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ORIGINAL PAPER

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Carrera unified formulation (CUF) for shells of revolution. I. Higher-order theory

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Abstract Here, higher order models of elastic shells of revolution are developed using the variational principle of virtual power for 3-D equations of the linear theory of elasticity and generalized series in the coordinates of the shell thickness. Following the Unified Carrera Formula (CUF), the stress and strain tensors, as well as the displacement vector, are expanded into series in terms of the coordinates of the shell thickness. As a result, all the equations of the theory of elasticity are transformed into the corresponding equations for the expansion coefficients in a series in terms of the coordinates of the shell thickness. All equations for shells of revolution of higher order are developed and presented here for cases whose middle surfaces can be represented analytically. The resulting equations can be used for theoretical analysis and calculation of the stress–strain state, as well as for modeling thin-walled structures used in science, engineering, and technology.

1 Introduction

Shell structures are important elements that are widely used in science, engineering, and technology. Theoretical analysis and modeling of shells has a long history. Many mathematical models of shells were developed in the last century. Shell structures can have a complex geometry, involving the parametrization of the middle surface of the shell and the use of a complex mathematical apparatus of differential geometry. There are many excellent books on complex geometry shells. Among others, we can recommend the books of Kilchevskiy [31], Kornishin et al. [36], Rekach and Krivoshapko [57], in which shells of complex geometry are considered, which are necessary for understanding mathematical information. For a deeper study of the differential geometry of surfaces and its application to shell theory, one can refer to Gray et al. [24], Kuhnel [44], von Seggern [67], which contain not only theoretical information, but also the Mathematica computer algebra program, which greatly simplifies theoretical analysis and applications.

Shells of revolution represent a very important class of structural elements that are very interesting for theoretical analysis and have many applications in science and technology. Of particular interest are shells, the middle surface of which can be represented in an analytical form, in the form of mathematical equations. A description and mathematical analysis of such surfaces can be found in Krivoshapko and Ivanov [42]. General information and analysis of some special classes of shells of revolution can be found in Koryakin et al. [35], Kovarik [37], Reddy [55, 56], Rekach and Krivoshapko [57], Timoshenko and Woinovsky-Krieger [64],

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Vlasov [66], Wan and Weinichke [68]. There are many books and thousands of articles on the various types of revolution shells. We cannot present and analyze them here. Therefore, we will mention here only those that we used in preparing this article.

The standard reference to classical plate theory is Timoshenko and Woinowsky-Krieger[64]. Cylindrical shells considered in numerous publications, see for example Kovarik [37], Reddy [55, 56], Timoshenko and Woinowsky-Krieger [64], Vlasov [66]. Conical shells considered in numerous publications, see for example Kovarik [37], Reddy [56], Vlasov [66]. Spherical shells considered in numerous publications, see for example Kovarik [37], Reddy [56], Timoshenko and Woinowsky-Krieger [64], Vlasov [66], Xie et al. [71]. Ellipsoidal shells considered in numerous publications, see for example Klochkov et al. [32–34], Krivoshapko [40], Meish [47], Meish and Maiborodina [48], Tangbanjongkij et al. [63]. Paraboloidal shells considered in numerous publications, see for example Al-Khatib et al. [1], Chernobryvko et all. [15, 16], Kang and Leissa [27, 28], Krivoshapko [41], Xie et al. [71]. Hyperboloidal shells considered for example in Kang and Leissa [29], Krivoshapko [39]. Toroidal shells considered in numerous publications, see for example Clark [17, 18], El-Raheb and Wagner [20, 21], Leung [45], Lutskaya et al. [46], Ming et al. [49], Naboulsi et al. [50], Senjanovic et al. [58, 59], Sun [26, 61], Sutcliffe [62], Wenmin et al. [70]. Elliptic toroidal shells considered in Zingoni A., et al. [72]. Catenoidal shells considered in Zun [60]. Pseudospherical shells considered for example in Gil-oulbe et al. [23], Krawczyk [38], Krivoshapko and Ivanov [43].

Most of the publications mentioned above used classical shell theories based on the Kirchhoff–Love and Midlin–Timoshenko hypotheses. There is another approach to the theory of shells, which consists in expanding the components of the stress–strain field into series of polynomials in thickness. This approach was first proposed by Cauchy [14] and Poisson [54], but at that time it did not find wide application and interest in the scientific community and did not receive further development for a long time. Significant extensions and developments of this approach for shells of arbitrary geometry were made by Kilchevskiy [31]. He created the so-called generalized tensor series for the expansion of three-dimensional equations of elasticity in terms of the thickness of the shell. Then the Legendre polynomials were proposed for the development of new theories of higher orders of Vekua plates and shells [65]. This approach has significant advantages, since the Legendre polynomials are orthogonal and, as a result, simpler equations are obtained. There are many books and research papers devoted to the application of the polynomial series to the development of higher order theories of bars, plates and shells. Among others, the books of Gulyaev et al. [25], Khoma [30], Pelekh and Lazko [51], Pelekh and Sukhorol'skii [52], Vekua [65] and the works of Czekanski and Zozulya [19], Zozulya [73–85], Zozulya and Saez [87, 88], Zozulya and Zhang [90]. Carrera's Unified Formulations (CUF) approach can be viewed as a generalization of the polynomial decomposition method for beams, plates and shells, including sandwich structures and multi-field loads. Hundreds of articles are available at (CUF) on the various extensions and applications of Carrera and many more. Among them, the following are mentioned here: Carrera [2–7, 13] deals with multilayer anisotropic plates and shells, and Carrera and Zozulya [10–14] and Zozulya and Carrera [86] deal with micropolar beams, plates, and shells. For more information and references related to the polynomial series approach for developing models of multilayer anisotropic composite plates and shells and their finite element analysis, see Carrera et al. [7, 53], as well as the works mentioned above.

In this work, 2-D models of elastic shells of rotation of higher order based on the 3-D theory of elasticity are developed. Higher-order models are based on the variational principle of virtual power with expansion of the equations of the 3-D theory of elasticity into generalized series in terms of cross-sectional coordinates in thickness. The equations of 2-D models of higher orders of shells of revolution are developed and presented, the middle surfaces of which can be represented analytically. The resulting equations can be used for theoretical analysis and calculation of the stress–strain state, as well as for modeling thin-walled structures that are used in science, engineering, and technology.

2 3-D theory of elasticity in coordinates related to the middle surface of shell

Here we are developing higher-order theories of elastic shells of revolution, which are based on polynomial expansion and CUF. Therefore, we consider first 3-D equations of the linear theory of elasticity. Let an elastic body occupy the domain $V = \Omega \times [-h, h]$, in a 3-D Euclidian space. Here Ω is the middle surface of the shell and $2h$ is the thickness of the shell.

For the purpose of the theories that are developed here, we introduce an orthogonal curvilinear system of coordinates $\mathbf{x}(x_1, x_2, x_3)$ related to the middle surface of the shell. The position vector of an arbitrary point is equal to $\mathbf{R}(x_1, x_2, x_3) = \mathbf{e}_i x_i$. Coordinates $\mathbf{x}_\alpha(x_1, x_2), \alpha = 1, 2$ are associated with the principle curvatures

k_1 and k_2 of the middle surface of the shell and coordinate x_3 is perpendicular to it. The position vector $\mathbf{R}(\mathbf{x})$ of any point in domain *V,* occupied by material points of the shell may be presented as

$$\mathbf{R}(\mathbf{x}) = \mathbf{r}(\mathbf{x}_\alpha) + x_3\mathbf{n}(\mathbf{x}_\alpha), \tag{1}$$

where $\mathbf{r}(\mathbf{x}_\alpha)$ is the position vector of the points located in the middle surface of the shell, and $\mathbf{n}(\mathbf{x}_\alpha)$ is a unit vector normal to the middle surface of the shell.

Unit orthogonal basic vectors and their derivatives with respect to space coordinates are equal to

$$\mathbf{e}_i = \frac{1}{H_i}\frac{\partial \mathbf{R}}{\partial x_i}, \quad \frac{\partial \mathbf{e}_i}{\partial x_j} = \Gamma_{ij}^k \mathbf{e}_k, \tag{2}$$

where H_i are Lamé coefficients and Γ_{ij}^k are Christoffel symbols. They are calculated by the equations

$$H_i = \left|\frac{\partial \mathbf{R}}{\partial x_i}\right| = \sqrt{\frac{\partial \mathbf{R}}{\partial x_i}\cdot\frac{\partial \mathbf{R}}{\partial x_i}}, \quad \Gamma_{ij}^k = -\frac{1}{H_i}\frac{\partial H_i}{\partial x_j}\delta_{ik} + \frac{1}{2H_iH_k}\left(\delta_{jk}\frac{\partial H_jH_k}{\partial x_i} + \delta_{ik}\frac{\partial H_iH_k}{\partial x_j} - \delta_{ij}\frac{\partial H_iH_j}{\partial x_k}\right). \tag{3}$$

In the orthogonal curvilinear coordinate system from the last equation it follows that $\Gamma_{ij}^k = 0$ for $i \neq j \neq k$ and

$$\Gamma_{ii}^k = -\frac{1}{H_k}\frac{\partial H_i}{\partial x_k}, \quad \Gamma_{ik}^k = \frac{1}{H_i}\frac{\partial H_k}{\partial x_i} \quad \text{for } i \neq k. \tag{4}$$

In this curvilinear system of coordinates the 3-D equations of elasticity can be simplified by considering that Lamé coefficients and their derivatives have the form

$$\begin{aligned} &H_\alpha = A_\alpha(1 + k_\alpha x_3) \text{ for } \alpha = 1, 2 \text{ and } H_3 = 1, \\ &\frac{\partial H_\beta}{\partial x_\alpha} = \frac{\partial A_\beta}{\partial x_\alpha}(1 + k_\alpha x_3), \ \frac{\partial H_\beta}{\partial x_3} = k_\beta A_\beta, \ \frac{\partial H_3}{\partial x_i} = 0. \end{aligned} \tag{5}$$

Here $A_\alpha(x_1, x_2) = \sqrt{\frac{\partial \mathbf{r}(x_1,x_2)}{\partial x_\alpha}\cdot\frac{\partial \mathbf{r}(x_1,x_2)}{\partial x_\alpha}}$ are the coefficients of the first quadratic form of a surface, k_α are the principal curvatures and $\alpha = 1, 2$.

Given that we have considered relatively thin shells, we can make the following assumptions:

$$1 + k_\alpha x_3 \approx 1 \rightarrow H_\alpha \approx A_\alpha, \quad \frac{\partial H_\beta}{\partial x_\alpha} = \frac{\partial A_\beta}{\partial x_\alpha}, \quad \frac{\partial H_\alpha}{\partial x_3} = k_\alpha A_\alpha, \quad \alpha, \beta = 1, 2. \tag{6}$$

Considering the assumptions (5) and (6) the Christoffel symbols (4) become

$$\Gamma_{\alpha\alpha}^\beta = \frac{1}{A_\alpha}\frac{\partial A_\beta}{\partial x_\alpha}, \quad \Gamma_{\alpha\alpha}^3 = -k_\alpha A_\alpha, \quad \Gamma_{3\alpha}^\alpha = k_\alpha A_\alpha, \quad \alpha, \beta = 1, 2. \tag{7}$$

The classical theory of elasticity assumes that the body consists of interconnected points and continuously fills the occupied volume. The position of a point during deformation is determined by the displacements vector $\mathbf{u}(x_1, x_2, x_3)$ as functions of their coordinates. Internal forces (the interaction between adjacent elements) are determined using a force stress tensor $\boldsymbol{\sigma}(x_1, x_2, x_3)$. Deformations are completely described by symmetric strain $\boldsymbol{\varepsilon}(x_1, x_2, x_3)$ tensor. An elastic body can be affected by surface and volume forces, which are represented by vectors $\mathbf{p}(x_1, x_2, x_3) = \boldsymbol{\sigma}(x_1, x_2, x_3)\cdot\mathbf{n}(x_1, x_2, x_3)$ and $\mathbf{b}(x_1, x_2, x_3)$ respectively. The tensors and vectors introduced above have the form

$$\sigma = \begin{vmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{vmatrix}, \quad \varepsilon = \begin{vmatrix} \varepsilon_{11} & \varepsilon_{21} & \varepsilon_{31} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{32} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{vmatrix}, \ \mathbf{u} = \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix}, \quad \mathbf{p} = \begin{vmatrix} p_1 \\ p_2 \\ p_3 \end{vmatrix}, \quad \mathbf{b} = \begin{vmatrix} b_1 \\ b_2 \\ b_3 \end{vmatrix}. \tag{8}$$

For convenience, we introduce vector notations here and represent the above functions that determine the stress–strain state of elastic media in the vector form. The classical force stress and strain tensors are symmetrical, so they can be presented as six component vectors:

$$\boldsymbol{\sigma} = \left|\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{xz}\right|^T, \quad \boldsymbol{\varepsilon} = \left|\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{xz}\right|^T. \tag{9}$$

Kinematic relations in the theory of elasticity relate the displacement vectors to the strain vector introduced in (8) and (9) by the following equations

$$\boldsymbol{\varepsilon} = \mathbf{D} \cdot \mathbf{u} \tag{10}$$

where $\mathbf{D}$ is a matrix differential operator of the form

$$\mathbf{D} = \begin{vmatrix} \frac{1}{A_1}\frac{\partial}{\partial x_1} & \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial x_2} & k_1 \\ \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial x_1} & \frac{1}{A_2}\frac{\partial}{\partial x_2} & k_2 \\ 0 & 0 & \frac{\partial}{\partial x_3} \\ \frac{1}{A_2}\frac{\partial}{\partial x_2} - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial x_2} & \frac{1}{A_1}\frac{\partial}{\partial x_1} - \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_3} - k_2 & \frac{1}{A_2}\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} - k_1 & 0 & \frac{1}{A_1}\frac{\partial}{\partial x_1} \end{vmatrix}. \tag{11}$$

Applying the matrix operator (11) to the displacement vector, the symmetric strain $\boldsymbol{\varepsilon}$ tensor can be presented in vector form:

$$\boldsymbol{\varepsilon} = \begin{vmatrix} \frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + \frac{u_2}{A_1 A_2}\frac{\partial A_1}{\partial x_2} + k_1 u_3 \\ \frac{1}{A_2}\frac{\partial u_2}{\partial x_2} + \frac{u_1}{A_1 A_2}\frac{\partial A_2}{\partial x_1} + k_2 u_3 \\ \frac{\partial u_3}{\partial x_3} \\ \frac{1}{A_1}\frac{\partial u_2}{\partial x_1} - \frac{u_1}{A_1 A_2}\frac{\partial A_2}{\partial x_2} + \frac{1}{A_1}\frac{\partial u_1}{\partial x_2} - \frac{u_2}{A_1 A_2}\frac{\partial A_2}{\partial x_1} \\ \frac{\partial u_2}{\partial x_3} - k_2 u_2 + \frac{1}{A_2}\frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} - k_1 u_1 + \frac{1}{A_1}\frac{\partial u_3}{\partial x_1} \end{vmatrix}. \tag{12}$$

Constitutive relations are usually established by introducing the potential energy density function. In the case of linear orthotropic elastic media, it can be presented in the general form

$$W(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon}^T \cdot \mathbf{C} \cdot \boldsymbol{\varepsilon}, \tag{13}$$

where $\mathbf{C}$ is the 6×6 matrix of elasticity moduli of the form

$$\mathbf{C} = \begin{vmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{vmatrix}. \tag{14}$$

Taking the derivative of the potential energy density function with respect to the strain $\boldsymbol{\varepsilon}$ tensor and substituting the kinematic relations (10) into the obtained result, the classical stress vector can be presented the following form:

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}} = \mathbf{C} \cdot \boldsymbol{\varepsilon} = \mathbf{C} \cdot \mathbf{D} \cdot \mathbf{u}. \tag{15}$$

Substituting the equations for the matrix of material constants (14), the operator (11) and the displacement vector into Eq. (15), we obtain equations for the classical stress vector expressed as a function of the displacement vector components.

In the case of isotropic material, the corresponding classical moduli of elasticity presented in Eq. (14) have the form

$$C_{11} = C_{22} = C_{33} = \lambda + 2\mu, \quad C_{12} = C_{13} = C_{23} = \lambda, \quad C_{44} = C_{55} = C_{66} = \mu, \tag{16}$$

where λ and μ are Lamé constants of classical elasticity.

The CUF approach for the development of higher order theories of plates and shells is based on the principle of virtual displacements (PVD), (see Washizu [69] for references), which in the general case of the 3-D linear elasticity can be presented in the form

$$\delta \int_V W(\boldsymbol{\varepsilon}) dV = \delta L_{ext}(\mathbf{b}, \mathbf{p}). \tag{17}$$

Here $W(\boldsymbol{\varepsilon})$ is the density of potential energy of the elastic body, $L_{ext}(\mathbf{b}, \mathbf{p})$ is the work of the external volume $\mathbf{b}$ and $\mathbf{p}$ surface loads.

We will show here how all the 3-D equations elasticity and natural boundary conditions can be found by calculating the corresponding variations of the potential energy density function. The equations obtained and the developed approach will be used to develop the theory of higher order elastic plates and shells.

