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ORIGINAL PAPER

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Attitude stabilization of a rigid body under disturbances with zero mean values

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Abstract The problem of attitude stabilization of a rigid body exposed to a nonstationary perturbing torque is investigated. The control torque consists of a restoring component and a dissipative one. Linear and nonlinear variants of restoring and perturbing torques are analyzed. Conditions of the asymptotic stability of the programmed orientation of the body are found with the use of the Lyapunov direct method and the averaging technique. The results of computer modeling, illustrating the conclusions obtained analytically, are presented.

List of symbols

A_1, A_2, A_3	Satellite principal central moments of inertia with respect to body frame axes x_1, x_2, x_3, $\mathrm{kg}\cdot\mathrm{m}^2$
a_1, a_2	Positive constants
$\mathbf{B}$	Constant symmetric and negative definite matrix
b_1, b_2	Positive constants
c	Positive constant
$c_1, \ldots, c_9$	Positive constants
$\mathbf{D}_1(t)$	Continuous and bounded matrix for $t \in [0, +\infty)$
$\mathbf{D}_2(t)$	Continuous and bounded matrix for $t \in [0, +\infty)$
$\mathbf{J}$	Satellite inertia tensor in body frame x_1, x_2, x_3, $\mathrm{kg}\cdot\mathrm{m}^2$
h	Positive parameter
h_0	Positive number
$\vec{L}$	Control torque vector in body frame, N·m
$\vec{L}_d$	Dissipative component of control torque, N·m
$\vec{L}_p$	Perturbing torque vector in body frame, N·m
$\vec{L}_r$	Restoring component of control torque, N·m
t	Time, s
V	Lyapunov function

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$V_1, \ldots, V_4$	Lyapunov functions
α	Positive constant
δ	Positive parameter
$\bar{\delta}$	Positive parameter
ε	Positive parameter such that $\varepsilon \in (0, 1)$
$\vec{\eta}_1, \vec{\eta}_2, \vec{\eta}_3$	Unit vectors of body-fixed frame
θ	"Aircraft" angle
λ	Auxiliary positive parameter
λ_0	Positive number
$\vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3$	Unit vectors of inertial frame
φ	"Aircraft" angle
ψ	"Aircraft" angle
$\vec{\omega}$	Angular velocity of satellite in inertial reference frame, rad/s

1 Introduction

The art of mathematical modeling of mechanical systems is based on the correct estimation of the acting forces and torques that affect the dynamics and provide qualitative and quantitative properties of motion. In those cases where the acting forces can be considered known and time invariant, the estimations are based on calculating the absolute values of forces and torques. In cases of time-varying forces and torques, such estimates are not enough. Important quantitative characteristics of variable force factors are their mean values. A comparison of the mean values of acting forces often reveals the main ones, and the rest can be classified as disturbances. However, numerous well-known examples of the analysis of the mechanical systems behavior indicate that disturbances with zero mean values are not necessarily insignificant. Therefore, neglecting such disturbances is unacceptable in many problems. At the same time, their account often significantly complicates analytical qualitative analysis of the mechanical system behavior [1–6]. Hence, on the one hand, there is a significant interest of specialists in problems of the dynamics of systems subjected to perturbations with zero mean values, and on the other hand, these complex problems are not well understood, and therefore the stream of publications on this topic continues [7–12]. Attitude stabilization of a spacecraft is one of the typical nonlinear problems, usually complicated by the presence of numerous nonstationary disturbances, including those with zero mean values. This problem is relevant in many astronautical and engineering applications [1,12–16]. This article is dedicated to this specific problem. It is worth mentioning that a similar problem was earlier considered in our paper [17], but with other assumptions concerning disturbances and control torques.

2 Statement of the problem

Let a rigid body rotating around its mass center O with angular velocity $\vec{\omega}$ be given. Denote by $Ox_1x_2x_3$ the principal central axes of inertia of the body. The attitude motion of the body under a control torque $\vec{L}$ is described by the Euler equations [1]

$$\mathbf{J}\dot{\vec{\omega}} + \vec{\omega} \times \mathbf{J}\vec{\omega} = \vec{L}. \tag{1}$$

Here, $\mathbf{J} = \mathrm{diag}\{A_1, A_2, A_3\}$ is inertia tensor of the body in the axes $Ox_1x_2x_3$.

Consider two right triples of mutually orthogonal unit vectors $\vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3$ and $\vec{\eta}_1, \vec{\eta}_2, \vec{\eta}_3$. Let vectors $\vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3$ be constant in the inertial frame, and vectors $\vec{\eta}_1, \vec{\eta}_2, \vec{\eta}_3$ be constant in the body-fixed frame. Thus, vectors $\vec{\xi}_1$, $\vec{\xi}_2, \vec{\xi}_3$ rotate with respect to the system $Ox_1x_2x_3$ with the angular velocity $-\vec{\omega}$. Hence, we obtain the Poisson kinematic equations

$$\dot{\vec{\xi}}_i = -\vec{\omega} \times \vec{\xi}_i, \quad i = 1, 2, 3. \tag{2}$$

It is worth noting that the systems (1), (2) may describe a wide variety of objects such as aircraft, satellite, submarine, missile, and quadcopter (Fig. 1) [1,18–20].

