



ORIGINAL PAPER

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Free flexural vibrations of homogeneous beams with symmetrically variable depths

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Abstract The subject of the paper are homogeneous beams of symmetrically variable depth and bisymmetrical cross sections. Free flexural vibrations of these beams are analytically and numerically studied. Based on Hamilton's principle, the differential equations of motion of these beams are obtained. The equations of motion are analytically solved with consideration of the bending lines of these beams subjected to their own weight. The fundamental natural frequency for exemplary beams is derived and presented in Tables and Figures.

1 Introduction

The variable depth beams are widely used, especially in building structures and transportation machinery. The vibration problems of these structures are significant for safety during their operation. Carrera et al. [1] proposed a new original unified approach to beam theory that includes practically all classical and advanced models for beams. According to the Carrera unified formulation (CUF), the error can be reduced by increasing the number of unknown variables. It is suitable for computer implementations and can deal with the most typical engineering challenges. The objective of the Ganesan and Zabihollah works [2, 3] was to perform an investigation of the free undamped vibration response of tapered composite beams, using the finite element method (a higher-order finite element formulation was developed). The finite element method was also used by Shahba et al. [4] for the structural analysis of axially functionally graded tapered Euler–Bernoulli beams. A beam element was proposed which takes advantage of the shape functions of homogeneous uniform beam elements. El-Sayed and El-Mongy [5] applied the modified variational iteration method (VIM) to solve the free vibration problem of a tapered Euler–Bernoulli beam mounted on two degrees of freedom mass-spring-damper subsystems. Both conical and wedge beams were investigated. Viglietti et al. [6] presented the free vibration analysis of tapered aircraft structures made of composite and metallic materials, taking into account global and local damage. A refined one-dimensional model was used to describe the structure in detail. Multicomponent aeronautical structures were modeled adopting Lagrange polynomials to evaluate the displacement field over the cross section. The model was assessed by comparing the results with classical FE models. The authors provided an accurate solution for the free vibration analyses of complex structures and are able to predict the consequences of global or local failure of a structural component. Banerjee and Ananthapuvirajah [7] investigated the free flexural vibration behaviour of a range of tapered beams by making use of the exact solutions of the governing differential equations and then imposing the necessary boundary conditions. It has been pointed out that an exact solution for the problem is possible by using Bessel functions rather than relying on a series solution which is somehow unnecessary and inefficient from a computational point of view.

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In another paper, Magnucki et al. [8] presented a three-point bending of an expanded-tapered beam with a rectangular cross section. The analytical model of the beam was formulated with consideration of a non-linear hypothesis of the cross-section deformation. The problem of shear stress distribution in the beam was analyzed. Zhou and Cheung [9] studied the vibrational characteristics of tapered beams with continuously varying rectangular cross sections of depth and breadth. The Euler–Bernoulli theory of bending was used to describe the motion of the beam. The eigenfrequency equation was obtained by the Rayleigh–Ritz method. Nijgh and Veljkovic [10] derived analytical prediction models for the elastic behaviour and the first eigenfrequency of non-prismatic composite beams with non-uniform shear connector arrangements. The approach was based on 6th and 2nd order differential equations used to define matrix equations for a finite number of linearized composite beam segments. The analytical models were validated using experimental and numerical results obtained with a simply supported tapered composite beam.

Demir et al. [11] studied the free vibration behavior of a multilayered symmetric sandwich beam made of functionally graded materials (FGMs) with variable cross-section resting on variable Winkler elastic foundation. It was assumed that the width of the beam varies exponentially along the length of the beam, and also the beam is resting on an elastic foundation, whose coefficient is variable along the length of the beam. Mahi et al. [12] proposed a new hyperbolic shear deformation theory applicable to bending and free vibration analysis of isotropic, functionally graded, sandwich, and laminated composite plates. The energy functional of the system was obtained using Hamilton's principle. Free vibration frequencies were accurately calculated using a set of boundary characteristic orthogonal polynomials associated with the Ritz method. Numerical comparisons were also carried out to verify and demonstrate the accuracy and efficiency of their theory. Chen et al. [13] investigated the free and forced vibration characteristics of functionally graded (FG) porous beams with non-uniform porosity distribution, whose elastic moduli and mass density are nonlinearly graded along the thickness direction. The authors derived the equation of motion within the framework of Timoshenko beam theory and by employing the Lagrange equation method together with Ritz trial functions.

Magnucki et al. [14] presented simply supported beams subjected to non-uniformly distributed loads. Shapes of bisymmetrical cross sections of the beams were expressed by special functions. The analytical model of the beams was formulated with consideration of the shear effect. A nonlinear hypothesis of deformation of a planar cross section of beams was assumed. The bending moment and the shear transverse force were formulated. The analytical model was validated using numerical FEM results. Magnucki and Lewinski [15] presented simply supported beams with symmetrically varying mechanical properties in the depth direction. Generalized load of the beams included the load types from uniformly distributed to point load (three-point bending). This load was described analytically by means of a certain function containing a dimensionless parameter. The individual nonlinear "polynomial" hypothesis was applied to describe the deformation of a planar cross section. Based on the definitions of the bending moment and the transverse shear force, the differential equation of equilibrium was obtained, and then analytically solved. In the paper, Magnucki et al. [16] presented a rectangular plate with symmetrically varying mechanical properties in the thickness direction. The nonlinear hypothesis of deformation of the straight line normal to the plate neutral surface was assumed. According to this hypothesis, the plate displacement field was formulated. Based on the Hamilton's principle, three differential equations of motion were obtained and then solved analytically. The critical loads and fundamental natural frequencies for exemplary plates were derived. The analytical model was verified by the numerical FEM method. Magnucki et al. [17] presented a beam with symmetrically varying mechanical properties in the depth direction. The proposed formulation of the functions of the properties makes a certain generalization in the research of functionally graded materials and allows to describe homogeneous, nonlinearly variable, and sandwich structures with the use of only one consistent analytical model.