The variation of the potential energy density of the elastic 3-D body is determined though the variations of the strain $\boldsymbol{\varepsilon}$ tensor in the form

$$\delta W = \delta \boldsymbol{\varepsilon} \cdot \boldsymbol{\sigma}. \tag{18}$$

Strain tensor variations can be represented in term of displacement vector variations. Substituting kinematic relations (10) into equation (18) one can obtain

$$\delta W = \mathbf{D} \cdot \delta \mathbf{u} \cdot \boldsymbol{\sigma}. \tag{19}$$

Variation of the work of the external volume and surface load in the case of elastic media can be calculated as follows:

$$\delta L_{ext}(\mathbf{b}_u, \mathbf{p}_u) = \int_V \mathbf{b}_u \cdot \delta \mathbf{u} dV + \int_{\partial V} \mathbf{p}_u \cdot \delta \mathbf{u} dS. \tag{20}$$

Now, substituting (19) and (20) into (17) the PVD for elastic media can be represented as

$$\int_V \mathbf{D} \cdot \delta \mathbf{u} \cdot \boldsymbol{\sigma} dV = \int_V \mathbf{b}_u \cdot \delta \mathbf{u} dV + \int_{\partial V} \mathbf{p}_u \cdot \delta \mathbf{u} dS. \tag{21}$$

This equation can be represented in the following form

$$\int_V \mathbf{D}^T \cdot (\boldsymbol{\sigma} \cdot \delta \mathbf{u}) dV - \int_{\partial V} (\mathbf{D}^T \cdot \boldsymbol{\sigma}) \cdot \delta \mathbf{u} dS = \int_V \mathbf{b}_u \cdot \delta \mathbf{u} dV + \int_{\partial V} \mathbf{p}_u \cdot \delta \mathbf{u} dS. \tag{22}$$

Here we use the following relations:

$$(\mathbf{D} \cdot \delta \mathbf{u}) \cdot \boldsymbol{\sigma} = (\mathbf{D}^T \cdot \boldsymbol{\sigma}) \cdot \delta \mathbf{u} - \mathbf{D}^T \cdot (\boldsymbol{\sigma} \cdot \delta \mathbf{u}). \tag{23}$$

Applying to the second term of the left-hand side of the variational equation (22) matrix analogy of the Gauss–Ostrogradsky divergence theorem and collecting the members that contain variations $\delta \mathbf{u}$ and $\delta \boldsymbol{\omega}$ separately, we obtain that the volume and surface integrals have the form

$$\int_V ((\mathbf{D}^T \cdot \boldsymbol{\sigma} + \mathbf{b}) \cdot \delta \mathbf{u} dV = \int_{\partial V} (\mathbf{D}_n^T \cdot \boldsymbol{\sigma} - \mathbf{p}) \cdot \delta \mathbf{u} dV, \tag{24}$$

where $\mathbf{D}_n^T$ is the matrix analogue of the vector normal to the boundary, it has the matrix form

$$\mathbf{D}_n^T = \begin{vmatrix} n_1 & 0 & 0 & n_2 & 0 & n_3 \\ 0 & n_2 & 0 & n_1 & n_3 & 0 \\ 0 & 0 & n_3 & 0 & n_2 & n_1 \end{vmatrix}. \tag{25}$$

Since the variations of $\delta \mathbf{u}$ and $\delta \boldsymbol{\omega}$ in (24) are arbitrary, in order to fulfill (24) it is necessary and sufficient that the corresponding volume and surface integrals are equal to zero:

$$\int_V (\mathbf{D}^T \cdot \boldsymbol{\sigma} + \mathbf{b}) dV = 0, \quad \int_{\partial V} (\mathbf{D}_n^T \cdot \boldsymbol{\sigma} - \mathbf{p}) \cdot dV = 0. \tag{26}$$

These integral equalities imply differential equations of equilibrium and natural boundary conditions for the linear theory of elasticity in the form

$$\mathbf{D}^T \cdot \boldsymbol{\sigma} + \mathbf{b} = 0, \quad \mathbf{D}_n^T \cdot \boldsymbol{\sigma} - \mathbf{p} = 0. \tag{27}$$

To obtain the equations of equilibrium for the linear theory of elasticity in the form of a displacement vector, we substitute the expressions for the classical stress vector from the generalized Hooke's law (15) into Eq. (27). As a result, the following differential equations are obtained:

$$\mathbf{D}^T \cdot \mathbf{C} \cdot \mathbf{D} \cdot \mathbf{u} + \mathbf{b} = 0. \tag{28}$$

In the same way, substituting the expressions for the classical stress vector from Eq. (15) one obtains natural boundary conditions for the linear theory of elasticity in the form of a displacement vector:

$$\mathbf{D}_n^T \cdot \mathbf{C} \cdot \mathbf{D} \cdot \mathbf{u} - \mathbf{p} = 0. \tag{29}$$

Here $\mathbf{p}$ is the classical traction vector given on the boundary.

In order for boundary value problem for differential equations of equilibrium of the linear theory of elasticity in a form of displacements vector (28) to have a unique solution, it is also necessary to set essential boundary conditions. In the case considered here, they can be taken in the form

$$\mathbf{u} - \mathbf{u}^0 = 0. \tag{30}$$

The differential equations of equilibrium (28) and the natural boundary conditions (29) can be represented in a compact form as

$$\mathbf{L} \cdot \mathbf{u} = \mathbf{b}, \quad \mathbf{B} \cdot \mathbf{u} = \mathbf{p}, \tag{31}$$

where $\mathbf{L}$ and $\mathbf{B}$ are the matrix differential operators, $\mathbf{u}$ is the vector of unknown functions and $\mathbf{b}$ and $\mathbf{p}$ are the vectors of external load and surface traction, respectively. They have the following form:

$$\begin{aligned} \mathbf{L} &= \begin{vmatrix} L_{u_1,u_1} & L_{u_1,u_2} & L_{u_1,u_3} \\ L_{u_2,u_1} & L_{u_2,u_2} & L_{u_2,u_3} \\ L_{u_3,u_1} & L_{u_3,u_2} & L_{u_3,u_3} \end{vmatrix}, \quad \mathbf{u} = \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix} \quad \mathbf{b} = \begin{vmatrix} b_1 \\ b_2 \\ b_3 \end{vmatrix}, \\ \mathbf{B} &= \begin{vmatrix} B_{u_1,u_1} & B_{u_1,u_2} & B_{u_1,u_3} \\ B_{u_2,u_1} & B_{u_2,u_2} & B_{u_2,u_3} \\ B_{u_3,u_1} & B_{u_3,u_2} & B_{u_3,u_3} \end{vmatrix}, \quad \mathbf{p} = \begin{vmatrix} p_1 \\ p_2 \\ p_3 \end{vmatrix}. \end{aligned} \tag{32}$$

Analytical expressions for matrix differential operators $\mathbf{L}$ and $\mathbf{B}$ for the linear orthotropic theory of elasticity in general case of the orthogonal curvilinear coordinates system are very complicate and are not given here. In the case of the Cartesian coordinates, they are presented in Carrera and Zozulya [8].

In following sections, the 3-D equations of the theory of elasticity presented here will be used to develop a higher-order theory of shells of revolution using the CUF approach.

3 CUF for elastic shells of higher order of arbitrary geometry

Following the CUF approach, the displacement fields, which are functions of curvilinear coordinates (x_1, x_2, x_3) are represented as series of functions of the coordinated x_3 directed orthogonally to the middle surface of the shell, in the form

$$\mathbf{u}(x_1, x_2, x_3) = \mathbf{F}_{u,\tau}(x_3) \cdot \mathbf{u}_\tau(x_1, x_2), \qquad \tau = 1, 2, \ \ldots, M, \tag{33}$$

where $\mathbf{F}_{\mathbf{u},\tau}(x_3)$ are the basic functions of the thickness coordinates and $\mathbf{u}_\tau(x_1, x_2)$ is the vector of displacements. They have the form

$$\mathbf{F}_{\mathbf{u},\tau}(x_3) = \begin{vmatrix} F_{u_1,\tau}(x_3) & 0 & 0 \\ 0 & F_{u_2,\tau}(x_3) & 0 \\ 0 & 0 & F_{u_3,\tau}(x_3) \end{vmatrix}, \quad \mathbf{u}_\tau(x_1, x_2) = \begin{vmatrix} u_{1,\tau}(x_1, x_2) \\ u_{2,\tau}(x_1, x_2) \\ u_{3,\tau}(x_1, x_2) \end{vmatrix}. \tag{34}$$

In (33), according to Einstein's notation, the repeated subscript τ indicates summation. In the general case, the choice of the number M and functions $\mathbf{F}_{u,\tau}(x_3)$ is arbitrary, i.e., for modeling the kinematic field of the shells along their thickness different base functions of any order can be considered. The final equation becomes simple if functions $\mathbf{F}_{u,\tau}$ are polynomials, especially orthogonal polynomials. The expansions coefficients $\mathbf{u}_\tau(x_1, x_2)$ as functions of the coordinates x_1 and x_2 coincided with the middle surface of the shell. The first subscript in basic functions $\mathbf{F}_{u,\tau}$ indicates the component of the displacement vector, the second index indicates the number of the function in the serial expansion.

Applying matrix differential operators (11) to the displacement vector represented by equations (33) one can obtain the strain vector in the form

$$\boldsymbol{\varepsilon} = \mathbf{D}_{u,\tau} \cdot \mathbf{u}_\tau, \tag{35}$$

where $\mathbf{D}_{a,\tau}$ is a matrix operator of the form

$$\mathbf{D}^T_{u,\tau} = \begin{vmatrix} \frac{F_{u_1,\tau}}{A_1}\frac{\partial}{\partial x_1} & \frac{F_{u_2,\tau}}{A_1A_2}\frac{\partial A_2}{\partial x_1} & 0 & \frac{F_{u_1,\tau}}{A_2}\frac{\partial}{\partial x_2} - \frac{F_{u_2,\tau}}{A_1A_2}\frac{\partial A_1}{\partial x_2} & 0 & \frac{\partial F_{u_1,\tau}}{\partial x_3} - F_{u_3,\tau}k_1 \\ \frac{F_{u_1,\tau}}{A_1A_2}\frac{\partial A_1}{\partial x_2} & \frac{F_{u_2,\tau}}{A_2}\frac{\partial}{\partial x_2} & 0 & \frac{F_{u_2,\tau}}{A_1}\frac{\partial}{\partial x_1} - \frac{F_{u_1,\tau}}{A_1A_2}\frac{\partial A_2}{\partial x_1} & \frac{\partial F_{u_2,\tau}}{\partial x_3} - F_{u_3,\tau}k_2 & 0 \\ F_{u_1,\tau}k_1 & F_{u_2,\tau}k_2 & \frac{\partial F_{u_3,\tau}}{\partial x_3} & 0 & \frac{F_{u_3,\tau}}{A_2}\frac{\partial}{\partial x_2} & \frac{F_{u_1,\tau}}{A_1}\frac{\partial}{\partial x_1} \end{vmatrix}. \tag{36}$$

Substituting the kinematic relations (35) into the generalized Hooke's law (15), the classical force stress vector can be presented as

$$\boldsymbol{\sigma} = \mathbf{C} \cdot \mathbf{D}_{u,\tau} \cdot \mathbf{u}_\tau. \tag{37}$$

Substituting the expression for the strain vector represented by equations (35) and the classical force stress vector represented by equation (37) into (18), we obtain a variation of the potential energy density in the form

$$\delta W = \mathbf{D}^T_{u,\tau} \cdot \mathbf{F}_{u,\tau}(x_3) \cdot \delta\mathbf{u}_\tau(x_1, x_2) \cdot \mathbf{C} \cdot (\mathbf{D}_{u,s} \cdot \mathbf{F}_{u,s}(x_3) \cdot \mathbf{u}_s(x_1, x_2). \tag{38}$$

The body force is a function of curvilinear coordinates $\mathbf{b}(x_1, x_2, x_3)$. They can also be represented as a series of functions of the thickness coordinate x_3 in the form

$$\mathbf{b}(x_1, x_2, x_3) = \mathbf{F}_{u,\tau}(x_3) \cdot \mathbf{b}_{u,\tau}(x_1, x_2), \qquad \tau = 1, 2, \ldots, M. \tag{39}$$

The surface load is the function only the coordinates x_1 and x_2, of the middle surface of shell and the thickness coordinate x_3 has specific values that correspond to points on the surfaces $x_3 = -h$ and $x_3 = h$.

Let us consider a variation of the work of the external body and surface load in the case of elastic media. Taking into account (39) and (20) the variation of the work of the external body and surface loads in the case of elastic media can be presented as

$$\delta L_{ext}(\mathbf{b}, \mathbf{p}) = \int_V \mathbf{F}_{u,\tau}(x_3) \cdot \delta\mathbf{u}_\tau(x_1, x_2) \cdot \mathbf{F}_{u,\tau}(x_3) \cdot \mathbf{b}_{u,\tau}(x_1, x_2) dV + \int_{\partial V} \mathbf{F}_{u,\tau}(x_3) \cdot \delta\mathbf{u}_\tau(x_1, x_2) \cdot \mathbf{p}(x_1, x_2) dS. \tag{40}$$

Now, using the CUF ear theory of elasticity approach can be represented in the form

$$\int_V (\mathbf{D}^T_{u,\tau} \cdot \mathbf{F}_{u,\tau}(x_3) \cdot \delta\mathbf{u}_\tau(x_1, x_2) dV = \int_V \mathbf{F}_{u,\tau}(x_3) \cdot \delta\mathbf{u}_\tau(x_1, x_2) \cdot \mathbf{F}_{u,\tau}(x_3) \cdot \mathbf{b}_{u,\tau}(x_1, x_2) dV + \int_{\partial V} \mathbf{F}_{u,\tau}(x_3) \cdot \delta\mathbf{u}_\tau(x_1, x_2) \cdot \mathbf{p}_u(x_1, x_2) dS. \tag{41}$$

For convenience, we split up the matrix operator $\mathbf{D}^T_{u,\tau}$ into two parts,

$$\mathbf{D}^T_{u,\tau} = \mathbf{D}^{y,T}_{u,\tau} + \mathbf{D}^{C,T}_{u,\tau}, \tag{42}$$

where

$$\mathbf{D}_{u,\tau}^{y,T} = \begin{vmatrix} \frac{F_{u_1,\tau}}{A_1}\frac{\partial}{\partial x_1} & \frac{F_{u_2,\tau}}{A_1A_2}\frac{\partial A_2}{\partial x_1} & 0 & \frac{F_{u_1,\tau}}{A_2}\frac{\partial}{\partial x_2} - \frac{F_{u_2,\tau}}{A_1A_2}\frac{\partial A_1}{\partial x_2} & 0 & -F_{u_3,\tau}k_1 \\ \frac{F_{u_1,\tau}}{A_1A_2}\frac{\partial A_1}{\partial x_2} & \frac{F_{u_2,\tau}}{A_2}\frac{\partial}{\partial x_2} & 0 & \frac{F_{u_2,\tau}}{A_1}\frac{\partial}{\partial x_1} - \frac{F_{u_1,\tau}}{A_1A_2}\frac{\partial A_2}{\partial x_1} & -F_{u_3,\tau}k_2 & 0 \\ F_{u_1,\tau}k_1 & F_{u_2,\tau}k_2 & 0 & 0 & \frac{F_{u_3,\tau}}{A_2}\frac{\partial}{\partial x_2} & \frac{F_{u_1,\tau}}{A_1}\frac{\partial}{\partial x_1} \end{vmatrix},$$
$$\mathbf{D}_{u,\tau}^{C,T} = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & \frac{\partial F_{u_1,\tau}}{\partial x_3} \\ 0 & 0 & 0 & 0 & \frac{\partial F_{u_2,\tau}}{\partial x_3} & 0 \\ 0 & 0 & \frac{\partial F_{u_3,\tau}}{\partial x_3} & 0 & 0 & 0 \end{vmatrix}. \tag{43}$$

The volume integral in equations (41) is transformed using the following relations:

$$\begin{aligned} &\mathbf{D}_{u,\tau}^{y,T} \cdot (\mathbf{F}_{u,\tau}(x_3) \cdot \delta\mathbf{u}_\tau(x_1,x_2)) \cdot \mathbf{C} \cdot (\mathbf{D}_{u,s} \cdot \mathbf{F}_{u,s}(x_3) \cdot \mathbf{u}_s(x_1,x_2) \\ &= \mathbf{D}_{u,\tau}^{y} \cdot (\mathbf{C} \cdot (\mathbf{D}_{u,s} \cdot \mathbf{F}_{u,s}(x_3) \cdot \mathbf{u}_s(x_1,x_2) - \mathbf{D}_{u,\tau}^{y} \cdot (\mathbf{C} \cdot (\mathbf{D}_{u,s} \cdot \mathbf{F}_{u,s}(x_3) \cdot \mathbf{u}_s(x_1,x_2). \end{aligned} \tag{44}$$

