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ORIGINAL PAPER

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Exact solutions for stochastic Bernoulli–Euler beams under deterministic loading

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Abstract This study deals with two general solutions for a simply supported linear elastic Bernoulli–Euler beam with a stochastic bending flexibility, subjected to a deterministic loading. Two model problems are considered. The first problem is associated with a trapezoidally distributed load, whereas the second problem treats a sinusoidally distributed load. The importance of the solution for the trapezoidal load lies in its practicality. The derivation of stochastic characteristics for random beams under a sinusoidal load is useful due to the expandability to generally distributed loads by a Fourier sine series expansion. Numerical results are reported for various cases illustrating the effect of stochasticity of the beam's properties on its flexural response.

1 Introduction

The importance of closed-form solutions in applied mechanics cannot be exaggerated. Such solutions can be used as benchmark problems against which the approximate solutions' veracity can be checked. Closed-form solutions for beams with stochastic properties or under stochastic loading, which are available in the literature, are indicated and summarized in Tables 1, 2, and 3. Specifically, Rzhanitsyn [11] was apparently the first who has found the correlation properties of a Bernoulli–Euler beam under a random load, represented as a random function of the axial coordinate. Elishakoff et al. [4] extended Rzhanitsyn's approach to include randomness in the beam's flexibility. Additional solutions were introduced by Impollonia and Elishakoff [6, 7] and by Elishakoff et al. [2]. Some of the above solutions were reviewed in the monograph by Elishakoff and Ren [3]. Shen et al. [12, 13] presented additional solutions including wide-sense stationary stochastic functions whose covariance functions do not possess explicit Fourier transforms. Such are the so-called Cauchy and Dagum functions, discussed in Gneiting and Schlacher [5] and in Porcu et al. [10], respectively. The absence of explicit Fourier transforms for these two functions is caused by the presence of long memory (Hurst) effects. Thus, the Fourier inversion is far from trivial. It should be remarked that while absolute integrability is a sufficient condition for the spectral density to exist, it is not a sufficient condition for the spectral density to be nonnegative and integrable, which is what one looks for when checking for positive definiteness of a candidate function.

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Table 1 Closed-form solutions for cantilever shear beam

Stochastic flexibility	Stochastic loading and Stochastic flexibility
Impollonia and Elishakoff [6]: Flexibility is a random field with a linear covariance function: $C_f = f_0^2 C^{\text{linear}}(r)$ Load is uniformly or triangularly distributed, and flexibility's expectation is constant: $\bar{f}(x) = f_0$ 1. With, $q(x) = q_0$ 2. With, $q(x) = \frac{q_1 x}{L}$ Shen et al. [13]: Flexibility is a Gaussian random field with either a linear, exponential, Matérn, Cauchy or Dagum covariance function. $C_f =$ $f_0^2 C^{\text{linear}}(r)$, $f_0^2 C^{\text{exponential}}(r)$, $f_0^2 C^{\text{Matern}}(r)$, $f_0^2 C^{\text{Cauchy}}(r)$ Load is uniformly distributed, and flexibility's expectation is uniformly distributed. $q(x) = q_0$ $\bar{f}(x) = f_0$	Impollonia and Elishakoff [7]: Load and flexibility are uncorrelated Gaussian random fields. Linear covariance function for the load and linear or exponential covariance function for the flexibility. Various shapes for the expectation functions of the load and the flexibility. Load and flexibility were taken as: $q(x) = q_0(x)\left[1 + \alpha_q(x)\right]$ $\bar{q}(x) = q_0(x)$; $C_{\alpha_q} = C^{\text{linear}}(r)$ $f(x) = f_0(x)\left[1 + \alpha_f(x)\right]$ $\bar{f}(x) = f_0(\mathrm{x})$ and the following cases were examined: 1. $\begin{cases} q_0(x) = q_0 \\ f_0(x) = f_0 \\ C_{\alpha_f} = C^{\text{exponential}}(r) \end{cases}$ 2. $\begin{cases} q_0(x) = U\left(-\frac{1}{5} + \frac{x}{L}\right) q_0 \\ f_0(x) = f_0 \\ C_{\alpha_f} = C^{\text{linear}}(r) \end{cases}$ U being the step function. 3. $\begin{cases} q_0(x) = q_0\left(1 - \frac{x}{2L}\right) \\ f_0(x) = f_0\left(1 - \frac{x}{2L}\right)^{-1} \\ C_{\alpha_f} = C^{\text{linear}}(r) \end{cases}$ Shen et al. [13]: Load and flexibility are uncorrelated Gaussian random fields. Solved for all combinations of: C_{f}/f_0^2, $C_q/\mathrm{q}_0^2 =$ $C^{\text{linear}}(r)$, $C^{\text{exponential}}(r)$, $C^{\text{Matern}}(r)$ Load's and flexibility's expectations are uniformly distributed. $\bar{q}(x) = q_0$ $\bar{f}(x) = f_0$