Let torque $\vec{L}$ be the sum of a dissipative component $\vec{L}_d$ and a restoring one $\vec{L}_r$: $\vec{L} = \vec{L}_d + \vec{L}_r$. We will assume that the dissipative torque is linear with respect to $\vec{\omega}$ [21,22] and it is defined by the formula

$$\vec{L}_d = h\mathbf{B}\vec{\omega}, \tag{3}$$

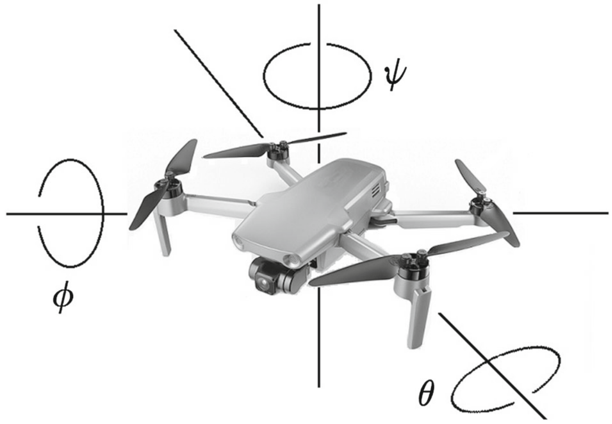

Fig. 1 Quadcopter attitude motion and angles ϕ, θ, ψ

where **B** is a constant symmetric and negative definite matrix, h is a positive parameter. The restoring torque $\vec{L}_r$ should be chosen such that the torque $\vec{L}$ provides triaxial stabilization of the body, i.e., the system of Eqs. (1), (2) should admit the asymptotically stable equilibrium position

$$\vec{\omega} = \vec{0}, \qquad \vec{\xi}_i = \vec{\eta}_i, \quad i = 1, 2, 3. \tag{4}$$

It is known (for example, see [19]), that the torque $\vec{L}_r$ can be defined by the formula

$$\vec{L}_r = -cf^{\nu}(\vec{\xi}_1, \vec{\xi}_2)\left(a_1\vec{\xi}_1 \times \vec{\eta}_1 + a_2\vec{\xi}_2 \times \vec{\eta}_2\right). \tag{5}$$

Here, c, a_1, a_2 are positive constants,

$$f(\vec{\xi}_1, \vec{\xi}_2) = \left(a_1\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + a_2\|\vec{\xi}_2 - \vec{\eta}_2\|^2\right)/2,$$

$\nu \geq 0$, and $\|\cdot\|$ is the Euclidean norm of a vector.

In the present paper, we consider the case where, along with the control torque $\vec{L}$, a nonstationary perturbing torque $\vec{L}_p$ acts on the body.

3 Construction of a strict Lyapunov function for the unperturbed system

Consider the unperturbed system composed of the Poisson kinematic Eq. (2) and the Euler dynamic equations

$$\mathbf{J}\dot{\vec{\omega}} + \vec{\omega} \times \mathbf{J}\vec{\omega} = \vec{L}_d + \vec{L}_r, \tag{6}$$

where dissipative and restoring torques are defined by the formulae (3) and (5), respectively.

Stability of the equilibrium position (4) for the system (2), (6) was proved in [19]. However, it is worth mentioning that results of [19] are based on the construction of a weak Lyapunov function. The derivative of this function along the solutions of the considered system is only nonnegative. Such Lyapunov functions are not well applicable to robustness analysis of nonlinear systems, since their negative semi-definite derivatives could become positive under arbitrarily small perturbations [23,24].

In [20,25], an approach was developed to transform the weak Lyapunov function constructed in [19] into a strict one (a function with negative definite derivative) [26,27]. At the same time, it should be noted that the approach of [20,25] can be used only for the case of linear restoring torque. Moreover, this approach is not effective for the investigation of the problem studied in the present paper. Therefore, we will propose another construction of a strict Lyapunov function for the system (2), (6).

Choose a Lyapunov function candidate as follows:

$$\begin{aligned} V\left(\vec{\omega}, \vec{\xi}_1, \vec{\xi}_2\right) = {} & \frac{\lambda}{2}\vec{\omega}^{\top}\mathbf{J}\vec{\omega} + \frac{a_1}{2}\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \frac{a_2}{2}\|\vec{\xi}_2 - \vec{\eta}_2\|^2 \\ & -\frac{1}{h}\left(a_1\vec{\xi}_1 \times \vec{\eta}_1 + a_2\vec{\xi}_2 \times \vec{\eta}_2\right)^{\top}\mathbf{B}^{-1}\mathbf{J}\vec{\omega}. \end{aligned} \tag{7}$$

Here, λ is an auxiliary positive parameter. Then,

$$\lambda c_1\|\vec{\omega}\|^2+\frac{a_1}{2}\|\vec{\xi}_1-\vec{\eta}_1\|^2+\frac{a_2}{2}\|\vec{\xi}_2-\vec{\eta}_2\|^2$$
$$-\frac{c_3}{h}\|\vec{\omega}\|\left(\|\vec{\xi}_1-\vec{\eta}_1\|+\|\vec{\xi}_2-\vec{\eta}_2\|\right)\leq V\left(\vec{\omega},\vec{\xi}_1,\vec{\xi}_2\right)$$
$$\leq \lambda c_2\|\vec{\omega}\|^2+\frac{a_1}{2}\|\vec{\xi}_1-\vec{\eta}_1\|^2+\frac{a_2}{2}\|\vec{\xi}_2-\vec{\eta}_2\|^2+\frac{c_3}{h}\|\vec{\omega}\|\left(\|\vec{\xi}_1-\vec{\eta}_1\|+\|\vec{\xi}_2-\vec{\eta}_2\|\right),$$

where c_1, c_2, c_3 are positive constants.

Differentiating the function (7) along the solutions of (2), (6), we obtain

$$\dot{V}=\lambda h\vec{\omega}^{\top}\mathbf{B}\vec{\omega}-\lambda c f^{\nu}\vec{\omega}^{\top}\left(a_1\vec{\xi}_1\times\vec{\eta}_1+a_2\vec{\xi}_2\times\vec{\eta}_2\right)$$
$$+\frac{1}{h}(a_1(\vec{\omega}\times\vec{\xi}_1)\times\vec{\eta}_1+a_2(\vec{\omega}\times\vec{\xi}_2)\times\vec{\eta}_2)^{\top}\mathbf{B}^{-1}\mathbf{J}\vec{\omega}$$
$$+\frac{1}{h}\left(a_1\vec{\xi}_1\times\vec{\eta}_1+a_2\vec{\xi}_2\times\vec{\eta}_2\right)^{\top}\mathbf{B}^{-1}(\vec{\omega}\times(\mathbf{J}\vec{\omega}))$$
$$+\frac{c}{h}f^{\nu}\left(a_1\vec{\xi}_1\times\vec{\eta}_1+a_2\vec{\xi}_2\times\vec{\eta}_2\right)^{\top}\mathbf{B}^{-1}\left(a_1\vec{\xi}_1\times\vec{\eta}_1+a_2\vec{\xi}_2\times\vec{\eta}_2\right).$$