Then, collecting the terms containing variations $\delta\mathbf{u}_\tau$ separately, we obtain that the following equations for the integrals over volume and surface, respectively:

$$\begin{aligned} &\int_V \Big((\mathbf{D}_{u,\tau}^{C,T} \cdot \mathbf{F}_{u,\tau}(x_3) \cdot \mathbf{C} \cdot \mathbf{D}_{u,s} \cdot \mathbf{F}_{u,s}(x_3) \cdot \mathbf{u}_s(x_1,x_2) - \mathbf{D}_{u,\tau}^{y,T} \cdot \mathbf{F}_{u,\tau}(x_3) \cdot \mathbf{C} \cdot (\mathbf{D}_{u,s} \cdot \mathbf{F}_{u,s}(x_3) \cdot \mathbf{u}_s(x_1,x_2) \\ &- \mathbf{F}_{u,\tau}(x_3) \cdot \mathbf{F}_{u,s}(x_3) \cdot \mathbf{b}_{u,s}(x_1,x_2)) \Big) \cdot \delta\mathbf{u}_\tau(x_1,x_2) dV = 0, \\ &\int_{\partial V} (\mathbf{D}_{n,\tau}^{u,T} \cdot \mathbf{C} \cdot \mathbf{D}_{u,s} \cdot \mathbf{F}_{u,s}(x_3) \cdot \mathbf{u}_s(x_1,x_2) - \mathbf{F}_{u,\tau}(x_3) \cdot \mathbf{p}_u(x_1,x_2) \cdot \delta\mathbf{u}_\tau(x_1,x_2)) dS = 0. \end{aligned} \tag{45}$$

Here, the volume integrals are transformed into integrals over the surface using the matrix analogy of Gauss-Ostrogradsky divergence theorem in the form

$$\int_V \mathbf{D}_{u,\tau}^{y} \cdot \mathbf{C} \cdot \mathbf{D}_{u,s} \cdot \mathbf{F}_{u,s}(x_3) \cdot \mathbf{u}_s(x_1,x_2) \cdot \delta\mathbf{u}_\tau(x_1,x_2) dV = \int_{\partial V} \mathbf{D}_{n,\tau}^{u,T} \cdot \mathbf{C} \cdot \mathbf{D}_{u,s} \cdot \mathbf{F}_{u,s}(x_3) \cdot \mathbf{u}_s(x_1,x_2) \cdot \delta\mathbf{u}_\tau(x_1,x_2) dS, \tag{46}$$

where $\mathbf{D}_{n,\tau}^{u,T}$ is the matrix analogy of the vector normal to the boundary, it has the form

$$\mathbf{D}_{n,\tau}^{u,T} = \begin{vmatrix} n_1F_{u_1,\tau} & 0 & n_2F_{u_1,\tau} & 0 & 0 & 0 \\ 0 & n_2F_{u_2,\tau} & n_1F_{u_2,\tau} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n_2F_{u_3,\tau} & n_1F_{u_3,\tau} \end{vmatrix}. \tag{47}$$

In fact, equations (45) are equations of equilibrium and natural boundary conditions for displacements of elastic higher order shells in the integral form, obtained using the CUF approach.

The integrals over volume and surface in (45) have the form

$$\int_V (\cdot) dV = \int_\Omega \int_{-h}^{h} (\cdot) dx_3 d\Omega, \quad \int_{\partial V} (\cdot) dV = \int_{\partial\Omega} \int_{-h}^{h} (\cdot) dx_3 dS. \tag{48}$$

Taking into account this decomposition and integrating the first and third equations in (45) over the shells thickness as well as that variations $\delta\mathbf{u}_\tau$ depend only on variables x_1 and x_2 differential equations for displacements of the higher order elastic shells can be represented in matrix form as

$$\mathbf{L}_M^G \cdot \mathbf{u}_M^G = \mathbf{b}_M^G, \tag{49}$$

where the global matrix operator $\mathbf{L}_n^G$, the vectors of unknown functions $\mathbf{u}_M^G$ and the right hand $\mathbf{b}_M^G$ side have the form

$$\mathbf{L}_M^G = \begin{vmatrix} \mathbf{L}_{1,1}^{loc} & \cdots & \mathbf{L}_{1,M}^{loc} \\ \vdots & \ddots & \vdots \\ \mathbf{L}_{M,1}^{loc} & \cdots & \mathbf{L}_{M,M}^{loc} \end{vmatrix}, \quad \mathbf{u}_M^G = \begin{vmatrix} \mathbf{u}_1^{loc} \\ \vdots \\ \mathbf{u}_M^{loc} \end{vmatrix}, \quad \mathbf{b}_M^G = \begin{vmatrix} \mathbf{b}_1^{loc} \\ \vdots \\ \mathbf{b}_M^{loc} \end{vmatrix}. \tag{50}$$

The matrices $\mathbf{L}_{\tau,s}^{loc}$ are the fundamental nuclei of the differential equations of equilibrium of elastic shells of higher orders. They, as well as the vectors of local unknown functions $\mathbf{u}_s^{loc}$ and local expression for external body and surface loads $\mathbf{b}_s^{loc}$ have the form

$$\mathbf{L}_{\tau,s}^{loc} = \begin{vmatrix} L_{u_1,u_1}^{\tau,s} & L_{u_1,u_2}^{\tau,s} & L_{u_1,u_3}^{\tau,s} \\ L_{u_2,u_1}^{\tau,s} & L_{u_2,u_2}^{\tau,s} & L_{u_2,u_3}^{\tau,s} \\ L_{u_3,u_1}^{\tau,s} & L_{u_3,u_2}^{\tau,s} & L_{u_3,u_3}^{\tau,s} \end{vmatrix}, \quad \mathbf{u}_s^{loc} = \begin{vmatrix} u_{1,s} \\ u_{2,s} \\ u_{3,s} \end{vmatrix}, \quad \mathbf{b}_s^{loc} = \begin{vmatrix} \tilde{b}_{u_1,\tau} \\ \tilde{b}_{u_2,\tau} \\ \tilde{b}_{u_3,\tau} \end{vmatrix}. \tag{51}$$

The components of the vector external body and surface loads $\mathbf{b}_s^{loc}$ have the form

$$\tilde{b}_{u_x,\tau} = J_{\tau,s}^{u_x,u_x} b_{u_x,s} + J_{\tau}^{u_x} p_{u_x}, \quad \tilde{b}_{u_y,\tau} = J_{\tau,s}^{u_y,u_y} b_{u_y,s} + J_{\tau}^{u_y} p_{u_y}, \quad \tilde{b}_{u_z,\tau} = J_{\tau,s}^{u_z,u_z} b_{u_z,s} + J_{\tau}^{u_z} p_{u_z}, \tag{52}$$

where

$$J_{\tau_\alpha,s_\beta}^{u_i,u_j} = \int_{-h}^{h} \frac{dF_{u_i,\tau}(x_3)}{dx_3} \frac{dF_{u_j,s}(x_3)}{dx_3} dx_3, \quad (i,j) \to (x_1,x_2,x_3). \tag{53}$$

As mentioned above, the natural boundary conditions are obtained from the second equation (45). After the integration over the shell thickness and by taking into account that the variations $\delta\mathbf{u}_\tau$ depend only on variables x_1 and x_2, the natural boundary conditions for the higher order elastic shells can be represented in matrix form as

$$\mathbf{B}_M^{N,G} \cdot \mathbf{u}_M^G = \mathbf{p}_M^G, \tag{54}$$

where the global matrix operator $\mathbf{B}_M^{N,G}$, the vectors of unknown functions $\mathbf{u}_M^G$ and the right-hand side $\mathbf{p}_M^G$ have the form

$$\mathbf{B}_M^{N,G} = \begin{vmatrix} \mathbf{B}_{1,1}^{loc} & \cdots & \mathbf{B}_{1,M}^{loc} \\ \vdots & \ddots & \vdots \\ \mathbf{B}_{M,1}^{loc} & \cdots & \mathbf{B}_{M,M}^{loc} \end{vmatrix}, \quad \mathbf{p}_M^G = \begin{vmatrix} \mathbf{p}_1^{loc} \\ \vdots \\ \mathbf{p}_M^{loc} \end{vmatrix}. \tag{55}$$

The matrices $\mathbf{B}_{\tau,s}^{loc}$ are the fundamental nuclei for the natural boundary for higher order elastic shells and $\mathbf{p}_s^{loc}$ are the vectors of local expression for the external load applied at the ends of the shells. They can be written as

$$\mathbf{B}_{\tau,s}^{loc} = \begin{vmatrix} B_{u_1,u_1}^{\tau,s} & B_{u_1,u_2}^{\tau,s} & B_{u_1,u_3}^{\tau,s} \\ B_{u_2,u_1}^{\tau,s} & B_{u_2,u_2}^{\tau,s} & B_{u_2,u_3}^{\tau,s} \\ B_{u_3,u_1}^{\tau,s} & B_{u_3,u_2}^{\tau,s} & B_{u_3,u_3}^{\tau,s} \end{vmatrix}, \quad \mathbf{p}_s^{loc} = \begin{vmatrix} J_{\tau,s}^{u_1,u_1} P_{u_1,s} \\ J_{\tau,s}^{u_2,u_2} P_{u_2,s} \\ J_{\tau,s}^{u_3,u_3} P_{u_3,s} \end{vmatrix}. \tag{56}$$

The essential boundary conditions for the higher order elastic shells can be represented in matrix form as

$$\mathbf{B}_M^{E,G} \cdot \mathbf{u}_M^G \Big|_0^L = \mathbf{u}_M^{0,G}, \tag{57}$$

where the global matrix operator $\mathbf{B}_M^{E,G}$, the vectors of the right-hand side have the form

$$\mathbf{B}_M^{E,G} = \begin{vmatrix} \mathbf{I} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{I} \end{vmatrix}, \quad \mathbf{u}_M^{0,G} = \begin{vmatrix} \mathbf{u}_1^{0,loc} \\ \vdots \\ \mathbf{u}_M^{0,loc} \end{vmatrix}. \tag{58}$$

Here $\mathbf{I}$ is the identity matrix and therefore the global matrix operator $\mathbf{B}_M^{E,G}$ is the identity matrix operator.

Coefficients of the fundamental nuclei $\mathbf{L}_{\tau,s}^{loc}$ and $\mathbf{B}_{\tau,s}^{loc}$ in general case of the orthogonal curvilinear system of coordinates are very complicate and will not be presented here. Instead, we will consider some special cases of shells of revolution.

4 CUF approach for higher order elastic shells of revolution

A simple concept of the surface was given by Euclid, considering it a figure that has only length and width. In fact, the concept of a surface is much more complex, since it covers the whole variety of forms created by nature and man, generalized into a geometric image.

Mathematically, a surface can be considered as 2-D manifold in 3-D space. The analytical representation of a surface can be done using its equation. From this point of view, a surface can be defined as the locus of points in space whose coordinates implicitly satisfy the given equation

$$F(x_1, x_2, x_3) = 0. \tag{59}$$

A more general analytical description of a surface is reduced to expressing the coordinates of its points in terms of two independent parameters and, usually denoted by the Latin letters u and v, in the form of a system of equations

$$x_1 = f_1(u, v), \quad x_2 = f_2(u, v), \quad x_3 = f_3(u, v). \tag{60}$$

If the coordinates x_1, $x_2.x_3$ of the surface point are referred to the radius vector, then we obtain an equivalent equation in vector form

$$\mathbf{r}(u, v) = f_1(u, v)\mathbf{e}_1 + f_2(u, v)\mathbf{e}_2 + f_3(u, v)\mathbf{e}_3. \tag{61}$$

The position of any point M on the surface can be specified by two parameters u and v. Therefore, a surface in three-dimensional space is said to be a 2-D set of points in space given by equations (60). The parameters u and v, that define a point on the surface are called the curvilinear coordinates of the surface.

Derivative of the vector function $\mathbf{r}(u, v)$ with respect to parameters u and v define the tangential to the surface vectors

$$\frac{\partial \mathbf{r}(u, v)}{\partial u} = \mathbf{r}_u(u, v), \quad \frac{\partial \mathbf{r}(u, v)}{\partial v} = \mathbf{r}_v(u, v). \tag{62}$$

The vectors $\mathbf{r}_u$ and $\mathbf{r}_u$ located in the tangential plane. Then coefficients of the first quadratic form of the surface, as well as the unit vector normal to the surface is determined though cross product in the form

$$A_u(x_1, x_2) = \sqrt{\frac{\partial \mathbf{r}(u, v)}{\partial x_u} \cdot \frac{\partial \mathbf{r}(u, v)}{\partial x_u}}, \quad A_v(x_1, x_2) = \sqrt{\frac{\partial \mathbf{r}(u, v)}{\partial x_v} \cdot \frac{\partial \mathbf{r}(u, v)}{\partial x_v}}, \quad \mathbf{n}(u, v) = \frac{\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v)}{|\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v)|}. \tag{63}$$

The principal curvatures of the surface are calculated using the following equations:

$$k_u = \frac{\mathbf{n}(u, v) \cdot \mathbf{r}_{uu}(u, v)}{\mathbf{r}_u(u, v)^2}, \quad k_v = \frac{\mathbf{n}(u, v) \cdot \mathbf{r}_{vv}(u, v)}{\mathbf{r}_v(u, v)^2}, \tag{64}$$

where

$$\frac{\partial^2 \mathbf{r}(u, v)}{\partial u^2} = \mathbf{r}_{uu}(u, v), \quad \frac{\partial^2 \mathbf{r}(u, v)}{\partial v^2} = \mathbf{r}_{vv}(u, v). \tag{65}$$

Further references related to the geometry of surfaces can be found in Gray et al. [24], Kuhnel [44], von Seggern [67] and applications to the complex shells theory in Galimov and Paimushin [22], Guliaev et al. [25], Kil'chevskiy [31], Kornishin et al. [36], Rekach and Krivoshapko [57].

4.1 Main relations for surfaces of revolution

The surface of revolution is formed by the rotation of a plane curve around a fixed straight line, called the axis of rotation. To derive parametric equations of the surface of revolution of a general form, we define the plane curve in the coordinate plane x_1, x_3 by the equations

$$x_1 = f_1(u), \quad x_3 = f_2(u). \tag{66}$$

Consider a point M with coordinated $x_1 = f_1(u)$, 0, $x_3 = f_2(u)$, after rotation around axis x_3 by the angle φ will receive new coordinated $x_1 = f_1(u)\cos(\varphi)$, $f_1(u)\sin(\varphi)$, $x_3 = f_2(u)$.

Therefore, the parametric equation of the surface of revolution can be presented in the following vector form

$$\mathbf{r}(u,\varphi) = f_1(u)\cos(\varphi)\mathbf{e}_1 + f_1(u)\sin(\varphi)\mathbf{e}_2 + f_2(u)\mathbf{e}_3. \tag{67}$$

The first derivatives of the vector $\mathbf{r}(u, \varphi)$ are given by the expressions

$$\begin{aligned} \frac{\partial \mathbf{r}(u,\varphi)}{\partial u} &= f_1(u)'\cos(\varphi)\mathbf{e}_1 + f_1(u)'\sin(\varphi)\mathbf{e}_2 + f_2(u)'\mathbf{e}_3, \\ \frac{\partial \mathbf{r}(u,\varphi)}{\partial \varphi} &= -f_1(u)\sin(\varphi)\mathbf{e}_1 + f_1(u)\mathrm{s}\cos(\varphi)\mathbf{e}_2 \end{aligned} \tag{68}$$

The coefficients of the first quadratic form of the surface, as well as the unit vector normal to the surface and the principal curvatures are calculated by the equations

$$\begin{aligned} &A_1(u) = \sqrt{(f_1(u)')^2 + (f_2(u)')^2}, \quad A_2(u) = f_1(u)', \\ &\mathbf{n}(u,v) = -\frac{f_2(u)'\cos(v)}{A_1(u)}\mathbf{e}_1 - \frac{f_2(u)'\sin(v)}{A_1(u)}\mathbf{e}_2 + \frac{f_1(u)'}{A_1(u)}\mathbf{e}_3, \\ &\kappa_1 = \frac{f_1(u)'f_2(u)'' - f_1(u)''f_2(u)'}{A_1^3}, \quad \kappa_2 = \frac{f_2(u)'}{A_1A_2}. \end{aligned} \tag{69}$$

Here $f_\alpha(u)'$ and $f_\alpha(u)''$ are the first and second derivatives of the function $f_1(u)$ with respect to u and $\alpha = 1, 2$.

More information related to the surfaces of revolution parametrization can be found in Gray et al. [23], Krivoshapko and Ivanov [42] and applications to the shells of revolution in Korjakin et al. [35], Kovarik [37], Reddy [56], Rekach and Krivoshapko [57], Vlasov [66], Wan and Weinitschke [68].

4.2 Circular plate in polar coordinates

The simplest surface of revolution is a plane formed by rotation around an axis x_3 of a straight line that passing though it. In this case, we have model of an elastic plate of the circular geometry in polar coordinates. Models of circular plates are very important and are often used in theoretical analysis as well as applications in sciences and engineering. The middle surface of the plate is a circle, the analytical representation of which in Cartesian coordinates x, y, z is given by the equation

$$x^2 + y^2 \le R^2, \quad z = 0. \tag{70}$$

We introduce polar coordinates where $x_1 = \rho$, $x_2 = \varphi$ and $x_3 = z$, $z \in [-h, h]$. The parametric equations of the surface of revolution (70) have the following vector form:

$$\mathbf{r}(\rho,\varphi) = \rho\cos(\varphi)\mathbf{e}_1 + \rho\sin(\varphi)\mathbf{e}_2. \tag{71}$$

In case the parameters ρ and φ or polar coordinates belong to intervals and $x \in [0, R]$ $\varphi \in [0, 2\pi]$ we have complete circle of radius R, otherwise a circular segment.

