = \begin{bmatrix} B_{u_1,u_1}^{\tau,s} & B_{u_1,u_2}^{\tau,s} & B_{u_1,u_3}^{\tau,s} \\ B_{u_2,u_1}^{\tau,s} & B_{u_2,u_2}^{\tau,s} & B_{u_2,u_3}^{\tau,s} \\ B_{u_3,u_1}^{\tau,s} & B_{u_3,u_2}^{\tau,s} & B_{u_3,u_3}^{\tau,s} \end{bmatrix}, \quad \mathbf{p}_s^{loc} = \begin{bmatrix} J_{\tau,s}^{u_1,u_1} P_{u_1,s} \\ J_{\tau,s}^{u_2,u_2} P_{u_2,s} \\ J_{\tau,s}^{u_3,u_3} P_{u_3,s} \end{bmatrix}. \quad (23)$$

The essential boundary conditions for the higher-order elastic shells can be represented in matrix form

$$\mathbf{B}_M^{E,G} \cdot \mathbf{u}_M^G \Big|_0^L = \mathbf{u}_M^{0,G} \quad (24)$$

where the global matrix operator $\mathbf{B}_M^{E,G}$ and the vectors of the right-hand side have the form

$$\mathbf{B}_M^{E,G} = \begin{vmatrix} \mathbf{I} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{I} \end{vmatrix}, \quad \mathbf{u}_M^{0,G} = \begin{vmatrix} \mathbf{u}_1^{0,loc} \\ \vdots \\ \mathbf{u}_M^{0,loc} \end{vmatrix} \quad (25)$$

where \mathbf{I} is the identity matrix, and therefore the global matrix operator $\mathbf{B}_M^{E,G}$ is the identity matrix operator.

In the following Sections, based on the here presented differential equations, we consider exact Navier solutions for shells of revolution.

3 Navier analytical solution for a cylindrical shell

Cylindrical shells are shells of revolution. They are formed by revolving around an axis of a straight line that is parallel to it. Models of elastic shells of the cylindrical geometry are very important and often used in the theoretical analysis, as well as applications in sciences and engineering. Therefore, we will develop equations for the higher-order theory of linear elastic cylindrical shells here. Let us introduce cylindrical coordinates where $x_1 = x$, $x_2 = \varphi$ and $x_3 = r$, $r \in [R - h, R + h]$. Coefficients of the first quadratic form of a surface and principal curvatures are equal to $A_1 = 1$, $A_2 = R$, and $k_1 = 0$, $k_2 = \frac{1}{R}$, respectively.

3.1 General 2-D case

Substituting these parameters into Eqs. (16–25), we obtain equations corresponding to the high-order linear elasticity theory for the case of a cylindrical shell.

The resulting differential equations have the same structure as (16), but the local matrices $\mathbf{L}_{\tau,s}^{loc}$ of the fundamental nucleus of differential equations of equilibrium for higher-order cylindrical elastic shells, as well as the vectors of local unknown functions \mathbf{u}_s^{loc} and the local expression for the external body and surface loads \mathbf{b}_s^{loc} , have the form

$$\mathbf{L}_{\tau,s}^{loc} = \begin{vmatrix} L_{u_x,u_x}^{\tau,s} & L_{u_x,u_\varphi}^{\tau,s} & L_{u_x,u_r}^{\tau,s} \\ L_{u_\varphi,u_x}^{\tau,s} & L_{u_\varphi,u_\varphi}^{\tau,s} & L_{u_\varphi,u_r}^{\tau,s} \\ L_{u_r,u_x}^{\tau,s} & L_{u_r,u_\varphi}^{\tau,s} & L_{u_r,u_r}^{\tau,s} \end{vmatrix}, \quad \mathbf{u}_s^{loc} = \begin{vmatrix} u_{x,s} \\ u_{\varphi,s} \\ u_{r,s} \end{vmatrix}, \quad \mathbf{b}_s^{loc} = \begin{vmatrix} \tilde{b}_{u_x,\tau} \\ \tilde{b}_{u_\varphi,\tau} \\ \tilde{b}_{u_r,\tau} \end{vmatrix}. \quad (26)$$

The local fundamental nucleus $\mathbf{L}_{\tau,s}^{loc}$ coefficients can be easily calculated using equations presented in the previous Sections. Their analytical expressions for the higher-order cylindrical elastic shells can be found in Carrera and Zozulya [13, 14].

The local matrices $\mathbf{B}_{\tau,s}^{loc}$ of the fundamental nucleus for natural boundary conditions, as well as \mathbf{p}_s^{loc} the vectors of the local expression for the external load applied to the shell boundary for the higher-order cylindrical shell, have the form

$$\mathbf{B}_{\tau,s}^{loc} = \begin{vmatrix} B_{u_x,u_x}^{\tau,s} & B_{u_x,u_\varphi}^{\tau,s} & B_{u_x,u_r}^{\tau,s} \\ B_{u_\varphi,u_x}^{\tau,s} & B_{u_\varphi,u_\varphi}^{\tau,s} & B_{u_\varphi,u_r}^{\tau,s} \\ B_{u_r,u_x}^{\tau,s} & B_{u_r,u_\varphi}^{\tau,s} & B_{u_r,u_r}^{\tau,s} \end{vmatrix}, \quad \mathbf{p}_s^{loc} = \begin{vmatrix} J_{\tau,s}^{u_x,u_x} P_{u_x,s} \\ J_{\tau,s}^{u_\varphi,u_\varphi} P_{u_\varphi,s} \\ J_{\tau,s}^{u_r,u_r} P_{u_r,s} \end{vmatrix}. \quad (27)$$

The fundamental nucleus $\mathbf{B}_{\tau,s}^{loc}$ coefficients can be easily calculated using the equations presented in the previous Section. Their analytical expressions can be found in Carrera and Zozulya [15].

Boundary value problems for higher-order cylindrical shells, which are formulated as a system of differential Eq. (16) and natural (21) and essential (24) boundary conditions, are usually solved numerically, using finite element methods. For details of the application of finite element methods and additional references, see Carrera et al. [11]. Only in the special case of homogeneous essential boundary conditions (24), such problems can be solved analytically, using Navier method.

Here, we consider Navier analytical solution for a higher-order cylindrical shell, which is based on the CUF. Generalizing the approach developed in Timoshenko and Woinowsky-Krieger [46] for the case of the higher-order cylindrical shell, we represent displacements fields in the form of a double Fourier series expansion as

$$\begin{aligned} u_x(x, \varphi, r) &= F_{u_x, \tau}(r) \sum_{n,m=0}^{\infty} U_{u_x, \tau}^{n,m} \cos\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{\varphi_0}\varphi\right), \\ u_\varphi(x, \varphi, r) &= F_{u_\varphi, \tau}(r) \sum_{n,m=0}^{\infty} U_{u_\varphi, \tau}^{n,m} \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{\varphi_0}\varphi\right), \\ u_r(x, \varphi, r) &= F_{u_r, \tau}(r) \sum_{n,m=0}^{\infty} U_{u_r, \tau}^{n,m} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{\varphi_0}\varphi\right) \end{aligned} \quad (28)$$

where $U_{u_x, \tau}^n$, $U_{u_\varphi, \tau}^n$, $U_{u_r, \tau}^n$ are the expansion coefficients in the Fourier series of functions $u_{x, \tau}(x, y)$, $u_{\varphi, \tau}(x, y)$, $u_{r, \tau}(x, y)$ of the form

$$\begin{aligned} U_{u_x, \tau}^{n,m} &= \frac{4}{L\varphi_0} \int_0^L \cos\left(\frac{n\pi}{L}x\right) \int_0^{\varphi_0} u_{x, \tau}(x, \varphi) \sin\left(\frac{m\pi}{\varphi_0}\varphi\right) d\varphi dx, \\ U_{u_\varphi, \tau}^{n,m} &= \frac{4}{L\varphi_0} \int_0^L \sin\left(\frac{n\pi}{L}x\right) \int_0^{\varphi_0} u_{\varphi, \tau}(x, \varphi) \cos\left(\frac{m\pi}{\varphi_0}\varphi\right) d\varphi dx, \\ U_{u_r, \tau}^{n,m} &= \frac{4}{L\varphi_0} \int_0^L \sin\left(\frac{n\pi}{L}x\right) \int_0^{\varphi_0} u_{r, \tau}(x, \varphi) \sin\left(\frac{m\pi}{\varphi_0}\varphi\right) d\varphi dx. \end{aligned} \quad (29)$$

The vector of the components of the external load is represented here in the form of an expansion into a Fourier series in the form

$$\begin{aligned} \tilde{b}_{u_x, \tau}(x, \varphi) &= \sum_{n,m=0}^{\infty} B_{u_x, \tau}^{n,m} \cos\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{\varphi_0}\varphi\right), \\ \tilde{b}_{u_\varphi, \tau}(x, \varphi) &= \sum_{n,m=0}^{\infty} B_{u_\varphi, \tau}^{n,m} \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{\varphi_0}\varphi\right), \\ \tilde{b}_{u_r, \tau}(x, \varphi) &= \sum_{n,m=0}^{\infty} B_{u_r, \tau}^{n,m} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{\varphi_0}\varphi\right) \end{aligned} \quad (30)$$

where $B_{u_x, \tau}^{n,m}$, $B_{u_\varphi, \tau}^{n,m}$, $B_{u_r, \tau}^{n,m}$ are the expansion coefficients in the Fourier series of functions $\tilde{b}_{u_x, \tau}(x, \varphi)$, $\tilde{b}_{u_\varphi, \tau}(x, \varphi)$, $\tilde{b}_{u_r, \tau}(x, \varphi)$, they have the form

$$\begin{aligned} B_{u_x, \tau}^{n,m} &= \frac{4}{L\varphi_0} \int_0^L \cos\left(\frac{n\pi}{L}x\right) \int_0^{\varphi_0} \tilde{b}_{u_x, \tau}(x, \varphi) \sin\left(\frac{m\pi}{\varphi_0}\varphi\right) d\varphi dx, \\ B_{u_\varphi, \tau}^{n,m} &= \frac{4}{L\varphi_0} \int_0^L \sin\left(\frac{n\pi}{L}x\right) \int_0^{\varphi_0} \tilde{b}_{u_\varphi, \tau}(x, \varphi) \cos\left(\frac{m\pi}{\varphi_0}\varphi\right) d\varphi dx, \\ B_{u_r, \tau}^{n,m} &= \frac{4}{L\varphi_0} \int_0^L \sin\left(\frac{n\pi}{L}x\right) \int_0^{\varphi_0} \tilde{b}_{u_r, \tau}(x, \varphi) \sin\left(\frac{m\pi}{\varphi_0}\varphi\right) d\varphi dx. \end{aligned} \quad (31)$$

Substituting the expansion in the Fourier series (28) for the components of the displacement vector into the system of differential equations for the cylindrical shell of a higher order (16) for arbitrary numbers n and m , we obtain a system of linear algebraic equations for the coefficients of the expansion in the Fourier series for displacements in the form

$$\mathbf{K}_{n,m,M}^G \cdot \mathbf{U}_{n,m,M}^G = \mathbf{B}_{n,m,M}^G \quad (32)$$

where the global matrix operator $\mathbf{K}_{n,m,M}^G$, the vectors of unknown coefficients $\mathbf{U}_{n,m,M}^G$, and the right-hand side $\mathbf{B}_{n,m,M}^G$ have the form

$$\mathbf{K}_{n,m,M}^G = \begin{vmatrix} \mathbf{K}_{1,1}^{n,m,loc} & \dots & \mathbf{K}_{1,M}^{n,m,loc} \\ \vdots & \ddots & \vdots \\ \mathbf{K}_{M,1}^{n,m,loc} & \dots & \mathbf{K}_{M,M}^{n,m,loc} \end{vmatrix}, \quad \mathbf{U}_{n,m,M}^G = \begin{vmatrix} \mathbf{U}_1^{n,m,loc} \\ \vdots \\ \mathbf{U}_M^{n,m,loc} \end{vmatrix}, \quad \mathbf{B}_{n,m,M}^G = \begin{vmatrix} \mathbf{B}_1^{n,m,loc} \\ \vdots \\ \mathbf{B}_M^{n,m,loc} \end{vmatrix}. \quad (33)$$