The matrix $\mathbf{B}$ is negative definite. Therefore, the inequality

$$\dot{V}\leq-\left(\lambda h c_4-\frac{c_5}{h}\right)\|\vec{\omega}\|^2-\frac{c_6}{h}f^{\nu}\left\|a_1\vec{\xi}_1\times\vec{\eta}_1+a_2\vec{\xi}_2\times\vec{\eta}_2\right\|^2$$
$$+\frac{c_7}{h}\|\vec{\omega}\|^2\left(\|\vec{\xi}_1-\vec{\eta}_1\|+\|\vec{\xi}_2-\vec{\eta}_2\|\right)+\lambda c_8\|\vec{\omega}\|\left(\|\vec{\xi}_1-\vec{\eta}_1\|+\|\vec{\xi}_2-\vec{\eta}_2\|\right)^{2\nu+1}$$

holds. Here, $c_i>0$, $i=4,\ldots,8$.

Choose a number $\varepsilon\in(0,1)$. In [28], it was proved that there exists $\delta>0$ such that

$$\left\|a_1\vec{\xi}_1\times\vec{\eta}_1+a_2\vec{\xi}_2\times\vec{\eta}_2\right\|^2\geq\varepsilon\left(a_1^2\|\vec{\xi}_1-\vec{\eta}_1\|^2+a_2^2\|\vec{\xi}_2-\vec{\eta}_2\|^2\right)$$

for $\|\vec{\xi}_1-\vec{\eta}_1\|^2+\|\vec{\xi}_2-\vec{\eta}_2\|^2<\delta^2$. Hence,

$$\dot{V}\leq-\left(\lambda h c_4-\frac{c_6}{h}\right)\|\vec{\omega}\|^2-\frac{c_9}{h}\left(\|\vec{\xi}_1-\vec{\eta}_1\|^2+\|\vec{\xi}_2-\vec{\eta}_2\|^2\right)^{\nu+1}$$
$$+\frac{c_7}{h}\|\vec{\omega}\|^2\left(\|\vec{\xi}_1-\vec{\eta}_1\|+\|\vec{\xi}_2-\vec{\eta}_2\|\right)+\lambda c_8\|\vec{\omega}\|\left(\|\vec{\xi}_1-\vec{\eta}_1\|+\|\vec{\xi}_2-\vec{\eta}_2\|\right)^{2\nu+1}$$

for $\|\vec{\xi}_1-\vec{\eta}_1\|^2+\|\vec{\xi}_2-\vec{\eta}_2\|^2<\delta^2$, where $c_9=\text{const}>0$.

As a result, we obtain that there exist positive numbers $\lambda, h, \bar{\delta}$ such that

$$\frac{1}{2}\lambda c_1\|\vec{\omega}\|^2+\frac{a_1}{4}\|\vec{\xi}_1-\vec{\eta}_1\|^2+\frac{a_2}{4}\|\vec{\xi}_2-\vec{\eta}_2\|^2$$
$$\leq V\left(\vec{\omega},\vec{\xi}_1,\vec{\xi}_2\right)\leq 2\lambda c_2\|\vec{\omega}\|^2+a_1\|\vec{\xi}_1-\vec{\eta}_1\|^2+a_2\|\vec{\xi}_2-\vec{\eta}_2\|^2,$$
$$\dot{V}\leq-\frac{1}{2}\lambda h c_4\|\vec{\omega}\|^2-\frac{c_9}{2h}\left(\|\vec{\xi}_1-\vec{\eta}_1\|^2+\|\vec{\xi}_2-\vec{\eta}_2\|^2\right)^{\nu+1}$$

for $\vec{\omega}\in\mathbb{R}^3$, $\|\vec{\xi}_1-\vec{\eta}_1\|^2+\|\vec{\xi}_2-\vec{\eta}_2\|^2<\bar{\delta}^2$.

It is worth noting that, in the case where $\nu=0$, value of λ should be sufficiently small and the value of h should be sufficiently large, whereas, in the case where $\nu>0$, h may be an arbitrary positive number and λ should be sufficiently large.

Thus, for an appropriate choice of λ and h, (7) is a strict Lyapunov function for the unperturbed system (2), (6).

In what follows, using the approach developed in [29–31] and taking into account structure and properties of the nonstationary torque $\vec{L}_p$, we will propose some modifications of the function (7) to derive conditions ensuring asymptotic stability of the equilibrium position (4) of the perturbed system.

4 Linear restoring and perturbing torques

Let $\nu = 0$ and $\vec{L}_p = \mathbf{D}_1(t)(\vec{\xi}_1 - \vec{\eta}_1) + \mathbf{D}_2(t)(\vec{\xi}_2 - \vec{\eta}_2)$. Here matrices $\mathbf{D}_1(t), \mathbf{D}_2(t) \in \mathbb{R}^{3\times 3}$ are continuous and bounded for $t \in [0, +\infty)$. Then, the system (1) takes the form

$$\mathbf{J}\dot{\vec{\omega}} + \vec{\omega} \times \mathbf{J}\vec{\omega} = h\mathbf{B}\vec{\omega} - a_1\vec{\xi}_1 \times \vec{\eta}_1 - a_2\vec{\xi}_2 \times \vec{\eta}_2 + \mathbf{D}_1(t)(\vec{\xi}_1 - \vec{\eta}_1) + \mathbf{D}_2(t)(\vec{\xi}_2 - \vec{\eta}_2). \tag{8}$$

Thus, we consider the case where restoring and perturbing torques are linear.

