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Galerkin-type solution for the Moore–Gibson–Thompson thermoelasticity theory

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Abstract It is prominent that the Galerkin-type representation plays a dominant role in probing various challenges of mathematical physics, continuum mechanics and occupies an important place in the field of partial differential equations (PDEs). Thus, the contemporary analysis of different boundary value problems (BVPs) in thermoelasticity theory commonly begins by analyzing the Galerkin-type representation of the field equations in terms of elementary functions (harmonic, biharmonic, and metaharmonic, etc). This work is aimed at formulating the representation of a Galerkin-type solution by means of elementary functions for the recently developed Moore–Gibson–Thompson (MGT) thermoelasticity theory. The MGT theory is a generalized form of the Lord–Shulman (LS) model as well as of the Green–Naghdi (GN) thermoelastic model. Here, we establish a theorem and derive the Galerkin-type solution for the basic governing equations under this theory. Later, the Galerkin representation of a system of equations for steady oscillations is derived. Based on this representation, we finally establish the general solution (GS) for the system of homogeneous equations of stable oscillation, neglecting the extrinsic body force and extrinsic heat supply.

1 Introduction

During the last few decades, various fields of science (geophysics, plasma physics, and acoustics) received significant attention for thermoelastic theory due to the development of pulsed lasers, rapid burst nuclear reactors, etc., which can supply heat pulses at a very instant time. The thermoelasticity theory is an extension of classical elasticity, which reveals the effect of thermal disturbances and mechanical effects on an elastic body. The uncoupled thermoelasticity theory suffers from two defects for an elastic body. First of all, this theory suffers from a physically unacceptable result related to infinite speed for thermal signals because of its heat equation having parabolic characteristics within the continuum field, which contradicts the theory of relativity. Secondly, it does not predict the temperature effect of the mechanical state of the elastic substance. For certain times, the solutions which are derived from the classical theory, differ slightly from those derived from the theory of coupled thermoelasticity. Biot [1] introduced a classical coupled thermoelasticity theory (CTE) based on Fourier's law which leads to a diffusion-type heat conduction model. This theory shows the mutual presence of the effect of temperature and strain on an elastic body. This theory suggests a finite speed for predominantly elastic disturbances but predicts an infinite speed for mostly thermal disturbances, which are coupled together. This paradox has aroused immense interest from the mathematical and technical point of view by the researchers, which suggested the classical theory needed an appropriate modification. The generalized Fourier's law of heat conduction came into existence due to the same. To prevail over the apparent drawback in the classical theory, many generalized thermoelasticity theories have been advocated by alternative formulations of the theory surpassing the classical thermoelasticity theory.

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The generalized thermoelasticity theories are considered to be more efficient than conventional thermoelasticity theories in the treatment of practical problems specially involving short time intervals and high heat fluxes because of their experimental evidence supporting the finite speed of propagation of heat waves. The first significant contribution to this new era of thermoelasticity theories was an observation of Lord and Shulman (LS) [2]. They employed a new theory dependent on the CV (Cattaneo–Vernotte) heat conduction model [3–5] in which Fourier's thermal conductivity law is modified by incorporating one new relaxation parameter along with the derivative of heat flux with respect to time. This modified theory was an attempt to remove the drawback of the Biot theory [1]. Later, some generalized thermoelasticity theories have been proposed without any alteration in Fourier's law. For example, Green and Lindsay (GL) [6] advocated an alternative theory involving the temperature rate term and two thermal relaxation time parameters in the constitutive relations. This theory is labeled as temperature-rate-dependent thermoelasticity theory (TRDTE) or GL theory. The linearized version of GL theory does not oppose the classical Fourier's law, whereas the stress–strain–temperature relation and the classical energy equation are modified. Due to the involvement of temperature rate terms in the constitutive relations, the heat conduction equation and the equation of motion under this theory are of hyperbolic type differential equations, and the propagation speed of a heat wave is predicted to be finite (see Refs. [6, 7]). The LS and GL models are found to have extensive uses in wave reflection problems for thermoelastic solids [8–10]. However, in some cases, GL theory fails to explain the existence of a discontinuity in the displacement field [11–15], which refutes the continuity hypothesis in continuum mechanics (see Refs. [7–15]). In order to overcome this jump discontinuity in the displacement field in the GL model, Yu et al. [16] have made an attempt and modified the Green and Lindsay (GL) [6] theory with the aid of the strain rate term in the basic equations and developed a new model of thermoelasticity theoretically. This modified Green–Lindsay (MGL) theory is also known as strain and temperature-rate-dependent thermoelasticity theory. Furthermore, by introducing thermal displacement in Fourier's law as a new variable, Green and Naghdi (GN) [17–19] made an alternative advancement in the thermoelasticity theory. Their theory is divided into three different parts which are subsequently termed as thermoelasticity theories of type GN-I, GN-II, and GN-III. For each of these theories, the constitutive assumptions for the heat flux vector are different. Here, we also refer to the dual-phase-lag (DPL) thermoelasticity theory introduced by Chandrasekharaiah [7]. This theory is based on a different heat conduction equation, called a dual-phase-lag model given by Tzou [20].