Matrices $\mathbf{K}_{\tau,s}^{n,m,loc}$ are the fundamental nucleus of algebraic Eq. (33) for the Navier analytical solution for the higher-order elastic cylindrical shell. These matrices and vectors of local unknown functions $\mathbf{U}_s^{n,m,loc}$ and local expression for external body forces $\mathbf{B}_s^{n,m,loc}$ have the form

$$\mathbf{K}_{\tau,s}^{n,m,loc} = \begin{vmatrix} K_{u_x, u_x}^{n,m,\tau,s} & K_{u_x, u_\varphi}^{n,m,\tau,s} & K_{u_x, u_r}^{n,m,\tau,s} \\ K_{u_\varphi, u_x}^{n,m,\tau,s} & K_{u_\varphi, u_\varphi}^{n,m,\tau,s} & K_{u_\varphi, u_r}^{n,m,\tau,s} \\ K_{u_r, u_x}^{n,m,\tau,s} & K_{u_r, u_\varphi}^{n,m,\tau,s} & K_{u_r, u_r}^{n,m,\tau,s} \end{vmatrix}, \quad \mathbf{U}_s^{n,loc} = \begin{vmatrix} U_{u_x, s}^{n,m} \\ U_{u_\varphi, s}^{n,m} \\ U_{u_r, s}^{n,m} \end{vmatrix}, \quad \mathbf{B}_s^{n,loc} = \begin{vmatrix} B_{u_x, \tau}^{n,m} \\ B_{u_\varphi, \tau}^{n,m} \\ B_{u_r, \tau}^{n,m} \end{vmatrix}. \quad (34)$$

The explicit expressions for the coefficients of the fundamental nucleus matrixes $\mathbf{K}_{\tau,s}^{n,m,loc}$ have the form

$$\begin{aligned} K_{u_x, u_x}^{n,m,\tau,s} &= C_{66} J_{\tau, s_r}^{u_x, u_x} + C_{11} J_{\tau, s}^{u_x, u_x} \frac{\pi^2 n^2}{L^2} + C_{44} J_{\tau, s}^{u_x, u_x} \frac{\pi^2 m^2}{R^2 \varphi_0^2}, \quad K_{u_x, u_\varphi}^{n,m,\tau,s} = (C_{12} + C_{44}) \frac{\pi^2 nm}{LR \varphi_0} J_{\tau, s}^{u_x, u_\varphi}, \\ K_{u_x, u_r}^{n,m,\tau,s} &= (C_{66} J_{\tau, s}^{u_x, u_r} - J_{\tau, s}^{u_x, u_r} C_{12} - C_{13} J_{\tau, s_r}^{u_x, u_r}) \frac{\pi n}{L}, \\ K_{u_\varphi, u_x}^{n,m,\tau,s} &= (C_{12} + C_{44}) J_{\tau, s}^{u_\varphi, u_x} \frac{\pi^2 nm}{LR \varphi_0}, \quad K_{u_\varphi, u_r}^{n,m,\tau,s} = - \frac{\pi m (J_{\tau, s}^{u_\varphi, u_r} C_{22} + J_{\tau, s_r}^{u_\varphi, u_r} RC_{23} + J_{\tau, s_r}^{u_\varphi, u_r} C_{55} - R J_{\tau, s}^{u_\varphi, u_r} C_{55})}{R^2 \varphi_0}, \\ K_{u_\varphi, u_\varphi}^{n,m,\tau,s} &= \left(J_{\tau, s_r}^{u_\varphi, u_\varphi} - R (J_{\tau, s_r}^{u_\varphi, u_\varphi} - J_{\tau, s}^{u_\varphi, u_\varphi}) + R^2 J_{\tau, s_r}^{u_\varphi, u_\varphi} \right) \frac{C_{55}}{R^2} + C_{44} J_{\tau, s}^{u_\varphi, u_\varphi} \frac{\pi^2 n^2}{L^2} + C_{22} J_{\tau, s}^{u_\varphi, u_\varphi} \frac{\pi^2 m^2}{R^2 \varphi_0^2}, \\ K_{u_r, u_x}^{n,m,\tau,s} &= \left(C_{55}^T J_{\tau, s_r}^{u_r, u_x} - C_{13} J_{\tau, s}^{u_r, u_x} + C_{12} J_{\tau, s}^{u_r, u_x} / R \right) \frac{\pi n}{L}, \\ K_{u_r, u_\varphi}^{n,m,\tau,s} &= \left(C_{55} (R J_{\tau, s_r}^{u_r, u_\varphi} - J_{\tau, s}^{u_r, u_\varphi}) - RC_{23} J_{\tau, s}^{u_r, u_\varphi} - C_{22} J_{\tau, s}^{u_r, u_\varphi} \right) \frac{\pi m}{R^2 \varphi_0}, \\ K_{u_r, u_r}^{n,m,\tau,s} &= \frac{C_{22} J_{\tau, s}^{u_r, u_r}}{R^2} + \frac{(J_{\tau, s}^{u_r, u_r} + J_{\tau, s_r}^{u_r, u_r}) C_{23}}{R} + C_{33} J_{\tau, s_r}^{u_r, u_r} + C_{66} J_{\tau, s}^{u_r, u_r} \frac{\pi^2 n^2}{L^2} + C_{55} J_{\tau, s}^{u_r, u_r} \frac{\pi^2 m^2}{R^2 \varphi_0^2}. \end{aligned} \quad (35)$$

If only two members are left in expansion (11), we will get a so-called completely linear expansion case (CLEC), which means that all functions describing stress–strain state of the shell change linearly along the shell thickness. All the equations become simple and automatically reduced from the general case. Therefore, we will not present them here. For more details of the CLEC, see Carrera and Zozulya [16].

To compare the results obtained using the higher-order theory with the results obtained using the classical shell theory, shell theory based on Timoshenko–Mindlin hypothesis is considered here. This theory can be obtained as a special case from the higher-order theory. In this case, the expansion (11) must be presented in the form

$$\mathbf{F}_{\mathbf{u},1} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \quad \mathbf{F}_{\mathbf{u},2} = \begin{vmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 0 \end{vmatrix}. \quad (36)$$

Then, all equations can be easily obtained from the CLEC. Therefore, we will not present them here. For more details of the theory based on the Timoshenko–Mindlin hypothesis, see Zozulya and Carrera [61].

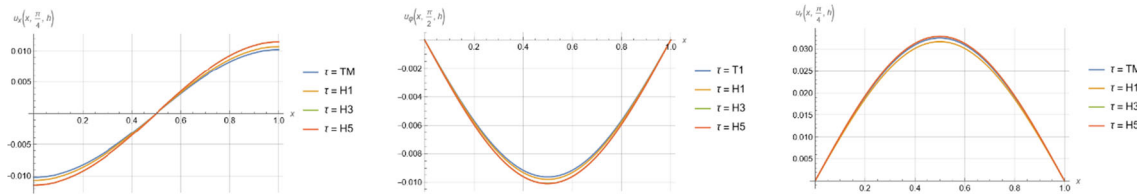


Fig. 1 Components u_x , u_φ , and u_r of the displacement vector in x direction

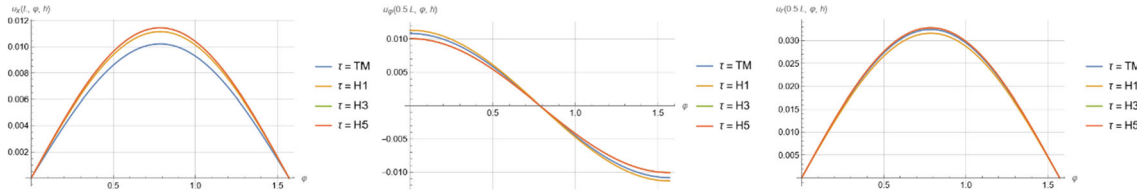


Fig. 2 Components u_x , u_φ , and u_r of the displacement vector in φ direction

To evaluate the developed theoretical models of a higher-order elastic cylindrical shell, let us consider an example of numerical calculation of displacements and stress tensor and briefly analyze the results obtained. Consider a section of a cylindrical shell supported at the edges and subjected to the mechanical loading applied to the upper surfaces of the shell. Following Timoshenko and Woinowsky-Krieger [46], the distribution of the applied mechanical load is presented as

$$\begin{aligned}\tilde{b}_{u_x,1}(x, y) &= B_{u_x,1}^{1,1} \cos\left(\frac{\pi}{L}x\right) \sin\left(\frac{\pi}{\varphi_0}\varphi\right), \\ \tilde{b}_{u_\varphi,1}(x, y) &= B_{u_\varphi,1}^{1,1} \sin\left(\frac{\pi}{L}x\right) \cos\left(\frac{\pi}{\varphi_0}\varphi\right), \\ \tilde{b}_{u_r,1}(x, y) &= B_{u_r,1}^{1,1} \sin\left(\frac{\pi}{L}x\right) \sin\left(\frac{\pi}{\varphi_0}\varphi\right)\end{aligned}\quad (37)$$

where L is the length of the shell side, φ_0 is the central angle subtended by the shell, $B_{u_x,1}^{1,1}$, $B_{u_\varphi,1}^{1,1}$, and $B_{u_r,1}^{1,1}$ are amplitudes of the mechanical loading.

For the numerical calculation, we use the Navier solutions in exact form for the higher-order theories represented by Eqs. (28–37). Calculations are made for the following data: Amplitude of mechanical loading is taken $B_{u_x,1}^{1,1} = B_{u_\varphi,1}^{1,1} = B_{u_r,1}^{1,1} = 10^6$ (N/m^2), geometrical parameters of the shell are $L = 1$ m, $R = L/2$, $h = L/10$, $\varphi_0 = \pi/2$. Material of the shell is considered isotropic, and mechanical properties of material are taken the same as in Carrera and Zozulya [14, 15], namely the Lamé constants are $\lambda = 762.6$ MPa and $\mu = 103.9$ MPa.

Figures 1 and 2 show the plots of the distribution of the components of displacement vector $\mathbf{u}(x, \varphi, r)$ in meridional and radial directions, respectively, for the third and fifth order—*H3*, *H5*, CLEC—*H1* and Timoshenko-Mindlin -*TM* shear deformation models. Figure 3 shows surface graphs of their distribution along surface x , φ for fifth order model—*H5* model.

For the CLEC—*H1* and Timoshenko-Mindlin -*TM* shear deformation models, following Carrera et al. [9] and Carrera et al. [10], we apply Poisson locking remedy, which consists of applying the modified elastic moduli in the form

$$\tilde{C}_{11} = \tilde{C}_{22} = C_{11} - \frac{C_{13}^2}{C_{33}} = \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu}, \quad \tilde{C}_{12} = C_{12} - \frac{C_{13}C_{23}}{C_{33}} = \frac{2\lambda\mu}{\lambda + 2\mu}. \quad (38)$$

Figures 4 and 5 show the plots of the distribution of the components of stress tensors $\sigma_{xx}(x, \varphi, r)$, $\sigma_{\varphi\varphi}(x, \varphi, r)$, $\sigma_{x\varphi}(x, \varphi, r)$ in meridional and radial directions, respectively, for the fifth order—*H5* model. Figure 6 shows density plot of their distribution along surface x , φ , for the fifth order—*H5* model.

The data presented in Figs. 1, 2, 3, 4, 5, and 6 provide qualitative and quantitative information on the behavior of the displacement vector components, as well as the components of the stress tensor. These results can be used as a benchmark example for finite element analysis of elastic cylindrical shells.