Let us determine conditions under which perturbations do not disturb asymptotic stability of the equilibrium position (4).

Consider the derivative of the Lyapunov function (7) with respect to the system (2), (8). If λ and $\bar{\delta}$ are sufficiently small and h is sufficiently large, then the inequalities

$$\begin{aligned}\dot{V} &\le -\frac{1}{2}\lambda h c_4\|\vec{\omega}\|^2 - \frac{c_9}{2h}\left(\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2\right) \\ &\quad + \lambda\vec{\omega}^\top\vec{L}_p - \frac{1}{h}\left(a_1\vec{\xi}_1 \times \vec{\eta}_1 + a_2\vec{\xi}_2 \times \vec{\eta}_2\right)^\top \mathbf{B}^{-1}\vec{L}_p \\ &\le -\frac{1}{3}\lambda h c_4\|\vec{\omega}\|^2 - \frac{c_9}{3h}\left(\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2\right) \\ &\quad - \frac{1}{h}\left(a_1\vec{\xi}_1 \times \vec{\eta}_1 + a_2\vec{\xi}_2 \times \vec{\eta}_2\right)^\top \mathbf{B}^{-1}\vec{L}_p\end{aligned} \tag{9}$$

hold for $\vec{\omega} \in \mathbb{R}^3$, $\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2 < \bar{\delta}^2$.

Theorem 1 *Let the matrices*

$$\int_0^t \mathbf{D}_i(s)\,ds, \quad i = 1, 2, \tag{10}$$

be bounded for $t \in [0, +\infty)$. Then, there exists a number $h_0 > 0$ such that the equilibrium position (4) of the system (2), (8) is uniformly asymptotically stable for any $h \ge h_0$.

Proof Modify the Lyapunov function (7) as follows:

$$V_1\left(t, \vec{\omega}, \vec{\xi}_1, \vec{\xi}_2\right) = V\left(\vec{\omega}, \vec{\xi}_1, \vec{\xi}_2\right) + \frac{1}{h}(a_1\vec{\xi}_1 \times \vec{\eta}_1 + a_2\vec{\xi}_2 \times \vec{\eta}_2)^\top \mathbf{B}^{-1} \sum_{i=1}^{2} \int_0^t \mathbf{D}_i(s)\,ds\,(\vec{\xi}_i - \vec{\eta}_i).$$

Using the results of the previous Section and the inequalities (9), it is easy to verify that one can choose and fix sufficiently small values of λ and $\bar{\delta}$ and after that find $h_0 > 0$ such that if $h \ge h_0$, $\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2 < \bar{\delta}^2$, then the function $V_1\left(t, \vec{\omega}, \vec{\xi}_1, \vec{\xi}_2\right)$ and its derivative with respect to the system (2), (8) satisfy the estimates

$$\begin{aligned}&\frac{1}{2}\lambda c_1\|\vec{\omega}\|^2 + \frac{a_1}{4}\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \frac{a_2}{4}\|\vec{\xi}_2 - \vec{\eta}_2\|^2 \\ &\quad - \frac{b_1}{h}\left(\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2\right) \le V_1\left(t, \vec{\omega}, \vec{\xi}_1, \vec{\xi}_2\right) \\ &\le 2\lambda c_2\|\vec{\omega}\|^2 + a_1\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + a_2\|\vec{\xi}_2 - \vec{\eta}_2\|^2 + \frac{b_1}{h}\left(\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2\right), \\ &\dot{V}_1 \le -\frac{1}{3}\lambda h c_4\|\vec{\omega}\|^2 - \frac{c_9}{3h}\left(\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2\right) \\ &\quad + \frac{b_2}{h}\|\vec{\omega}\|\left(\|\vec{\xi}_1 - \vec{\eta}_1\| + \|\vec{\xi}_2 - \vec{\eta}_2\|\right),\end{aligned}$$

where b_1, b_2 are positive constants.

Hence, for sufficiently large values of h_0, the inequalities

$$\frac{1}{2}\lambda c_1\|\vec{\omega}\|^2 + \frac{a_1}{8}\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \frac{a_2}{8}\|\vec{\xi}_2 - \vec{\eta}_2\|^2$$
$$\leq V_1\left(t, \vec{\omega}, \vec{\xi}_1, \vec{\xi}_2\right) \leq 2\left(\lambda c_2\|\vec{\omega}\|^2 + a_1\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + a_2\|\vec{\xi}_2 - \vec{\eta}_2\|^2\right),$$
$$\dot{V}_1 \leq -\frac{1}{4}\lambda h c_4\|\vec{\omega}\|^2 - \frac{c_9}{4h}\left(\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2\right)$$

hold for $h \geq h_0$, $t \geq 0$, $\vec{\omega} \in \mathbb{R}^3$, $\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2 < \bar{\delta}^2$.

Thus, all the assumptions of the theorem on the uniform asymptotic stability (see [32]) are fulfilled for the function $V_1\left(t, \vec{\omega}, \vec{\xi}_1, \vec{\xi}_2\right)$. □

Remark 1 For instance, the assumption of Theorem 1 on boundedness of the matrices (10) is fulfilled if entries of these matrices are periodic functions with zero mean values.

The next theorem gives us stability conditions for a wider class of perturbed systems.

Theorem 2 *Let*

$$\frac{1}{T}\int_t^{t+T} \mathbf{D}_i(s)\,ds \to \mathbf{0} \quad as \quad T \to +\infty, \quad i = 1, 2, \tag{11}$$

uniformly with respect to $t \in [0, +\infty)$. *Then, there exists a number* $h_0 > 0$ *such that the equilibrium position* (4) *of the system* (2), (8) *is uniformly asymptotically stable for any* $h \geq h_0$.