The range of parameters for which these functions are positive definite and as such can be used as covariance functions is discussed in Lim and Teo [8] and Berg et al. [1].

In Tables 1, 2 and 3:

$r \equiv |x - y|$

$C^{\text{linear}}(r) = 1 - r/L$

$C^{\text{exponential}}(r) = e^{-r/L}$

$C^{\text{Materm}}(r) = (1 + r/L)e^{-r/L}$

$C^{\text{Cauchy}}(r) = (1 + r/L)^{-1}$

q_0+q_1+f_0+EI$_0$,a,P_0 = Constants.

L = Beams length.

C_i = Correlation function of variable i.

$\overline{\square}$ = Expectation operator.

While the aforementioned works do provide closed-form solutions to several important cases involving Bernoulli–Euler beams, solutions to two specific cases are absent in the literature to the best of the authors' knowledge. The first is the trapezoidal load, and the second is the sinusoidal load. The trapezoidal load case is a practical example, while the sinusoidal one is important since any load shape can be represented by a sine and cosine series. Closed-form solutions that are currently available in the literature were all developed for a specific load shape, whereas the solution for the sinusoidal load expands the boundaries of obtainable

Table 2 Closed-form solutions for simply supported Bernoulli–Euler beam

Stochastic loading	Stochastic flexibility	Stochastic loading and Stochastic flexibility
Elishakoff et al. [2]: Load is a random field with a zero expectation and an exponential covariance function. $C_q = a^2 C^{\text{exponential}}(r)$ $\bar{q}(x) = 0$ Flexibility is constant. Shen et al. [12]: Load is a Gaussian random field with either a linear, exponential or Matérn covariance function: $C_f = C^{\text{linear}}(r)$ $C_f = C^{\text{exponential}}(r)$ $C_f = C^{\text{Matern}}(r)$ Load's and flexibility's expectations are uniformly distributed. $\bar{q}(x) = q_0$ $\bar{f}(x) = f_0$	Elishakoff et al. [4]: Stiffness is a random field with an exponential covariance function: $EI(x) = EI_0[1 + \alpha_s(x)]$ $C_{\alpha_s} = C^{\text{exponential}}(r)$ Load is uniform and stiffness's expectation is constant. $\bar{q}(x) = q_0$ $\overline{EI}(x) = EI_0$	Elishakoff et al. [2]: Flexibility is taken as: $f(x) = f_0 \left[1 + \alpha_f\right] \bar{f}(x) = f_0$ 1. α_f is a single random variable: $C_f(x, y) = f_0^2 Var\left[\alpha_f\right]$ and the load is a random field with a second-order auto-regressive correlation function[a]: $C_q = a^2(1 + r/L)e^{-\text{r/L}}$ $\bar{q}(x) = 0$ 2. α_f is a random field with an exponential covariance function: $C_{\alpha_f}(x, y) = C^{\text{exponential}}(r)$ and the load depends on a single random variable: $q(x) = \alpha_q P_0; \overline{\alpha_q} = 0$ $C_q = P_0^2 Var\left[\alpha_q\right]$ 3. α_f and the load are random fields with a triangular and exponential covariance functions, respectively: $C_{\alpha_f}(x, y) = C^{\text{exponential}}(r)$ $\bar{q}(x) = 0; C_q = a^2 C^{linear}(r)$ Shen et al. [12]: Flexibility is taken as $f(x) = f_0 \left[1 + \alpha_f\right] \bar{f}(x) = f_0$ 1. α_f is a single random variable: $C_f(x, y) = f_0^2 Var\left[\alpha_f\right]$ and the load is a random field: $\bar{q}(x) = q_0$ $C_q = C^{linear}(r), C^{\text{exponential}}(r), C^{\text{Matern}}(r)$ 2. α_f is a random field with an exponential covariance function: $C_f(x, y) = f_0^2 C_{\alpha_f}(x, y)$ $C_{\alpha_f} = C^{\text{linear}}(r), C^{\text{exponential}}(r), C^{\text{Matern}}(r), C^{\text{Cauchy}}(r)$ and the load depends on a single random variable: $q(x) = \alpha_q P_0; \overline{\alpha_q} = 0$ $C_q(x, y) = P_0^2 Var\left[\alpha_q\right]$ 3. α_f and the load are random fields: $C_f(x, y) = f_0^2 C_{\alpha_f}(x, y); \bar{q}(x) = q_0$ Solved for all combinations of: $C_{\alpha_f}, C_q = C^{\text{linear}}(r), C^{\text{exponential}}(r), C^{\text{Matern}}(r)$ and also for: $C_{\alpha_f} = C^{\text{Cauchy}}(r)$with: $C_q = C^{\text{linear}}(r)$