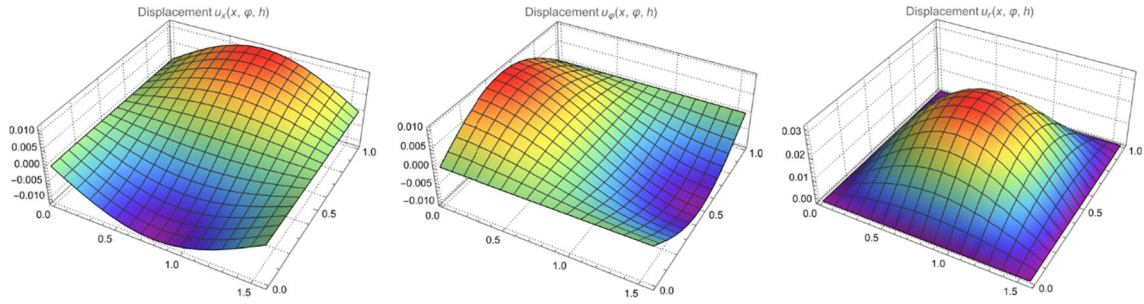


Fig. 3 Components u_x , u_φ , and u_r of the displacement vector in x and φ directions

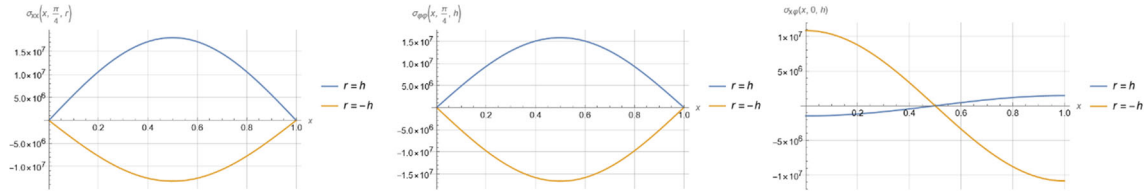


Fig. 4 Components σ_{xx} , $\sigma_{\varphi\varphi}$, and $\sigma_{x\varphi}$ of the force stress tensor in x direction

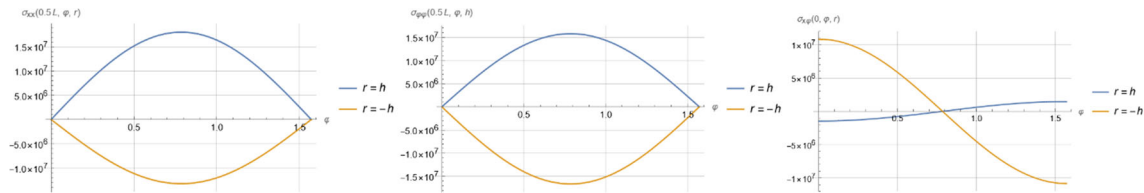


Fig. 5 Components σ_{xx} , $\sigma_{\varphi\varphi}$, and $\sigma_{x\varphi}$ of the force stress tensor in φ direction

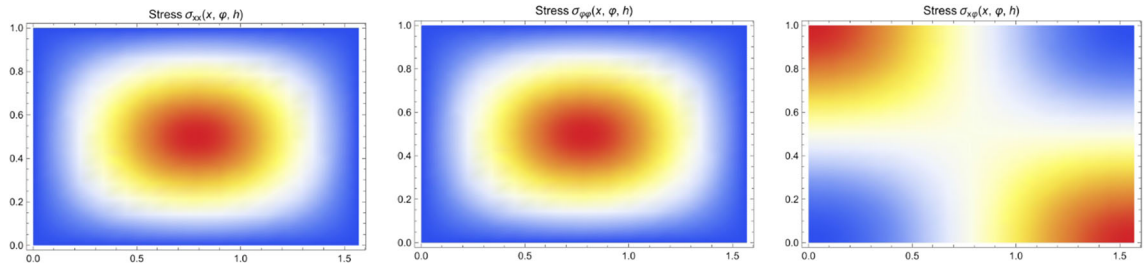


Fig. 6 Components σ_{xx} , $\sigma_{\varphi\varphi}$, and $\sigma_{x\varphi}$ of the force stress tensor in x and φ direction

3.2 Axisymmetric 1-D case

Obtained above equations can be applied for the case of external loading applied axisymmetrically. But in this case, all functions describing stress–strain state of the shell are 1-D, and the resulting equations become much simpler. Therefore, we consider this case separately here.

Due to the fact that all functions describing the stress–strain state of the shell do not depend on coordinate φ , all derivatives with respect to φ vanish, and the displacement in the radial direction is equal to zero. Therefore, tensors of stress $\boldsymbol{\sigma}(x, r)$ and strain $\boldsymbol{\epsilon}(x, r)$ tensors, as well as the displacements $\mathbf{u}(x, r)$, body $\mathbf{b}(x, r)$ and surface $\mathbf{p}(x, r)$ forces vectors, introduced in (4) take the form

$$\begin{aligned} \boldsymbol{\sigma} &= |\sigma_{xx}, \sigma_{\varphi\varphi}, \sigma_{rr}, \sigma_{xr}|^T, & \boldsymbol{\epsilon} &= |\epsilon_{xx}, \epsilon_{\varphi\varphi}, \epsilon_{rr}, \epsilon_{xr}|^T, \\ \mathbf{u} &= |u_x, u_r|^T, & \mathbf{b} &= |b_x, b_r|^T, & \mathbf{p} &= |p_x, p_r|^T. \end{aligned} \quad (39)$$

Substituting these parameters in Eqs. (16–25), taking into account that the displacement vector components are represented in the form (11), we obtain equations corresponding to the high-order linear elasticity theory for the case of an axisymmetric cylindrical shell.

The resulting differential equations have the same structure as (16), but the local matrices $\mathbf{L}_{\tau,s}^{loc}$ of the fundamental nucleus of differential equations of equilibrium for the higher-order axisymmetric cylindrical elastic shells, as well as the vectors of local unknown functions \mathbf{u}_s^{loc} and the local expression for the external body and surface loads \mathbf{b}_s^{loc} have the form

$$\mathbf{L}_{\tau,s}^{loc} = \begin{vmatrix} L_{u_x, u_x}^{\tau,s} & L_{u_x, u_r}^{\tau,s} \\ L_{u_r, u_x}^{\tau,s} & L_{u_r, u_r}^{\tau,s} \end{vmatrix}, \quad \mathbf{u}_s^{loc} = \begin{vmatrix} u_{x,s} \\ u_{r,s} \end{vmatrix}, \quad \mathbf{b}_s^{loc} = \begin{vmatrix} \tilde{b}_{u_x, \tau} \\ \tilde{b}_{u_r, \tau} \end{vmatrix}. \quad (40)$$

The coefficients of the fundamental nucleus $\mathbf{L}_{\tau,s}^{loc}$ can be easily calculated using the equations presented in the previous Section for the general case of higher-order cylindrical elastic shells.

The local matrices $\mathbf{B}_{\tau,s}^{loc}$ of the fundamental nucleus for natural boundary conditions, as well as \mathbf{p}_s^{loc} the vectors of the local expression for the external load applied to the ends of the shell for an axisymmetric cylindrical shell of higher order, have the form

$$\mathbf{B}_{\tau,s}^{loc} = \begin{vmatrix} B_{u_x, u_x}^{\tau,s} & B_{u_x, u_r}^{\tau,s} \\ B_{u_r, u_x}^{\tau,s} & B_{u_r, u_r}^{\tau,s} \end{vmatrix}, \quad \mathbf{p}_s^{loc} = \begin{vmatrix} J_{\tau,s}^{u_x, u_x} P_{u_x, s} \\ J_{\tau,s}^{u_r, u_r} P_{u_r, s} \end{vmatrix}. \quad (41)$$

The coefficients of the fundamental nucleus $\mathbf{B}_{\tau,s}^{loc}$ can be easily calculated using the equations presented in the previous Sections for the general case of higher-order cylindrical elastic shells.

Boundary value problems for a higher-order axisymmetric cylindrical shell, which can be formulated as a system of ordinary differential equations and corresponding boundary conditions, are usually solved numerically. For the special case of homogeneous essential boundary conditions (24), the problem can be solved analytically, using Fourier series expansion method.

Here, we demonstrate how to solve problems for an axisymmetric higher-order cylindrical shell using the Fourier series expansion method solving. In the same way as before, we represent displacements in the form of an expansion in a Fourier series of the form

$$u_x(x, r) = F_{u_x, \tau}(r) \sum_{n=0}^{\infty} U_{u_x, \tau}^n \cos\left(\frac{n\pi}{L}x\right), \quad u_r(x, r) = F_{u_r, \tau}(r) \sum_{n=0}^{\infty} U_{u_r, \tau}^n \sin\left(\frac{n\pi}{L}x\right) \quad (42)$$

where $U_{u_x, \tau}^n$, $U_{u_r, \tau}^n$ are the expansion coefficients in the Fourier series of functions $u_x, \tau(x)$, $u_r, \tau(x)$ in the form

$$U_{u_x, \tau}^n = \frac{2}{L} \int_0^L \cos\left(\frac{n\pi}{L}x\right) dx, \quad U_{u_r, \tau}^n = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi}{L}x\right) dx. \quad (43)$$

The components of the external load vector will be represented here as a Fourier series expansion in the form

$$\tilde{b}_{u_x, \tau}(x) = \sum_{n=0}^{\infty} B_{u_x, \tau}^n \cos\left(\frac{n\pi}{L}x\right), \quad \tilde{b}_{u_r, \tau}(x) = \sum_{n=0}^{\infty} B_{u_r, \tau}^n \sin\left(\frac{n\pi}{L}x\right), \quad (44)$$

where $B_{u_x, \tau}^n$, $B_{u_r, \tau}^n$ are the expansion coefficients in the Fourier series of functions $\tilde{b}_{u_x, \tau}(x)$, $\tilde{b}_{u_r, \tau}(x)$, that have the form

$$B_{u_x, \tau}^n = \frac{2}{L} \int_0^L \cos\left(\frac{n\pi}{L}x\right) dx, \quad B_{u_r, \tau}^n = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi}{L}x\right) dx. \quad (45)$$

Substituting the Fourier series expansion (42) for the components of the displacement vector into the system of ordinary differential equations for an axisymmetric higher-order cylindrical shell for arbitrary numbers

n , we obtain a system of linear algebraic equations for the coefficients of the Fourier series expansion for displacements in the form

$$\mathbf{K}_{n,M}^G \cdot \mathbf{U}_{n,M}^G = \mathbf{B}_{n,M}^G \quad (46)$$

where the global matrix operator $\mathbf{K}_{n,M}^G$, the vectors of unknown coefficients $\mathbf{U}_{n,M}^G$, and the right-hand side $\mathbf{B}_{n,M}^G$ have the form

$$\mathbf{K}_{n,M}^G = \begin{bmatrix} \mathbf{K}_{1,1}^{n,loc} & \cdots & \mathbf{K}_{1,M}^{n,loc} \\ \vdots & \ddots & \vdots \\ \mathbf{K}_{M,1}^{n,loc} & \cdots & \mathbf{K}_{M,M}^{n,loc} \end{bmatrix}, \quad \mathbf{U}_{n,M}^G = \begin{bmatrix} \mathbf{U}_1^{n,loc} \\ \vdots \\ \mathbf{U}_M^{n,loc} \end{bmatrix}, \quad \mathbf{B}_{n,M}^G = \begin{bmatrix} \mathbf{B}_1^{n,loc} \\ \vdots \\ \mathbf{B}_M^{n,loc} \end{bmatrix}. \quad (47)$$

Matrices $\mathbf{K}_{\tau,s}^{n,loc}$ are the fundamental nucleus of algebraic Eq. (47) for the analytical solution based on the expansion in the Fourier series for an axisymmetric higher-order cylindrical shell. These matrices and vectors of local unknown functions $\mathbf{U}_s^{n,loc}$ and the local expression for external body forces $\mathbf{B}_s^{n,loc}$ have the form

$$\mathbf{K}_{\tau,s}^{n,loc} = \begin{bmatrix} K_{u_x, u_x}^{n,\tau,s} & K_{u_x, u_r}^{n,\tau,s} \\ K_{u_r, u_x}^{n,\tau,s} & K_{u_r, u_r}^{n,\tau,s} \end{bmatrix}, \quad \mathbf{U}_s^{n,loc} = \begin{bmatrix} U_{u_x, s}^{n,loc} \\ U_{u_r, s}^{n,loc} \end{bmatrix}, \quad \mathbf{B}_s^{n,loc} = \begin{bmatrix} B_{u_x, \tau}^{n,loc} \\ B_{u_r, \tau}^{n,loc} \end{bmatrix}. \quad (48)$$