Proof In this case, we will use the following modification of the Lyapunov function (7):

$$V_2\left(t, \vec{\omega}, \vec{\xi}_1, \vec{\xi}_2\right) = V\left(\vec{\omega}, \vec{\xi}_1, \vec{\xi}_2\right) + \frac{1}{h}(a_1\vec{\xi}_1 \times \vec{\eta}_1$$
$$+ a_2\vec{\xi}_2 \times \vec{\eta}_2)^\top \mathbf{B}^{-1}\sum_{i=1}^{2}\int_0^t e^{\alpha(s-t)}\mathbf{D}_i(s)\,ds\,(\vec{\xi}_i - \vec{\eta}_i),$$

where α is a positive parameter.

Under an appropriate choice of λ, h_0, $\bar{\delta}$, we obtain

$$\frac{1}{2}\lambda c_1\|\vec{\omega}\|^2 + \frac{a_1}{4}\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \frac{a_2}{4}\|\vec{\xi}_2 - \vec{\eta}_2\|^2$$
$$-\frac{b_3}{\alpha h}\left(\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2\right) \leq V_2\left(t, \vec{\omega}, \vec{\xi}_1, \vec{\xi}_2\right)$$
$$\leq 2\lambda c_2\|\vec{\omega}\|^2 + a_1\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + a_2\|\vec{\xi}_2 - \vec{\eta}_2\|^2 + \frac{b_3}{\alpha h}\left(\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2\right),$$
$$\dot{V}_2 \leq -\frac{1}{3}\lambda h c_4\|\vec{\omega}\|^2 - \frac{c_9}{3h}\left(\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2\right)$$
$$+\frac{b_4}{\alpha h}\|\vec{\omega}\|\left(\|\vec{\xi}_1 - \vec{\eta}_1\| + \|\vec{\xi}_2 - \vec{\eta}_2\|\right)$$
$$+\frac{\alpha b_5}{h}\sum_{i=1}^{2}\left\|\int_0^t e^{\alpha(s-t)}\mathbf{D}_i(s)\,ds\right\|\left(\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2\right),$$

where b_3, b_4, b_5 are positive constants.

In [33], it was proved that

$$\alpha\int_0^t e^{\alpha(s-t)}\mathbf{D}_i(s)\,ds \to \mathbf{0} \quad \text{as} \quad \alpha \to 0, \quad i = 1, 2,$$

uniformly with respect to $t \in [0, +\infty)$. Therefore, there exists $\alpha > 0$ such that

$$6b_5\alpha \sum_{i=1}^{2} \left\| \int_0^t e^{\alpha(s-t)} \mathbf{D}_i(s)\, ds \right\| < c_9$$

for $t \in [0, +\infty)$.

Then, for fixed values of $\lambda, \bar{\delta}, \alpha$, one can find a sufficiently large number h_0 such that

$$\begin{aligned} &\frac{1}{2}\lambda c_1 \|\vec{\omega}\|^2 + \frac{a_1}{8}\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \frac{a_2}{8}\|\vec{\xi}_2 - \vec{\eta}_2\|^2 \\ &\quad \le V_2\left(t, \vec{\omega}, \vec{\xi}_1, \vec{\xi}_2\right) \le 2\left(\lambda c_2 \|\vec{\omega}\|^2 + a_1\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + a_2\|\vec{\xi}_2 - \vec{\eta}_2\|^2\right), \\ &\quad \dot{V}_2 \le -\frac{1}{4}\lambda h c_4 \|\vec{\omega}\|^2 - \frac{c_9}{8h}\left(\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2\right) \end{aligned}$$

for $h \ge h_0$, $t \ge 0$, $\vec{\omega} \in \mathbb{R}^3$, $\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2 < \bar{\delta}^2$. □

Remark 2 The conditions (11) are fulfilled if entries of the matrices $\mathbf{D}_1(t), \mathbf{D}_2(t)$ are almost periodic functions with zero mean values. It is known (see [34]), that, for such matrices, the integrals (10) may be unbounded.

Remark 3 It is worth noting that Theorem 1 is a special case of Theorem 2. However, Theorem 1 possesses own meaning, since the proof of the theorem gives us less conservative restrictions on the parameter h than those in the proof of Theorem 2.

5 Purely nonlinear restoring and perturbing torques

Next, assume that $\nu > 0$ and the perturbing torque has the form $\vec{L}_p = \mathbf{D}(t)\, \vec{G}(\vec{\xi}_1 - \vec{\eta}_1, \vec{\xi}_2 - \vec{\eta}_2)$, where the matrix $\mathbf{D}(t) \in \mathbb{R}^{3\times m}$ is continuous and bounded for $t \in [0, +\infty)$ and components of the vector $\vec{G}(\vec{u}, \vec{v}) \in \mathbb{R}^m$ are continuously differentiable for $\vec{u}, \vec{v} \in \mathbb{R}^3$ homogeneous functions of the order $2\nu + 1$. Hence, we consider the system

$$\begin{aligned} \mathbf{J}\dot{\vec{\omega}} + \vec{\omega} \times \mathbf{J}\vec{\omega} = &-cf^{\nu}(\vec{\xi}_1, \vec{\xi}_2)\left(a_1\vec{\xi}_1 \times \vec{\eta}_1 + a_2\vec{\xi}_2 \times \vec{\eta}_2\right) \\ &+h\mathbf{B}\vec{\omega} + \mathbf{D}(t)\vec{G}(\vec{\xi}_1 - \vec{\eta}_1, \vec{\xi}_2 - \vec{\eta}_2). \end{aligned} \tag{12}$$

In this case, restoring and perturbing torques are purely nonlinear and homogeneous vector functions, and the homogeneity order of $\vec{L}_r$ coincides with that of $\vec{L}_p$.

Remark 4 It is known (see [35–38]) that, in numerous models of mechanical systems, strong nonlinear restoring forces with real-valued powers should be taken into consideration. Such forces can be related both to physical configurations and purely nonlinear material properties [3,39]. In addition, power-law characteristics of restoring forces provide smooth approximations of non-smooth forces [38].