Elishakoff [18] described the problem of the cross-correlation effect in random vibrations of discrete systems, beams, plates, and shells. It is demonstrated that natural frequencies in beams on elastic foundations as well as cylindrical or spherical shells might cluster together, resulting in a substantial percentage error if cross-correlations are ignored. In the paper [19], Magnucki considered the problem of free axisymmetric flexural vibration of a circular plate with a clamped edge supported on an elastic foundation. The mechanical properties also vary in the depth direction. Uzny et al. [20] presented the boundary problem of column vibrations in which the longitudinal inertia of the mass element, which loads the slender system, is taken into account. The column was analyzed as a simply supported structure. Experimental verification of the adopted mathematical model was also carried out. Good agreement between the numerical and experimental tests was obtained. Nikkhoo et al. [21] proposed a fast computation of beam-type dynamic response to a force or a mass moving across. Dynamics of a single-span beam was computed with a semi-analytical procedure based on characteristic orthogonal polynomials. Yuan et al. [22] developed a method to simplify the governing equations for the free

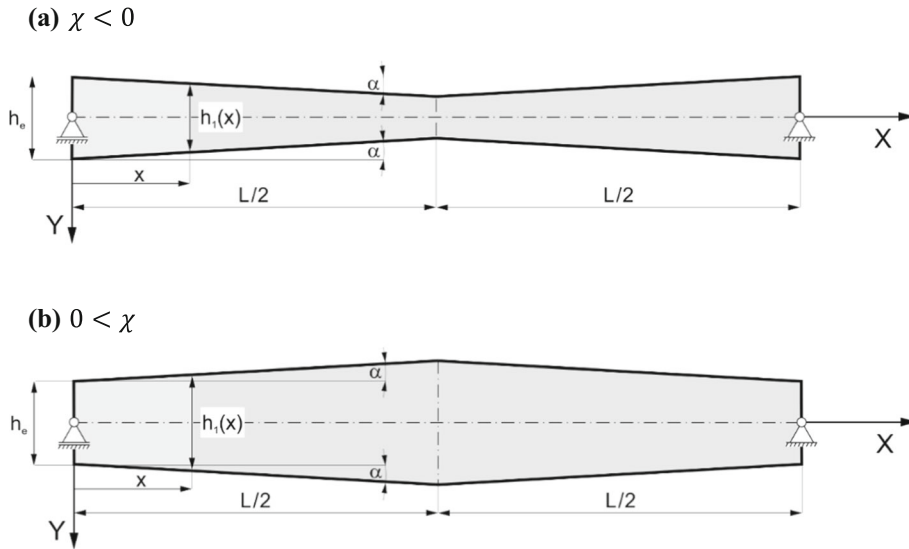


Fig. 1 Schemes of beams with linearly varying depths—expanded-tapered beams

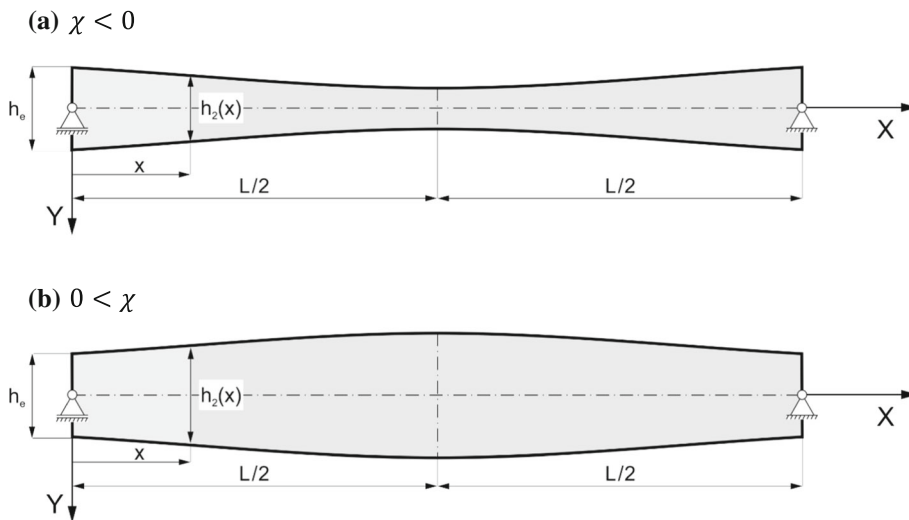


Fig. 2 Schemes of beams with nonlinearly varying depths

vibration of Timoshenko beams with both geometrical non-uniformity and material inhomogeneity along the beam axis. A series of exact analytical solutions was derived from the reduced equations. In [23], due to many works on beam vibrations dealing with analytical and numerical techniques, the authors (Guo and Zhang) presented a procedure of a spreading residue harmonic balance for approximating the periodic behavior of a tapered beam. Tan et al. [24] presented an exact approach to investigate the flexural free vibrations of multistep non-uniform beams. The authors used the transfer matrix method, the exact general solutions of one-step beam and iterative method to determine the natural frequencies and modal shapes of a multistep beam with variable cross section. The results were verified by the finite element method. Zingoni [25] considered the problem of the free vibration of plates and developed an efficient group-theoretic formulation for the solution of the problem by the method of finite differences.

The subject of the studies are two types of simply supported homogeneous beams of symmetrically variable depth and bisymmetrical cross sections. The first type beams have a linearly variable depth (Fig. 1), while the second type concerns a nonlinearly variable depth (Fig. 2).

The linearly variable depth of the beam is

$$h_1(\xi) = h_e \tilde{h}_1(\xi), \tag{1}$$

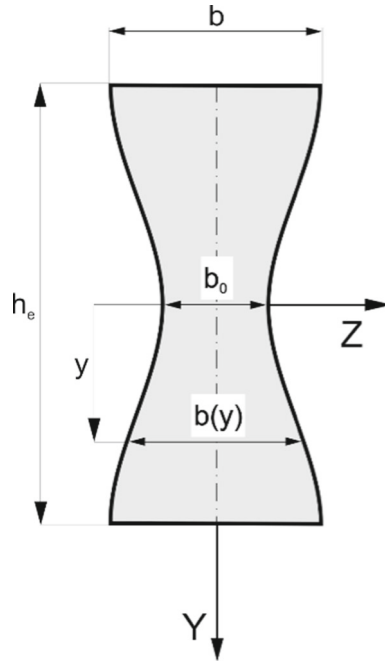


Fig. 3 Scheme of the shape of the bisymmetrical cross section of the beam ends

with the dimensionless depth

$$\tilde{h}_1(\xi) = 1 + 2\chi\xi, \quad (2)$$

and $\xi = x/L$ dimensionless coordinate ($0 \leq \xi \leq 1/2$), $\chi = \lambda \tan \alpha$ —dimensionless coefficient, $\lambda = L/h_e$ —relative length of the beam, L —length of the beam, h_e —depth of the beam ends.