The explicit expressions for the coefficients of the fundamental nucleus matrices $\mathbf{K}_{\tau,s}^{n,loc}$ have the form

$$\begin{aligned} K_{u_x, u_x}^{n,\tau,s} &= C_{55} J_{\tau, s}^{u_x, u_x} + C_{11} J_{\tau, s}^{u_x, u_x} \frac{\pi^2 n^2}{L^2}, \quad K_{u_x, u_r}^{n,\tau,s} = (C_{55} J_{\tau, s}^{u_x, u_r} - J_{\tau, s}^{u_x, u_r} C_{12}/R - C_{13} J_{\tau, s}^{u_x, u_r}) \frac{\pi n}{L}, \\ K_{u_r, u_x}^{n,\tau,s} &= (C_{55}^T J_{\tau, s}^{u_r, u_x} - C_{13} J_{\tau, s}^{u_r, u_x} + C_{12} J_{\tau, s}^{u_r, u_x}/R) \frac{\pi n}{L}, \\ K_{u_r, u_r}^{n,\tau,s} &= \frac{C_{22} J_{\tau, s}^{u_r, u_r}}{R^2} + \frac{(J_{\tau, s}^{u_r, u_r} + J_{\tau, s}^{u_r, u_r}) C_{23}}{R} + C_{33} J_{\tau, s}^{u_r, u_r} + C_{55} J_{\tau, s}^{u_r, u_r} \frac{\pi^2 n^2}{L^2}. \end{aligned}$$

As in the general case 2-D of shell models of higher order, here, we consider the third and fifth order—*H3*, *H5*, *CLEC–H1* and Timoshenko-Mindlin—*TM* shear deformation models. In the case of the Timoshenko-Mindlin—*TM* shear deformation model, the solution based on the Fourier series expansion can be presented as

$$\begin{aligned} u_x(x, r) &= \frac{\sum_{n=0}^{\infty} \left((3C_{22}C_{66}L^6 + \pi^4 C_{11}C_{66}h^2R^2L^2 + \pi^2 C_{11}C_{22}h^2L^4) P_1^n + (3\pi C_{66}C_{12}RL^5 + \pi^3 C_{11}C_{12}h^2RL_1^3) P_2^n \right) \cos(\pi nx/L)}{2\pi h \left(C_{11}^2 C_{66} \pi^4 h^2 R^2 + C_{11}^2 C_{22} \pi^2 h^2 L^2 - 3C_{12}^2 C_{66} L^4 - C_{11} C_{12}^2 \pi^2 h^2 L^2 + 3C_{11} C_{22} C_{66} L^4 \right)} \\ &\quad - \frac{3C_{66}RL^3r \sum_{n=0}^{\infty} (C_{12}LP_1^n + C_{11}\pi RP_2^n) \cos(\pi nx/L)}{2h \left(C_{11}^2 C_{66} \pi^4 h^2 R^2 + C_{11}^2 C_{22} \pi^2 h^2 L^2 - 3C_{12}^2 C_{66} L^4 - C_{11} C_{12}^2 \pi^2 h^2 L^2 + 3C_{11} C_{22} C_{66} L^4 \right)}, \\ u_r(x) &= \frac{(C_{11}h^2\pi^2 + 3C_{66}L_1^2)RL_1^2 \sum_{n=1}^{\infty} (C_{12}L_1P_1^n + C_{11}\pi RP_1^n) \sin(\pi nx/L)}{2\pi h \left(C_{11}^2 C_{66} \pi^4 h^2 R^2 + C_{11}^2 C_{22} \pi^2 h^2 L^2 - 3C_{12}^2 C_{66} L^4 - C_{11} C_{12}^2 \pi^2 h^2 L^2 + 3C_{11} C_{22} C_{66} L^4 \right)}. \quad (49) \end{aligned}$$

For the numerical calculation, we use the Fourier series solutions in exact form for the higher-order theories represented by Eqs. (42–49). Calculations are made for the following data: Amplitude of mechanical loading is taken $B_{u_x, 1}^{1,1} = B_{u_r, 1}^{1,1} = 10^6$ (N/m²), geometrical parameters of the shell are $L = 1$ m, $R = L/2$, $h = L/10$. Material of the shell is considered isotropic, and mechanical properties of material are taken the same as in Carrera and Zozulya [14, 15], namely the Lamé constants are $\lambda = 762.6$ MPa and $\mu = 103.9$ MPa.

Figure 7 shows the plots of the distribution of the components of displacement vector $\mathbf{u}(x, r)$ in the x axis direction, for the third and fifth order—*H3*, *H5*, *CLEC–H1* and Timoshenko-Mindlin—*TM* shear deformation models.

Figures 8, 9, and 10 show the plots of the distribution of the components of stress tensors $\sigma_{xx}(x, r)$, $\sigma_{\varphi\varphi}(x, r)$, $\sigma_{x\varphi}(x, r)$ in meridional and radial directions, respectively, for the fifth order—*H5* model. Figure 11 shows a density plot of their distribution along plane x, r , for the fifth order—*H5* model.

The data presented in Figs. 7, 8, 9, 10, 11, and 12 provide qualitative and quantitative information on the behavior of the displacement vector components, as well as the components of the stress tensor. These results can be used as a benchmark example for finite element analysis of elastic cylindrical shells in the case of axisymmetric loading.

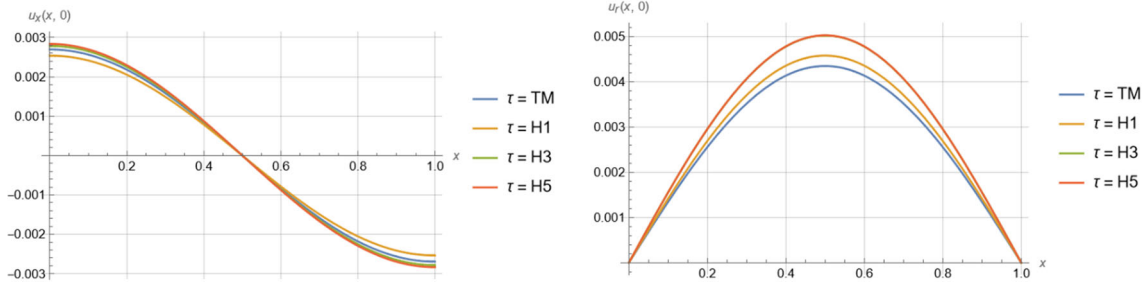


Fig. 7 Components u_x and u_r of the displacement vector in x direction

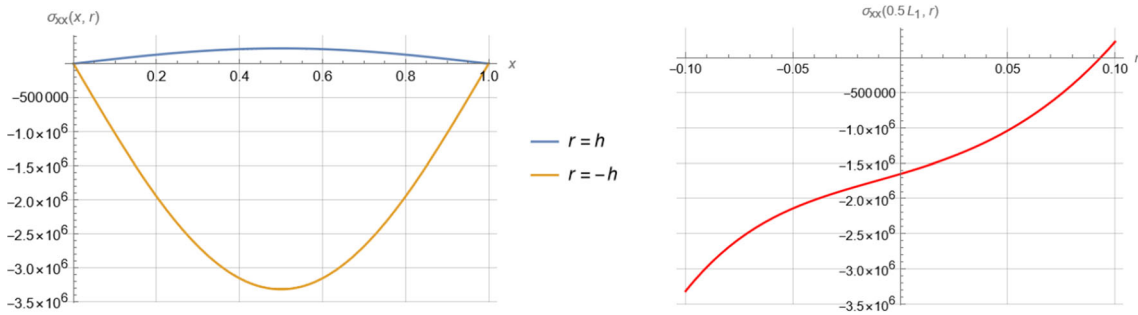


Fig. 8 Component σ_{xx} of the stress tensor in x and r direction, respectively

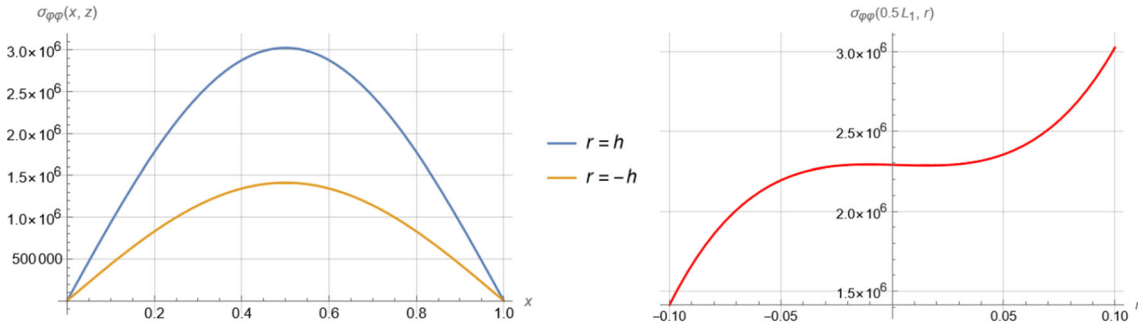


Fig. 9 Component $\sigma_{\phi\phi}$ of the stress tensor in x and r direction, respectively

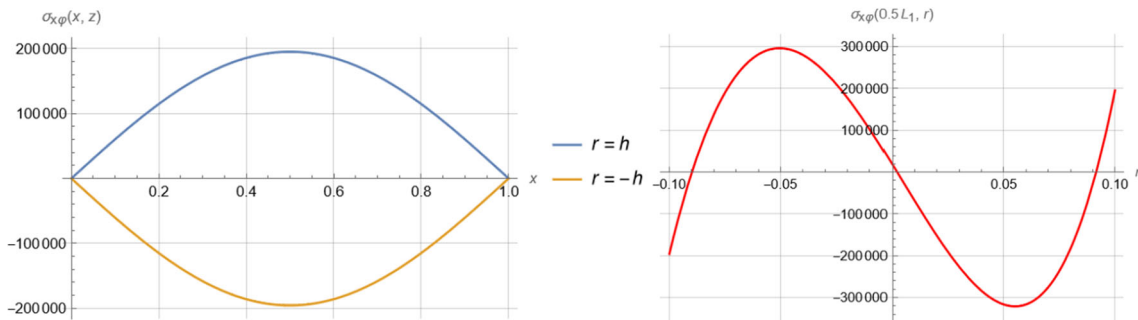


Fig. 10 Component $\sigma_{x\phi}$ of the stress tensor in x and r direction, respectively

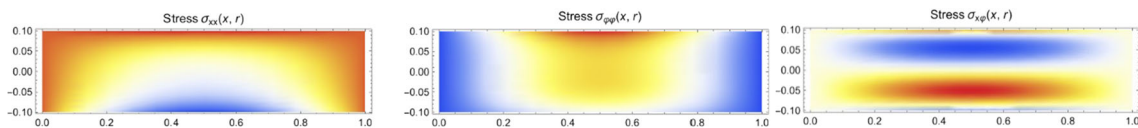


Fig. 11 Components σ_{xx} , $\sigma_{\phi\phi}$, and $\sigma_{x\phi}$ of the stress tensor in x , and r directions, respectively

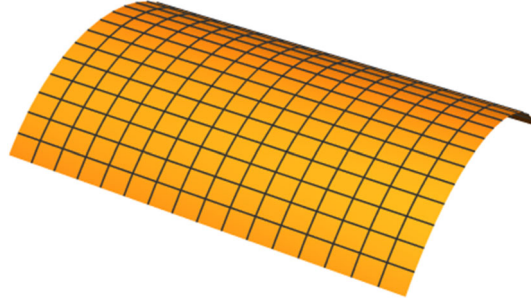


Fig. 12 Surface of a part of a circular cylinder

4 Navier analytical solution for a shallow shell with rectangular planform

Here, we consider some aspects of the theory of shallow shells of revolution, with an emphasis on the possibility of applying the Navier close form solution method. First, following Ambartsumyan [1], Leissa and Qatu [33], Nazarov [34], Qatu [41], and Vlasov [51], we recall the main geometric hypothesis commonly used in classical shell theory to simplify the basic equations.