The coefficients of the first quadratic form of the circle, as well as the unit vector normal and the principal curvatures are equal to

$$A_1 = 1, \ A_2 = \rho, \mathbf{n}(x,\varphi) = \mathbf{e}_3, \kappa_1 = 0, \quad \kappa_2 = 0, \tag{72}$$

respectively.

The coefficients Lamé for a circular plate have the form

$$H_1 = 1, H_1 = \rho \text{ and } H_3 = 1. \tag{73}$$

Substituting these parameters into equations (49)–(58), we obtain equations corresponding to the higher order theory of a circular plate. The final equations have the form (50), and the essential boundary conditions have the form (57).

The matrices $\mathbf{L}^{loc}_{\tau,s}$ in (51) are the fundamental nuclei of differential equations of equilibrium for higher order circular elastic plates. They, as well as the vectors of local unknown functions $\mathbf{u}^{loc}_{s}$ and the local expression for the external body and surface loads $\mathbf{b}^{loc}_{s}$ have the form

$$\mathbf{L}^{loc}_{\tau,s} = \begin{vmatrix} L^{\tau,s}_{u_\rho,u_\rho} & L^{\tau,s}_{u_\rho,u_\varphi} & L^{\tau,s}_{u_\rho,u_z} \\ L^{\tau,s}_{u_\varphi,u_\rho} & L^{\tau,s}_{u_\varphi,u_\varphi} & L^{\tau,s}_{u_\varphi,u_z} \\ L^{\tau,s}_{u_z,u_\rho} & L^{\tau,s}_{u_z,u_\varphi} & L^{\tau,s}_{u_z,u_z} \end{vmatrix}, \quad \mathbf{u}^{loc}_{s} = \begin{vmatrix} u_{\rho,s} \\ u_{\varphi,s} \\ u_{z,s} \end{vmatrix}, \quad \mathbf{b}^{loc}_{s} = \begin{vmatrix} \tilde{b}_{u_\rho,\tau} \\ \tilde{b}_{u_\varphi,\tau} \\ \tilde{b}_{u_\rho,\tau} \end{vmatrix}. \tag{74}$$

The fundamental nuclei $\mathbf{L}^{loc}_{\tau,s}$ coefficients can be easy calculated using equations presented in the previous sections. Their analytical expressions can be found in Carrera and Zozulya [11, 12].

The matrices $\mathbf{B}^{loc}_{\tau,s}$ of the fundamental nuclei for natural boundary conditions, as well as $\mathbf{p}^{loc}_{s}$ the vectors of local the expression for the external load applied to the shell ends for the higher order circular plate have the form

$$\mathbf{B}^{loc}_{\tau,s} = \begin{vmatrix} B^{\tau,s}_{u_\rho,u_\rho} & B^{\tau,s}_{u_\rho,u_\varphi} & B^{\tau,s}_{u_\rho,u_z} \\ B^{\tau,s}_{u_\varphi,u_\rho} & B^{\tau,s}_{u_\varphi,u_\varphi} & B^{\tau,s}_{u_\varphi,u_z} \\ B^{\tau,s}_{u_z,u_\rho} & B^{\tau,s}_{u_z,u_\varphi} & B^{\tau,s}_{u_z,u_z} \end{vmatrix}, \quad \mathbf{p}^{loc}_{s} = \begin{vmatrix} J^{u_\rho,u_\rho}_{\tau,s} P_{u_\rho,s} \\ J^{u_\varphi,u_\varphi}_{\tau,s} P_{u_\varphi,s} \\ J^{u_z,u_z}_{\tau,s} P_{u_z,s} \end{vmatrix}. \tag{75}$$

Coefficients of the fundamental nuclei $\mathbf{B}^{loc}_{\tau,s}$ can be easy calculated using equations presented in the previous sections. Their analytical expressions can be found in Carrera and Zozulya [11, 12].

The standard reference to classical plate theory is Timoshenko and Woinowsky-Krieger [64].

4.3 Cylindrical shell

Models of elastic shells of the cylindrical geometry are very important and are often used in theoretical analysis as well as applications in sciences and engineering. Consider a cylindrical shell formed by rotation around an axis x_3 of a straight line parallel to it and located at a distance R from it. The middle surface of the shell is a cylinder, the analytical representation of which in Cartesian coordinates x, y, z is given by the equation

$$x^2 + y^2 = R^2. \tag{76}$$

We introduce cylindrical coordinates where $x_1 = x, x_2 = \varphi$ and $x_3 = r, r \in [R-h,\ R+h]$. The parametric equations of the surface of revolution (76) have the following vector form

$$\mathbf{r}(x,\varphi) = R\cos(\varphi)\mathbf{e}_1 + R\sin(\varphi)\mathbf{e}_2 + x\mathbf{e}_3. \tag{77}$$

If the parameters x and φ or cylindrical coordinates belong to intervals and $x \in [0,\ H]$ $\varphi \in [0,\ 2\pi]$, we have a closed cylinder of length H, otherwise a cylindrical segment, as shown in Fig. 1.

The coefficients of the first quadratic form of a cylindrical surface, as well as the unit vector normal to the surface and the principal curvatures are equal to

$$A_1 = 1,\ A_2 = R,\ \mathbf{n}(x,\varphi) = -\cos(\varphi)\mathbf{e}_1 - \sin(\varphi)\mathbf{e}_2 + \mathbf{e}_3,\ \kappa_1 = 0, \quad \kappa_2 = \frac{1}{R}, \tag{78}$$

respectively.

The coefficients Lamé for a cylindrical shell have the form

$$H_1 = 1,\ H_2 = R\left(1 + \frac{r}{R}\right) \text{ and } H_3 = 1. \tag{79}$$

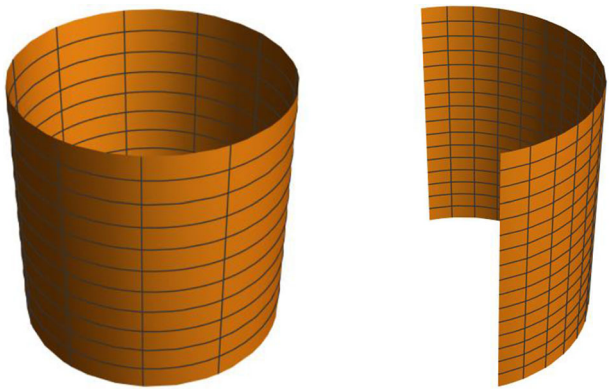

Fig. 1 Two cylindrical surfaces of revolution

Substituting these parameters into equations (49)–(58), we obtain equations corresponding to the higher order theory of the linear theory for a cylindrical shell. The final equations have the form (50), and the essential boundary conditions have the form (57).

The matrices $\mathbf{L}_{\tau,s}^{loc}$ in (51) are the fundamental nuclei of differential equations of equilibrium for higher order cylindrical elastic shells. They, as well as the vectors of local unknown functions $\mathbf{u}_s^{loc}$ and the local expression for the external body and surface loads $\mathbf{b}_s^{loc}$, have the form

$$\mathbf{L}_{\tau,s}^{loc} = \begin{vmatrix} L_{u_x,u_x}^{\tau,s} & L_{u_x,u_\varphi}^{\tau,s} & L_{u_x,u_r}^{\tau,s} \\ L_{u_\varphi,u_x}^{\tau,s} & L_{u_\varphi,u_\varphi}^{\tau,s} & L_{u_\varphi,u_r}^{\tau,s} \\ L_{u_r,u_x}^{\tau,s} & L_{u_r,u_\varphi}^{\tau,s} & L_{u_r,u_r}^{\tau,s} \end{vmatrix}, \quad \mathbf{u}_s^{loc} = \begin{vmatrix} u_{x,s} \\ u_{\varphi,s} \\ u_{r,s} \end{vmatrix}, \quad \mathbf{b}_s^{loc} = \begin{vmatrix} \tilde{b}_{u_x,\tau} \\ \tilde{b}_{u_\varphi,\tau} \\ \tilde{b}_{u_r,\tau} \end{vmatrix}. \tag{80}$$

The fundamental nuclei $\mathbf{L}_{\tau,s}^{loc}$ coefficients can be easy calculated using the equations presented in the previous sections. Their analytical expressions can be found in Carrera and Zozulya [13, 14].

The matrices $\mathbf{B}_{\tau,s}^{loc}$ of the fundamental nuclei for natural boundary conditions, as well as the vectors $\mathbf{p}_s^{loc}$ of the local external load applied to the shell ends for the higher order cylindrical shell have the form

$$\mathbf{B}_{\tau,s}^{loc} = \begin{vmatrix} B_{u_x,u_x}^{\tau,s} & B_{u_x,u_\varphi}^{\tau,s} & B_{u_x,u_r}^{\tau,s} \\ B_{u_\varphi,u_x}^{\tau,s} & B_{u_\varphi,u_\varphi}^{\tau,s} & B_{u_\varphi,u_r}^{\tau,s} \\ B_{u_r,u_x}^{\tau,s} & B_{u_r,u_\varphi}^{\tau,s} & B_{u_r,u_r}^{\tau,s} \end{vmatrix}, \quad \mathbf{p}_s^{loc} = \begin{vmatrix} J_{\tau,s}^{u_x,u_x} P_{u_x,s} \\ J_{\tau,s}^{u_\varphi,u_\varphi} P_{u_\varphi,s} \\ J_{\tau,s}^{u_r,u_r} P_{u_r,s} \end{vmatrix}. \tag{81}$$

The fundamental nuclei $\mathbf{B}_{\tau,s}^{loc}$ coefficients can be easy calculated using equations presented in the previous sections. Their analytical expressions can be found in Carrera and Zozulya [11, 12].

Cylindrical shells considered in numerous publications, see for example Kovarik [37], Reddy [56], Timoshenko and Woinowsky-Krieger [64], Vlasov [66].

4.4 Conical shell

Models of elastic shells of conical geometry are very important and are often used in theoretical analysis as well as applications in sciences and engineering. Consider a conical shell formed by rotation around an axis x_3 of a straight line, not parallel, but forming constant angle ψ with it. The middle surface of the shell is a cone, the analytical representation of which in Cartesian coordinates x, y, z is given by the equation

$$x^2 + y^2 = z^2 \cos(\psi)^2. \tag{82}$$

We introduce cylindrical coordinates where $x_1 = x$, $x_2 = \varphi$ and $x_3 = r$, $r \in [x - h,\ x\cos(\psi) + h]$. The parametric equations of the surface of revolution (82) have the following vector form:

$$\mathbf{r}(x, \varphi) = x \sin(\psi)\cos(\varphi)\mathbf{e}_1 + x \sin(\psi)\sin(\varphi)\mathbf{e}_2 + x \cos(\psi)\mathbf{e}_3. \tag{83}$$

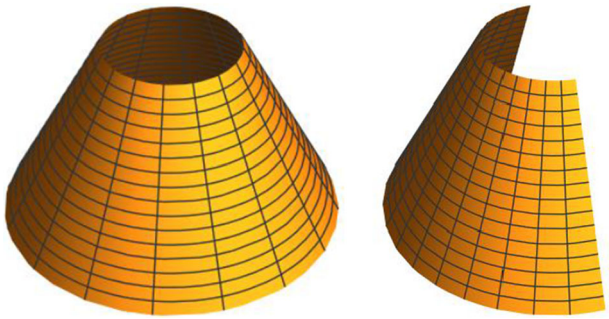

Fig. 2 Two conical surfaces of revolution

If the parameters x and φ or the cylindrical coordinates belong to intervals and $x \in [0,\ H]\ \varphi \in [0,\ 2\pi]$ we have a closed cone of length H, otherwise a conical segment, as shown in Fig. 2.

The coefficients of the first quadratic form of a conical surface, as well as the unit vector normal to the surface and the principal curvatures are equal to

$$\begin{aligned} &A_1 = 1, \quad A_2 = x\sin(\psi), \\ &\mathbf{n}(x,\varphi) = -\cos(\psi)\cos(\varphi)\mathbf{e}_1 - \cos(\psi)\sin(\varphi)\mathbf{e}_2 + \sin(\psi)\mathbf{e}_3, \\ &\kappa_1 = 0, \quad \kappa_2 = \frac{\cot(\psi)}{x}, \end{aligned} \tag{84}$$

respectively.

The coefficients Lamé for a conical shell have the form

$$H_1 = 1, \quad H_2 = x\sin(\psi)\left(1 + \frac{r\cot(\psi)}{x}\right) \text{ and } H_3 = 1. \tag{85}$$

Substituting these parameters into the equations (49)–(58), we obtain equations corresponding to the higher order theory of the linear theory for a conical shell. The final equations have the form (50), and the essential boundary conditions have the form (57).

The matrices $\mathbf{L}^{loc}_{\tau,s}$ in (51) are the fundamental nuclei of differential equations of equilibrium for higher order conical elastic shells. They, as well as the vectors of local unknown functions $\mathbf{u}^{loc}_{s}$ and the local expression for the external body and surface loads $\mathbf{b}^{loc}_{s}$ have the form

$$\mathbf{L}^{loc}_{\tau,s} = \begin{vmatrix} L^{\tau,s}_{u_x,u_x} & L^{\tau,s}_{u_x,u_\varphi} & L^{\tau,s}_{u_x,u_r} \\ L^{\tau,s}_{u_\varphi,u_x} & L^{\tau,s}_{u_\varphi,u_\varphi} & L^{\tau,s}_{u_\varphi,u_r} \\ L^{\tau,s}_{u_r,u_x} & L^{\tau,s}_{u_r,u_\varphi} & L^{\tau,s}_{u_r,u_r} \end{vmatrix}, \quad \mathbf{u}^{loc}_{s} = \begin{vmatrix} u_{x,s} \\ u_{\varphi,s} \\ u_{r,s} \end{vmatrix}, \quad \mathbf{b}^{loc}_{s} = \begin{vmatrix} \tilde{b}_{u_x,\tau} \\ \tilde{b}_{u_\varphi,\tau} \\ \tilde{b}_{u_r,\tau} \end{vmatrix}. \tag{86}$$

The fundamental nuclei $\mathbf{L}^{loc}_{\tau,s}$ coefficients can be easy calculated using equations presented in the previous sections. Their analytical expressions can be found in Carrera and Zozulya [11, 12].

The matrices $\mathbf{B}^{loc}_{\tau,s}$ of the fundamental nuclei for natural boundary conditions, as well as the vectors $\mathbf{p}^{loc}_{s}$ of the local external load applied to the shell ends for the higher order conical shell have the form

$$\mathbf{B}^{loc}_{\tau,s} = \begin{vmatrix} B^{\tau,s}_{u_x,u_x} & B^{\tau,s}_{u_x,u_\varphi} & B^{\tau,s}_{u_x,u_r} \\ B^{\tau,s}_{u_\varphi,u_x} & B^{\tau,s}_{u_\varphi,u_\varphi} & B^{\tau,s}_{u_\varphi,u_r} \\ B^{\tau,s}_{u_r,u_x} & B^{\tau,s}_{u_r,u_\varphi} & B^{\tau,s}_{u_r,u_r} \end{vmatrix}, \quad \mathbf{p}^{loc}_{s} = \begin{vmatrix} J^{u_x,u_x}_{\tau,s} P_{u_x,s} \\ J^{u_\varphi,u_\varphi}_{\tau,s} P_{u_\varphi,s} \\ J^{u_r,u_r}_{\tau,s} P_{u_r,s} \end{vmatrix}. \tag{87}$$

The fundamental nuclei $\mathbf{B}^{loc}_{\tau,s}$ coefficients can be easy calculated using equations presented in the previous sections. Their analytical expressions can be found in Carrera and Zozulya [11, 12].

Conical shells considered in numerous publications, see for example Kovarik [37], Reddy [56], Vlasov [66].

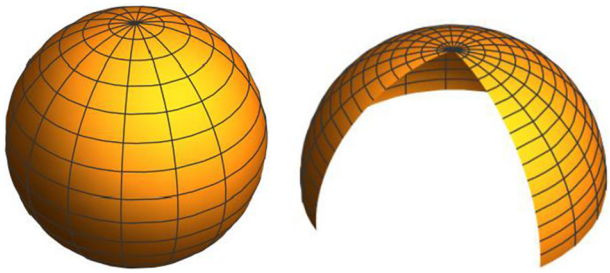

Fig. 3 Two spherical surfaces of revolution

4.5 Spherical shell

Models of elastic shells of spherical geometry are very important and are often used in theoretical analysis as well as applications in sciences and engineering. Consider a spherical shell formed by rotation around the axis x_3 of the circle $x_1^2 + x_3^2 = R^2$. The middle surface of the shell is a sphere, the analytical representation of which in Cartesian coordinates $x,\ y,\ z$ is given by the equation

$$x^2 + y^2 + z^2 = R^2. \tag{88}$$

We introduce spherical coordinates where $x_1 = \varphi, x_2 = \psi$ and $x_3 = r, r \in [R-h,\ R+h]$. The parametric equations of the surface of revolution (88) have the following vector form:

$$\mathbf{r}(\psi, \varphi) = R\cos(\psi)\cos(\varphi)\mathbf{e}_1 + R\cos(\psi)\sin(\varphi)\mathbf{e}_2 + R\sin(\psi)\mathbf{e}_3. \tag{89}$$

If the parameters φ and ψ or the spherical coordinates belong to intervals $\varphi \in [0,\ 2\pi]$ and $\psi \in [-\pi/2,\ \pi/2]$ we have a complete spherical surface, otherwise a spherical segment, as shown in Fig. 3.