The aim of the present Section is to show that, for purely nonlinear restoring and disturbing torques, the asymptotic stability of the equilibrium position (4) can be guaranteed under less conservative conditions than for linear torques.

For an arbitrarily chosen $h > 0$, one can find $\lambda_0 > 0$ and $\bar{\delta} > 0$ such that the derivative of the Lyapunov function (7) with respect to the system (2), (12) satisfies for $\lambda \ge \lambda_0$, $\vec{\omega} \in \mathbb{R}^3$, $\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2 < \bar{\delta}^2$ the inequality

$$\begin{aligned} \dot{V} \le &-\frac{1}{3}\lambda h c_4 \|\vec{\omega}\|^2 - \frac{c_9}{3h}\left(\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2\right)^{\nu+1} \\ &-\frac{1}{h}\left(a_1\vec{\xi}_1 \times \vec{\eta}_1 + a_2\vec{\xi}_2 \times \vec{\eta}_2\right)^{\top} \mathbf{B}^{-1}\vec{L}_p. \end{aligned}$$

Theorem 3 *Let the matrix* $\int_0^t \mathbf{D}(s)\, ds$ *be bounded for* $t \in [0, +\infty)$. *Then the equilibrium position* (4) *of the system* (2), (12) *is uniformly asymptotically stable for any* $h > 0$.

Proof Choose and fix an arbitrary positive value of the parameter h. Construct a Lyapunov function by the formula

$$V_3\left(t,\vec{\omega},\vec{\xi}_1,\vec{\xi}_2\right)=V\left(\vec{\omega},\vec{\xi}_1,\vec{\xi}_2\right)+\frac{1}{h}\left(a_1\vec{\xi}_1\times\vec{\eta}_1+a_2\vec{\xi}_2\times\vec{\eta}_2\right)^{\top}\mathbf{B}^{-1}\int\limits_0^t\mathbf{D}(s)\,ds\,\vec{G}(\vec{\xi}_1-\vec{\eta}_1,\vec{\xi}_2-\vec{\eta}_2).$$

If λ is sufficiently large and $\bar{\delta}$ is sufficiently small, then the function $V_1\left(t,\vec{\omega},\vec{\xi}_1,\vec{\xi}_2\right)$ and its derivative with respect to the system (2), (8) satisfy the estimates

$$\begin{aligned}&\frac{1}{2}\lambda c_1\|\vec{\omega}\|^2+\frac{a_1}{4}\|\vec{\xi}_1-\vec{\eta}_1\|^2+\frac{a_2}{4}\|\vec{\xi}_2-\vec{\eta}_2\|^2\\&\quad-\frac{b_1}{h}\left(\|\vec{\xi}_1-\vec{\eta}_1\|^2+\|\vec{\xi}_2-\vec{\eta}_2\|^2\right)^{\nu+1}\le V_3\left(t,\vec{\omega},\vec{\xi}_1,\vec{\xi}_2\right)\\&\quad\le 2\lambda c_2\|\vec{\omega}\|^2+a_1\|\vec{\xi}_1-\vec{\eta}_1\|^2+a_2\|\vec{\xi}_2-\vec{\eta}_2\|^2+\frac{b_1}{h}\left(\|\vec{\xi}_1-\vec{\eta}_1\|^2+\|\vec{\xi}_2-\vec{\eta}_2\|^2\right)^{\nu+1},\\&\quad\dot{V}_3\le-\frac{1}{3}\lambda hc_4\|\vec{\omega}\|^2-\frac{c_9}{3h}\left(\|\vec{\xi}_1-\vec{\eta}_1\|^2+\|\vec{\xi}_2-\vec{\eta}_2\|^2\right)^{\nu+1}\\&\quad+\frac{b_2}{h}\|\vec{\omega}\|\left(\|\vec{\xi}_1-\vec{\eta}_1\|+\|\vec{\xi}_2-\vec{\eta}_2\|\right)^{2\nu+1}\end{aligned}$$

for $t\ge 0$, $\vec{\omega}\in\mathbb{R}^3$, $\|\vec{\xi}_1-\vec{\eta}_1\|^2+\|\vec{\xi}_2-\vec{\eta}_2\|^2<\bar{\delta}^2$. Here b_1 and b_2 are positive constants.

Using properties of homogeneous functions (see [40,41]), it can be proved the existence of a number $\delta_0>0$ such that the estimates

$$\begin{aligned}&\frac{1}{2}\lambda c_1\|\vec{\omega}\|^2+\frac{a_1}{8}\|\vec{\xi}_1-\vec{\eta}_1\|^2+\frac{a_2}{8}\|\vec{\xi}_2-\vec{\eta}_2\|^2\\&\quad\le V_3\left(t,\vec{\omega},\vec{\xi}_1,\vec{\xi}_2\right)\le 2\left(\lambda c_2\|\vec{\omega}\|^2+a_1\|\vec{\xi}_1-\vec{\eta}_1\|^2+a_2\|\vec{\xi}_2-\vec{\eta}_2\|^2\right),\\&\quad\dot{V}_3\le-\frac{1}{4}\lambda hc_4\|\vec{\omega}\|^2-\frac{c_9}{4h}\left(\|\vec{\xi}_1-\vec{\eta}_1\|^2+\|\vec{\xi}_2-\vec{\eta}_2\|^2\right)^{\nu+1}\end{aligned}$$

hold for $t\ge 0$, $\vec{\omega}\in\mathbb{R}^3$, $\|\vec{\xi}_1-\vec{\eta}_1\|^2+\|\vec{\xi}_2-\vec{\eta}_2\|^2<{\delta_0}^2$. □

Theorem 4 *Let*

$$\frac{1}{T}\int\limits_t^{t+T}\mathbf{D}(s)\,ds\to\mathbf{0}\quad as\quad T\to+\infty$$

uniformly with respect to $t\in[0,+\infty)$. *Then, the equilibrium position* (4) *of the system* (2), (12) *is uniformly asymptotically stable for any* $h>0$.