4.1 Basic relations for shallow shells with rectangular planform

The shallowness can be measured in various ways; for example, the angle of the tangent to the planform γ , the span-to radius ratio L/R , or the rise-to-span ratio f/L should be relatively small. For the shallow shells, the arrow shell's rise above the planform satisfies the inequality $f/L \leq 1/5$. Shallow shells can have various types of curvature (e.g., circular cylindrical, spherical, ellipsoidal, conical, hyperbolic paraboloidal, etc.) and various types of planforms (rectangular, triangular, trapezoidal, circular, elliptical, and others).

When constructing the theory of shallow shells, it is assumed that the internal geometry of the middle surface of the shell, regardless of the value of the Gaussian curvature $K = k_1 k_2$, coincides with the geometry of the planform, i.e., the expression for the first quadratic form of the surface

$$dS^2 = A_1^2 dx_1^2 + A_2^2 dx_2^2 \quad (50)$$

is identified with the analogous expression for the first quadratic-quadratic form in the planform. In this case, the Rodriguez-Gauss equation is replaced by the approximate equality

$$\frac{A_1 A_2}{R_1 R_2} \approx 0. \quad (51)$$

For the shallow shell under consideration, represented in the curvilinear orthogonal coordinate system x_1, x_2, x_3 , the first quadratic form of the middle surface can be represented with a sufficiently high accuracy by the following equation:

$$dS^2 = dx_1^2 + dx_2^2 \quad (52)$$

which is equivalent to the assumption that the coefficients of the first quadratic form $A_1 \approx 1$ and $A_2 \approx 1$.

In the general case, when the coordinates x_1 and x_2 are dimensionless, and the coefficients A_1 and A_2 are not equal to one and are functions of coordinates x_1 and x_2 , i.e., $A_1(x_1, x_2)$ and $A_2(x_1, x_2)$, with an accuracy of the accepted geometrical assumptions, it must be assumed that A_1 and A_2 behave as constants when differentiating. With the same accuracy, it should be assumed that the main curvatures of the middle surface $k_1 = k_1(x_1, x_2)$ and $k_2 = k_2(x_1, x_2)$ behave as constants when differentiating.

Next, consider the shallow shells of a rectangular planform. In this case, one can introduce a Cartesian coordinate system x, y, z and represent the middle surface of the shell as a function of the form $z = f(x, y)$. Then, the parametrized vector equation of the middle surface has the form

$$\mathbf{r}(x, y) = x\mathbf{e}_1 + y\mathbf{e}_2 + f(x, y)\mathbf{e}_3. \quad (53)$$

The first and second derivatives of the radius-vector (53) with respect to coordinates can be presented in the form

$$\begin{aligned}\frac{\partial \mathbf{r}(x, y)}{\partial x} &= \mathbf{e}_1 + \frac{\partial f(x, y)}{\partial x} \mathbf{e}_3, & \frac{\partial^2 \mathbf{r}(x, y)}{\partial x^2} &\approx \frac{\partial^2 f(x, y)}{\partial x^2} \mathbf{e}_3, \\ \frac{\partial \mathbf{r}(x, y)}{\partial y} &= \mathbf{e}_2 + \frac{\partial f(x, y)}{\partial x} \mathbf{e}_3, & \frac{\partial^2 \mathbf{r}(x, y)}{\partial y^2} &\approx \frac{\partial^2 f(x, y)}{\partial y^2} \mathbf{e}_3.\end{aligned}\quad (54)$$

The equations for the coefficients of the first quadratic form, the unit vector normal to the surface, and the principal curvatures of the middle surface of the shallow shell are

$$\begin{aligned}A_1(x, y) &= \sqrt{\frac{\partial \mathbf{r}(x, y)}{\partial x} \cdot \frac{\partial \mathbf{r}(x, y)}{\partial x}} = \sqrt{1 + \left(\frac{\partial f(x, y)}{\partial x}\right)^2} \approx 1, & A_2(x, y) &= \sqrt{\frac{\partial \mathbf{r}(x, y)}{\partial y} \cdot \frac{\partial \mathbf{r}(x, y)}{\partial y}} = \sqrt{1 + \left(\frac{\partial f(x, y)}{\partial y}\right)^2} \approx 1 \\ \mathbf{n}(x, y) &= \frac{\mathbf{r}_x(x, y) \times \mathbf{r}_y(x, y)}{|\mathbf{r}_x(x, y) \times \mathbf{r}_y(x, y)|} \approx -\frac{\partial f(x, y)}{\partial x} \mathbf{e}_1 - \frac{\partial f(x, y)}{\partial y} \mathbf{e}_2 + \mathbf{e}_3, \\ k_x(x, y) &= \frac{\mathbf{n}(x, y) \cdot \mathbf{r}_{xx}(x, y)}{\mathbf{r}_x(x, y)^2} \approx \frac{\partial^2 f(x, y)}{\partial x^2}, & k_y(x, y) &= \frac{\mathbf{n}(x, y) \cdot \mathbf{r}_{yy}(x, y)}{\mathbf{r}_y(x, y)^2} \approx \frac{\partial^2 f(x, y)}{\partial y^2}.\end{aligned}\quad (55)$$

In these equations, we take into consideration assumption that the angle of the tangent to the planform γ is small and that the derivatives $\frac{\partial f(x, y)}{\partial x}$ and $\frac{\partial f(x, y)}{\partial y}$ belong to tangential plane and that is why they are small, too. Therefore, their square can be neglected.

Taking into account Eq. (55) and assumption that the coefficients of the first quadratic form of a surface A_x and A_y , and the principal curvatures k_x and k_y behave as constants when differentiating, we can simplify the Lamé coefficients and their derivatives and take them in the form

$$\begin{aligned}H_x &= 1 + k_x z, & H_y &= 1 + k_y z & \text{and } H_3 &= 1, \\ \frac{\partial H_\beta}{\partial x} &= (1 + k_\beta z), & \frac{\partial H_\beta}{\partial x_\beta} &= k_\beta, & \frac{\partial H_3}{\partial x_i} &= 0, & \beta = x, y, & x_i = (x, y, z).\end{aligned}\quad (56)$$

In the next Section, we will show that for shallow shells with rectangular planform the Navier close form solution method can be applied. Let us now show that most of the shells of revolution considered in Carrera and Zozulya [17] can be considered in rectangular planform, and Cartesian coordinates can be used to parametrize their middle surface, and the Navier close form solution method can be applied. For details of parametrization of the second-order analytical surfaces, see Rogers and Adams [44].

4.1.1 Shallow cylindrical shell

The middle surface of the cylindrical shell is a part of a circular cylinder, see Fig. 12.

The equation for a circular cylinder in Cartesian coordinates is

$$x^2 + z^2 = R^2. \quad (57)$$

Then, the parametrized vector equation of the middle surface has the form

$$\mathbf{r}(x, y) = x \mathbf{e}_1 + y \mathbf{e}_2 + \sqrt{R^2 - x^2} \mathbf{e}_3. \quad (58)$$

The coefficients of the first quadratic form, the unit vector normal to the surface, and the principal curvatures of the middle surface of the shallow cylindrical shell are equal to

$$A_1 = 1, \quad A_2 = 1, \quad \mathbf{n}(x) = \frac{x}{R} \mathbf{e}_1 + \mathbf{e}_3, \quad k_x = 1/R, \quad k_y = 1. \quad (59)$$

We take into account here that the shell is shallow and relation x/R is small, so its square exponent can be neglected.

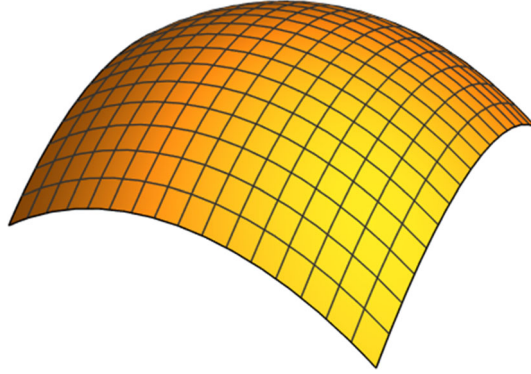


Fig. 13 Surface of a part of a sphere

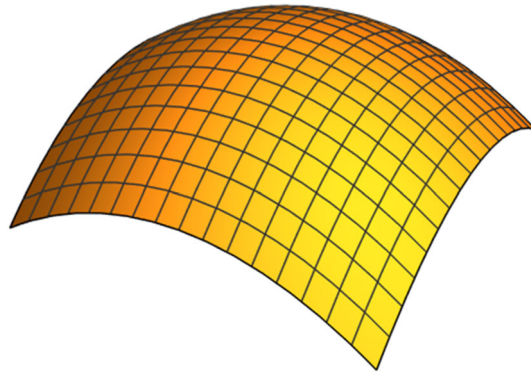


Fig. 14 Surface of a part of an ellipsoid

4.1.2 Shallow spherical shell

The middle surface of the spherical shell is a part of a sphere, see Fig. 13.

The equation for a sphere in Cartesian coordinates is

$$x^2 + y^2 + z^2 = R^2. \quad (60)$$

Then, the parametrized vector equation of the middle surface has the form

$$\mathbf{r}(x, y) = x\mathbf{e}_1 + y\mathbf{e}_2 + \sqrt{R^2 - x^2 - y^2}\mathbf{e}_3. \quad (61)$$

The coefficients of the first quadratic form, the unit vector normal to the surface, and the principal curvatures of the middle surface of the shallow cylindrical shell are equal to

$$A_1 = 1, \quad A_2 = 1, \quad \mathbf{n}(x) = \frac{x}{R}\mathbf{e}_1 + \frac{y}{R}\mathbf{e}_2 + \mathbf{e}_3, \quad k_x = 1/R, \quad k_y = 1/R. \quad (62)$$

We take into account here that the shell is shallow and relation x/R is small, so its square exponent can be neglected.

4.1.3 Shallow ellipsoidal shell

The middle surface of the paraboloidal shell is a part of an ellipsoid, see Fig. 14.

The ellipsoid equation in Cartesian coordinates has the form

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 0. \quad (63)$$

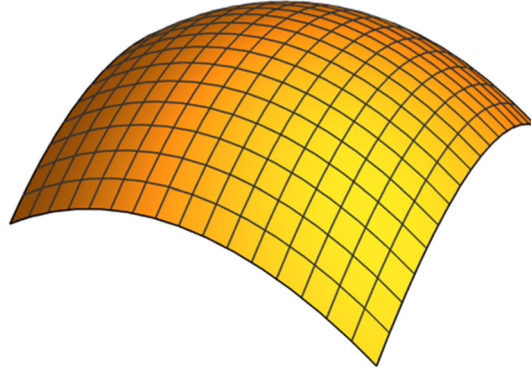


Fig. 15 Surface of a part of a paraboloid

Then, the parametrized vector equation of the middle surface has the form

$$\mathbf{r}(x, y) = x\mathbf{e}_1 + y\mathbf{e}_2 + b\sqrt{1 - \frac{x^2 + y^2}{a^2}}\mathbf{e}_3. \quad (64)$$

The coefficients of the first quadratic form, the unit vector normal to the surface, and the principal curvatures of the middle surface of the shallow cylindrical shell are equal to

$$A_1 = 1, \quad A_2 = 1, \quad \mathbf{n}(x) = \frac{bx}{a}\mathbf{e}_1 + \frac{by}{a}\mathbf{e}_2 + \mathbf{e}_3, \quad k_x = b/a^2, \quad k_y = b/a^2. \quad (65)$$

We take into account here that the shell is shallow and relation x/a is small, so its square exponent can be neglected.

4.1.4 Shallow paraboloidal shell

The middle surface of the paraboloidal shell is a part of a paraboloid, see Fig. 15.