In recent years, the area of fluid mechanics [21] has gained a lot of interest for the Moore–Gibson–Thompson (MGT) heat conduction equation, which is considered by the adjoining energy equation to the equation $\overrightarrow{q} + \tau \frac{\partial \overrightarrow{q}}{\partial t} = -\left(K \overrightarrow{\nabla} \Theta + K^* \overrightarrow{\nabla} \nu\right)$, where $\overrightarrow{q}$, τ, Θ, and ν are heat flux, relaxation time parameter, temperature, and thermal displacement, respectively. Here, K denotes thermal conductivity and K^* is termed as thermal conductivity rate. In 2019, Quintanilla [22] has derived the constitutive equations for coupled thermoelasticity theory based on this MGT heat conduction equation and developed an alternative theory called as the Moore–Gibson–Thompson (MGT) thermoelasticity theory. The uniqueness of the solution and exponential stability of this generalized thermoelasticity theory are also discussed by Quintanilla [22]. The MGT theory can be seen as a fusion of both LS [2] and GN-III [19] thermoelasticity theories. Under the MGT thermoelasticity theory, Pellicer and Quintanilla [23] proved the uniqueness and instability of some thermomechanical problems. The domain of influence results for this MGT theory are recently discussed by Jangid and Mukhopadhyay [24, 25]. Further, the theoretical aspects and other practical applicabilities of the MGT thermoelasticity theory can be found in Refs. [26–28].

When studied to some problems of continuum mechanics, we obtain the solutions of BVPs in the context of the convolution type integral by the potential method. Solving for various boundary value problems in mathematical physics and continuum mechanics, the potential method is a powerful and refined tool. A theoretical tool of this method is for proving the existence and construction of solutions for BVPs, and a practical tool is to construct analytical and numerical solutions. The potential method's significant utilization is to reduce three-dimensional (3D) boundary value problems to a lower-dimensional boundary integral equation of Fredholm's type. For specific domains, some conventional methods gave exact elaboration of a numerical solution but fraught with few difficulties for an arbitrary domain. The fundamental solutions gained a special place in the partial differential equation for the investigation of various BVPs. In the studies of elasticity and thermoelasticity theories, the contemporary treatment for several BVPs normally involves the construction of the Galerkin-type representation [29] of field equations by means of various elementary functions like harmonic, metaharmonic, and biharmonic, etc. These are the foundations to obtain the fundamental solution of the theory. These elementary functions are also known as the solution of Helmholtz's equation. On the basis of the classical theory of elasticity, some representations of solution-related dynamical problems can be found in [30–33] and the references therein. In the context of the isothermal theory, the Boussinesq–Papkovitch–Neuber (BPN) [31], Green–Lamé (GL) [32], and Cauchy–Kovalevski–Somigliana (CKS) [33] types solutions for materials with voids were elaborated by Chandrasekharaiah [34, 35]. Ciarletta [36] discussed a Galerkin-type representation of the solution for the linear theory of micropolar thermoelasticity ([37–40]) by considering the GN-II theory. For a Kelvin–Voigt material with void, Svanadze [41] established the representation of the solution in case of the linear theory of thermo-viscoelasticity. Scalia and Svanadze [42] presented the Galerkin-type representation of thermoelasticity theory with micro-temperatures. Iacovache [43] presented the Galerkin-type solution of the equations in the field of elastokinetics. Svanadze and de Boer [44] established the Galerkin-type representation for an incompressible solid skeleton in the linear theory of the liquid-saturated porous medium. Ciarletta [45] derived fundamental solutions and general solutions for the dynamical theory of the binary mixture of an elastic solid. The Galerkin-type solution of the equation for the three-phase-lag (TPL) thermoelasticity theory was established by Mukhopadhyay et al. [46]. Recently, Gupta and Mukhopadhyay [47] and Singh et al. [48] derived the Galerkin-type representation and fundamental solutions for the modified Green–Lindsay (MGL) thermoelasticity theory [16].

The present work is aimed at deriving the Galerkin-type representation of the solution in the context of the recently developed MGT thermoelasticity theory. The work is presented as follows. For isotropic elastic material, the field equations for the Moore–Gibson–Thompson thermoelasticity theory [22] are formed in Sect. 2. We derive the Galerkin-type solution of basic governing equations in terms of the elementary functions in Sect. 3. In Sect. 4, we form a theorem representing the Galerkin-type solution of equations for the stable oscillations. Finally, we develop the general solution of the system of equations in the case of stable oscillations in terms of elementary functions in Sect. 5.

2 Governing equations

We consider an arbitrary point $\boldsymbol{x} = (x_1, x_2, x_3)$ in 3D Euclidean space Ξ^3. With the time variable t, we examine an isotropic elastic material that occupies the region W. By following Quintanilla [22], the field equations for the Moore–Gibson–Thompson thermoelasticity theory in the presence of body force and heat source are considered in the form

$$\mu(\Delta \boldsymbol{u}) - \alpha \text{ grad } \Theta + (\lambda + \mu)\{\text{grad div } \boldsymbol{u}\} + \rho' \boldsymbol{f} = \rho' \ddot{\boldsymbol{u}}, \tag{1}$$

$$K\Delta\dot{\Theta} + K^*\Delta\Theta - \alpha\Theta_0\{\text{div } \ddot{\boldsymbol{u}} + \tau \text{ div } \dddot{\boldsymbol{u}}\} - \rho' c_E\left\{\ddot{\Theta} + \tau\dddot{\Theta}\right\} = -\left(1 + \tau\frac{\partial}{\partial t}\right)r, \tag{2}$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$, Θ, $\boldsymbol{f}$, and $r = \rho'\dot{w}$ are the displacement vector, temperature measured from the constant reference temperature $\Theta_0(> 0)$, body force vector, and heat source, respectively. ω is an external rate of heat supply. Δ is the Laplacian operator, and $\alpha = (3\lambda + 2\mu)\alpha'$ is the thermoelasticity constant with α' being the coefficient of linear thermal expansion. λ, μ, K, and K^* are constitutive coefficients. Here, c_E represents the specific heat capacity, ρ'(>0) is the reference mass density, and τ denotes the relaxation time parameter. The superposed dot represents the derivative with respect to time.