The nonlinearly variable depth of the beam is

$$h_2(\xi) = h_e \tilde{h}_2(\xi), \quad (3)$$

with dimensionless depth

$$\tilde{h}_2(\xi) = 1 + \chi \sin(\pi\xi), \quad (4)$$

and: $\xi = x/L$ —dimensionless coordinate ($0 \leq \xi \leq 1$), χ —dimensionless coefficient, L —length of the beam, h_e —depth of the beam ends.

Taking into account the papers [14], the shape of the bisymmetrical cross section of these beams is assumed as shown in Fig. 3.

The width of the cross section symmetrically varies in the depth direction is

$$b(\eta) = b\tilde{b}(\eta), \quad (5)$$

with the dimensionless width

$$\tilde{b}(\eta) = \beta_0 + (1 - \beta_0)(5\eta^2 - 256\eta^{10})^{k_c}, \quad (6)$$

and $\eta = y/h_e$ —dimensionless coordinate ($-1/2 \leq \eta \leq 1/2$), $\beta_0 = b_0/b$ —parameter, k_c —exponent (real number).

The shape of the cross section (Fig. 3) is controlled by values of the parameter β_0 and exponent k_c . The area of the bisymmetrical cross section of the beam ends is

$$A_e = bh_e \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{b}(\eta) d\eta. \quad (7)$$

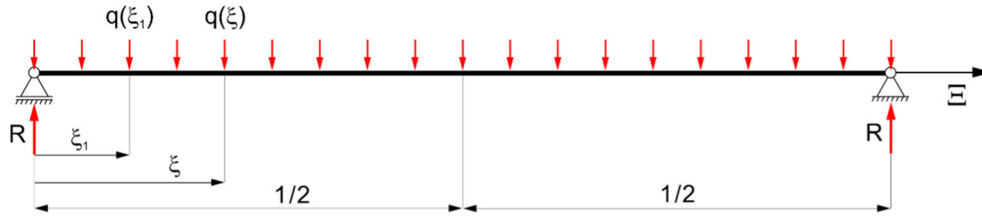


Fig. 4 Scheme of the load of the beam with linearly varying depth under the own weight

The second moment—moment of inertia of the bisymmetrical cross section of the beam ends is

$$J_{ze} = bh_e^3 \int_{-\frac{1}{2}}^{\frac{1}{2}} \eta^2 \tilde{b}(\eta) d\eta. \quad (8)$$

An analytical model presented in this paper involves only the bisymmetrical shape of the beam.

2 Beams with linearly varying depths: analytical studies

The load of the beam with linearly varying depth under its own weight is shown in Fig. 4.

The intensity of the load is

$$q(\xi) = q_e \tilde{h}_1(\xi), \quad (9)$$

where $q_e = g\rho A_e$, $g = 9.81 \text{ m/s}^2$ —acceleration of gravity, ρ —mass density.

The reactions of the supports are

$$R = q_e L \int_0^{\frac{1}{2}} \tilde{h}_1(\xi) d\xi = \frac{1}{2} \left(1 + \frac{1}{2} \chi \right) q_e L. \quad (10)$$

The bending moment is

$$M_b(\xi) = RL\xi - q_e L^2 \int_0^{\xi} (\xi - \xi_1) \tilde{h}_1(\xi_1) d\xi_1 \quad (11)$$

where ξ_1 —dimensionless coordinate ($0 \leq \xi_1 \leq \xi$).

Consequently, after integration and simple transformation, one obtains

$$M_b(\xi) = \tilde{M}_b(\xi) q_e L^2, \quad (12)$$

where the dimensionless bending moment is as follows:

$$\tilde{M}_b(\xi) = \frac{1}{2}(\xi - \xi^2) + \left(\frac{1}{4}\xi - \frac{1}{3}\xi^3 \right) \chi. \quad (13)$$

The differential equation of the beam deflection curve, according to the Euler–Bernoulli beam theory, is in the following form:

$$\frac{E J_z(\xi)}{L^2} \cdot \frac{d^2 v}{d\xi^2} = -M_b(\xi), \quad (14)$$

where $J_z(\xi) = J_{ze} \tilde{h}_1^3(\xi)$ —moment of inertia of the bisymmetrical cross section of the beam, $v(\xi)$ —deflection of the beam.

Substituting the expressions (12) into Eq. (14), one obtains

$$\frac{d^2 v}{d\xi^2} = -\frac{\tilde{M}_b(\xi)}{\tilde{h}_1^3(\xi)} \cdot \frac{q_e L^4}{E J_{ze}}. \quad (15)$$

This equation, after integration, is in the following form:

$$\frac{dv}{d\xi} = \left\{ C_1 + \frac{4+6\chi+3\chi^2+4\chi[3(1+\chi)^2+\tilde{h}_1^2(\xi)]\xi}{96\chi^3\tilde{h}_1^2(\xi)} \right\} \frac{q_e L^4}{E J_{ze}}, \tag{16}$$

where

$C_1 = -\frac{4+6\chi+3\chi^2+8\chi(1+\chi)^2}{96\chi^3(1+\chi)^2}$ —integration constant resulting from the condition $dv/d\xi|_{1/2} = 0$. Equation (16) after integration is as follows:

$$v(\xi) = \tilde{v}(\xi) \frac{q_e L^4}{E J_{ze}}, \tag{17}$$

with the dimensionless deflection curve of the beam

$$\tilde{v}(\xi) = C_2 + C_1 \xi + \frac{2+6\chi+3\chi^2+2[2\chi^2\xi^2+3(1+\chi)^2 \ln \tilde{h}_1(\xi)]\tilde{h}_1(\xi)}{192\chi^4\tilde{h}_1(\xi)}, \tag{18}$$

and $C_2 = -\frac{2+6\chi+3\chi^2}{192\chi^4}$ —integration constant calculated from the condition $v(0) = 0$.