The paraboloid equation in Cartesian coordinates has the form

$$\frac{x^2 + y^2}{R^2} = z. \quad (66)$$

Then, the parametrized vector equation of the middle surface has the form

$$\mathbf{r}(x, y) = x\mathbf{e}_1 + y\mathbf{e}_2 + \frac{x^2 + y^2}{R^2}\mathbf{e}_3. \quad (67)$$

The coefficients of the first quadratic form, the unit vector normal to the surface, and the principal curvatures of the middle surface of the shallow cylindrical shell are equal to

$$A_1 = 1, \quad A_2 = 1, \quad \mathbf{n}(x) = \frac{2x}{R}\mathbf{e}_1 + \frac{2y}{R}\mathbf{e}_2 + \mathbf{e}_3, \quad k_x = 2/R, \quad k_y = 2/R. \quad (68)$$

We take into account here that the shell is shallow and relation x/R is small, so its square exponent can be neglected.

4.1.5 Shallow hyperboloidal shell

The middle surface of the hyperboloidal shell is a part of a one-sheeted hyperboloid, see Fig. 16.

The paraboloid equation in Cartesian coordinates has the form

$$\frac{x^2 + z^2}{a^2} - \frac{y^2}{b^2} = 1. \quad (69)$$

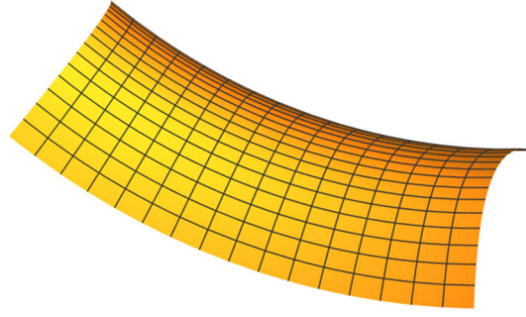


Fig. 16 Surface of a part of a one-sheeted hyperboloid

Then, the parametrized vector equation of the middle surface has the form

$$\mathbf{r}(x, y) = x\mathbf{e}_1 + y\mathbf{e}_2 + b\sqrt{1 + \frac{x^2 - y^2}{a^2}}\mathbf{e}_3. \quad (70)$$

The coefficients of the first quadratic form, the unit vector normal to the surface, and the principal curvatures of the middle surface of the shallow cylindrical shell are equal to

$$A_1 = 1, \quad A_2 = 1, \quad \mathbf{n}(x) = \frac{2x}{R}\mathbf{e}_1 - \frac{2y}{R}\mathbf{e}_2 + \mathbf{e}_3, \quad k_x = b/a^2, \quad k_y = -b/a^2. \quad (71)$$

We take into account here that the shell is shallow and relation x/a is small, so its square exponent can be neglected.

4.2 Navier analytical solution for shallow shells with rectangular planform

For shallow shells, in the general case, the coefficients of the first quadratic form of the middle surface and the principal curvatures are equal to $A_1 = 1$, $A_2 = 1$, and $k_1 = \frac{1}{R_1}$, $k_2 = \frac{1}{R_2}$, respectively. Here, we show that in this case the Navier analytical solution method can be used.

Substituting into Eqs. (16–25) the coefficients of the first quadratic form of the middle surface and principal curvatures presented above, we obtain equations corresponding to the high-order linear elasticity theory for the case of a shallow spherical shell.

The resulting differential equations have the same structure as (16), but the local matrices $\mathbf{L}_{\tau,s}^{loc}$ of the fundamental nucleus of differential equations of equilibrium for higher-order shallow spherical elastic shells, as well as the vectors of local unknown functions \mathbf{u}_s^{loc} and the local expression for the external body and surface loads \mathbf{b}_s^{loc} , have the form

$$\mathbf{L}_{\tau,s}^{loc} = \begin{vmatrix} L_{u_x, u_x}^{\tau,s} & L_{u_x, u_y}^{\tau,s} & L_{u_x, u_z}^{\tau,s} \\ L_{u_y, u_x}^{\tau,s} & L_{u_y, u_y}^{\tau,s} & L_{u_y, u_z}^{\tau,s} \\ L_{u_z, u_x}^{\tau,s} & L_{u_z, u_y}^{\tau,s} & L_{u_z, u_z}^{\tau,s} \end{vmatrix}, \quad \mathbf{u}_s^{loc} = \begin{vmatrix} u_{x,s} \\ u_{y,s} \\ u_{z,s} \end{vmatrix}, \quad \mathbf{b}_s^{loc} = \begin{vmatrix} \tilde{b}_{u_x, \tau} \\ \tilde{b}_{u_y, \tau} \\ \tilde{b}_{u_z, \tau} \end{vmatrix}. \quad (72)$$

The local fundamental nucleus $\mathbf{L}_{\tau,s}^{loc}$ coefficients can be easily calculated using equations presented in the previous Sections. Their analytical expressions for the higher-order shallow spherical elastic shells can be easily calculated using the method explained in detail in Carrera and Zozulya [15, 16].

The local matrices $\mathbf{B}_{\tau,s}^{loc}$ of the fundamental nucleus for natural boundary conditions, as well as \mathbf{p}_s^{loc} the vectors of the local expression for the external load applied to the shell boundary for the higher-order shallow spherical shell, have the form

$$\mathbf{B}_{\tau,s}^{loc} = \begin{vmatrix} B_{u_x, u_x}^{\tau,s} & B_{u_x, u_y}^{\tau,s} & B_{u_x, u_z}^{\tau,s} \\ B_{u_y, u_x}^{\tau,s} & B_{u_y, u_y}^{\tau,s} & B_{u_y, u_z}^{\tau,s} \\ B_{u_z, u_x}^{\tau,s} & B_{u_z, u_y}^{\tau,s} & B_{u_z, u_z}^{\tau,s} \end{vmatrix}, \quad \mathbf{p}_s^{loc} = \begin{vmatrix} J_{\tau,s}^{u_x, u_x} P_{u_x, s} \\ J_{\tau,s}^{u_y, u_y} P_{u_y, s} \\ J_{\tau,s}^{u_z, u_z} P_{u_z, s} \end{vmatrix}. \quad (73)$$

The fundamental nucleus $\mathbf{B}_{\tau,s}^{loc}$ coefficients can be easily calculated using the equations presented in the previous Sections. Their analytical expressions for the higher-order shallow spherical elastic shells can be easily calculated using the method explained in detail in Carrera and Zozulya [15, 16].

Let us consider the analytical Navier solution for a higher-order shallow spherical shell, which is based on the CUF. Generalizing the approach developed in Timoshenko and Woinowsky-Krieger [46] and Vlasov [51] for the case of the higher-order shallow spherical shell, we represent displacements fields in the form of a double Fourier series expansion as

$$\begin{aligned} u_x(x, y, z) &= F_{u_x,\tau}(z) \sum_{n,m=0}^{\infty} U_{u_x,\tau}^{n,m} \cos\left(\frac{n\pi}{L_1}x\right) \sin\left(\frac{m\pi}{L_2}y\right), \\ u_y(x, y, z) &= F_{u_y,\tau}(z) \sum_{n,m=0}^{\infty} U_{u_y,\tau}^{n,m} \sin\left(\frac{n\pi}{L_1}x\right) \cos\left(\frac{m\pi}{L_2}y\right), \\ u_z(x, y, z) &= F_{u_z,\tau}(z) \sum_{n,m=0}^{\infty} U_{u_z,\tau}^{n,m} \sin\left(\frac{n\pi}{L_1}x\right) \sin\left(\frac{m\pi}{L_2}y\right) \end{aligned} \quad (74)$$

where L_1 and L_2 are dimensions of the rectangle, and $U_{u_x,\tau}^n$, $U_{u_y,\tau}^n$, $U_{u_z,\tau}^n$ are the expansion coefficients in the Fourier series of functions $u_{x,\tau}(x, y)$, $u_{y,\tau}(x, y)$, $u_{z,\tau}(x, y)$ of the form

$$\begin{aligned} U_{u_x,\tau}^{n,m} &= \frac{4}{L_1 L_2} \int_0^{L_1} \cos\left(\frac{n\pi}{L_1}x\right) \int_0^{L_2} u_{x,\tau}(x, y) \sin\left(\frac{m\pi}{L_2}y\right) dy dx, \\ U_{u_y,\tau}^{n,m} &= \frac{4}{L_1 L_2} \int_0^{L_1} \sin\left(\frac{n\pi}{L_1}x\right) \int_0^{L_2} u_{y,\tau}(x, y) \cos\left(\frac{m\pi}{L_2}y\right) dy dx, \\ U_{u_z,\tau}^{n,m} &= \frac{4}{L_1 L_2} \int_0^{L_1} \sin\left(\frac{n\pi}{L_1}x\right) \int_0^{L_2} u_{z,\tau}(x, y) \sin\left(\frac{m\pi}{L_2}y\right) dy dx. \end{aligned} \quad (75)$$

The vector of the components of the external load is represented here in the form of an expansion into a Fourier series in the form

$$\begin{aligned} \tilde{b}_{u_x,\tau}(x, y) &= \sum_{n,m=0}^{\infty} B_{u_x,\tau}^{n,m} \cos\left(\frac{n\pi}{L_1}x\right) \sin\left(\frac{m\pi}{L_2}y\right), \\ \tilde{b}_{u_y,\tau}(x, y) &= \sum_{n,m=0}^{\infty} B_{u_y,\tau}^{n,m} \sin\left(\frac{n\pi}{L_1}x\right) \cos\left(\frac{m\pi}{L_2}y\right), \\ \tilde{b}_{u_z,\tau}(x, y) &= \sum_{n,m=0}^{\infty} B_{u_z,\tau}^{n,m} \sin\left(\frac{n\pi}{L_1}x\right) \sin\left(\frac{m\pi}{L_2}y\right), \end{aligned} \quad (76)$$

where $B_{u_x,\tau}^{n,m}$, $B_{u_y,\tau}^{n,m}$, $B_{u_z,\tau}^{n,m}$ are the expansion coefficients in the Fourier series of functions $\tilde{b}_{u_x,\tau}(x, y)$, $\tilde{b}_{u_y,\tau}(x, y)$, $\tilde{b}_{u_z,\tau}(x, y)$, they have the form

$$\begin{aligned} B_{u_x,\tau}^{n,m} &= \frac{4}{L_1 L_2} \int_0^{L_1} \cos\left(\frac{n\pi}{L_1}x\right) \int_0^{L_2} \tilde{b}_{u_x,\tau}(x, y) \sin\left(\frac{m\pi}{L_2}y\right) dy dx, \\ B_{u_y,\tau}^{n,m} &= \frac{4}{L_1 L_2} \int_0^{L_1} \sin\left(\frac{n\pi}{L_1}x\right) \int_0^{L_2} \tilde{b}_{u_y,\tau}(x, y) \cos\left(\frac{m\pi}{L_2}y\right) dy dx, \\ B_{u_z,\tau}^{n,m} &= \frac{4}{L_1 L_2} \int_0^{L_1} \sin\left(\frac{n\pi}{L_1}x\right) \int_0^{L_2} \tilde{b}_{u_z,\tau}(x, y) \sin\left(\frac{m\pi}{L_2}y\right) dy dx. \end{aligned} \quad (77)$$

Substituting the expansion in the Fourier series (28) for the components of the displacement vector into the system of differential equations for the shallow spherical shell of a higher order (16) for arbitrary numbers n and m , we obtain a system of linear algebraic equations for the coefficients of the expansion in the Fourier series for displacements in the form

$$\mathbf{K}_{n,m,M}^G \cdot \mathbf{U}_{n,m,M}^G = \mathbf{B}_{n,m,M}^G \quad (78)$$

where the global matrix operator $\mathbf{K}_{n,m,M}^G$, the vectors of unknown coefficients $\mathbf{U}_{n,m,M}^G$, and the right-hand side $\mathbf{B}_{n,m,M}^G$ have the form

$$\mathbf{K}_{n,m,M}^G = \begin{bmatrix} \mathbf{K}_{1,1}^{n,m,loc} & \cdots & \mathbf{K}_{1,M}^{n,m,loc} \\ \vdots & \ddots & \vdots \\ \mathbf{K}_{M,1}^{n,m,loc} & \cdots & \mathbf{K}_{M,M}^{n,m,loc} \end{bmatrix}, \quad \mathbf{U}_{n,m,M}^G = \begin{bmatrix} \mathbf{U}_1^{n,m,loc} \\ \vdots \\ \mathbf{U}_M^{n,m,loc} \end{bmatrix}, \quad \mathbf{B}_{n,m,M}^G = \begin{bmatrix} \mathbf{B}_1^{n,m,loc} \\ \vdots \\ \mathbf{B}_M^{n,m,loc} \end{bmatrix}. \quad (79)$$