We introduce the following notations:

$$\begin{aligned}
&n_1 = \left(\frac{\lambda + \mu}{\rho'}\right), \ n_2 = \frac{\mu}{\rho'}, \ n_3 = \frac{\alpha}{\rho'},\\
&g_1(\Delta, T) = n_2\Delta - T^2, \ g_2(\Delta, T) = \left(K\,T + K^*\right)\Delta - \rho' c_E \Box_1,\\
&T^k = \frac{\partial^k}{\partial t^k} \text{ for } k = 1, 2, 3,\\
&\Box_1 = \left(T^2 + \tau\, T^3\right).
\end{aligned}$$

Therefore, Eqs. (1) and (2) take the following form:

$$n_1 \text{ grad div } \boldsymbol{u} + g_1\boldsymbol{u} - n_3 \text{ grad } \Theta = -\boldsymbol{f}, \tag{3}$$

$$g_2\Theta - \alpha\Theta_0\Box_1\text{div } \boldsymbol{u} = -(1 + \tau T)r. \tag{4}$$

3 Galerkin-type solution of the equations of motion

We use the matrix differential operators as:

$$
\begin{aligned}
&\mathbf{\Gamma}[\boldsymbol{D}_{\mathrm{x}}, T] = \begin{bmatrix} \mathbf{\Gamma}^{(1)} & \mathbf{\Gamma}^{(2)} \\ \mathbf{\Gamma}^{(3)} & \mathbf{\Gamma}^{(4)} \end{bmatrix}_{4\times 4},\\
&\mathbf{\Gamma}^{(1)}(\boldsymbol{D}_{\mathrm{x}}, T) = \left[\Gamma^{(1)}_{kj}\right]_{3\times 3},\quad \mathbf{\Gamma}^{(2)} = \left[\Gamma^{(2)}_{k1}\right]_{3\times 1},\quad \mathbf{\Gamma}^{(3)} = \left[\Gamma^{(3)}_{1j}\right]_{1\times 3},\quad \mathbf{\Gamma}^{(4)} = [\Gamma_{44}]_{1\times 1},\\
&\Gamma^{(1)}_{kj}(\boldsymbol{D}_{\mathrm{x}}, T) = g_1\delta_{kj} + n_1\frac{\partial^2}{\partial x_k \partial x_j},\\
&\Gamma^{(2)}_{k1}(\boldsymbol{D}_{\mathrm{x}}, T) = -n_3\frac{\partial}{\partial x_k},\\
&\Gamma^{(3)}_{1j}(\boldsymbol{D}_{\mathrm{x}}, T) = -(\alpha\Theta_0\Box_1)\frac{\partial}{\partial x_j},\\
&\Gamma_{44}(\boldsymbol{D}_{\mathrm{x}}, T) = g_2 \text{ and } \boldsymbol{D}_{\mathrm{x}} = \left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right\},
\end{aligned}
\tag{5}
$$

where δ_{kj} represents the Kronecker delta for $k, j = 1, 2, 3$.

After implementing the above operators, we write Eqs. (3) and (4) as:

$$\mathbf{\Gamma}(\boldsymbol{D}_{\mathrm{x}}, T)\boldsymbol{U}(\mathrm{x}, t) = \boldsymbol{F}(\mathrm{x}, t), \tag{6}$$

with $\boldsymbol{U} = (\boldsymbol{u}, \Theta)$, $\boldsymbol{F} = [-\boldsymbol{f}, -(1 + \tau T)r]$, where $(\mathrm{x}, t) \in W \times (0, \infty)$.

Now, we set the system of equations as follows:

$$g_1\boldsymbol{u} + n_1 \text{ grad div } \boldsymbol{u} - \alpha\Theta_0\Box_1\text{grad } \Theta = \boldsymbol{F}', \tag{7}$$

$$g_2\Theta - n_3 \text{ div } \boldsymbol{u} = F_0, \tag{8}$$

where $\boldsymbol{F}' = (F_1', F_2', F_3')$ and F_0 are the vector component and scalar function, respectively, on the domain $W \times (0, \infty)$.

Using the matrix operator as defined above, Eqs. (7) and (8) can be reformulated as:

$$\boldsymbol{\Gamma}^{\boldsymbol{T}}(\boldsymbol{D}_{\mathrm{x}}, T)\boldsymbol{U}(\mathrm{x}, T) = \boldsymbol{M}(\mathrm{x}, t), \tag{9}$$

where $\boldsymbol{\Gamma}^{\boldsymbol{T}}$ represents the transpose of the matrix $\mathbf{\Gamma}$ and $\boldsymbol{M} = \left(\boldsymbol{F}', F_0\right)$.

Implementing the operator "divergence" to Eq. (7) yields

$$\varOmega_1 \text{ div } \boldsymbol{u} - \alpha\Theta_0\Box_1\Delta\Theta = \text{div } \boldsymbol{F}', \tag{10}$$

where $\varOmega_1 = \left(\frac{\lambda+2\mu}{\rho'}\right)\Delta - T^2$.