The dimensionless deflection curve of the beam with constant depth ($\chi = 0$ —the particular case) is as follows:

$$\lim_{\chi \rightarrow 0} \tilde{v}(\xi) = \frac{1}{24}(1 - 2\xi^2 + \xi^3)\xi. \tag{19}$$

Hamilton’s principle reads

$$\delta \int_{t_1}^{t_2} (U_k - U_\varepsilon) dt = 0, \tag{20}$$

where

- The kinetic energy:

$$U_k = \varrho A_e L \int_0^{\frac{1}{2}} \tilde{h}_1(\xi) \left(\frac{\partial v}{\partial t} \right)^2 d\xi, \tag{21}$$

- The elastic strain energy:

$$U_\varepsilon = \frac{E J_{ze}}{L^3} \int_0^{\frac{1}{2}} \tilde{h}_1^3(\xi) \left(\frac{\partial^2 v}{\partial \xi^2} \right)^2 d\xi. \tag{22}$$

Therefore, based on Hamilton’s principle (20), the differential equation of motion is in the following form:

$$\tilde{h}_1(\xi) \frac{\partial^2 v}{\partial t^2} + \tilde{h}_1(\xi) \left\{ 24\chi^2 \frac{\partial^2 v}{\partial \xi^2} + 12\chi \tilde{h}_1(\xi) \frac{\partial^3 v}{\partial \xi^3} + \tilde{h}_1^2(\xi) \frac{\partial^4 v}{\partial \xi^4} \right\} \frac{E J_{ze}}{\varrho A_e L^4} = 0. \tag{23}$$

The differential Eq. (23) is approximately solved with the use of the assumed function

$$v(\xi, t) = \tilde{v}(\xi) v_a(t), \tag{24}$$

where $\tilde{v}(\xi)$ —dimensionless deflection curve of the beam (18), $v_a(t)$ —function of time t .

Substituting this function into Eq. (23) one obtains

$$\tilde{h}_1(\xi) \tilde{v}(\xi) \frac{d^2 v_a}{dt^2} + \tilde{h}_1(\xi) \frac{E J_{ze}}{\varrho A_e L^4} v_a(t) = 0. \tag{25}$$

Then, after applying the Galerkin method, this equation is approximately solved,

$$\int_0^{\frac{1}{2}} \Phi_1(\xi) \tilde{v}(\xi) d\xi = 0, \tag{26}$$

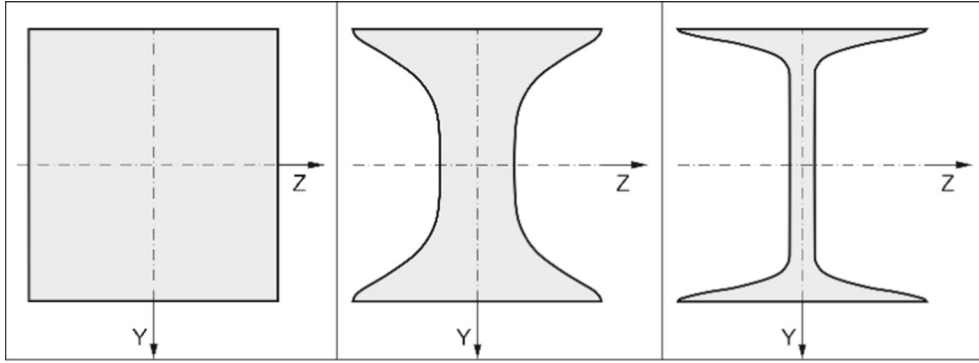


Fig. 5 Three example cross sections of the family beams

Table 1 Values of the dimensionless coefficient C_{ω}

χ	- 0.75	- 0.50	- 0.25	0	0.25	0.50	0.75
$C_{\omega 1}$	12.675	36.2.6	64.963	97.548	133.351	172.024	213.339

Table 2 Values of the fundamental natural frequency of the beam **CS-1** ($\beta_0 = 1, k_c = 1$)

χ	- 0.75	- 0.50	- 0.25	0	0.25	0.50	0.75
$\omega^{(An)}$ [rad/s]	72.05	121.77	163.11	199.88	233.70	265.43	295.59
$\omega^{(An)}/2\pi$ [Hz]	11.47	19.38	25.96	31.81	37.19	42.24	47.01

where $\Phi_1(\xi)$ —the left part of Eq. (25).

Integrating the expressions (26), with consideration of the function (18) and after simple transformation, one obtains

$$\frac{d^2 v_a}{dt^2} + C_{\omega 1} \frac{E J_{ze}}{\rho A_e L^4} v_a(t) = 0, \tag{27}$$

with the dimensionless coefficient $C_{\omega 1} = J_{11}/J_{01}$, and

$$J_{01} = \int_0^{\frac{1}{2}} \tilde{h}_1(\xi) \tilde{v}^2(\xi) d\xi, J_{11} = \int_0^{\frac{1}{2}} \tilde{h}_1(\xi) \tilde{v}(\xi) d\xi = 0.$$

Equation (27) is approximately solved with the use of the assumed function

$$v_a(t) = v_a \sin(\omega t), \tag{28}$$

where v_a —amplitude of the flexural vibration, ω —fundamental natural frequency.

After substituting this function into Eq. (27), the fundamental natural frequency is obtained,

$$\omega = \frac{10^6}{L^2} \sqrt{C_{\omega 1} \frac{E J_{ze}}{\rho A_e}} \text{ [rad/s]}, \tag{29}$$

where L [mm]—length of the beam, ρ [kg/m³]—mass density, A_e [mm²]—area of the cross section of the beam ends, J_{ze} [mm⁴]—moment of inertia of the cross section of the beam ends. Three exemplary cross sections of the beam family are assumed for the studies (Fig. 5).

The detailed calculations of the exemplary beams with linearly varying depths are carried out for the following data: length $L = 3600$ mm, width of the beam $b = 80$ mm, depth of the beam ends $h_e = 180$ mm, material constants $\rho = 7850$ kg/m³ and $E = 2 \cdot 10^5$ MPa. The results of the calculations are specified in Tables 1, 2, 3, and 4.