Proof Let h be a fixed positive number. Consider the Lyapunov function

$$V_4\left(t,\vec{\omega},\vec{\xi}_1,\vec{\xi}_2\right)=V\left(\vec{\omega},\vec{\xi}_1,\vec{\xi}_2\right)+\frac{1}{h}\left(a_1\vec{\xi}_1\times\vec{\eta}_1+a_2\vec{\xi}_2\times\vec{\eta}_2\right)^{\top}\mathbf{B}^{-1}\int\limits_0^t e^{\alpha(s-t)}\mathbf{D}(s)\,ds\,\vec{G}(\vec{\xi}_1-\vec{\eta}_1,\vec{\xi}_2-\vec{\eta}_2),$$

where $\alpha=\text{const}>0$.

If λ is sufficiently large and $\bar{\delta}$ is sufficiently small, then

$$\begin{aligned}
&\frac{1}{2}\lambda c_1\|\vec{\omega}\|^2 + \frac{a_1}{4}\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \frac{a_2}{4}\|\vec{\xi}_2 - \vec{\eta}_2\|^2 \\
&\quad - \frac{b_3}{\alpha h}\left(\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2\right)^{\nu+1} \le V_4\left(t, \vec{\omega}, \vec{\xi}_1, \vec{\xi}_2\right) \\
&\le 2\lambda c_2\|\vec{\omega}\|^2 + a_1\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + a_2\|\vec{\xi}_2 - \vec{\eta}_2\|^2 + \frac{b_3}{\alpha h}\left(\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2\right)^{\nu+1}, \\
&\dot{V}_4 \le -\frac{1}{3}\lambda h c_4\|\vec{\omega}\|^2 - \frac{c_9}{3h}\left(\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2\right)^{\nu+1} \\
&\quad + \frac{b_4}{\alpha h}\|\vec{\omega}\|\left(\|\vec{\xi}_1 - \vec{\eta}_1\| + \|\vec{\xi}_2 - \vec{\eta}_2\|\right)^{2\nu+1} \\
&\quad + \frac{\alpha b_5}{h}\left\|\int_0^t e^{\alpha(s-t)}\mathbf{D}(s)\,ds\right\|\left(\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2\right)^{\nu+1}.
\end{aligned}$$

Here, b_3, b_4, b_5 are positive constants.

Similarly to the proof of Theorem 2, choose $\alpha > 0$ such that

$$6b_5\alpha\left\|\int_0^t e^{\alpha(s-t)}\mathbf{D}(s)\,ds\right\| < c_9$$

for $t \in [0, +\infty)$. Then, for sufficiently small values of $\bar{\delta}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\lambda c_1\|\vec{\omega}\|^2 + \frac{a_1}{8}\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \frac{a_2}{8}\|\vec{\xi}_2 - \vec{\eta}_2\|^2 \\
&\quad \le V_4\left(t, \vec{\omega}, \vec{\xi}_1, \vec{\xi}_2\right) \le 2\left(\lambda c_2\|\vec{\omega}\|^2 + a_1\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + a_2\|\vec{\xi}_2 - \vec{\eta}_2\|^2\right), \\
&\quad \dot{V}_4 \le -\frac{1}{4}\lambda h c_4\|\vec{\omega}\|^2 - \frac{c_9}{4h}\left(\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2\right)^{\nu+1}
\end{aligned}$$

for $t \ge 0$, $\vec{\omega} \in \mathbb{R}^3$, $\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2 < \bar{\delta}^2$. □

Remark 5 Theorem 3 is a special case of Theorem 4. However, the proof of Theorem 3 permits us to derive a wider estimate of attraction domain of the equilibrium position than that which can be obtained with the aid of the proof of Theorem 4.

Remark 6 Compared with Theorems 1 and 2, Theorems 3 and 4 guarantee the asymptotic stability of the equilibrium position (4) for any $h > 0$.

6 Computer modeling and discussion

The aim of the present paper is to provide a constructive approach to robustness analysis in the problem of attitude control for a rigid body subjected to nonstationary disturbing torques with zero mean values. It is worth noting that the disturbing torques (linear and nonlinear) are not assumed to be small in magnitude. For this reason, the obtained results seem to be attractive from the practical point of view.

The suggested approach is based on construction of a strict Lyapunov functions for the system governing the rigid body attitude dynamics. Theorems 1–4 ensure conditions under which perturbations do not disturb asymptotic stability of the programmed attitude motion.

In this Section, we illustrate Theorems 1–4 by means of a numerical simulation with the use of Maple-2019 tools for the numerical integration of differential equations.

Let the inertial parameters of a rigid body be given as: $A_1 = 20$, $A_2 = 24$, $A_3 = 16$. Here and in what follows all parameters are taken in International System of Units. The programmed orientation (4) of the body