Then the above Eqs. (8) and (10) take the matrix form as

$$\boldsymbol{\varOmega}(\Delta, T)\boldsymbol{V} = \widetilde{\boldsymbol{F}}, \tag{11}$$

where

$$
\begin{aligned}
&\boldsymbol{V} = (\text{div } \boldsymbol{u}, \Theta),\ \widetilde{\boldsymbol{F}} = (f_1, f_2) = \left(\text{div } \boldsymbol{F}', F_0\right),\ \text{with } \boldsymbol{\varOmega}(\Delta, T) = \left[\varOmega_{kj}(\Delta, T)\right]_{2\times 2}\\
&= \begin{bmatrix} \varOmega_1 & -\alpha\Theta_0\Box_1\Delta \\ -n_3 & g_2 \end{bmatrix}_{2\times 2}.
\end{aligned}
$$

Now system (11) yields

$$\ell_1(\Delta, T)\boldsymbol{V} = \boldsymbol{\Phi}, \tag{12}$$

where

$$\boldsymbol{\Phi} = (\Phi_1, \Phi_2), \ \ \Phi_j = \sum_{k=1}^{2} \Omega^*_{kj} f_k, \ \ \ell_1(\Delta, T) = \det \boldsymbol{\Omega}(\Delta, T), \tag{13}$$

for $j = 1, 2$, and Ω^*_{kj} are the co-factors of the elements of the matrix $\boldsymbol{\Omega}$.

With the use of operator $\ell_1(\Delta, T)$ to Eq. (7), and utilizing Eq. (12), we find

$$\ell_1(\Delta, T) g_1 \boldsymbol{u} = \boldsymbol{\Phi}', \tag{14}$$

where

$$\boldsymbol{\Phi}' = \ell_1 \boldsymbol{F}' - \operatorname{grad}(n_1 \Phi_1 - \alpha \Theta_0 \Box_1 \Phi_2). \tag{15}$$

Next, in view of Eqs. (12) and (14), we obtain

$$\boldsymbol{\ell}(\Delta, T) \boldsymbol{U}(\mathrm{x}, T) = \widetilde{\boldsymbol{\Phi}}, \tag{16}$$

where $\widetilde{\boldsymbol{\Phi}} = \left(\boldsymbol{\Phi}', \Phi_2\right)$ and

$$\begin{aligned} &\boldsymbol{\ell}(\Delta, T) = \left[\ell_{kj}(\Delta, T)\right]_{4\times 4}, \\ &\ell_{yy} = \ell_1(\Delta, T) g_1, \ \text{ for } y = 1, 2, 3, \\ &\ell_{44} = \ell_1(\Delta, T), \ \ \ell_{kj} = 0, \ \text{ for } k, j = 1, 2, 3, 4, \ \ k \neq j. \end{aligned} \tag{17}$$

Further, introducing the new operators as

$$\begin{aligned} &\eta_{k1}(\Delta, T) = -\left(n_1 \Omega^*_{k1} - \alpha \Theta_0 \Box_1 \Omega^*_{k2}\right), \\ &\eta_{k2}(\Delta, T) = \Omega^*_{k2}, \ k = 1, 2, \end{aligned} \tag{18}$$

it follows from Eqs. (13) and (15) that

$$\boldsymbol{\Phi'} = (\ell_1 \boldsymbol{I} + \eta_{11} \operatorname{grad} \operatorname{div}) \boldsymbol{F}' + \eta_{21} \operatorname{grad} F_0, \tag{19}$$

$$\Phi_2 = \eta_{12} \operatorname{div} \boldsymbol{F}' + \eta_{22} F_0, \tag{20}$$

where $\boldsymbol{I}$ denotes the identity matrix.

Clearly, in view of Eqs. (19) and (20), we get

$$\widetilde{\boldsymbol{\Phi}}(\mathrm{x}, t) = \boldsymbol{Z}^{\boldsymbol{T}}(\boldsymbol{D}_{\mathrm{x}}, T) \boldsymbol{M}(\mathrm{x}, t), \tag{21}$$

where

$$\begin{aligned} &\boldsymbol{Z}[\boldsymbol{D}_{\mathrm{x}}, T] = \begin{bmatrix} \boldsymbol{Z}^{(1)} & \boldsymbol{Z}^{(2)} \\ \boldsymbol{Z}^{(3)} & \boldsymbol{Z}^{(4)} \end{bmatrix}_{4\times 4}, \\ &\boldsymbol{Z}^{(1)}(\boldsymbol{D}_{\mathrm{x}}, T) = \left[Z^{(1)}_{kj}\right]_{3\times 3}, \ \ \boldsymbol{Z}^{(2)} = \left[Z^{(2)}_{k1}\right]_{3\times 1}, \ \ \boldsymbol{Z}^{(3)} = \left[Z^{(3)}_{1j}\right]_{1\times 3}, \ \ \boldsymbol{Z}^{(4)} = [Z_{44}]_{1\times 1}, \\ &Z^{(1)}_{kj}(\boldsymbol{D}_{\mathrm{x}}, T) = \ell_1(\Delta, T) \delta_{kj} + \eta_{11}(\Delta, T) \frac{\partial^2}{\partial x_k \partial x_j}, \\ &Z^{(2)}_{k1}(\boldsymbol{D}_{\mathrm{x}}, T) = \eta_{12}(\Delta, T) \frac{\partial}{\partial x_k}, \\ &Z^{(3)}_{1j}(\boldsymbol{D}_{\mathrm{x}}, T) = \eta_{21}(\Delta, T) \frac{\partial}{\partial x_j}, \\ &Z_{44} = \eta_{22}(\Delta, T), \ \text{ for } k, j = 1, 2, 3. \end{aligned} \tag{22}$$