Table 3 Values of the fundamental natural frequency of the beam **CS-2** ($\beta_0 = 0.3$, $k_c = 2$)

χ	− 0.75	− 0.50	− 0.25	0	0.25	0.50	0.75
$\omega^{(An)}$ [rad/s]	85.56	144.60	193.69	237.35	277.51	315.19	351.00
$\omega^{(An)}/2\pi$ [Hz]	13.62	23.01	30.83	37.78	44.17	50.16	55.86

Table 4 Values of the fundamental natural frequency of the beam **CS-3** ($\beta_0 = 0.1$, $k_c = 9$)

χ	− 0.75	− 0.50	− 0.25	0	0.25	0.50	0.75
$\omega^{(An)}$ [rad/s]	96.34	162.82	218.10	267.26	312.48	354.91	395.24
$\omega^{(An)}/2\pi$ [Hz]	15.33	25.91	34.71	42.54	49.73	56.49	62.90

3 Beams with nonlinearly varying depths: analytical studies

The load of the beam with nonlinearly varying depth under its own weight is similar to that shown in Fig. 4. The intensity of the load is

$$q(\xi) = q_e \tilde{h}_2(\xi), \quad (30)$$

where $q_e = g\rho A_e$.

The reactions of the supports are

$$R = q_e L \int_0^{\frac{1}{2}} \tilde{h}_2(\xi) d\xi = \frac{1}{2} \left(1 + \frac{2}{\pi} \chi \right) q_e L. \quad (31)$$

The bending moment is

$$M_b(\xi) = RL\xi - q_e L^2 \int_0^{\xi} (\xi - \xi_1) \tilde{h}_2(\xi_1) d\xi_1. \quad (32)$$

Thus, after integration and simple transformation, one obtains

$$M_b(\xi) = \tilde{M}_b(\xi) q_e L^2, \quad (33)$$

where the dimensionless bending moment is as follows:

$$\tilde{M}_b(\xi) = \frac{1}{2} (\xi - \xi^2) + \frac{1}{\pi^2} \chi \sin(\pi \xi). \quad (34)$$

The differential equation of the deflection curve of the beam, according to the Euler–Bernoulli beam theory, with consideration of the expression (34) is in the following form:

$$\frac{d^2 v}{d\xi^2} = - \frac{\tilde{M}_b(\xi)}{\tilde{h}_2^3(\xi)} \cdot \frac{q_e L^4}{EJ_{ze}}. \quad (35)$$

Solving this equation in a similar way as for a beam with linearly varying depth, the dimensionless deflection curve of the beam is obtained in the following form:

$$\tilde{v}(\xi) = C_1 \xi - \iint \frac{\tilde{M}_b(\xi)}{\tilde{h}_2^3(\xi)} d\xi^2, \quad (36)$$

where $C_1 = \int_0^{1/2} \frac{\tilde{M}_b(\xi)}{\tilde{h}_2^3(\xi)} d\xi$ —integration constant calculated from the condition $dv/d\xi|_{1/2} = 0$.

Based on Hamilton's principle (20), the differential equation of motion is in the following form:

$$\tilde{h}_2(\xi) \frac{\partial^2 v}{\partial t^2} + \tilde{h}_2(\xi) \left\{ f_2(\xi) \frac{\partial^2 v}{\partial \xi^2} + 6\pi \chi \cos(\pi \xi) \tilde{h}_2(\xi) \frac{\partial^3 v}{\partial \xi^3} + \tilde{h}_2^2(\xi) \frac{\partial^4 v}{\partial \xi^4} \right\} \frac{EJ_{ze}}{\rho A_e L^4} = 0, \quad (37)$$

Table 5 Values of the coefficient $C_{\omega 2}$

χ	− 0.75	− 0.50	− 0.25	0	0.25	0.50	0.75
$C_{\omega 2}$	8.370	29.464	59.677	97.548	142.259	193.264	250.170

Table 6 Values of the fundamental natural frequency of the beam **CS-1** ($\beta_0 = 1, k_c = 1$)

χ	− 0.75	− 0.50	− 0.25	0	0.25	0.50	0.75
$\omega^{(An)}$ [rad/s]	58.55	109.85	156.34	199.88	241.38	281.34	320.09
$\omega^{(An)}/2\pi$ [Hz]	9.32	17.48	24.88	31.81	38.42	44.78	50.94

Table 7 Values of the fundamental natural frequency of the beam **CS-2** ($\beta_0 = 0.3, k_c = 2$)

χ	− 0.75	− 0.50	− 0.25	0	0.25	0.50	0.75
$\omega^{(An)}$ [rad/s]	69.52	130.44	185.64	237.35	286.63	334.08	380.10
$\omega^{(An)}/2\pi$ [Hz]	11.06	20.76	29.55	37.78	45.62	53.17	60.49

where $f_2(\xi) = 3\pi^2\chi \left[2\chi \cos^2(\pi\xi) - \sin(\pi\xi)\tilde{h}_2(\xi) \right]$.

The differential Eq. (37) is approximately solved with the use of the assumed function

$$v(\xi, t) = \tilde{v}(\xi)v_a(t), \quad (38)$$

where $\tilde{v}(\xi)$ —dimensionless deflection curve of the beam (36), $v_a(t)$ —function of time t .

Substituting this function into Eq. (37) one obtains

$$\tilde{h}_2(\xi)\tilde{v}(\xi)\frac{d^2v_a}{dt^2} + \tilde{h}_2(\xi)\frac{EJ_{ze}}{\rho A_e L^4}v_a(t) = 0, \quad (39)$$

This equation is approximately solved with the use of the Galerkin method,

$$\int_0^1 \Phi_2(\xi)\tilde{v}(\xi)d\xi = 0, \quad (40)$$

where $\Phi_2(\xi)$ —the left part of Eq. (39).

Integrating the expressions (40), with consideration of the function (36) and after simple transformation, one obtains

$$\frac{d^2v_a}{dt^2} + C_{\omega 2}\frac{EJ_{ze}}{\rho A_e L^4}v_a(t) = 0, \quad (41)$$

with dimensionless coefficient $C_{\omega 2} = J_{12}/J_{02}$, and

$$J_{02} = \int_0^{1/2} \tilde{h}_2(\xi)\tilde{v}^2(\xi)d\xi, \quad J_{12} = \int_0^{1/2} \tilde{h}_2(\xi)\tilde{v}(\xi)d\xi.$$

Equation (41) is approximately solved with the use of the function (28), and the fundamental natural frequency is obtained as

$$\omega = \frac{10^6}{L^2} \sqrt{C_{\omega 2} \frac{EJ_{ze}}{\rho A_e}} \cdot [\text{rad/s}]. \quad (42)$$

The detailed calculations of the exemplary beams with nonlinearly varying depths are carried out for the same date as for beams with linearly varying depths. The results of the calculations are specified in Tables 5, 6, 7, and 8.