The coefficients of the first quadratic form of a spherical surface, as well as the unit vector normal to the surface and the principal curvatures are equal to

$$\begin{aligned} &A_1 = R, \quad A_2 = R\sin(\psi), \\ &\mathbf{n}(\psi, \varphi) = -\cos(\psi)\cos(\varphi)\mathbf{e}_1 - \cos(\psi)\sin(\varphi)\mathbf{e}_2 - \sin(\psi)\mathbf{e}_3, \\ &\kappa_1 = \frac{1}{R}, \quad \kappa_2 = \frac{1}{R}, \end{aligned} \tag{90}$$

respectively.

The coefficients Lamé for a spherical shell have the form

$$H_1 = R\left(1 + \frac{r}{R}\right), \quad H_2 = R\sin(\psi)\left(1 + \frac{r}{R}\right) \text{ and } H_3 = 1. \tag{91}$$

Substituting these parameters into equations (49)–(58), we obtain equations corresponding to the higher order theory of the linear theory for a spherical shell. The final equations have the form (50), and the essential boundary conditions have the form (57).

Matrices $\mathbf{L}_{\tau,s}^{loc}$ in (51) are the fundamental nuclei of the differential equations of equilibrium for the higher order spherical elastic shells. They as well as vectors of local unknown functions $\mathbf{u}_s^{loc}$ and local expression for external body and surface loads $\mathbf{b}_s^{loc}$ have the form

$$\mathbf{L}_{\tau,s}^{loc} = \begin{vmatrix} L_{u_\varphi,u_\varphi}^{\tau,s} & L_{u_\varphi,u_\psi}^{\tau,s} & L_{u_\varphi,u_r}^{\tau,s} \\ L_{u_\psi,u_\varphi}^{\tau,s} & L_{u_\psi,u_\psi}^{\tau,s} & L_{u_\psi,u_r}^{\tau,s} \\ L_{u_r,u_\varphi}^{\tau,s} & L_{u_r,u_\psi}^{\tau,s} & L_{u_r,u_r}^{\tau,s} \end{vmatrix}, \quad \mathbf{u}_s^{loc} = \begin{vmatrix} u_{\varphi,s} \\ u_{\psi,s} \\ u_{r,s} \end{vmatrix}, \quad \mathbf{b}_s^{loc} = \begin{vmatrix} \tilde{b}_{u_\varphi,\tau} \\ \tilde{b}_{u_\psi,\tau} \\ \tilde{b}_{u_r,\tau} \end{vmatrix}. \tag{92}$$

The fundamental nuclei $\mathbf{L}_{\tau,s}^{loc}$ coefficients can be easy calculated using equations presented in the previous sections. Their analytical expressions can be found in Carrera and Zozulya [11, 12].

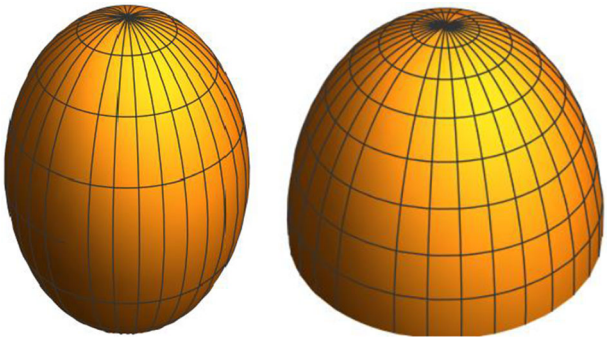

Fig. 4 Two ellipsoidal surfaces of revolution

The matrices $\mathbf{B}^{loc}_{\tau,s}$ of the fundamental nuclei for natural boundary conditions, as well as the vectors $\mathbf{p}^{loc}_{s}$ of the local external load applied to the shell ends for the higher order spherical elastic shells have the form

$$\mathbf{B}^{loc}_{\tau,s} = \begin{vmatrix} B^{\tau,s}_{u_\varphi,u_\varphi} & B^{\tau,s}_{u_\varphi,u_\psi} & B^{\tau,s}_{u_\varphi,u_r} \\ B^{\tau,s}_{u_\psi,u_\varphi} & B^{\tau,s}_{u_\psi,u_\psi} & B^{\tau,s}_{u_\psi,u_r} \\ B^{\tau,s}_{u_r,u_\varphi} & B^{\tau,s}_{u_r,u_\psi} & B^{\tau,s}_{u_r,u_r} \end{vmatrix}, \quad \mathbf{p}^{loc}_{s} = \begin{vmatrix} J^{u_\varphi,u_\varphi}_{\tau,s} P_{u_{\varphi,s}} \\ J^{u_\psi,u_\psi}_{\tau,s} P_{u_{\psi,s}} \\ J^{u_r,u_r}_{\tau,s} P_{u_{r,s}} \end{vmatrix}. \tag{93}$$

The fundamental nuclei $\mathbf{B}^{loc}_{\tau,s}$ coefficients can be easy calculated using equations presented in the previous sections. Their analytical expressions can be found in Carrera and Zozulya [13, 14].

Spherical shells considered in numerous publications, see for example Kovarik [37], Reddy [56], Timoshenko and Woinowsky-Krieger [64], Vlasov [66], Xie et al. [71].

4.6 Ellipsoidal shell

Models of elastic shells of ellipsoidal geometry are very important and are often used in theoretical analysis as well as applications in sciences and engineering. Consider an ellipsoidal shell formed by rotation around the axis x_3 of the ellipse $\frac{x_1^2}{a^2} + \frac{x_3^2}{b^2} = 1$. The middle surface of the shell is an ellipsoid, the analytical representation of which in Cartesian coordinates $x,\ y,\ z$ is given by the equation

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1. \tag{94}$$

We introduce spherical coordinates where $x_1 = \varphi$, $x_2 = \psi$ and $x_3 = r$, $r \in [-h,\ h]$. The parametric equations of the surface of revolution (94) have the following vector form:

$$\mathbf{r}(\varphi, \psi) = a\cos(\psi)\cos(\varphi)\mathbf{e}_1 + a\cos(\psi)\sin(\varphi)\mathbf{e}_2 + b\sin(\psi)\mathbf{e}_3. \tag{95}$$

If parameters φ and ψ or spherical coordinates belong to intervals $\varphi \in [0,\ 2\pi]$ and $\psi \in [-\pi/2,\ \pi/2]$ we have a complete ellipsoidal surface, otherwise a ellipsoidal segment, as shown in Fig. 4.

The coefficients of the first quadratic form of a ellipsoidal surface, as well as the unit vector normal to the surface and the principle curvatures are equal to

$$\begin{aligned} &A_1 = \sqrt{b^2\cos(\psi)^2 + a^2\sin(\psi)^2}, \quad A_2 = a\cos(\psi), \\ &\mathbf{n}(\varphi, \psi) = -\frac{b\cos(\psi)\cos(\varphi)}{\sqrt{b^2\cos(\psi)^2 + a^2\sin(\psi)^2}}\mathbf{e}_1 - \frac{b\cos(\psi)\sin(\varphi)}{\sqrt{b^2\cos(\psi)^2 + a^2\sin(\psi)^2}}\mathbf{e}_2 - \frac{a\sin(\psi)}{\sqrt{b^2\cos(\psi)^2 + a^2\sin(\psi)^2}}\mathbf{e}_3, \\ &\kappa_1 = \frac{ab}{\left(b^2\cos(\psi)^2 + a^2\sin(\psi)^2\right)^{3/2}}, \quad \kappa_2 = \frac{b}{a\sqrt{b^2\cos(\psi)^2 + a^2\sin(\psi)^2}}, \end{aligned} \tag{96}$$

respectively.

The coefficients Lamé for an ellipsoidal shell have the form

$$\begin{aligned} H_1 &= \sqrt{b^2\cos(\psi)^2 + a^2\sin(\psi)^2}\left(1 + \frac{abr}{\left(b^2\cos(\psi)^2 + a^2\sin(\psi)^2\right)^{3/2}}\right), \\ H_2 &= a\cos(\psi)\left(1 + \frac{br}{a\sqrt{b^2\cos(\psi)^2 + a^2\sin(\psi)^2}}\right), \\ H_3 &= 1. \end{aligned} \tag{97}$$

Substituting these parameters into equations (49)–(58), we obtain equations corresponding to the higher order theory of the linear theory for an ellipsoidal. The final equations have the form (50), and the essential boundary conditions have the form (57).

The matrices $\mathbf{L}_{\tau,s}^{loc}$ in (51) are the fundamental nuclei of differential equations of equilibrium for higher order ellipsoidal elastic shells. They, as well as the vectors of local unknown functions $\mathbf{u}_s^{loc}$ and the local expression for the external body and surface loads $\mathbf{b}_s^{loc}$ have the form

$$\mathbf{L}_{\tau,s}^{loc} = \begin{vmatrix} L_{u_\varphi,u_\varphi}^{\tau,s} & L_{u_\varphi,u_\psi}^{\tau,s} & L_{u_\varphi,u_r}^{\tau,s} \\ L_{u_\psi,u_\varphi}^{\tau,s} & L_{u_\psi,u_\psi}^{\tau,s} & L_{u_\psi,u_r}^{\tau,s} \\ L_{u_r,u_\varphi}^{\tau,s} & L_{u_r,u_\psi}^{\tau,s} & L_{u_r,u_r}^{\tau,s} \end{vmatrix}, \quad \mathbf{u}_s^{loc} = \begin{vmatrix} u_{\varphi,s} \\ u_{\psi,s} \\ u_{r,s} \end{vmatrix}, \quad \mathbf{b}_s^{loc} = \begin{vmatrix} \tilde{b}_{u_\varphi,\tau} \\ \tilde{b}_{u_\psi,\tau} \\ \tilde{b}_{u_r,\tau} \end{vmatrix}. \tag{98}$$

The fundamental nuclei $\mathbf{L}_{\tau,s}^{loc}$ coefficients can be easy calculated using equations presented in the previous sections. Their analytic expressions are complicated, and we do not present them here.

The matrices $\mathbf{B}_{\tau,s}^{loc}$ of the fundamental nuclei for natural boundary conditions, as well as the vectors $\mathbf{p}_s^{loc}$ of the local external load applied to the shell ends for the higher order ellipsoidal elastic shells have the form

$$\mathbf{B}_{\tau,s}^{loc} = \begin{vmatrix} B_{u_\varphi,u_\varphi}^{\tau,s} & B_{u_\varphi,u_\psi}^{\tau,s} & B_{u_\varphi,u_r}^{\tau,s} \\ B_{u_\psi,u_\varphi}^{\tau,s} & B_{u_\psi,u_\psi}^{\tau,s} & B_{u_\psi,u_r}^{\tau,s} \\ B_{u_r,u_\varphi}^{\tau,s} & B_{u_r,u_\psi}^{\tau,s} & B_{u_r,u_r}^{\tau,s} \end{vmatrix}, \quad \mathbf{p}_s^{loc} = \begin{vmatrix} J_{\tau,s}^{u_\varphi,u_\varphi} P_{u_{\varphi,s}} \\ J_{\tau,s}^{u_\psi,u_\psi} P_{u_{\psi,s}} \\ J_{\tau,s}^{u_r,u_r} P_{u_{r,s}} \end{vmatrix}. \tag{99}$$

The fundamental nuclei $\mathbf{B}_{\tau,s}^{loc}$ coefficients can be easy calculated using equations presented in the previous sections. Their analytic expressions are complicated, and we do not present them here.

Ellipsoidal shells considered in numerous publications, see for example Klochkov et al. [32–34], Krivoshapko [40], Meish [47], Meish and Maiborodina [48], Tangbanjongkij et al. [63].

4.7 Paraboloidal shell

Models of elastic shells of paraboloidal geometry are very important and are often used in theoretical analysis as well as applications in sciences and engineering. Consider a paraboloidal shell formed by rotating around the axis x_3 of a parabolic curing line $\frac{x_1^2}{a^2} = x_3$. The middle surface of the shell is a paraboloid, the analytical representation of which in Cartesian coordinates x, y, z is given by the equation

$$\frac{x^2 + y^2}{a^2} = z. \tag{100}$$

We introduce curvilinear coordinates where $x_1 = x$, $x_2 = \varphi$ and $x_3 = r$, $r \in [-h, h]$. The parametric equations of the surface of revolution (100) have the following vector form:

$$\mathbf{r}(x, \varphi) = ax\cos(\varphi)\mathbf{e}_1 + ax\sin(\varphi)\mathbf{e}_2 + ax^2\mathbf{e}_3. \tag{101}$$

If parameters x and φ or spherical coordinates belong to intervals $x \in [0,\ H]$ and $\varphi \in [0,\ 2\pi]$ we have a complete paraboloidal surface, otherwise a paraboloidal segment, as shown in Fig. 5.

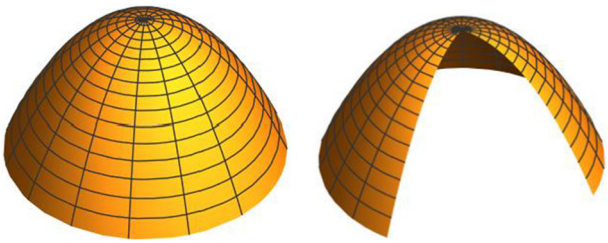

Fig. 5 Two paraboloidal surfaces of revolution

The coefficients of the first quadratic form of a paraboloidal surface, as well as the unit vector normal to the surface and the principal curvatures are equal to

$$\begin{aligned} &A_1 = a\sqrt{1+4x^2},\ A_2 = ax, \\ &\mathbf{n}(\varphi,\psi) = -\frac{2ax\cos(\varphi)}{a\sqrt{1+4x^2}}\mathbf{e}_1 - \frac{2ax\sin(\varphi)}{a\sqrt{1+4x^2}}\mathbf{e}_2 - \frac{a}{a\sqrt{1+4x^2}}\mathbf{e}_3, \\ &\kappa_1 = \frac{2}{a\left(1+4x^2\right)^{3/2}},\quad \kappa_2 = \frac{2}{a\sqrt{1+4x^2}}, \end{aligned} \tag{102}$$

respectively.

The coefficients Lamé for a paraboloidal shell have the form

$$H_1 = a\sqrt{1+4x^2}\left(1+\frac{2r}{a\left(1+4x^2\right)^{3/2}}\right),\ H_2 = ax\left(1+\frac{2r}{a\sqrt{1+4x^2}}\right) \text{ and } H_3 = 1. \tag{103}$$

Substituting these parameters into equations (49)–(58), we obtain equations corresponding to the higher order theory of the linear theory for a paraboloidal shell. The final equations have the form (50), and the essential boundary conditions have the form (57).

The matrices $\mathbf{L}^{loc}_{\tau,s}$ in (51) are the fundamental nuclei of differential equations of equilibrium for higher order paraboloidal elastic shells. They, as well as the vectors of local unknown functions $\mathbf{u}^{loc}_{s}$ and the local expression for external body and surface loads $\mathbf{b}^{loc}_{s}$ have the form

$$\mathbf{L}^{loc}_{\tau,s} = \begin{vmatrix} L^{\tau,s}_{u_x,u_x} & L^{\tau,s}_{u_x,u_\varphi} & L^{\tau,s}_{u_x,u_r} \\ L^{\tau,s}_{u_\varphi,u_x} & L^{\tau,s}_{u_\varphi,u_\varphi} & L^{\tau,s}_{u_\varphi,u_r} \\ L^{\tau,s}_{u_r,u_x} & L^{\tau,s}_{u_r,u_\varphi} & L^{\tau,s}_{u_r,u_r} \end{vmatrix},\quad \mathbf{u}^{loc}_{s} = \begin{vmatrix} u_{x,s} \\ u_{\varphi,s} \\ u_{r,s} \end{vmatrix},\quad \mathbf{b}^{loc}_{s} = \begin{vmatrix} \tilde{b}_{u_x,\tau} \\ \tilde{b}_{u_\varphi,\tau} \\ \tilde{b}_{u_r,\tau} \end{vmatrix}. \tag{104}$$

The fundamental nuclei $\mathbf{L}^{loc}_{\tau,s}$ coefficients can be easy calculated using equations presented in the previous sections. Their analytic expressions are complicated, and we do not present them here.