Matrices $\mathbf{K}_{\tau,s}^{n,m,loc}$ are the fundamental nucleus of algebraic Eq. (33) for the Navier analytical solution for the higher-order elastic shallow spherical shell. These matrices and vectors of local unknown functions $\mathbf{U}_s^{n,m,loc}$ and local expression for external body forces $\mathbf{B}_s^{n,m,loc}$ have the form

$$\mathbf{K}_{\tau,s}^{n,m,loc} = \begin{bmatrix} K_{u_x,u_x}^{n,m,\tau,s} & K_{u_x,u_y}^{n,m,\tau,s} & K_{u_x,u_z}^{n,m,\tau,s} \\ K_{u_y,u_x}^{n,m,\tau,s} & K_{u_y,u_y}^{n,m,\tau,s} & K_{u_y,u_z}^{n,m,\tau,s} \\ K_{u_z,u_x}^{n,m,\tau,s} & K_{u_z,u_y}^{n,m,\tau,s} & K_{u_z,u_z}^{n,m,\tau,s} \end{bmatrix}, \quad \mathbf{U}_s^{n,m,loc} = \begin{bmatrix} U_{u_x,s}^{n,m} \\ U_{u_y,s}^{n,m} \\ U_{u_z,s}^{n,m} \end{bmatrix}, \quad \mathbf{B}_s^{n,m,loc} = \begin{bmatrix} B_{u_x,\tau}^{n,m} \\ B_{u_y,\tau}^{n,m} \\ B_{u_z,\tau}^{n,m} \end{bmatrix}. \quad (80)$$

The explicit expressions for the coefficients of the fundamental nucleus matrices $\mathbf{K}_{\tau,s}^{n,m,loc}$ have the form

$$\begin{aligned} K_{u_x,u_x}^{n,m,\tau,s} &= C_{11} J_{\tau,s}^{u_x,u_x} \frac{\pi^2 n^2}{L_1^2} + C_{44} J_{\tau,s}^{u_x,u_x} \frac{\pi^2 m^2}{L_2^2} + C_{66} \left(\frac{J_{\tau,s}^{u_x,u_x}}{R_1^2} + J_{\tau_r,s_r}^{u_x,u_x} - \frac{J_{\tau_r,s}^{u_x,u_x} + J_{\tau,s_r}^{u_x,u_x}}{R_1} \right), \\ K_{u_x,u_y}^{n,m,\tau,s} &= (C_{12} + C_{44}) \frac{\pi^2 nm}{L_1 L_2} J_{\tau,s}^{u_x,u_y}, \\ K_{u_x,u_z}^{n,m,\tau,s} &= \left((C_{66} J_{\tau_z,s}^{u_x,u_z} - C_{13} J_{\tau_r,s_r}^{u_x,u_z}) R_1 R_2 - J_{\tau,s}^{u_x,u_z} (C_{11} R_1 + (C_{12} + C_{55}) R_2) \right) \frac{\pi n}{L_1 R_1 R_2}, \\ K_{u_y,u_x}^{n,m,\tau,s} &= (C_{12} + C_{44}) J_{\tau,s}^{u_y,u_x} \frac{\pi^2 nm}{L_1 L_2}, \\ K_{u_y,u_y}^{n,m,\tau,s} &= \left(J_{\tau_r,s_r}^{u_y,u_y} - R_1 \left(J_{\tau,s}^{u_y,u_y} - J_{\tau_r,s}^{u_y,u_y} \right) + R_1^2 J_{\tau_r,s_r}^{u_y,u_y} \right) \frac{C_{55}}{R_1^2} + C_{44} J_{\tau,s}^{u_y,u_y} \frac{\pi^2 n^2}{L_1^2} + C_{22} J_{\tau,s}^{u_y,u_y} \frac{\pi^2 m^2}{L_2^2}, \\ K_{u_y,u_z}^{n,m,\tau,s} &= - \frac{\pi m \left(J_{\tau,s}^{u_y,u_z} (C_{12} R_2 + C_{22} R_1 + C_{55} R_1) + J_{\tau_r,s_r}^{u_y,u_z} R_1 R_2 C_{23} - R_1 R_2 J_{\tau_r,s}^{u_y,u_z} C_{55} \right)}{L_2 R_1 R_2}, \\ K_{u_z,u_x}^{n,m,\tau,s} &= \left((C_{66} J_{\tau_r,s_r}^{u_z,u_x} - C_{13} J_{\tau_r,s_r}^{u_z,u_x}) R_1 R_2 + ((C_{11} + C_{66}) R_2 + C_{12} R_1) J_{\tau,s}^{u_z,u_x} \right) \frac{\pi n}{L_1 R_1 R_2}, \\ K_{u_z,u_y}^{n,m,\tau,s} &= \left((C_{55} R J_{\tau_r,s_r}^{u_z,u_y} - C_{23} J_{\tau_r,s}^{u_z,u_y}) R_1 R_2 - (R_2 C_{12} + R_1 (C_{22} + C_{55})) J_{\tau,s}^{u_z,u_y} \right) \frac{\pi m}{L_2 R_1 R_2}, \\ K_{u_z,u_z}^{n,m,\tau,s} &= C_{33} J_{\tau_r,s_r}^{u_z,u_z} + \left(\frac{n^2 \pi^2 C_{66}}{L_1^2} + \frac{m^2 \pi^2 C_{55}}{L_2^2} + \frac{C_{11}}{R_1^2} + \frac{C_{22}}{R_2^2} + \frac{2C_{12}}{R_1 R_2} \right) J_{\tau,s}^{u_z,u_z} \\ &\quad + \frac{(J_{\tau_r,s}^{u_z,u_z} + J_{\tau_r,s_r}^{u_z,u_z}) (R_2 C_{13} + R_1 C_{23})}{R_1 R_2}. \end{aligned} \quad (81)$$

We have obtained here a system of linear algebraic equations in matrix form (78) with coefficients (79, 80 and 81), which represent the solution of the problem for shallow shells of the higher order with supported edges by the Navier analytical solution method. In the next Section, we solve the problem for the shallow spherical shell with rectangular planform.

4.3 Navier analytical solution for shallow spherical shells with rectangular planform

To illustrate the Navier analytical solution method, consider a higher-order model of a linear elastic spherical shell. Spherical shells are shells of revolution. They are formed by revolving around the axis of the circular segment. In the general case of a spherical shell, the coefficients of the first quadratic form of the middle surface and the principal curvatures are equal to $A_1 = R$, $A_2 = R \sin(\psi)$, and $k_1 = \frac{1}{R}$, $k_2 = \frac{1}{R}$, and spherical coordinates $x_1 = \varphi$, $x_2 = \psi$ and $x_3 = r$, $r \in [R - h, R + h]$ are used.

Unfortunately, in the general case of the spherical linear elastic shells considered above, Navier exact solution method is inapplicable. Therefore, we consider here shallow spherical shells with square planform. In this case, Cartesian coordinates x , y , z can be introduced to parametrize the middle surface of the shell. The parametric equations of the middle surface of the shell in this case have the form (61), the coefficients of the first quadratic form, the unit vector normal to the surface, and the principal curvatures of the middle surface of the shell are presented by Eq. (62).

The approach presented in the previous Section can also be applied here for the shallow spherical shell, only instead of a rectangular planform with dimensions L_1 and L_2 , we have a square planform with dimension L . Therefore, Eqs.(74–77) must be replaced by the following.

The displacements fields as

$$\begin{aligned} u_x(x, y, z) &= F_{u_x, \tau}(z) \sum_{n,m=0}^{\infty} U_{u_x, \tau}^{n,m} \cos\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}y\right), \\ u_y(x, y, z) &= F_{u_y, \tau}(z) \sum_{n,m=0}^{\infty} U_{u_y, \tau}^{n,m} \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{L}y\right), \\ u_z(x, y, z) &= F_{u_z, \tau}(z) \sum_{n,m=0}^{\infty} U_{u_z, \tau}^{n,m} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}y\right) \end{aligned} \quad (82)$$

where $U_{u_x, \tau}^n$, $U_{u_y, \tau}^n$, $U_{u_z, \tau}^n$ are the expansion coefficients in the Fourier series of functions $u_{x, \tau}(x, y)$, $u_{y, \tau}(x, y)$, $u_{z, \tau}(x, y)$ of the form

$$\begin{aligned} U_{u_x, \tau}^{n,m} &= \frac{4}{L^2} \int_0^L \cos\left(\frac{n\pi}{L}x\right) \int_0^L u_{x, \tau}(x, y) \sin\left(\frac{m\pi}{L}y\right) dy dx, \\ U_{u_y, \tau}^{n,m} &= \frac{4}{L^2} \int_0^L \sin\left(\frac{n\pi}{L}x\right) \int_0^L u_{y, \tau}(x, y) \cos\left(\frac{m\pi}{L}y\right) dy dx, \\ U_{u_z, \tau}^{n,m} &= \frac{4}{L^2} \int_0^L \sin\left(\frac{n\pi}{L}x\right) \int_0^L u_{z, \tau}(x, y) \sin\left(\frac{m\pi}{L}y\right) dy dx. \end{aligned} \quad (83)$$

The vector of the components of the external load is

$$\begin{aligned} \tilde{b}_{u_x, \tau}(x, y) &= \sum_{n,m=0}^{\infty} B_{u_x, \tau}^{n,m} \cos\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}y\right), \\ \tilde{b}_{u_y, \tau}(x, y) &= \sum_{n,m=0}^{\infty} B_{u_y, \tau}^{n,m} \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{L}y\right), \\ \tilde{b}_{u_z, \tau}(x, y) &= \sum_{n,m=0}^{\infty} B_{u_z, \tau}^{n,m} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}y\right) \end{aligned} \quad (84)$$

where $B_{u_x, \tau}^{n, m}$, $B_{u_y, \tau}^{n, m}$, $B_{u_z, \tau}^{n, m}$ are the expansion coefficients in the Fourier series of functions $\tilde{b}_{u_x, \tau}(x, y)$, $\tilde{b}_{u_y, \tau}(x, y)$, $\tilde{b}_{u_z, \tau}(x, y)$, they have the form

$$\begin{aligned} B_{u_x, \tau}^{n, m} &= \frac{4}{L^2} \int_0^L \cos\left(\frac{n\pi}{L}x\right) \int_0^L \tilde{b}_{u_x, \tau}(x, y) \sin\left(\frac{m\pi}{L}y\right) dy dx, \\ B_{u_y, \tau}^{n, m} &= \frac{4}{L^2} \int_0^{L_1} \sin\left(\frac{n\pi}{L}x\right) \int_0^L \tilde{b}_{u_y, \tau}(x, y) \cos\left(\frac{m\pi}{L}y\right) dy dx, \\ B_{u_z, \tau}^{n, m} &= \frac{4}{L^2} \int_0^L \sin\left(\frac{n\pi}{L}x\right) \int_0^L \tilde{b}_{u_z, \tau}(x, y) \sin\left(\frac{m\pi}{L}y\right) dy dx. \end{aligned} \quad (85)$$