Table 8 Values of the fundamental natural frequency of the beam **CS-3** ($\beta_0 = 0.1$, $k_c = 9$)

χ	- 0.75	- 0.50	- 0.25	0	0.25	0.50	0.75
$\omega^{(An)}$ [rad/s]	78.28	146.88	209.04	267.26	322.75	376.18	428.00
$\omega^{(An)}/2\pi$ [Hz]	12.46	23.38	33.27	42.54	51.37	59.87	68.12

**Fig. 6** Exemplary FEM model of the beam with linearly varying depth for numerical studies**Table 9** Values of the fundamental natural frequency of the beam **CS-1** ($\beta_0 = 1$, $k_c = 1$)

χ	- 0.75	- 0.50	- 0.25	0	0.25	0.50	0.75
$\omega^{(FEM)}$ [rad/s]	11.44	19.32	25.86	31.65	36.95	41.90	46.56

Table 10 Values of the fundamental natural frequency of the beam **CS-2** ($\beta_0 = 0.3$, $k_c = 2$)

χ	- 0.75	- 0.50	- 0.25	0	0.25	0.50	0.75
$\omega^{(FEM)}$ [rad/s]	13.45	22.84	30.56	37.40	43.59	49.39	54.79

Table 11 Values of the fundamental natural frequency of the beam **CS-3** ($\beta_0 = 0.1$, $k_c = 9$)

χ	- 0.75	- 0.50	- 0.25	0	0.25	0.50	0.75
$\omega^{(FEM)}$ [rad/s]	15.14	25.97	34.74	42.35	49.34	55.87	62.09

4 Beams with linearly varying depths: numerical FEM studies

The FEM model of the beam with linearly varying depth is developed with the use of the Abaqus 6.12. Taking into account the symmetry of the beam, the model relates to a quarter of the beam (Fig. 6).

The FEM calculations are performed for the data adopted above for analytical studies. The results of the calculations are specified in Tables 9, 10, and 11.

The analytical (Tables 2, 3, 4) and numerical (Tables 9, 10, 11) results are graphically compared in Fig. 7. Comparison of the results indicates very good convergence of both series of the results. The differences do not exceed 1.95%.

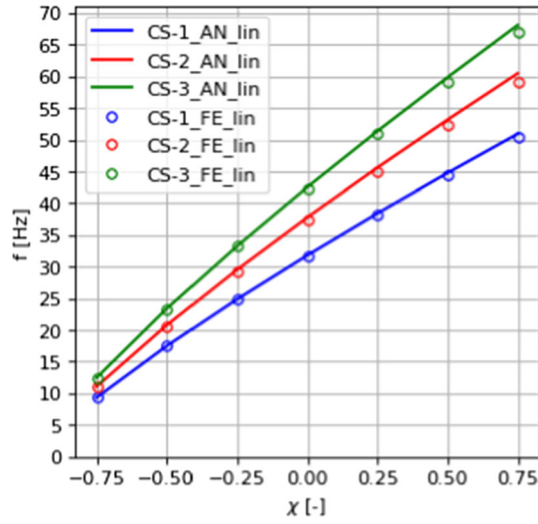


Fig. 7 Comparison of the values of the fundamental natural frequency of the beams with linearly varying depths calculated analytically (AN) and numerically (FE)

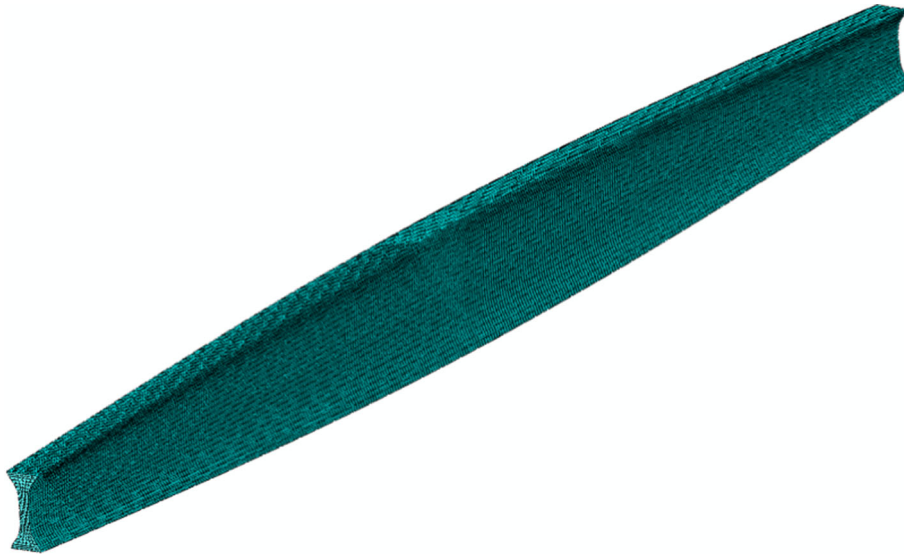


Fig. 8 Exemplary FEM model of the beam with linearly varying depth for numerical studies

Table 12 Values of the fundamental natural frequency of the beam **CS-1** ($\beta_0 = 1, k_c = 1$)

χ	- 0.75	- 0.50	- 0.25	0	0.25	0.50	0.75
$\omega^{(FEM)}$ [rad/s]	9.30	17.44	24.79	31.65	38.15	44.37	50.34

Table 13 Values of the fundamental natural frequency of the beam **CS-2** ($\beta_0 = 0.3, k_c = 2$)

χ	- 0.75	- 0.50	- 0.25	0	0.25	0.50	0.75
$\omega^{(FEM)}$ [rad/s]	11.00	20.64	29.24	37.40	44.99	52.24	59.16