The matrices $\mathbf{B}^{loc}_{\tau,s}$ of the fundamental nuclei for natural boundary conditions, as well as the vectors $\mathbf{p}^{loc}_{s}$ of the local external load applied to the shell ends for the higher order paraboloidal elastic shells have the form

$$\mathbf{B}^{loc}_{\tau,s} = \begin{vmatrix} B^{\tau,s}_{u_x,u_x} & B^{\tau,s}_{u_x,u_\varphi} & B^{\tau,s}_{u_x,u_r} \\ B^{\tau,s}_{u_\varphi,u_x} & B^{\tau,s}_{u_\varphi,u_\varphi} & B^{\tau,s}_{u_\varphi,u_r} \\ B^{\tau,s}_{u_r,u_x} & B^{\tau,s}_{u_r,u_\varphi} & B^{\tau,s}_{u_r,u_r} \end{vmatrix},\quad \mathbf{p}^{loc}_{s} = \begin{vmatrix} J^{u_x,u_x}_{\tau,s} P_{u_x,s} \\ J^{u_\varphi,u_\varphi}_{\tau,s} P_{u_\varphi,s} \\ J^{u_r,u_r}_{\tau,s} P_{u_r,s}. \end{vmatrix} \tag{105}$$

The fundamental nuclei coefficients $\mathbf{B}^{loc}_{\tau,s}$ can be easy calculated using equations presented in the previous sections. Their analytic expressions are complicated, and we do not present them here.

Paraboloidal shells considered in numerous publications, see for example Al-Khatib et al. [1], Chernobryvko et all. [15, 16], Kang and Leissa [27, 28], Krivoshapko [41], Xie et al. [71].

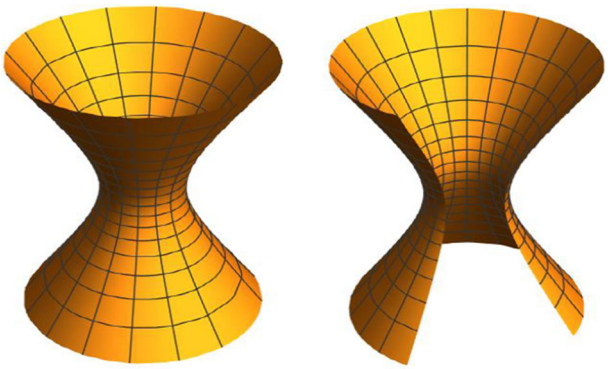

Fig. 6 Two hyperboloidal surfaces of revolution

4.8 Hyperboloidal shell

Models of elastic shells of hyperboloidal geometry are very important and are often used in theoretical analysis as well as applications in sciences and engineering. Consider a hyperboloidal shell formed by rotation around the axis x_3 of a hyperbola $\frac{x_1^2}{a^2} - \frac{x_3^2}{b^2} = 1$. The middle surface of the shell is a hyperboloid, the analytical representation of which in Cartesian coordinates x, y, z is given by the equation

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{b^2} = 1. \tag{106}$$

We introduce curvilinear coordinates where $x_1 = x$, $x_2 = \varphi$ and $x_3 = r$, $r \in [-h,\ h]$. The parametric equations of the surface of revolution (106) have the following vector form:

$$\mathbf{r}(x, \varphi) = a\cosh(x)\cos(\varphi)\mathbf{e}_1 + a\cosh(x)\sin(\varphi)\mathbf{e}_2 + b\sinh(x)\mathbf{e}_3. \tag{107}$$

If the parameters x and φ or the spherical coordinates belong to intervals $x \in [0,\ H]$ and $\varphi \in [0,\ 2\pi]$ we have a complete hyperboloidal surface otherwise a hyperboloidal segment, as shown in Fig. 6.

The coefficients of the first quadratic form of a hyperboloidal surface, as well as the unit vector normal to the surface and the principal curvatures are equal to

$$\begin{aligned}
&A_1 = \sqrt{a^2\sinh(x)^2 + b^2\cosh(x)^2},\ A_2 = a\cosh(x),\\
&\mathbf{n}(\varphi, \psi) = -\frac{b\cos(\varphi)\cosh(x)}{\sqrt{a^2\sinh(x)^2 + b^2\cosh(x)^2}}\mathbf{e}_1 - \frac{b\cosh(x)\sin(\varphi)}{\sqrt{a^2\sinh(x)^2 + b^2\cosh(x)^2}}\mathbf{e}_2 + \frac{a\sinh(x)}{\sqrt{a^2\sinh(x)^2 + b^2\cosh(x)^2}}\mathbf{e}_3,\\
&\kappa_1 = -\frac{ab}{\left(a^2\sinh(x)^2 + b^2\cosh(x)^2\right)^{3/2}},\quad \kappa_2 = \frac{b}{a\sqrt{a^2\sinh(x)^2 + b^2\cosh(x)^2}},
\end{aligned} \tag{108}$$

respectively.

The coefficients Lamé for a hyperboloidal shell have the form

$$\begin{aligned}
H_1 &= \sqrt{a^2\sinh(x)^2 + b^2\cosh(x)^2}\left(1 - \frac{abr}{\left(a^2\sinh(x)^2 + b^2\cosh(x)^2\right)^{3/2}}\right),\\
H_2 &= a\cosh(x)\left(1 + \frac{br}{a\sqrt{a^2\sinh(x)^2 + b^2\cosh(x)^2}}\right),\\
H_3 &= 1.
\end{aligned} \tag{109}$$

Substituting these parameters into equations (49)–(58), we obtain equations corresponding to the higher order theory of the linear theory for a hyperboloidal. The final equations have the form (50), and the essential boundary conditions have the form (57).

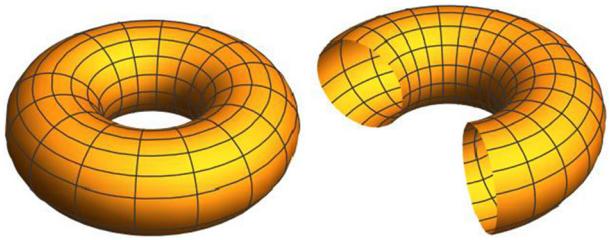

Fig. 7 Two toroidal surfaces of revolution

The matrices $\mathbf{L}^{loc}_{\tau,s}$ in (51) are the fundamental nuclei of differential equations of equilibrium for higher order hyperboloidal elastic shells. They, as well as the vectors of local unknown functions $\mathbf{u}^{loc}_{s}$ and the local expression for the external body and surface loads $\mathbf{b}^{loc}_{s}$ have the form

$$\mathbf{L}^{loc}_{\tau,s} = \begin{vmatrix} L^{\tau,s}_{u_x,u_x} & L^{\tau,s}_{u_x,u_\varphi} & L^{\tau,s}_{u_x,u_r} \\ L^{\tau,s}_{u_\varphi,u_x} & L^{\tau,s}_{u_\varphi,u_\varphi} & L^{\tau,s}_{u_\varphi,u_r} \\ L^{\tau,s}_{u_r,u_x} & L^{\tau,s}_{u_r,u_\varphi} & L^{\tau,s}_{u_r,u_r} \end{vmatrix}, \quad \mathbf{u}^{loc}_{s} = \begin{vmatrix} u_{x,s} \\ u_{\varphi,s} \\ u_{r,s} \end{vmatrix}, \quad \mathbf{b}^{loc}_{s} = \begin{vmatrix} \tilde{b}_{u_x,\tau} \\ \tilde{b}_{u_\varphi,\tau} \\ \tilde{b}_{u_r,\tau} \end{vmatrix}. \tag{110}$$

The fundamental nuclei $\mathbf{L}^{loc}_{\tau,s}$ coefficients can be easy calculated using equations presented in the previous sections. Their analytic expressions are complicated, and we do not present them here.

The matrices $\mathbf{B}^{loc}_{\tau,s}$ of the fundamental nuclei for natural boundary conditions, as well as the vectors $\mathbf{p}^{loc}_{s}$ of the local external load applied to the shell ends for the higher order hyperboloidal elastic shells have the form

$$\mathbf{B}^{loc}_{\tau,s} = \begin{vmatrix} B^{\tau,s}_{u_x,u_x} & B^{\tau,s}_{u_x,u_\varphi} & B^{\tau,s}_{u_x,u_r} \\ B^{\tau,s}_{u_\varphi,u_x} & B^{\tau,s}_{u_\varphi,u_\varphi} & B^{\tau,s}_{u_\varphi,u_r} \\ B^{\tau,s}_{u_r,u_x} & B^{\tau,s}_{u_r,u_\varphi} & B^{\tau,s}_{u_r,u_r} \end{vmatrix}, \quad \mathbf{p}^{loc}_{s} = \begin{vmatrix} J^{u_x,u_x}_{\tau,s} P_{u_{x,s}} \\ J^{u_\varphi,u_\varphi}_{\tau,s} P_{u_{\varphi,s}} \\ J^{u_r,u_r}_{\tau,s} P_{u_{r,s}} \end{vmatrix}. \tag{111}$$

The fundamental nuclei $\mathbf{B}^{loc}_{\tau,s}$ coefficients can be easy calculated using equations presented in the previous sections. Their analytic expressions are complicated, and we do not present them here.

Hyperboloidal shells considered for example in Kang and Leissa [29], Krivoshapko [39].

4.9 Toroidal shell

Models of elastic shells of toroidal geometry are very important and are often used in theoretical analysis as well as applications in sciences and engineering. Consider a toroidal shell formed by rotation around the axis x_3 of a circle of radius r and the center of the circle located outside the axis at a distance R: $(x_1 - R)^2 + x_3^2 = r^2$. The middle surface of the shell is a torus, the analytical representation of which in Cartesian coordinates x, y, z is given by the equation

$$\left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 = r^2. \tag{112}$$

We introduce curvilinear coordinates where $x_1 = \varphi$, $x_2 = \psi$ and $x_3 = z$, $z \in [-h, h]$. The parametric equations of the surface of revolution (112) have the following vector form:

$$\mathbf{r}(\psi, \varphi) = (R + r\cos(\psi))\cos(\varphi)\mathbf{e}_1 + (R + r\cos(\psi))\sin(\varphi)\mathbf{e}_2 + r\sin(\psi)\mathbf{e}_3. \tag{113}$$

If parameters φ and ψ or curvilinear coordinates belong to intervals $\varphi \in [0, 2\pi]$ and $\psi \in [-\pi, \pi]$ we have a complete toroidal surface otherwise a toroidal segment, as shown in Fig. 7.

The coefficients of the first quadratic form of a toroidal surface, as well as the unit vector normal to the surface and the principal curvatures are equal to

$$\begin{aligned} &A_1 = r, \ A_2 = R + r\cos(\psi), \\ &\mathbf{n}(\varphi, \psi) = -\cos(\psi)\cos(\varphi)\mathbf{e}_1 - \cos(\psi)\sin(\varphi)\mathbf{e}_2 - \sin(\psi)\mathbf{e}_3, \\ &\kappa_1 = \frac{1}{r}, \quad \kappa_2 = \frac{\cos(\psi)}{R + r\cos(\psi)}, \end{aligned} \tag{114}$$

respectively.

The coefficients Lamé for a toroidal shell have the form

$$H_1 = r\left(1+\frac{z}{r}\right), \quad H_2 = (R + r\cos(\psi))\left(1+\frac{z\cos(\psi)}{R+r\cos(\psi)}\right), \quad H_3 = 1. \tag{115}$$

Substituting these parameters into equations (49)–(58), we obtain equations corresponding to the higher order theory of the linear theory for a toroidal shell. The final equations have the form (50), and the essential boundary conditions have the form (57).

The matrices $\mathbf{L}^{loc}_{\tau,s}$ in (51) are the fundamental nuclei of differential equations of equilibrium for higher order toroidal elastic shells. They, as well as the vectors of local unknown functions $\mathbf{u}^{loc}_{s}$ and the local expression for the external body and surface loads $\mathbf{b}^{loc}_{s}$ have the form

$$\mathbf{L}^{loc}_{\tau,s} = \begin{vmatrix} L^{\tau,s}_{u_\varphi,u_\varphi} & L^{\tau,s}_{u_\varphi,u_\psi} & L^{\tau,s}_{u_\varphi,u_r} \\ L^{\tau,s}_{u_\psi,u_\varphi} & L^{\tau,s}_{u_\psi,u_\psi} & L^{\tau,s}_{u_\psi,u_r} \\ L^{\tau,s}_{u_r,u_\varphi} & L^{\tau,s}_{u_r,u_\psi} & L^{\tau,s}_{u_r,u_r} \end{vmatrix}, \quad \mathbf{u}^{loc}_{s} = \begin{vmatrix} u_{\varphi,s} \\ u_{\psi,s} \\ u_{r,s} \end{vmatrix}, \quad \mathbf{b}^{loc}_{s} = \begin{vmatrix} \tilde{b}_{u_\varphi,\tau} \\ \tilde{b}_{u_\psi,\tau} \\ \tilde{b}_{u_r,\tau} \end{vmatrix}. \tag{116}$$

The fundamental nuclei $\mathbf{L}^{loc}_{\tau,s}$ coefficients can be easy calculated using equations presented in the previous sections. Their analytic expressions are complicated, and we do not present them here.

The matrices $\mathbf{B}^{loc}_{\tau,s}$ of the fundamental nuclei for natural boundary conditions, as well as the vectors $\mathbf{p}^{loc}_{s}$ of the local external load applied to the shell ends for the higher order toroidal elastic shells have the form

$$\mathbf{B}^{loc}_{\tau,s} = \begin{vmatrix} B^{\tau,s}_{u_\varphi,u_\varphi} & B^{\tau,s}_{u_\varphi,u_\psi} & B^{\tau,s}_{u_\varphi,u_r} \\ B^{\tau,s}_{u_\psi,u_\varphi} & B^{\tau,s}_{u_\psi,u_\psi} & B^{\tau,s}_{u_\psi,u_r} \\ B^{\tau,s}_{u_r,u_\varphi} & B^{\tau,s}_{u_r,u_\psi} & B^{\tau,s}_{u_r,u_r} \end{vmatrix}, \quad \mathbf{p}^{loc}_{s} = \begin{vmatrix} J^{u_\varphi,u_\varphi}_{\tau,s} P_{u_{\varphi,s}} \\ J^{u_\psi,u_\psi}_{\tau,s} P_{u_{\psi,s}} \\ J^{u_r,u_r}_{\tau,s} P_{u_{r,s}} \end{vmatrix}. \tag{117}$$

The fundamental nuclei $\mathbf{B}^{loc}_{\tau,s}$ coefficients can be easy calculated using equations presented in the previous sections. Their analytic expressions are complicated, and we do not present them here.

Toroidal shells considered in numerous publications, see for example Clark [17, 18], El-Raheb and Wagner [20, 21], Leung [45], Lutskaya et al. [46], Ming et al. [49], Naboulsi et al. [50], Senjanovic et al. [58, 59], Sun [26, 61], Sutcliffe [62], Wenmin et al. [70].

4.10 Elliptic toroidal shell

Models of elastic shells of elliptic toroidal geometry are very important and are often used in theoretical analysis as well as applications in sciences and engineering. Consider an elliptic toroidal shell formed by rotation around the axis x_3 of a ellipse with axes a, b and the center of the ellipse located outside the axis at a distance R: $\frac{(x_1-R)^2}{a^2} + \frac{x_3^2}{b^2} - 1 = 0$. The middle surface of the shell is an elliptic torus, the analytical representation of which in Cartesian coordinates x, y, z is given by the equation

$$\frac{\left(\sqrt{x^2+y^2}-R\right)^2}{a^2} + \frac{z^2}{b^2} - 1 = 0. \tag{118}$$

We introduce curvilinear coordinates where $x_1 = \varphi$, $x_2 = \psi$ and $x_3 = r$, $r \in [-h, h]$. The parametric equations of the surface of revolution (118) have the following vector form:

$$\mathbf{r}(\psi,\varphi) = (a + b\cos(\psi))\cos(\varphi)\mathbf{e}_1 + (a + b\cos(\psi))\sin(\varphi)\mathbf{e}_2 + R\sin(\psi)\mathbf{e}_3. \tag{119}$$

If parameters φ and ψ or curvilinear coordinates belong to intervals $\varphi \in [0, 2\pi]$ and $\psi \in [-\pi, \pi]$ we have a complete elliptic toroidal surface otherwise an elliptic toroidal segment, as shown in Fig. 8.

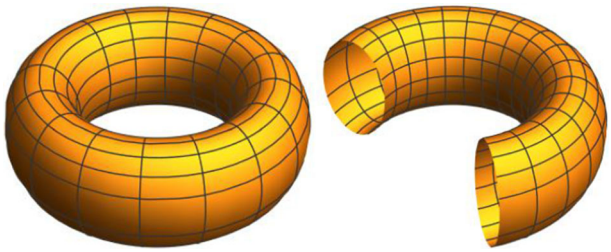

Fig. 8 Two elliptic toroidal surfaces of revolution

The coefficients of the first quadratic form of a toroidal surface, as well as the unit vector normal to the surface and the principal curvatures are equal to

$$
\begin{aligned}
&A_1 = \sqrt{R^2\cos(\psi)^2 + b^2\sin(\psi)^2},\ A_2 = a + b\cos(\psi),\\
&\mathbf{n}(\psi,\varphi) = -\frac{R\cos(\psi)\cos(\varphi)}{\sqrt{R^2\cos(\psi)^2 + b^2\sin(\psi)^2}}\mathbf{e}_1 - \frac{R\cos(\psi)\sin(\varphi)}{\sqrt{R^2\cos(\psi)^2 + b^2\sin(\psi)^2}}\mathbf{e}_2 - \frac{b\sin(\psi)}{\sqrt{R^2\cos(\psi)^2 + b^2\sin(\psi)^2}}\mathbf{e}_3,\\
&\kappa_1 = \frac{bR}{\left(R^2\cos(\psi)^2 + b^2\sin(\psi)^2\right)^{3/2}},\quad \kappa_2 = \frac{R\cos(\psi)}{(a + b\cos(\psi))\sqrt{R^2\cos(\psi)^2 + b^2\sin(\psi)^2}},
\end{aligned}
\tag{120}
$$

respectively.