The explicit expressions for the coefficients of the fundamental nucleus matrices $\mathbf{K}_{\tau, s}^{n, m, loc}$ are

$$\begin{aligned} K_{u_x, u_x}^{n, m, \tau, s} &= C_{11} J_{\tau, s}^{u_x, u_x} \frac{\pi^2 n^2}{L^2} + C_{44} J_{\tau, s}^{u_x, u_x} \frac{\pi^2 m^2}{L^2} + C_{66} \left(\frac{J_{\tau, s}^{u_x, u_x}}{R^2} + J_{\tau_r, s_r}^{u_x, u_x} - \frac{J_{\tau, s}^{u_x, u_x} + J_{\tau_r, s_r}^{u_x, u_x}}{R} \right), \\ K_{u_x, u_\varphi}^{n, m, \tau, s} &= (C_{12} + C_{44}) \frac{\pi^2 nm}{L^2} J_{\tau, s}^{u_x, u_\varphi}, \quad K_{u_x, u_r}^{n, m, \tau, s} = ((C_{66} J_{\tau_z, s}^{u_x, u_r} - C_{13} J_{\tau, s_r}^{u_x, u_r})R - J_{\tau, s}^{u_x, u_r} (C_{11} + C_{12} + C_{66})) \frac{\pi n}{LR}, \\ K_{u_\varphi, u_x}^{n, m, \tau, s} &= (C_{12} + C_{44}) J_{\tau, s}^{u_\varphi, u_x} \frac{\pi^2 nm}{L^2}, \\ K_{u_\varphi, u_\varphi}^{n, m, \tau, s} &= \left(J_{\tau_r, s_r}^{u_\varphi, u_\varphi} - R \left(J_{\tau, s_r}^{u_\varphi, u_\varphi} - J_{\tau_r, s}^{u_\varphi, u_\varphi} \right) + R^2 J_{\tau_r, s_r}^{u_\varphi, u_\varphi} \right) \frac{C_{55}}{R^2} + C_{44} J_{\tau, s}^{u_\varphi, u_\varphi} \frac{\pi^2 n^2}{L^2} + C_{22} J_{\tau, s}^{u_\varphi, u_\varphi} \frac{\pi^2 m^2}{L^2}, \\ K_{u_\varphi, u_r}^{n, m, \tau, s} &= - \frac{\pi m \left(J_{\tau, s}^{u_\varphi, u_r} (C_{12} + C_{22} + C_{55}) + J_{\tau_r, s_r}^{u_\varphi, u_r} RC_{23} - R J_{\tau_r, s}^{u_\varphi, u_r} C_{55} \right)}{RL}, \\ K_{u_r, u_x}^{n, m, \tau, s} &= (C_{66} J_{\tau, s_r}^{u_r, u_x} - C_{13} J_{\tau_r, s}^{u_r, u_x} + (C_{11} + C_{12} + C_{66}) J_{\tau, s}^{u_r, u_x} / R) \frac{\pi n}{L}, \\ K_{u_r, u_\varphi}^{n, m, \tau, s} &= \left(C_{55} R J_{\tau, s_r}^{u_r, u_\varphi} - RC_{23} J_{\tau_r, s}^{u_r, u_\varphi} - (C_{12} + C_{22} + C_{55}) J_{\tau, s}^{u_r, u_\varphi} \right) \frac{\pi m}{RL}, \\ K_{u_r, u_r}^{n, m, \tau, s} &= C_{33} J_{\tau_r, s_r}^{u_r, u_r} + \frac{(C_{11} + 2C_{12} + C_{22}) J_{\tau, s}^{u_r, u_r}}{R^2} + \frac{(J_{\tau_r, s}^{u_r, u_r} + J_{\tau_r, s_r}^{u_r, u_r}) (C_{13} + C_{23})}{R} \\ &\quad + \left(C_{66} \frac{\pi^2 n^2}{L^2} + C_{55} \frac{\pi^2 m^2}{L^2} \right) J_{\tau, s}^{u_r, u_r}. \end{aligned} \quad (86)$$

For the numerical calculation, we use the Navier solutions in exact form for the higher-order theories represented by Eqs. (78–80) and (82–86). Calculations are made for the following data: Amplitude of mechanical loading is taken $B_{u_x, 1}^{1, 1} = B_{u_\varphi, 1}^{1, 1} = B_{u_r, 1}^{1, 1} = 10^6$ (N/m²), geometrical parameters of the shell are $L = 1$ m, $R = 10L$, $h = L/10$. Material of the shell is considered isotropic, and mechanical properties of material are taken the same as in Carrera and Zozulya [14, 15], namely the Lamé constants are $\lambda = 762.6$ MPa and $\mu = 103.9$ MPa.

Figure 17 shows the plots of the distribution of the components of displacement vector $\mathbf{u}(x, y, z)$ in x axis direction for the third and fifth order— $H3$, $H5$, CLEC— $H1$ and Timoshenko-Mindlin— TM shear deformation models. Due to symmetry distribution in axis, x direction is the same and is not presented here. Figure 18 shows surface graphs of their distribution along plane x, y for fifth order model— $H5$ model.

Figure 19 shows the plots of the distribution of the components of stress tensors $\sigma_{xx}(x, y)$, $\sigma_{xy}(x, y)$ in x axis direction for the fifth order— $H5$ model. Figures 20 and 21 show the density plot of their distribution along plane x, y and y, z , respectively, for the fifth order— $H5$ model.

The data presented in Figs. 17, 18, 19, 20, and 21 provide qualitative and quantitative information on the behavior of the displacement vector components, as well as the components of the stress tensor. These results can be used as a benchmark example for finite element analysis of elastic shallow spherical shells.

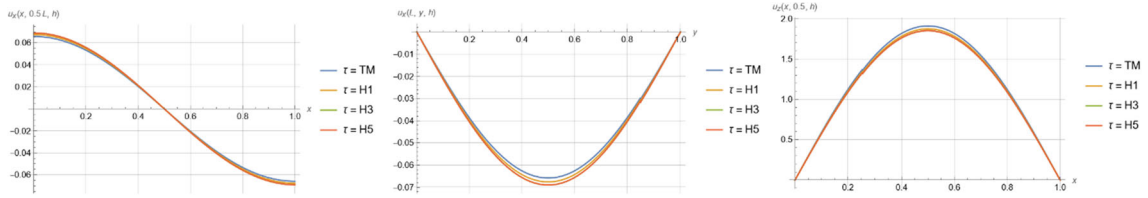


Fig. 17 Components u_x , u_y , and u_z of the displacement vector in x direction

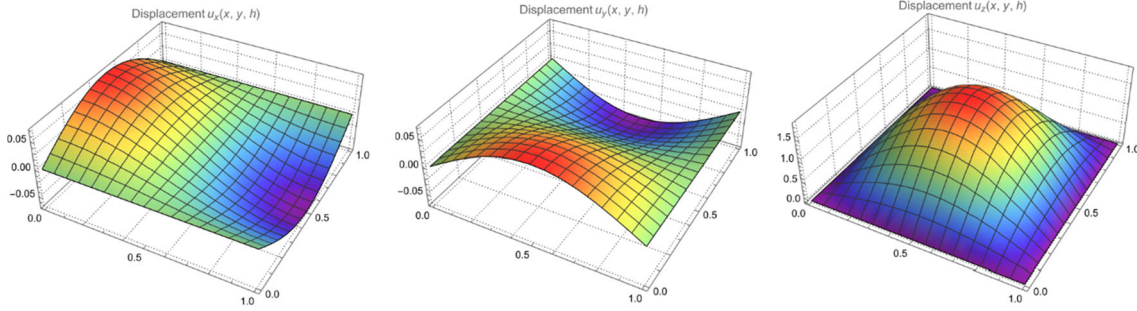


Fig. 18 Components u_x , u_y , and u_z of the displacement vector in x and y directions

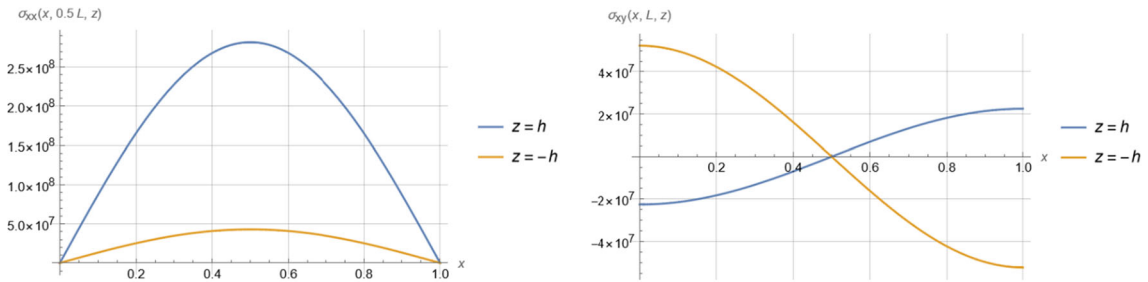


Fig. 19 Components σ_{xx} and σ_{xy} of the stress tensor in x direction

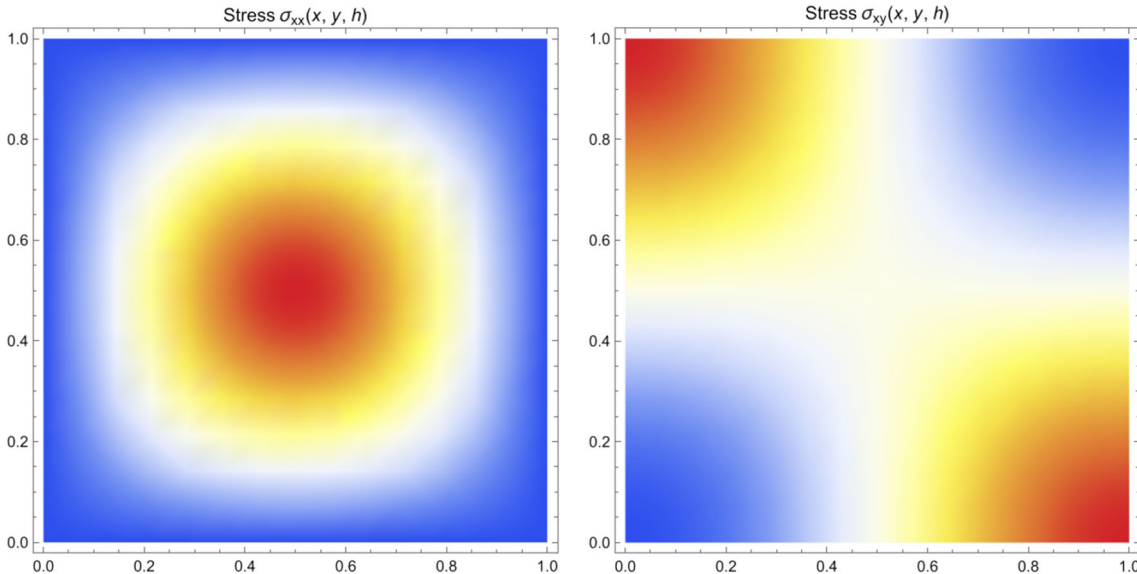


Fig. 20 Components σ_{xx} and σ_{xy} of the stress tensor in x and y directions

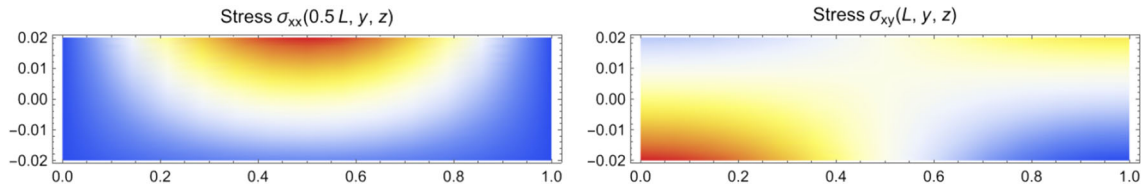


Fig. 21 Components σ_{xx} and σ_{xy} of the stress tensor in y and z directions

5 Conclusions

The Navier exact analytical solution method for the case of a freely supported cylindrical shell and shallow shells of revolution are applied here for the higher-order theory based on the CUF. All the coefficients for the corresponding matrix operators, that are included in the Navier close form solution, are presented in an analytical form.

To test the developed approach and the obtained equations, a segment of the higher-order cylindrical shell and axisymmetric cylindrical shell supported on the edges, as well as shallow spherical shells with rectangular planform, are considered here. Numerical calculations were performed using the computer algebra software Mathematica. The results of the displacements and stress calculations are presented in the form of graphs and surfers and density plots.

The resulting equations can be used for theoretical analysis and numerical calculation of the stress–strain state, as well as for modeling thin-walled structures that are used in science, engineering, and technology. These numerical results can be used as a benchmark example for finite element analysis of the elastic higher-order shells.

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