Now, by virtue of Eqs. (9), (16), and (21), we obtain

$$\boldsymbol{\ell}\, \boldsymbol{U} = \boldsymbol{Z}^T \boldsymbol{\Gamma}^T \boldsymbol{U}.$$

Hence, we get

$$\boldsymbol{\Gamma}(\boldsymbol{D}_{\mathrm{x}}, T) \boldsymbol{Z}(\boldsymbol{D}_{\mathrm{x}}, T) = \boldsymbol{\ell}(\Delta, T), \ \text{ as } \boldsymbol{Z}^{\boldsymbol{T}} \boldsymbol{\Gamma}^{\boldsymbol{T}} = \boldsymbol{\ell}. \tag{23}$$

Therefore, we have proved the following lemma.

Lemma A The matrix differential operators $\boldsymbol{\Gamma}$, $\boldsymbol{Z}$, and $\boldsymbol{\ell}$ satisfy Eq. (23), where $\boldsymbol{\Gamma}$, $\boldsymbol{Z}$, and $\boldsymbol{\ell}$ are defined by Eqs. (5), (22), and (17), respectively.

Now, $M'_y(\mathrm{x}, t)$ for $y = 1, 2, 3$ and $h(\mathrm{x}, t)$ are functions on the region $W \times (0, \infty)$ with $\boldsymbol{M}' = M'_y$ for $y = 1, 2, 3$ and $\widetilde{\boldsymbol{M}} = (\boldsymbol{M}', h)$.

Then we establish the following theorem that gives the Galerkin-type solution of Eqs. (3) and (4):

Theorem A Let

$$\boldsymbol{u} = \boldsymbol{Z}^{(1)}\boldsymbol{M}' + \boldsymbol{Z}^{(2)}h, \tag{24}$$

$$\Theta = \boldsymbol{Z}^{(3)}\boldsymbol{M}' + Z^{(4)}h, \tag{25}$$

where the M'_y, h are fields of class C^7 and C^5, respectively, and also satisfy the following equations:

$$\ell_1(\Delta, T)g_1\boldsymbol{M}' = -\boldsymbol{f}, \tag{26}$$

$$\ell_1(\Delta, T)h = -(1 + \tau T)r \tag{27}$$

on the region $W \times (0, \infty)$. Then, $\boldsymbol{U} = (\boldsymbol{u}, \Theta)$ yields the solution of Eqs. (3) and (4).

Proof From Eqs. (24) and (25), we have

$$\boldsymbol{U}(\mathrm{x}, t) = \boldsymbol{Z}(\boldsymbol{D}_{\mathrm{x}}, T)\widetilde{\boldsymbol{M}}(\mathrm{x}, t). \tag{28}$$

Also, from Eqs. (26) and (27), we get

$$\boldsymbol{\ell}(\Delta, T)\widetilde{\boldsymbol{M}}(\mathrm{x}, t) = \boldsymbol{F}(\Delta, T). \tag{29}$$

In view of Eqs. (23), (28), and (29), we obtain $\boldsymbol{\Gamma}\boldsymbol{U} = \boldsymbol{\Gamma}\boldsymbol{Z}\widetilde{\boldsymbol{M}} = \boldsymbol{\ell}\widetilde{\boldsymbol{M}} = \boldsymbol{F}$.

This completes the proof of the theorem. □

4 Galerkin-type solution of the system of equations for steady oscillations

If we assume

$$\boldsymbol{u}(\mathrm{x}, t) = \mathrm{Re}\left[\widetilde{\boldsymbol{u}}(\mathbf{x})e^{-i\omega t}\right], \qquad \Theta(\mathbf{x}, t) = \mathrm{Re}\left[\widetilde{\Theta}(\mathrm{x})e^{-i\omega t}\right],$$
$$\boldsymbol{f}(\mathrm{x}, t) = \mathrm{Re}\left[\widetilde{\boldsymbol{f}}(\mathrm{x})e^{-i\omega t}\right], \qquad r(\boldsymbol{x}, t) = \mathrm{Re}\left[\widetilde{r}(\mathrm{x})e^{-i\omega t}\right],$$

then, from Eqs. (1) and (2), the system of equations of the stable oscillations for the Moore–Gibson–Thompson thermoelasticity theory is obtained as follows:

$$\mu(\Delta\widetilde{\boldsymbol{u}}) + (\lambda + \mu)\{\mathrm{grad\ div}\ \widetilde{\boldsymbol{u}}\} - \alpha\ \mathrm{grad}\widetilde{\Theta} + \rho'\widetilde{\boldsymbol{f}} = -\omega^2\rho'\widetilde{\boldsymbol{u}}, \tag{30}$$

$$\begin{aligned}&\left\{K\Delta(-i\omega) + K^*\Delta + \rho' c_E\left(\omega^2 + i\tau\omega^3\right)\right\}\widetilde{\Theta} + \alpha\Theta_0\left\{\omega^2\ \mathrm{div}\ \widetilde{\boldsymbol{u}} + i\tau\omega^3\ \mathrm{div}\ \widetilde{\boldsymbol{u}}\right\}\\ &= -(1 - i\tau\omega)\widetilde{r},\end{aligned} \tag{31}$$

where $(\mathrm{x}, t) \in W \times (0, \infty)$, $i = \sqrt{-1}$, and $\omega(> 0)$ represents the frequency of oscillation.