The coefficients Lamé for an elliptic toroidal shell have the form

$$
\begin{aligned}
H_1 &= \sqrt{R^2\cos(\psi)^2 + b^2\sin(\psi)^2}\left(1 + \frac{bRr}{\left(R^2\cos(\psi)^2 + b^2\sin(\psi)^2\right)^{3/2}}\right),\\
H_2 &= (a + b\cos(\psi))\left(1 + \frac{Rr\cos(\psi)}{(a + b\cos(\psi))\sqrt{R^2\cos(\psi)^2 + b^2\sin(\psi)^2}}\right),\ H_3 = 1
\end{aligned}
\tag{121}
$$

Substituting these parameters into equations (49)–(58), we obtain equations corresponding to the higher order theory of the linear theory for an elliptic toroidal shell. The final equations have the form (50), and the essential boundary conditions have the form (57).

The matrices $\mathbf{L}^{loc}_{\tau,s}$ in (51) are the fundamental nuclei of differential equations of equilibrium for higher order elliptic toroidal elastic shells. They, as well as the vectors of local unknown functions $\mathbf{u}^{loc}_s$ and the local expression for the external body and surface loads $\mathbf{b}^{loc}_s$ have the form

$$
\mathbf{L}^{loc}_{\tau,s} = \begin{vmatrix} L^{\tau,s}_{u_\varphi,u_\varphi} & L^{\tau,s}_{u_\varphi,u_\psi} & L^{\tau,s}_{u_\varphi,u_r} \\ L^{\tau,s}_{u_\psi,u_\varphi} & L^{\tau,s}_{u_\psi,u_\psi} & L^{\tau,s}_{u_\psi,u_r} \\ L^{\tau,s}_{u_r,u_\varphi} & L^{\tau,s}_{u_r,u_\psi} & L^{\tau,s}_{u_r,u_r} \end{vmatrix},\quad \mathbf{u}^{loc}_s = \begin{vmatrix} u_{\varphi,s} \\ u_{\psi,s} \\ u_{r,s} \end{vmatrix},\quad \mathbf{b}^{loc}_s = \begin{vmatrix} \tilde{b}_{u_\varphi,\tau} \\ \tilde{b}_{u_\psi,\tau} \\ \tilde{b}_{u_r,\tau} \end{vmatrix}.
\tag{122}
$$

The fundamental nuclei $\mathbf{L}^{loc}_{\tau,s}$ coefficients can be easy calculated using equations presented in the previous sections. Their analytic expressions are complicated, and we do not present them here.

The matrices $\mathbf{B}^{loc}_{\tau,s}$ of the fundamental nuclei for natural boundary conditions, as well as the vectors $\mathbf{p}^{loc}_s$ of the local external load applied to the shell ends for the higher order elliptic toroidal elastic shells have the form

$$
\mathbf{B}^{loc}_{\tau,s} = \begin{vmatrix} B^{\tau,s}_{u_\varphi,u_\varphi} & B^{\tau,s}_{u_\varphi,u_\psi} & B^{\tau,s}_{u_\varphi,u_r} \\ B^{\tau,s}_{u_\psi,u_\varphi} & B^{\tau,s}_{u_\psi,u_\psi} & B^{\tau,s}_{u_\psi,u_r} \\ B^{\tau,s}_{u_r,u_\varphi} & B^{\tau,s}_{u_r,u_\psi} & B^{\tau,s}_{u_r,u_r} \end{vmatrix},\quad \mathbf{p}^{loc}_s = \begin{vmatrix} J^{u_\varphi,u_\varphi}_{\tau,s} P_{u_{\varphi,s}} \\ J^{u_\psi,u_\psi}_{\tau,s} P_{u_{\psi,s}} \\ J^{u_r,u_r}_{\tau,s} P_{u_{r,s}} \end{vmatrix}.
\tag{123}
$$

The fundamental nuclei $\mathbf{B}^{loc}_{\tau,s}$ coefficients can be easy calculated using equations presented in the previous sections. Their analytic expressions are complicated, and we do not present them here.

Elliptic toroidal shells considered in Zingoni A., et al. [72].

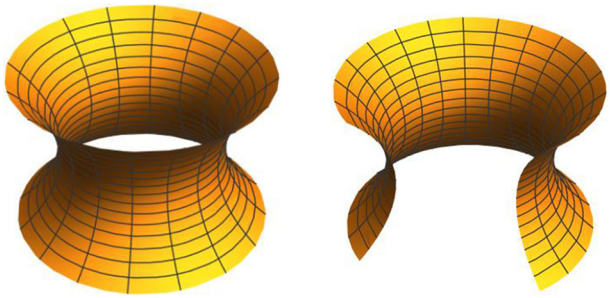

Fig. 9 Two catenoidal surfaces of revolution

4.11 Catenoidal shell

Models of elastic shells of catenoidal geometry are very important and are often used in theoretical analysis as well as applications in sciences and engineering. Consider a catenoidal shell formed by rotation around the axis x_3 of a catenary $a\cosh(x_3/a) = x_1$ curve line segment. The middle surface of the shell is a catenoid, the analytical representation of which in Cartesian coordinates $x,\ y,\ z$ is given by the equation

$$x^2 + y^2 = a\cosh(z/a). \tag{124}$$

We introduce curvilinear coordinates where $x_1 = x$, $x_2 = \varphi$ and $x_3 = r$, $r \in [-h,\ h]$. The parametric equations of the surface of revolution (124) have the following vector form:

$$\mathbf{r}(x,\varphi) = a\cosh(x/a)\cos(\varphi)\mathbf{e}_1 + a\cosh(x/a)\sin(\varphi)\mathbf{e}_2 + x\mathbf{e}_3. \tag{125}$$

If the parameters x and φ or cylindrical coordinates belong to intervals $x \in [0,\ H]$ and $\varphi \in [0,\ 2\pi]$ we have a complete catenoidal surface otherwise a catenoidal segment, as shown in Fig. 9.

The coefficients of the first quadratic form of a catenoidal surface, as well as the unit vector normal to the surface and the principal curvatures are equal to

$$\begin{aligned} &A_1 = \cosh(x/a),\ A_2 = a\cosh(x/a),\\ &\mathbf{n}(x,\varphi) = -\frac{\cos(\varphi)}{\cosh(x/a)}\mathbf{e}_1 - \frac{\sin(\varphi)}{\cosh(x/a)}\mathbf{e}_2 + \operatorname{sech}(x/a)^2\mathbf{e}_3,\\ &\kappa_1 = -\frac{\operatorname{sech}(x/a)^2}{a},\quad \kappa_2 = \frac{\operatorname{sech}(x/a)^2}{a}, \end{aligned} \tag{126}$$

respectively.

The coefficients Lamé for a catenoidal shell have the form

$$\begin{aligned} H_1 &= \cosh(x/a)\left(1 - \frac{r}{a\cosh(x/a)^2}\right),\\ H_2 &= a\cosh(x/a)\left(1 + \frac{r}{a\cosh(x/a)^2}\right) \text{ and } H_3 = 1 \end{aligned} \tag{127}$$

Substituting these parameters into equations (49)–(58), we obtain equations corresponding to the higher order theory of the linear theory for a catenoidal shell. The final equations have the form (50), and the essential boundary conditions have the form (57).

The matrices $\mathbf{L}^{loc}_{\tau,s}$ in (51) are the fundamental nuclei of differential equations of equilibrium for higher order catenoidal elastic shells. They, as well as the vectors of local unknown functions $\mathbf{u}^{loc}_s$ and the local expression for the external body and surface loads $\mathbf{b}^{loc}_s$ have the form

$$\mathbf{L}^{loc}_{\tau,s} = \begin{vmatrix} L^{\tau,s}_{u_x,u_x} & L^{\tau,s}_{u_x,u_\varphi} & L^{\tau,s}_{u_x,u_r}\\ L^{\tau,s}_{u_\varphi,u_x} & L^{\tau,s}_{u_\varphi,u_\varphi} & L^{\tau,s}_{u_\varphi,u_r}\\ L^{\tau,s}_{u_r,u_x} & L^{\tau,s}_{u_r,u_\varphi} & L^{\tau,s}_{u_r,u_r} \end{vmatrix},\quad \mathbf{u}^{loc}_s = \begin{vmatrix} u_{x,s}\\ u_{\varphi,s}\\ u_{r,s} \end{vmatrix},\quad \mathbf{b}^{loc}_s = \begin{vmatrix} \tilde{b}_{u_x,\tau}\\ \tilde{b}_{u_\varphi,\tau}\\ \tilde{b}_{u_r,\tau} \end{vmatrix}. \tag{128}$$

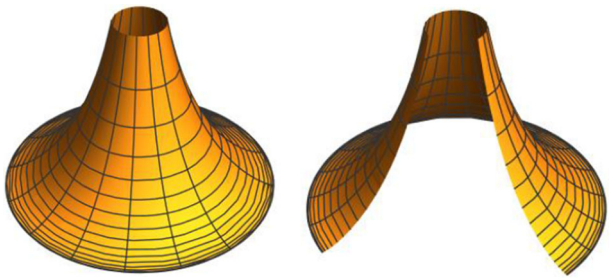

Fig. 10 Two pseudospherical surfaces of revolution

The fundamental nuclei $\mathbf{L}^{loc}_{\tau,s}$ coefficients can be easy calculated using equations presented in the previous sections. Their analytic expressions are complicated, and we do not present them here.

The matrices $\mathbf{B}^{loc}_{\tau,s}$ of the fundamental nuclei for natural boundary conditions, as well as the vectors $\mathbf{p}^{loc}_{s}$ of the local external load applied to the shell ends for the higher order catenoidal elastic shells have the form

$$\mathbf{B}^{loc}_{\tau,s} = \begin{vmatrix} B^{\tau,s}_{u_x,u_x} & B^{\tau,s}_{u_x,u_\varphi} & B^{\tau,s}_{u_x,u_r} \\ B^{\tau,s}_{u_\varphi,u_x} & B^{\tau,s}_{u_\varphi,u_\varphi} & B^{\tau,s}_{u_\varphi,u_r} \\ B^{\tau,s}_{u_r,u_x} & B^{\tau,s}_{u_r,u_\varphi} & B^{\tau,s}_{u_r,u_r} \end{vmatrix}, \quad \mathbf{p}^{loc}_{s} = \begin{vmatrix} J^{u_x,u_x}_{\tau,s} P_{u_{x,s}} \\ J^{u_\varphi,u_\varphi}_{\tau,s} P_{u_{\varphi,s}} \\ J^{u_r,u_r}_{\tau,s} P_{u_{r,s}} \end{vmatrix}. \tag{129}$$

The fundamental nuclei $\mathbf{B}^{loc}_{\tau,s}$ coefficients can be easy calculated using equations presented in the previous sections. Their analytic expressions are complicated, and we do not present them here.

Catenoidal shells considered in Zun [60].

4.12 Pseudospherical shell

Models of elastic shells of pseudospherical geometry are very important and are often used in theoretical analysis as well as applications in sciences and engineering. Consider a pseudospherical shell formed by rotation around the axis x_3 of an involute of the catenary $a\cosh(x_3/a) = x_1$ curved line segment. The middle surface of the shell is a pseudosphere, the analytical representation of which in Cartesian coordinates x, y, z is given by the equation

$$a\ln(a + \sqrt{a^2 - x^2 - y^2}) - \sqrt{a^2 - x^2 - y^2} = z. \tag{130}$$

We introduce curvilinear coordinates where $x_1 = x$, $x_2 = \varphi$ and $x_3 = r$, $r \in [-h, h]$. The parametric equations of the surface of revolution (130) have the following vector form

$$\mathbf{r}(x,\varphi) = a\sin(x)\cos(\varphi)\mathbf{e}_1 + a\sin(x)\sin(\varphi)\mathbf{e}_2 + a(\cos(x) + \log(\tan(x/2))\mathbf{e}_3. \tag{131}$$

If the parameters x and φ or curvilinear coordinates belong to intervals $x \in [0,\ H]$ and $\varphi \in [0,\ 2\pi]$ we have a complete pseudospherical surface otherwise a pseudospherical segment, as shown in Fig. 10.

The coefficients of the first quadratic form of a pseudospherical surface, as well as the unit vector normal to the surface and the principal curvatures are equal to

$$\begin{aligned} &A_1 = a\cot(x),\ A_2 = a\sin(x), \\ &\mathbf{n}(x,\varphi) = -\cos(x)\cos(\varphi)\mathbf{e}_1 - \cos(x)\sin(\varphi)\mathbf{e}_2 + \frac{\cos(x)}{\cot(x)}\mathbf{e}_3, \\ &\kappa_1 = -\frac{1}{a\cot(x)},\quad \kappa_2 = \frac{\cot(x)}{a}, \end{aligned} \tag{132}$$

respectively.

The Lamé coefficients for a pseudospherical shell have the form

$$H_1 = a\cot(x), \left(1 - \frac{r}{a\cot(x)}\right),\quad H_2 = a\sin(x)\left(1 + \frac{r\cot(x)}{a}\right),\ H_3 = 1. \tag{133}$$

Substituting these parameters into equations (49)–(58), we obtain equations corresponding to the higher order theory of the linear theory for a pseudospherical shell. The final equations have the form (50), and the essential boundary conditions have the form (57).

The matrices $\mathbf{L}_{\tau,s}^{loc}$ in (51) are the fundamental nuclei of differential equations of equilibrium for higher order pseudospherical elastic shells. They, as well as the vectors of local unknown functions $\mathbf{u}_s^{loc}$ and the local expression for external the body and surface loads $\mathbf{b}_s^{loc}$ have the form

$$\mathbf{L}_{\tau,s}^{loc} = \begin{vmatrix} L_{u_x,u_x}^{\tau,s} & L_{u_x,u_\varphi}^{\tau,s} & L_{u_x,u_r}^{\tau,s} \\ L_{u_\varphi,u_x}^{\tau,s} & L_{u_\varphi,u_\varphi}^{\tau,s} & L_{u_\varphi,u_r}^{\tau,s} \\ L_{u_r,u_x}^{\tau,s} & L_{u_r,u_\varphi}^{\tau,s} & L_{u_r,u_r}^{\tau,s} \end{vmatrix}, \quad \mathbf{u}_s^{loc} = \begin{vmatrix} u_{x,s} \\ u_{\varphi,s} \\ u_{r,s} \end{vmatrix}, \quad \mathbf{b}_s^{loc} = \begin{vmatrix} \tilde{b}_{u_x,\tau} \\ \tilde{b}_{u_\varphi,\tau} \\ \tilde{b}_{u_r,\tau} \end{vmatrix}. \tag{134}$$

The fundamental nuclei $\mathbf{L}_{\tau,s}^{loc}$ coefficients can be easy calculated using equations presented in the previous sections. Their analytic expressions are complicated, and we do not present them here.

The matrices $\mathbf{B}_{\tau,s}^{loc}$ of the fundamental nuclei for natural boundary conditions, as well as the vectors $\mathbf{p}_s^{loc}$ of the local external load applied to the shell ends for the higher order pseudospherical elastic shells have the form

$$\mathbf{B}_{\tau,s}^{loc} = \begin{vmatrix} B_{u_x,u_x}^{\tau,s} & B_{u_x,u_\varphi}^{\tau,s} & B_{u_x,u_r}^{\tau,s} \\ B_{u_\varphi,u_x}^{\tau,s} & B_{u_\varphi,u_\varphi}^{\tau,s} & B_{u_\varphi,u_r}^{\tau,s} \\ B_{u_r,u_x}^{\tau,s} & B_{u_r,u_\varphi}^{\tau,s} & B_{u_r,u_r}^{\tau,s} \end{vmatrix}, \quad \mathbf{p}_s^{loc} = \begin{vmatrix} J_{\tau,s}^{u_x,u_x} P_{u_x,s} \\ J_{\tau,s}^{u_\varphi,u_\varphi} P_{u_\varphi,s} \\ J_{\tau,s}^{u_r,u_r} P_{u_r,s} \end{vmatrix}. \tag{135}$$

The fundamental nuclei $\mathbf{B}_{\tau,s}^{loc}$ coefficients can be easy calculated using equations presented in the previous sections. Their analytic expressions are complicated, and we do not present them here.

Pseudospherical shells considered for example in Gil-oulbe et al. [23], Krawczyk [38], Krivoshapko and Ivanov [43].

5 Conclusion

Higher-order theories for elastic shells of revolution have been developed here using the CUF approach which is based on the series expansion of general 3-D equations of linear theory of elasticity into a series expansion with respect to shell thickness. In the expansion mentioned above, the 2-D higher order shells of revolution theories are developed from general 3-D equations of linear theory of elasticity using the principle of virtual power. All the functions that define the stress–strain state of the shell including classical tensors as well as strain tensors, vectors of displacements and body forces have been expressed in terms of the coefficients of that general series expansion with respect to the shell thickness coordinate. Thereby, all equations of linear theory of elasticity including generalized Hooke's law have been transformed to the corresponding equations for the coefficients of the series expansion accordance with the (CUF) approach. The system of the equations of equilibrium in terms of the series expansion of displacement vectors coefficients has been obtained.