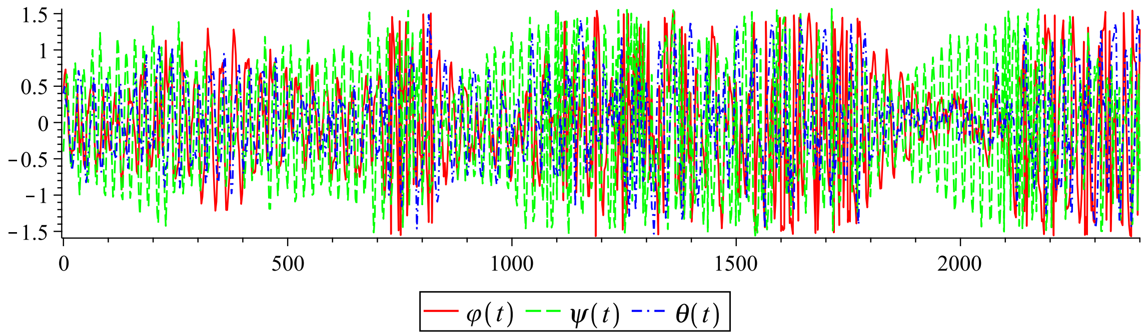

Fig. 2 Angles time history, $h = 0.1$, $\nu = 0$

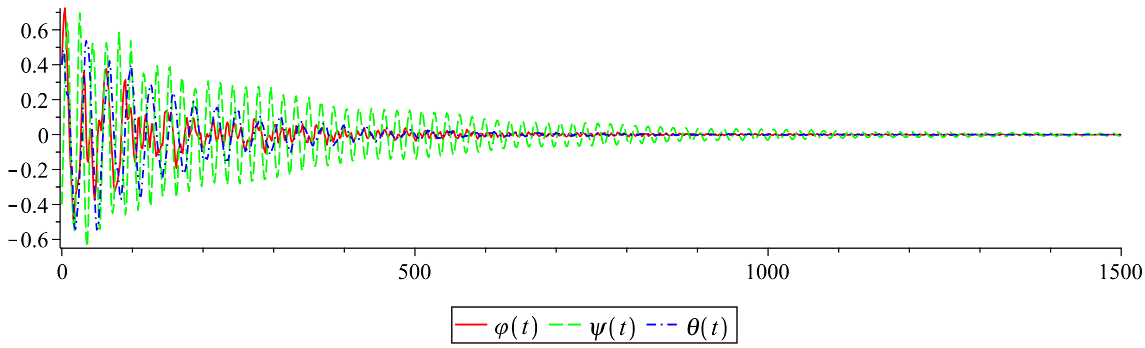

Fig. 3 Angles time history, $h = 0.4$, $\nu = 0$

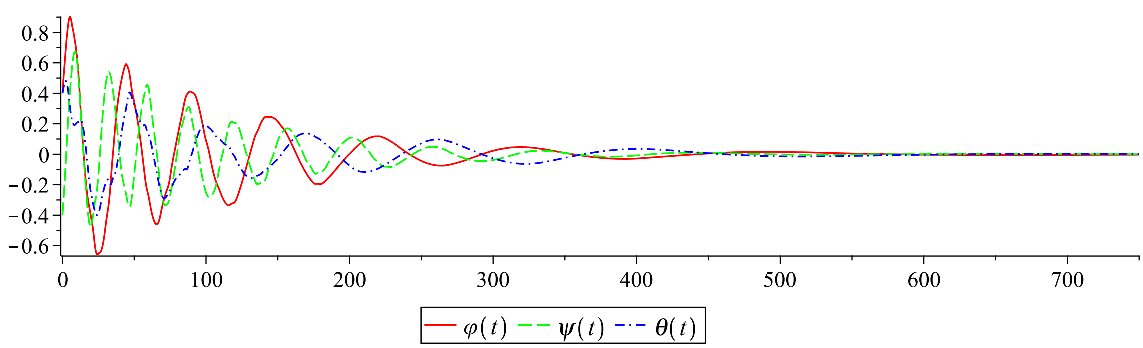

Fig. 4 Angles time history, $h = 0.1$, $\nu = 0.5$

is such that "aircraft" angles φ, θ, ψ in the inertial coordinate system are all equal to zero. The disturbing torque is taken in the form $\vec{L}_p = \mathbf{D}(t)\,\vec{G}(\vec{\xi}_1 - \vec{\eta}_1, \vec{\xi}_2 - \vec{\eta}_2)$, where

$$\mathbf{D}(t) = \mathrm{diag}\{\sin t + \cos(\sqrt{3}\,t),\, \sin 2t + \cos(\sqrt{2}\,t),\, \sin 3t + \cos t\},$$

$$\vec{G}(\vec{\xi}_1 - \vec{\eta}_1, \vec{\xi}_2 - \vec{\eta}_2) = \left(\left(\|\vec{\xi}_1 - \vec{\eta}_1\|^2 + \|\vec{\xi}_2 - \vec{\eta}_2\|^2\right)/2\right)^{\nu} (\vec{\xi}_1 - \vec{\eta}_1 + \vec{\xi}_2 - \vec{\eta}_2).$$

Choose the matrix $\mathbf{B}$ of dissipative torque in the form $\mathbf{B} = -\mathrm{diag}\{1, 1, 1\}$. Let $a_1 = 1$, $a_2 = 1$, $c = 1$. Consider the control process governed by the system (2), (12) for different values of h and ν and the same initial conditions $\varphi(0) = 0.4$, $\theta(0) = 0.4$, $\psi(0) = -0.4$, $\omega_1(0) = \omega_2(0) = \omega_3(0) = 0.2$.

First, we take $h = 0.1$ and $\nu = 0$. In this case disturbing and control torques are linear, the dissipative torque is small, and the process doesn't converge to the programmed motion as can be seen from Fig. 2.

In accordance with Theorems 1 and 2, there exists a number $h_0 > 0$ such that the programmed motion is uniformly asymptotically stable for any $h \geq h_0$. In our case $h_0 = 0.4$ is appropriate as it can be seen from Fig. 3. where the stabilization process is shown.

At the same time, Theorems 3 and 4 give us the possibility to reach the goal of a stabilization process without dissipative torque increasing. This possibility is based on applying the nonlinear restoring torque ($\nu > 0$). As is shown in Fig. 4. asymptotic stability is achieved at $\nu = 0.5$ even for $h = 0.1$.

We believe that our approach to the Lyapunov stability analysis in the problem of attitude control for a rigid body subjected to nonstationary disturbing torques with zero mean values is rather effective, and it can be exploited for the problem of satellite attitude stabilization with the use of electrodynamic attitude control system [28,42,43]. As is known, a satellite that moves in the Earth's gravitational and magnetic fields [44,45] is subjected to a lot of disturbing torques [18,46–48]. From the mathematical point of view, the majority of these torques can be modeled by almost periodic functions of time with zero mean values [49]. The magnitudes of these torques are often close to each other, and, generally speaking, they are not negligibly small [1]. For this reason, the usage of well-known perturbation methods faces difficulties in such problems, and the methods based on the application of Lyapunov functions seem to be promising.

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