Table 14 Values of the fundamental natural frequency of the beam **CS-3** ($\beta_0 = 0.1, k_c = 9$)

χ	- 0.75	- 0.50	- 0.25	0	0.25	0.50	0.75
$\omega^{(FEM)}$ [rad/s]	12.28	23.37	33.27	42.35	50.94	59.14	67.00

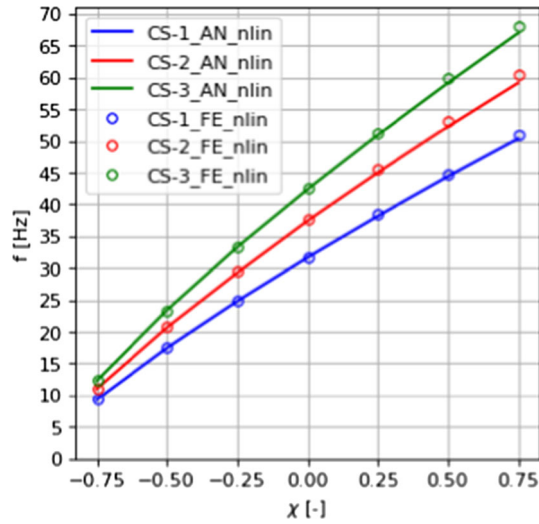


Fig. 9 Comparison of the values of the fundamental natural frequency of the beams with nonlinearly varying depths calculated analytically (AN) and numerically (FE)

5 Beams with nonlinearly varying depths: numerical FEM studies

The FEM model of the beam with linearly varying depth is developed with the use of the Abaqus 6.12. Taking into account the symmetry of the beam, the model refers to a quarter of the beam (Fig. 8).

The FEM calculations are carried out for the data adopted above for analytical studies. The results of the calculations are specified in Tables 12, 13, and 14.

The analytical (Tables 6, 7, 8) and numerical (Tables 12, 13, 14) results are graphically compared in Fig. 9. Comparison of the results indicates very good convergence of both series of the results. The differences do not exceed 2.25%.

6 Analytical study based on a simplified model of the beams

The simplified model of the studied beams is formulated taking into account the constant average value of their depth. Therefore,

- The average area of the bisymmetrical cross section of the beams is

$$A_{av} = bh_{av}\tilde{A}, \quad (43)$$

where: h_{av} —average depth, $\tilde{A} = \int_{-1/2}^{1/2} \tilde{b}(\eta)d\eta$ —dimensionless area.

The intensity of the load (9), in this case, is constant,

$$q_0 = g\rho A_{av}. \quad (44)$$

Thus, the bending moment (12) is as follows:

$$M_b(\xi) = \frac{1}{2}(\xi - \xi^2)q_0L^2, \quad (45)$$

and the deflection curve of the beams (17) is in the following form:

$$v(\xi) = \tilde{v}(\xi) \frac{5q_0L^4}{384EJ_{z,av}}, \quad (46)$$

with the dimensionless deflection curve

$$\tilde{v}(\xi) = \frac{16}{5}(\xi - 2\xi^3 + \xi^4). \quad (47)$$

The function $\sin(\pi\xi)$ is equivalent to this function (47); therefore, for further studies it is assumed that

$$\tilde{v}(\xi) = \sin(\pi\xi). \quad (48)$$

The elastic strain energy (22) is as follows:

$$U_\varepsilon = \frac{EJ_{z,av}}{L^3} \int_0^{\frac{1}{2}} \left(\frac{\partial^2 v}{\partial \xi^2} \right)^2 d\xi. \quad (49)$$

and the kinetic energy (21) is in the following form:

$$U_k = \rho A_{av} L \int_0^{\frac{1}{2}} \left(\frac{\partial v}{\partial t} \right)^2 d\xi. \quad (50)$$

Based on Hamilton's principle (20), the differential equation of motion is as follows:

$$\frac{\partial^2 v}{\partial t^2} + \frac{EJ_{z,av}}{\rho A_{av} L^4} \frac{\partial^4 v}{\partial \xi^4} = 0. \quad (51)$$

Equation (51) is approximately solved with the use of the assumed function

$$v(\xi, t) = v_a(t) \sin(\pi\xi), \quad (52)$$

where $v_a(t)$ —function of time.

Substituting this function into Eq. (51) one obtains

$$\frac{d^2 v_a}{dt^2} + \pi^4 \frac{EJ_{z,av}}{\rho A_{av} L^4} v_a(t) = 0. \quad (53)$$

Equation (53) is approximately solved with the use of the assumed function

$$v_a(t) = v_a \sin(\omega t), \quad (54)$$

where v_a —amplitude of the flexural vibration, ω —fundamental natural frequency.

After substituting this function into Eq. (53) one obtains the fundamental natural frequency for the simplified model of the beam as

$$\omega^{(S)} = \frac{\pi^2 10^6}{L^2} \sqrt{\frac{EJ_{z,av}}{\rho A_{av}}} \text{[rad/s]}. \quad (55)$$

The average dimensionless depth of the beams with linearly varying depths (2) is as follows:

$$\tilde{h}_{1,av} = \int_0^{\frac{1}{2}} \tilde{h}_1(\xi) d\xi = 1 + \frac{1}{2} \chi. \quad (56)$$

The results of the detailed calculations of the fundamental natural frequency (55) for the exemplary beams with consideration of the average depth (56) are specified in Tables 15, 16, and 17 and presented in Fig. 10. The average dimensionless depth of the beams with nonlinearly varying depths (4) is as follows:

$$\tilde{h}_{2,av} = \int_0^{\frac{1}{2}} \tilde{h}_2(\xi) d\xi = 1 + \frac{2}{\pi} \chi. \quad (57)$$

The results of the detailed calculations of the fundamental natural frequency (55) for the exemplary beams with consideration of the average depth (57) are specified in Tables 18, 19, and 20 and presented in Fig. 11.

The relative difference between the calculated fundamental natural frequency values for exact and simplified models of the beams is as follows:

$$\Delta = \frac{\omega^{(S)} - \omega^{(An)}}{\omega^{(An)}} \text{[%]}. \quad (58)$$

The values of the relative differences (58) for the beams with linearly varying depths are specified in Table 21. The values of the relative differences (58) for the beams with nonlinearly varying are specified in Table 22.