Now, we use the following notations:

$$\begin{aligned}\boldsymbol{\Upsilon}(\Delta) &= \left[\Upsilon_{kj}(\Delta)\right]_{2\times 2}\\ &= \begin{bmatrix}\omega^2\rho' + (\lambda + 2\mu)\Delta & \alpha\Theta_0\left(\omega^2 + i\tau\omega^3\right)\Delta\\ -\alpha & (K^* - i\omega K)\Delta + \rho' c_E\left(\omega^2 + i\tau\omega^3\right)\end{bmatrix}_{2\times 2},\\ \widetilde{\ell}_1(\Delta) &= \det\Upsilon(\Delta),\\ e_{k1}(\Delta) &= -\left[(\lambda + \mu)\Upsilon^*_{k1} + \alpha\Theta_0\left(\omega^2 + i\tau\omega^3\right)\Upsilon^*_{k2}\right],\\ e_{k2}(\Delta) &= \Upsilon^*_{k2}\quad \text{for } k = 1, 2.\end{aligned}$$

If the equation $\widetilde{\ell}_1(-\lambda) = 0$ has two roots, namely λ_1^2 and λ_2^2, then $\widetilde{\ell}_1(\Delta)$ can be written as

$$\widetilde{\ell}_1(\Delta) = \left(\Delta + \lambda_1^2\right)\left(\Delta + \lambda_2^2\right).$$

Further, we introduce the matrix differential operators $\boldsymbol{R}$ and $\widetilde{\boldsymbol{\ell}}$, defined by

(a)

$$\begin{aligned}
&\boldsymbol{R}[\boldsymbol{D}_{\mathrm{x}}, T] = \begin{bmatrix} \boldsymbol{R}^{(1)} & \boldsymbol{R}^{(2)} \\ \boldsymbol{R}^{(3)} & \boldsymbol{R}^{(4)} \end{bmatrix}_{4\times 4},\\
&\boldsymbol{R}^{(1)}(\boldsymbol{D}_{\mathrm{x}}, T) = \left[R^{(1)}_{kj}\right]_{3\times 3},\ \ \boldsymbol{R}^{(2)} = \left[R^{(2)}_{k1}\right]_{3\times 1},\ \ \boldsymbol{R}^{(3)} = \left[R^{(3)}_{1k}\right]_{1\times 3},\ \ \boldsymbol{R}^{(4)} = [R_{44}]_{1\times 1},\\
&R^{(1)}_{kj}(\boldsymbol{D}_{\boldsymbol{X}}) = \widetilde{\ell}_1(\Delta)\delta_{kj} + e_{11}(\Delta)\frac{\partial^2}{\partial x_k \partial x_j},\\
&R^{(2)}_{k1}(\boldsymbol{D}_{\boldsymbol{X}}) = e_{12}(\Delta)\frac{\partial}{\partial x_k},\\
&R^{(3)}_{1k}(\boldsymbol{D}_{\boldsymbol{X}}) = e_{21}(\Delta)\frac{\partial}{\partial x_k},\\
&R_{44} = e_{22}(\Delta),\ \ \text{for } k, j = 1, 2, 3.
\end{aligned} \tag{32}$$

(b)

$$\begin{aligned}
&\widetilde{\boldsymbol{\ell}}(\Delta, T) = \left[\widetilde{\ell}_{kj}(\Delta)\right]_{4\times 4},\\
&\widetilde{\ell}_{yy} = \widetilde{\ell}_1(\Delta)\left[\omega^2\rho' + \mu\Delta\right],\quad y = 1, 2, 3,\\
&\widetilde{\ell}_{44} = \widetilde{\ell}_1(\Delta),\quad \widetilde{\ell}_{kj} = 0,\ \ k, j = 1, 2, 3, 4,\ \ k \neq j.
\end{aligned} \tag{33}$$

Let $\widetilde{Q}_y$, for $y = 1, 2, 3$ and s be functions on W. Here $\widetilde{\boldsymbol{Q}} = \left(\widetilde{Q}_1, \widetilde{Q}_2, \widetilde{Q}_3\right)$ and $\boldsymbol{Q} = \left(\widetilde{\boldsymbol{Q}}, s\right)$. Hence, by taking into account the Theorem A, we conclude the Galerkin-type solution to the system of equations for steady oscillations by the following theorem for the system of equations given by (30) and (31):

Theorem B Let

$$\widetilde{\boldsymbol{u}} = \boldsymbol{R}^{(1)}\widetilde{\boldsymbol{Q}} + \boldsymbol{R}^{(2)}s, \tag{34}$$

$$\widetilde{\Theta} = \boldsymbol{R}^{(3)}\widetilde{\boldsymbol{Q}} + R^{(4)}s, \tag{35}$$

where the $\widetilde{Q}_y$ and s are fields of class C^6 and C^4, respectively, and also satisfy the following equations:

$$\widetilde{\ell}_1(\Delta)\left[\omega^2\rho' + \mu\Delta\right]\widetilde{\boldsymbol{Q}} = -\widetilde{\boldsymbol{f}}, \tag{36}$$

$$\widetilde{\ell}_1(\Delta)s = -(1 - i\tau\,\omega)\widetilde{r} \tag{37}$$

on W. Then, $\left(\widetilde{\boldsymbol{u}}, \widetilde{\Theta}\right)$ is the solution of Eqs. (30) and (31).