Notice that the second-order auto-regressive correlation function used in Elishakoff et al. [4] and also used in Elishakoff et al. [2] is identical to the Matérn function used in Shen et al. [12, 13]

Table 3 Closed-form solutions for cantilever Bernoulli–Euler beam

Stochastic loading	Stochastic flexibility	Stochastic loading and Stochastic flexibility
Shen et al. [12]: Load is a Gaussian random field with either a linear, exponential or Matérn covariance function. $C_f = C^{\text{linear}}(r)$ $C_f = C^{\text{exponential}}(r)$ $C_f = C^{\text{Matern}}(r)$ Load's and flexibility's expectation are uniformly distributed. $\bar{q}(x) = q_0$ $\bar{f}(x) = f_0$	Elishakoff et al. [4]: Stiffness is taken as: $EI(x) = EI_0[1+\alpha_s(x)]$ $\overline{EI}(x) = EI_0$ where $\alpha_s(x)$ is a random field. 1. With a linear correlation function: $C_{\alpha_s} = C^{linear}(r)$ and a load distributed trapezoidally: $q(x) = q_0 + q_1\frac{x}{L}$ 2. With a second-order auto-regressive correlation function[b]: $C_{\alpha_s} = (1 + r/L)e^{-\mathrm{r/L}}$ and loaded by an end moment.	Shen et al. [12]: Flexibility is taken as: $f(x) = f_0[1+\alpha_f]; \quad \bar{f}(x) = f_0$ 1. α_f is a single random variable: $C_f(x, y) = f_0^2\mathrm{Var}[\alpha_f]$ and the load is a random field: $\bar{q}(x) = q_0$ $C_q = C^{\text{linear}}(r)$, $C^{\text{exponential}}(r)$, $C^{\text{Matern}}(r)$ 2. α_f is a random field with an exponential covariance function: $C_{\alpha_f} = C^{\text{linear}}(r)$, $C^{\text{exponential}}(r)$, $C^{\text{Matern}}(r)$, $C^{\text{Cauchy}}(r)$ and the load depends on a single random variable: $q(x) = \alpha_q P_0; \overline{\alpha_q} = 0$ $C_q(x, y) = P_0^2 Var[\alpha_q]$ 3. α_f and the load are random fields: $C_f(x, y) = f_0^2 C_{\alpha_f}(x, y); \bar{q}(x) = q_0$ Solved for all combinations of: $C_{\alpha_f}, C_q =$ $C^{linear}(r)$, $C^{\text{exponential}}(r)$, $C^{\text{Matern}}(r)$ and also for: $C_{\alpha_f} = C^{\text{Cauchy}}(r)$with: $C_q = C^{linear}(r)$

Notice that the second-order auto-regressive correlation function used in Elishakoff et al. [4] and also used in Elishakoff et al. [2] is identical to the Matérn function used in Shen et al. [12, 13]

solutions and enables more versatility from that point of view. Finding exact closed-form solutions for various load shapes would enlarge the collection of benchmark problems for evaluating the accuracy of approximate and numerical methods.