A significant relative difference is noticeable for the beams with a negative dimensionless coefficient χ (the concave beams), whereas for the beams with a positive dimensionless coefficient χ (the convex beams) the relative difference is insignificant from a practical point of view.

Table 15 Values of the fundamental natural frequency of the beam **CS-1** ($\beta_0 = 1, k_c = 1$)

χ	- 0.75	- 0.50	- 0.25	0	0.25	0.50	0.75
$\omega^{(s)}$ [rad/s]	124.84	149.80	174.77	199.88	224.70	249.67	274.64
$\omega^{(s)}/2\pi$ [Hz]	19.87	23.84	27.82	31.81	35.76	39.74	43.71

Table 16 Values of the fundamental natural frequency of the beam **CS-2** ($\beta_0 = 0.3, k_c = 2$)

χ	- 0.75	- 0.50	- 0.25	0	0.25	0.50	0.75
$\omega^{(An)}$ [rad/s]	148.24	177.88	207.53	237.35	266.83	296.47	326.12
$\omega^{(An)}/2\pi$ [Hz]	23.59	28.31	33.03	37.78	42.47	47.19	51.90

Table 17 Values of the fundamental natural frequency of the beam **CS-3** ($\beta_0 = 0.1, k_c = 9$)

χ	- 0.75	- 0.50	- 0.25	0	0.25	0.50	0.75
$\omega^{(An)}$ [rad/s]	166.92	200.30	233.68	267.26	300.45	333.45	367.22
$\omega^{(An)}/2\pi$ [Hz]	26.57	31.88	37.19	42.54	47.82	53.13	58.44

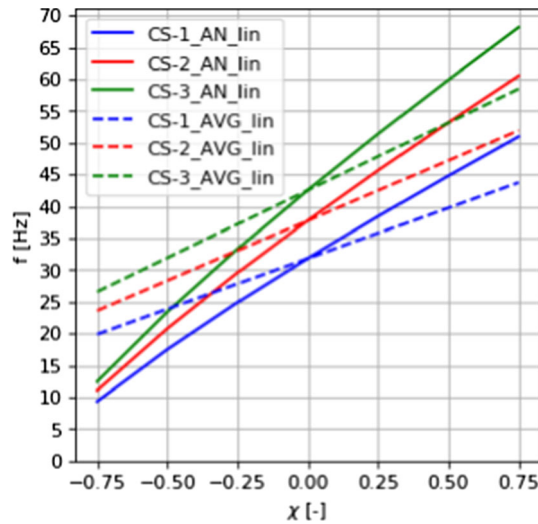


Fig. 10 Comparison of the values of the fundamental natural frequency of the beams with linearly varying depths calculated accurately (AN) and approximately (AVG)

Table 18 Values of the fundamental natural frequency of the beam **CS-1** ($\beta_0 = 1, k_c = 1$)

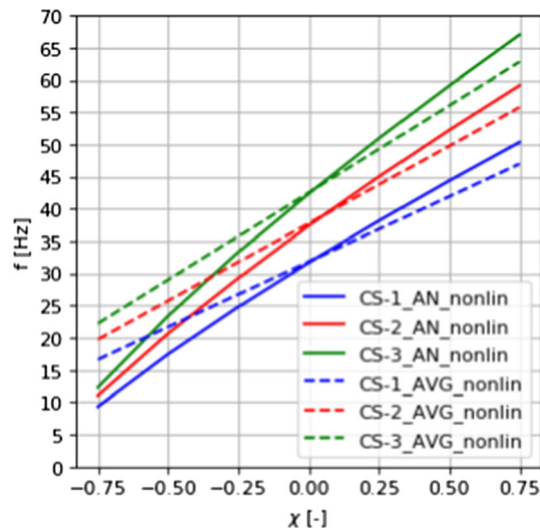
χ	- 0.75	- 0.50	- 0.25	0	0.25	0.50	0.75
$\omega^{(An)}$ [rad/s]	104.37	136.16	167.95	199.88	231.53	263.31	295.10
$\omega^{(An)}/2\pi$ [Hz]	16.61	21.67	26.73	31.81	36.85	41.91	46.97

Table 19 Values of the fundamental natural frequency of the beam **CS-2** ($\beta_0 = 0.3, k_c = 2$)

χ	- 0.75	- 0.50	- 0.25	0	0.25	0.50	0.75
$\omega^{(An)}$ [rad/s]	123.93	161.68	199.43	237.35	274.93	312.68	350.42
$\omega^{(An)}/2\pi$ [Hz]	19.72	25.73	31.74	37.78	43.76	49.76	55.77

Table 20 Values of the fundamental natural frequency of the beam CS-3 ($\beta_0 = 0.1$, $k_c = 9$)

χ	-0.75	-0.50	-0.25	0	0.25	0.50	0.75
$\omega^{(An)}$ [rad/s]	139.55	182.06	224.56	267.26	309.57	352.08	394.58
$\omega^{(An)}/2\pi$ [Hz]	22.21	28.98	35.74	42.54	49.27	56.04	62.80

**Fig. 11** Comparison of the values of the fundamental natural frequency of the beams with nonlinearly varying depths calculated accurately (AN) and approximately (AVG)**Table 21** Values of the relative differences (58) for the beams with linearly varying depths

χ	-0.75	-0.50	-0.25	0	0.25	0.50	0.75
Δ [%]	73.3	23.0	7.1	0	-3.9	-5.9	-7.1

Table 22 Values of the relative differences (58) for the beams with nonlinearly varying depths

χ	-0.75	-0.50	-0.25	0	0.25	0.50	0.75
Δ [%]	78.3	24.0	7.4	0	-4.1	-6.4	-7.8

7 Conclusions

In this paper, the two types of simply supported homogeneous beams with symmetrically variable depth and bisymmetrical cross sections were considered. The mathematical model of analyzed beams was proposed, and the formulas for the natural frequencies were derived. The results obtained in the analytical study were compared with finite element ones from Abaqus. The difference between results from analytical and numerical (FEM) approach is slightly below 2% in the case of beams with linearly varying depths and slightly above 2% in the case of beams with nonlinearly varying depths. It is shown that in engineering practice it is acceptable using a constant average depth (approximate solution) only for the convex beams.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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