5 General solution of the system of equations for steady oscillations

If external body force $\widetilde{\boldsymbol{f}}$ and external heat source $\widetilde{r}$ are assumed to be neglected, then Eqs. (30) and (31) can be written as

$$\left(\omega^2\rho' + \mu\Delta\right)\widetilde{\boldsymbol{u}} + (\lambda + \mu)\{\text{grad div }\widetilde{\boldsymbol{u}}\} - \alpha \text{ grad }\widetilde{\Theta} = 0, \tag{38}$$

$$\left\{\left(K^* - i\omega K\right)\Delta + \rho' c_E\left(\omega^2 + i\tau\omega^3\right)\right\}\widetilde{\Theta} + \alpha\Theta_0\left\{\omega^2 + i\,\tau\,\omega^3\right\}\text{div}\,\widetilde{\boldsymbol{u}} = 0. \tag{39}$$

Then, we propose the, subsequent lemma for the above system of equations:

Lemma B If $\left(\widetilde{\boldsymbol{u}}, \widetilde{\Theta}\right)$ is yielded as a solution of Eqs. (38) and (39), then

$$\widetilde{\ell}_1(\Delta)\text{div }\widetilde{\boldsymbol{u}} = 0, \tag{40}$$

$$\widetilde{\ell}_1(\Delta)\widetilde{\Theta} = 0, \tag{41}$$

$$\left(\omega^2\rho' + \mu\Delta\right)\text{curl}\,\widetilde{\boldsymbol{u}} = 0. \tag{42}$$

Proof By applying the operator "divergence" to Eq. (38), we get

$$\left\{\omega^2\rho' + (\lambda + 2\mu)\Delta\right\}\text{div}\, \widetilde{\boldsymbol{u}} - \alpha\, \Delta\widetilde{\Theta} = 0. \tag{43}$$

After eliminating $\widetilde{\Theta}$ from Eqs. (39) and (43), we obtain

$$\widetilde{\ell}_1 \text{div}\, \widetilde{\boldsymbol{u}} = 0.$$

Again, eliminating div $\widetilde{\boldsymbol{u}}$ from Eqs. (39) and (43) gives

$$\widetilde{\ell}_1 \widetilde{\Theta} = 0.$$

Additionally, by applying the "curl" operator to Eq. (38), we obtain

$$\left(\omega^2\rho' + \mu\Delta\right)\text{curl}\, \widetilde{\boldsymbol{u}} = 0.$$

Thus, this proves Lemma B. □

Theorem C If $\left(\widetilde{\boldsymbol{u}}, \widetilde{\Theta}\right)$ is evaluated as a solution of Eqs. (38) and (39), then

$$\widetilde{\boldsymbol{u}}(\mathbf{x}) = \alpha \text{ grad} \sum_{k=1}^{2} \phi_k(\mathbf{x}) + \boldsymbol{\Psi}(\mathbf{x}), \tag{44}$$

$$\widetilde{\Theta}(\mathbf{x}) = \sum_{k=1}^{2} a_k \phi_k(\mathbf{x}), \tag{45}$$

where ϕ_k $(k = 1, 2)$ and $\boldsymbol{\Psi} = (\boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2, \boldsymbol{\Psi}_3)$ satisfy the following equations:

$$\left(\Delta + \lambda_k^2\right)\phi_k(\mathbf{x}) = 0, \tag{46}$$

$$\left(\Delta + \frac{\omega^2\rho'}{\mu}\right)\boldsymbol{\Psi}(\mathbf{x}) = 0, \qquad \mathbf{x} \in W, \tag{47}$$

$$\text{div}\, \boldsymbol{\Psi}(\mathbf{x}) = 0, \tag{48}$$

and

$$a_k = -(\lambda + 2\mu)\lambda_k^2 + \omega^2\rho', \ \text{ for } k = 1, 2. \tag{49}$$

Proof Suppose $\left(\widetilde{\boldsymbol{u}}, \widetilde{\Theta}\right)$ to be a solution of Eqs. (38) and (39). Then, by taking into consideration $\Delta\widetilde{\boldsymbol{u}} =$ grad div$\widetilde{\boldsymbol{u}}$ − curl curl $\widetilde{\boldsymbol{u}}$, from Eq. (38), we have

$$\widetilde{\boldsymbol{u}} = \frac{1}{\omega^2\rho'}\left[\text{grad}\left\{-(\lambda + 2\mu)\text{div}\, \widetilde{\boldsymbol{u}} + \alpha\widetilde{\Theta}\right\} + \mu \text{ curl curl } \widetilde{\boldsymbol{u}}\right]. \tag{50}$$

Now, introducing the notation $\boldsymbol{\Psi}(\mathbf{x})$ as

$$\boldsymbol{\Psi}(\mathbf{x}) = \frac{\mu}{\omega^2\rho'}\text{curl curl } \widetilde{\boldsymbol{u}} \tag{51}$$

and with Eq. (42), by using div curl $\widetilde{\boldsymbol{u}} = 0$ for $\mathbf{x} \in W$, we see that Eqs. (47) and (48) can be easily obtained.