The equations of 2-D models of higher orders of shells of revolution are developed and presented here, for the cases the middle surfaces of which can be represented analytically. More specifically, we represent here a higher order theory for plates in polar coordinates, cylindrical. conical, spherical, elliptical, parabolical, hyperboloidal, toroidal, elliptic toroidal, catenoidal and pseudospherical shells.

The obtained equations can be used for theoretical analysis and calculation of the stress–strain state, as well as for modeling thin-walled structures that are used in science, engineering, and technology.

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References

1. Al-Khatib, O.J., Buchanan, G.R.: Free vibration of a paraboloidal shell of revolution including shear deformation and rotary inertia. Thin-Walled Struct. **48**, 223–232 (2010)

2. Carrera, E.: A class of two-dimensional theories for anisotropic multilayered plates analysis. In: *Atti Della Accademia Delle Scienze di Torino. Classe di Scienze Fisiche Matematiche e Naturali*, vol. 19–20, pp. 1–39 (1995)
3. Carrera, E.: Multilayered shell theories that account for a layer-wise mixed description. Part II: numerical evaluations. AIAA J. **37**, 1117–1124 (1999)
4. Carrera, E.: Multilayered shell theories that account for a layer-wise mixed description. Part I: governing equations. AIAA J. **37**, 1107–1116 (1999)
5. Carrera, E.: Developments, ideas and evaluations based upon the Reissner's mixed variational theorem in the modeling of multilayered plates and shells. Appl. Mech. Rev. **54**, 301–329 (2001)
6. Carrera, E.: Theories and finite elements for multilayered plates and shells. Arch. Comput. Methods Eng. **9**(2), 87–140 (2002)
7. Carrera, E.: Theories and finite elements for multilayered plates and shells: a unified compact formulation with numerical assessment and benchmarking. Arch. Comput. Methods Eng. **10**(3), 215–296 (2003)
8. Carrera, E., Zozulya, V.V.: Carrera unified formulation for the micropolar plates. Mech. Adv. Mater. Struct. **29**(22), 3163–3186 (2022). https://doi.org/10.1080/15376494.2021.1889726
9. Carrera, E., Zozulya, V.V.: Closed-form solution for the micropolar plates: Carrera unified formulation (CUF) approach. Arch. Appl. Mech. **91**, 91–116 (2021)
10. Carrera, E., Zozulya, V.V.: Carrera unified formulation (CUF) for the micropolar beams: analytical solutions. Mech. Adv. Mater. Struct. **28**(6), 583–607 (2021)
11. Carrera, E., Zozulya, V.V.: Carrera unified formulation (CUF) for the micropolar plates and shells. I. Higher order theory. Mech. Adv. Mater. Struct. **29**(6), 773–795 (2022)
12. Carrera, E., Zozulya, V.V.: Carrera unified formulation (CUF) for the micropolar plates and shells. II. Complete linear expansion case. Mech. Adv. Mater. Struct. **29**(6), 796–815 (2022)
13. Carrera, E., Elishakoff, I., Petrolo, M.: Who needs refined structural theories? Compos. Struct. **264**, 113671 (2021)
14. Cauchy, L.: Sur l'equilibre et le mouvernent d'une plaque solide. Exerc. Mat. **3**, 328–355 (1828)
15. Chernobryvko, M.V., Avramov, K.V.: Natural vibrations of parabolic shells. J. Math. Sci. **217**, 229–238 (2016)
16. Chernobryvko, M.V., Avramov, K.V., Romanenko, V.N., Batutina, T.J., Tonkonogenko, A.M.: Free linear vibrations of thin axisymmetric parabolic shells. Meccanica **49**, 2839–2845 (2014)
17. Clark, R.A.: On the theory of thin elastic toroidal shells. J. Math. Phys. **29**, 146–178 (1950)
18. Clark, R.A.: Asymptotic solutions of toroidal shell problems. Q. Appl. Math. **16**, 47–60 (1958)
19. Czekanski, A., Zozulya, V.V.: Dynamics of vibrating beams using first-order theory based on Legendre polynomial expansion. Arch. Appl. Mech. **90**, 789–814 (2020)
20. El-Raheb, M., Wagner, P.: Harmonic response of cylindrical and toroidal shells to an internal acoustic field. Part II. Results. J. Acoust. Soc. Am. **78**(2), 747–757 (1985)
21. El-Raheb, M., Wagner, P.: Harmonic response of cylindrical and toroidal shells to an internal acoustic field. Part I. Theory. J. Acoust. Soc. Am. **78**(2), 738–746 (1985)
22. Galimov, K.Z., Paimushin, V.N.: Theory of shells of complex geometry, p. 164. Kazan University Press, Kazan (1985)
23. Gil-Oulbé, M., Ndomilep, I.J.A., Ngandu, P.: Pseudospheric shells in the construction. RUDN J. Eng. Res. **22**(1), 84–99 (2021)
24. Gray, A., Abbena, E., Salamon, S.: Modern differential geometry of curves and surfaces with mathematica, 3rd edn., p. 1016. Chapman and Hall/CRC, New York (2006)
25. Guliaev, V.I., Bazhenov, V.A., Lizunov, P.P.: Non-Classical Shell Theory and its Application to Solving of Engineering Problems. L'vov, Vyscha Shkola, 192p (1978)
26. Sun, B. (ed.): Toroidal Shells. Nova Science Publishers Inc., 186 p (2012)
27. Kang, J.H., Leissa, A.W.: Three-dimensional vibration analysis of thick Paraboidal shells. Int. J. JSME **45**(1), 2–7 (2002)
28. Kang, J.-H., Leissa, A.W.: Free vibration analysis of complete paraboloidal shells of revolution with variable thickness and solid paraboloids from a three-dimensional theory. Comput. Struct. **83**, 2594–2608 (2005)
29. Kang, J.H., Leissa, A.W.: Three-dimensional vibration analysis of thick hyperboloidal shells of revolution. J. Sound Vib. **282**, 277–296 (2005)
30. Khoma, I.Y.: Generalized Theory of Anisotropic Shells. Naukova Dumka, Kiev, 172p (1987) (in Russian)
31. Kil'chevskiy, N.A.: Fundamentals of the Analytical Mechanics of Shells. NASA TT, F-292, Washington, 361p (1965)
32. Klochkov, Y.V., Nikolaev, A.P., Kiseleva, T.A., Marchenko, S.S.: Comparative analysis of the results of finite element calculations based on an ellipsoidal shell. J. Mach. Manuf. Reliab. **45**(4), 328–336 (2016)
33. Klochkov, Y.V., Nikolaev, A.P., Sobolevskaya, T.A., Vakhnina, O.V., Klochkov, MYu.: Comparative analysis of the results of finite element calculations based on an ellipsoidal shell. Lobach. J. Math. **41**(3), 373–381 (2020)
34. Klochkov, Y.V., Nikolaev, A.P., Ishchanov, T.R., Andreev, A.S.: Comparative analysis of the results of finite element calculations based on an ellipsoidal shell. J. Mach. Manuf. Reliab. **49**(4), 301–307 (2020)
35. Korjakin, A., RikardS, R., Altenbach, H., Chate, A.: Free damped vibrations of sandwich shells of revolution. J. Sandwich Struct. Mater. **3**, 171–196 (2001)
36. Kornishin, M.S., Paimushin, V.N., Snigirev, V.F.: Computational Geometry in Problems of Shell Mechanics, p. 208. Nauka, Moscow (1989)
37. Kovarik, V.: Stresses in Layered Shells of Revolution, p. 432. Elsevier, Amsterdam (1989)
38. Krawczyk, J.: Infinitesimal isometric deformations of a pseudospherical shell. J. Math. Sci. **109**, 1312–1320 (2002)
39. Krivoshapko, S.N.: Static, vibration, and buckling analyses and applications to one-sheet hyperboloidal shells of revolution. Appl. Mech. Rev. **55**(3), 241–270 (2002)
40. Krivoshapko, S.N.: Research on general and axisymmetric ellipsoidal shells used as domes, pressure vessels, and tanks. Appl. Mech. Rev. **60**(11), 336–355 (2007)
41. Krivoshapko, S.N.: On application of parabolic shells of revolution in civil engineering in 2000–2017. Struct. Mech. Eng. Constr. Build. **4**, 4–14 (2017)
42. Krivoshapko, S.N., Ivanov, V.N.: Encyclopedia of Analytical Surfaces, p. 761. Springer, New York (2015)
43. Krivoshapko, S.N., Ivanov, V.N.: Pseudospherical shells in building industry. Build. Reconstr. **2**, 32–40 (2018)

44. Kuhnel, W.: Differential Geometry: Curves, Surfaces, Manifolds, 3rd edn., p. 418. American Mathematical Society, Providence (2016)
45. Leung, A.Y.T., Kwok, N.T.C.: Free vibration analysis of a toroidal shell. Thin-Walled Struct. **18**, 317–332 (1994)
46. Lutskaya, I.V., Maximuk, V.A., Chernyshenko, I.S.: Modeling the deformation of orthotropic toroidal shells with elliptical cross-section based on mixed functionals. Int. Appl. Mech. **54**, 660–665 (2018). https://doi.org/10.1007/s10778-018-0920-0
47. Meish, V.F.: Numerical solution of dynamic problems for reinforced ellipsoidal shells under nonstationary loads. Int. Appl. Mech. **41**(4), 386–391 (2005)
48. Meish, V.F., Maiborodina, N.V.: Stress state of discretely stiffened ellipsoidal shells under a nonstationary normal load. Int. Appl. Mech. **54**(6), 675–686 (2018)
49. Ming, R.S., Pan, J., Norton, M.P.: Free vibrations of elastic circular toroidal shells. Appl. Acoust. **63**, 513–528 (2002)
50. Naboulsi, S.K., Palazotto, A.N., Greer, J.M.: Static-dynamic analyses of toroidal shells. J. Aerosp. Eng. **13**(3), 110–121 (2000)
51. Pelekh, B.L., Lazko, V.A.: Laminated Anisotropic Plates and Shells with Stress Concentrators. Naukova Dumka, Kiev, 296p (1982)
52. Pelekh, B.L., Sukhorol'skii, M.A.: Contact Problems of the Theory of Elastic Anisotropic Shells. Naukova Dumka, Kiev, 216p (1980) (in Russian)
53. Carrera, E., Cinefra, M., Petrolo, M., Zappino, E.: Finite Element Analysis of Structures through Unified Formulation. Wiley, New Delhi, p 385 (2014)
54. Poissons, D.: Memoire sur l'equilibre et le mouvement des corps elastique. In: Memoires de l'Academie Royale des Sciences, vol. VIII, pp. 357–570, 623–627 (1829)
55. Reddy, J.N.: Mechanics of Laminated Composite Plates and Shells: Theory and Analysis, 2nd edn. CRC Press LLC, 855p (2004)
56. Reddy, J.N.: Theory and Analysis of Elastic Plates and Shells, 2nd edn., p. 561. CRC Press LLC, New York (2006)
57. Rekach, V.G., Krivoshapko, S.N.: Calculation of Shells of Complex Geometry, p. 176. UDN University Press, Moscow (1988)
58. Senjanovic, I., Alujevic, N., Catipovic, I.: Cakmak D, Vladimir N Vibration analysis of rotating toroidal shell by the Rayleigh–Ritz method and Fourier series. Eng. Struct. **173**, 870–891 (2018)
59. Senjanovic, I., Alujevic, N., Catipovic, I., Cakmak, D., Vladimir, N., Cho, D.-S.: Natural vibration analysis of pressurised and rotating toroidal shell segment by Rayleigh–Ritz. Eng. Model. **2**(4), 57–81 (2019)
60. Sun, B.: Deformation and stress analysis of catenary shell of revolution. Preprints (2021). https://doi.org/10.20944/preprints202104.0494.v1
61. Sun, B.: Closed-form solution of axisymmetric slender elastic toroidal shells. J. Eng. Mech. ASCE **136**(10), 1281–1288 (2010)
62. Sutcliffe, W.J.: Stress analysis of toroidal shells of elliptical cross-section. Int. J. Mech. Sci. **13**(11), 951–958 (1971)
63. Tangbanjongkij, C., Chucheepsakul, S., Jiammeepreecha, W.: Analytical and numerical analyses for a variety of submerged hemi-ellipsoidal shells. J. Eng. Mech. ASCE **146**(7), 1–15 (2020)
64. Timoshenko, S., Woinowsky-Krieger, S.: Theory of Plates and Shells, 2nd edn., p. 611. McGraw-Hill Book Company, Paris (1959)
65. Vekua, I.N.: Shell Theory, General Methods of Construction, p. 287. Pitman Advanced Publishing Program, Boston (1986)
66. Vlasov, V.Z.: General Theory of Shells and Its Application in Engineering. Published by NASA-TT-F-99, 913p (1964)
67. von Seggern, D.H.: CRC Standard Curves and Surfaces with Mathematica, 2nd edn., p. 660. CRC Press Taylor & Francis Group, Boca Raton (2016)
68. Wan, F.Y.M., Weinitschke, H.J.: On shells of revolution with the Love–Kirchhoff hypotheses. J. Eng. Math. **22**, 285–334 (1988)
69. Washizu, K.: Variational Methods in Elasticity and Plasticity, 3rd edn., p. 630. Pergamon Press, New York (1982)
70. Wenmin, R., Wenguo, L., Wei, Z.: A survey of works on the theory of toroidal shells and curved tubes. Acta Mech. Sin. (English Ser.) **15**(3), 225–234 (1999)
71. Xie, X., Zheng, H., Jin, G.: Free vibration of four-parameter functionally graded spherical and parabolic shells of revolution with arbitrary boundary conditions. Compos. B **77**, 59–73 (2015)
72. Zingoni, A., et al.: Equatorial bending of an elliptic toroidal shell. Thin-Walled Struct. **96**, 286–294 (2015)
73. Zozulya, V.V.: A high order theory for linear thermoelastic shells: comparison with classical theories. J. Eng. (2013)
74. Zozulya, V.V.: The combines problem of thermoelastic contact between two plates through a heat conducting layer. J. Appl. Math. Mech. **53**(5), 622–627 (1989)
75. Zozulya, V.V.: Contact cylindrical shell with a rigid body through the heat-conducting layer in transitional temperature field. Mech. Solids **2**, 160–165 (1991)
76. Zozulya, V.V.: Laminated shells with debonding between laminas in temperature field. Int. Appl. Mech. **42**(7), 842–848 (2006)
77. Zozulya, V.V.: Mathematical modeling of pencil-thin nuclear fuel rods. In: Gupta, A. (ed.) Structural Mechanics in Reactor Technology, pp. C04-C12. Canada, Toronto (2007)
78. Zozulya, V.V.: A high-order theory for functionally graded axially symmetric cylindrical shells. Arch. Appl. Mech. **83**(3), 331–343 (2013)
79. Zozulya, V.V.: A higher order theory for shells, plates and rods. Int. J. Mech. Sci. **103**, 40–54 (2015)
80. Zozulya, V.V.: Nonlocal theory of curved rods. 2-D, high order, Timoshenko's and Euler–Bernoulli models. Curve. Layer. Struct. **4**, 221–236 (2017)
81. Zozulya, V.V.: Couple stress theory of curved rods. 2-D, high order, Timoshenko's and Euler–Bernoulli models. Curve. Layer. Struct. **4**, 1192–2132 (2017)
82. Zozulya, V.V.: Micropolar curved rods. 2-D, high order, Timoshenko's and Euler–Bernoulli models. Curve. Layer. Struct. **4**, 104–118 (2017)

83. Zozulya, V.V.: Higher order theory of micropolar curved rods. In: Altenbach, H., Öchsner, A. (eds.) Encyclopedia of continuum mechanics, pp. 1–11. Springer, Berlin, Heidelberg (2018)
84. Zozulya, V.V.: Higher order couple stress theory of plates and shells. J. Appl. Math. Mech. (ZAMM) **98**(10), 1834–1863 (2018)
85. Zozulya, V.V.: Higher order theory of micropolar plates and shells. J. Appl. Math. Mech. (ZAMM) **98**(6), 886–918 (2018)
86. Zozulya, V.V., Carrera, E.: Carrera unified formulation (CUF) for the micropolar plates and shells. III. Classical models. Mech. Adv. Mater. Struct. (2021). https://doi.org/10.1080/15376494.2021.1975855
87. Zozulya, V.V., Saez, A.: High-order theory for arched structures and its application for the study of the electrostatically actuated MEMS devices. Arch. Appl. Mech. **84**(7), 1037–1055 (2014)
88. Zozulya, V.V., Saez, A.: A high order theory of a thermo elastic beams and its application to the MEMS/NEMS analysis and simulations. Arch. Appl. Mech. **86**(7), 1255–1272 (2015)
89. Zozulya, V.V., Zhang, Ch.: A high order theory for functionally graded axisymmetric cylindrical shells. Int. J. Mech. Sci. **60**(1), 12–22 (2012)