Now, let

$$\phi_y = b_y \left[\prod_{\substack{k=1 \\ k \neq y}}^{2} \left(\Delta + \lambda_k^2\right)\right]\widetilde{\Theta}, \tag{52}$$

$$b_y = \left[a_y \prod_{\substack{k=1 \\ k \neq y}}^{2} \left(\lambda_k^2 - \lambda_y^2\right) \right]^{-1}, \quad y = 1, 2. \tag{53}$$

Hence, in view of Eqs. (41), (52) yields Eqs. (45) and (46).

By using Eqs. (38), (45), (46), and (49), we obtain

$$\operatorname{div} \widetilde{\boldsymbol{u}} = -\alpha \sum_{k=1}^{2} \lambda_k^2 \phi_k. \tag{54}$$

Hence, Eq. (50) yields

$$\widetilde{\boldsymbol{u}} = \frac{1}{\omega^2 \rho'} \left[\operatorname{grad} \left\{ (\lambda + 2\mu)\alpha \sum_{k=1}^{2} \lambda_k^2 \phi_k + \alpha \widetilde{\Theta} \right\} + \mu \operatorname{curl} \operatorname{curl} \widetilde{\boldsymbol{u}} \right]. \tag{55}$$

With the help of Eqs. (49) and (51), Eq. (55) yields

$$\widetilde{\boldsymbol{u}}(\mathbf{x}) = \alpha \operatorname{grad} \sum_{i=1}^{2} \phi_k(\mathbf{x}) + \boldsymbol{\Psi}(\mathbf{x}). \tag{56}$$

It satisfy the complete proof of Theorem C.

Theorem D If $(\widetilde{\boldsymbol{u}}, \widetilde{\Theta})$ is provided as in Eqs. (44) and (45), where ϕ_k and $\boldsymbol{\Psi}$ satisfy Eqs. (46)–(48), then $(\widetilde{\boldsymbol{u}}, \widetilde{\Theta})$ will be considered as the solution of Eqs. (38) and (39) on W.

Proof With the help of Eqs. (46) and (47), (44) can be presented as:

$$\Delta \widetilde{\boldsymbol{u}} = -\alpha \operatorname{grad} \sum_{k=1}^{2} \lambda_k^2 \phi_k - \frac{\omega^2 \rho'}{\mu} \boldsymbol{\Psi},$$

$$\operatorname{grad} \operatorname{div} \widetilde{\boldsymbol{u}} = -\alpha \operatorname{grad} \sum_{k=1}^{2} \lambda_k^2 \phi_k. \tag{57}$$

Firstly, we change $\widetilde{\boldsymbol{u}}$ and $\widetilde{\Theta}$ as delivered in Eqs. (44) and (45) on the left side of Eq. (38). Then, by using Eqs. (46), (49), and (57), we get

$$\begin{aligned}
&\left(\omega^2 \rho' + \mu \Delta\right) \widetilde{\boldsymbol{u}} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \widetilde{\boldsymbol{u}} - \alpha \operatorname{grad} \widetilde{\Theta} \\
&\quad = \omega^2 \rho' \left(\alpha \operatorname{grad} \sum_{k=1}^{2} \phi_k + \boldsymbol{\Psi} \right) - \alpha \operatorname{grad} \sum_{k=1}^{2} \left\{ (\lambda + 2\mu) \lambda_k^2 + a_k \right\} \phi_k - \omega^2 \rho' \boldsymbol{\Psi}.
\end{aligned}$$

After rearranging the above expression, we obtain

$$\left(\omega^2 \rho' + \mu \Delta\right) \widetilde{\boldsymbol{u}} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \widetilde{\boldsymbol{u}} - \alpha \operatorname{grad} \widetilde{\Theta} = 0,$$

which is the field Eq. (38).

Similarly, substituting $\widetilde{\boldsymbol{u}}$ and $\widetilde{\Theta}$ on the left-hand side of Eq. (39) as given in (44) and (45) and utilizing (46), (49), and (54), finally we get

$$\begin{aligned}
&\left\{ \left(K^* - i\omega K\right) \Delta + \rho' c_E \left(\omega^2 + i\tau\omega^3\right) \right\} \widetilde{\Theta} \\
&\quad + \alpha \Theta_0 \left(\omega^2 + i\tau\omega^3\right) \operatorname{div} \widetilde{\boldsymbol{u}} \\
&\quad = \left\{ \left(K^* - i\omega K\right) \Delta + \rho' c_E \left(\omega^2 + i\tau\omega^3\right) \right\} \left(\sum_{k=1}^{2} a_k \phi_k \right)
\end{aligned}$$

$$
\begin{aligned}
& +\alpha^2\Theta_0\big(\omega^2+i\tau\omega^3\big)\Bigg(-\sum_{k=1}^{2}\lambda_k^2\phi_k\Bigg)\\
&=\sum_{k=1}^{2}\Big[a_k\big\{\big(K^*-i\omega K\big)(-\lambda_k^2)+\rho' c_E\big(\omega^2+i\tau\omega^3\big)\big\}\\
&\quad+\alpha^2\Theta_0\big(\omega^2+i\tau\omega^3\big)(-\lambda_k^2)\Big]\phi_k\\
&=0\,(\text{by using } \widetilde{\ell_1}\big(-\lambda_k^2\big)=0 \text{ for } k=1,2).
\end{aligned}
$$

Hence, the field equation (39) is satisfied.

Therefore, we obtain the general solution of the system of Eqs. (38) and (39) in terms of the metaharmonic functions ϕ_k and $\boldsymbol{\Psi}$. □

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