In this paper, two general solutions for a simply supported Bernoulli–Euler beam with a stochastic bending flexibility under deterministic loading are presented. The first problem looks at a trapezoidally distributed load, whereas the second problem deals with the sinusoidally distributed load. These two cases are solved for an exponential and for a Matérn covariance function for the beam's bending flexibility, which is modeled as a wide-sense stationary stochastic function. The solutions consist of exact expressions for the expectation and covariance functions of the beam's deflection. The obtained new solutions aim to expand the available spectrum of closed-form stochastic solutions and to explore further aspects of the stochastic physical problem at hand.

2 Statically determinate stochastic Bernoulli–Euler beam subject to deterministic loading

Expectation function for the deflection

For a statically determinate Bernoulli–Euler beam of length L, subjected to a distributed deterministic load *q(x),* the bending moment *m(x)* satisfies the following differential equation:

$$\frac{\partial^2 m(x)}{\partial x^2} = q(x). \tag{1}$$

Hence,

$$m(x) = \int_0^x \int_0^v q(u)\mathrm{d}u\mathrm{d}v + V_0 x + m_0 \tag{2}$$

where V_0 and m_0 are integration constants which represent the shear force and the bending moment at $x = 0$. Note that $q(x)$ can take any deterministic distribution along the beam.

The deflection of the beam satisfies the following differential equation:

$$\frac{1}{f(x)} \frac{\partial^2 w(x)}{\partial x^2} = m(x) \tag{3}$$

which can be rewritten as:

$$\frac{\partial^2 w(x)}{\partial x^2} = f(x)m(x) \tag{4}$$

where $f(x) = \frac{1}{\mathrm{EI}(x)}$ is the beam's bending flexibility.

Assume that the beam flexibility is a random function with given mean and covariance functions. We now apply the expectation operator to both sides of Eq. (4):

$$\begin{gathered} E\left[\frac{\partial^2 w(x)}{\partial x^2}\right] = E[f(x)m(x)] \\ \int_{-\infty}^{\infty} \frac{\partial^2 w(x)}{\partial x^2} p_w \mathrm{d}p = \int_{-\infty}^{\infty} f(x)m(x)p_f \mathrm{d}p \\ \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} w(x)p_w \mathrm{d}p = m(x) \int_{-\infty}^{\infty} f(x)p_f \mathrm{d}p \end{gathered} \tag{5}$$

where p_w and p_f are the probability density functions of the deflection and the bending flexibility, respectively. Equation (5) results in:

$$\frac{\partial^2 \bar{w}(x)}{\partial x^2} = \bar{f}(x)m(x) \tag{6}$$

where $\bar{w}(x)$ and $\bar{f}(x)$ are, respectively, the expectation functions of the deflection and the bending flexibility.

Equation (6) formulates the differential equation for the deflection's expectation. Note that Eq. (6) is identical in form to that of a beam with an equivalent deterministic bending flexibility $\bar{f}(x)$.

Covariance function for the deflection

Subtracting Eq. (6) from Eq. (4) yields:

$$\begin{gathered} \frac{\partial^2 w(x)}{\partial x^2} - \frac{\partial^2 \bar{w}(x)}{\partial x^2} = f(x)m(x) - \bar{f}(x)m(x), \\ \frac{\partial^2}{\partial x^2}[w(x) - \bar{w}(x)] = m(x)\big[f(x) - \bar{f}(x)\big]. \end{gathered} \tag{7}$$

Evaluating Eq. (7) at the argument y reads:

$$\frac{\partial^2}{\partial y^2}[w(y) - \bar{w}(y)] = m(y)\big[f(y) - \bar{f}(y)\big]. \tag{8}$$

Multiplying Eq. (7) by Eq. (8) gives:

$$\frac{\partial^2[w(x) - \bar{w}(x)]}{\partial x^2} \frac{\partial^2[w(y) - \bar{w}(y)]}{\partial y^2} = m(x)\big[f(x) - \bar{f}(x)\big]m(y)\big[f(y) - \bar{f}(y)\big]. \tag{9}$$

Finally, applying the expectation operator to both sides of the result we arrive at:

$$\frac{\partial^4 \mathrm{Cov}_w(x, y)}{\partial x^2 \partial y^2} = E\big\{m(x)m(y)\big[f(x) - \bar{f}(x)\big]\big[f(y) - \bar{f}(y)\big]\big\} = m(x)m(y)\mathrm{Cov}_f(x, y) \tag{10}$$

where $\mathrm{Cov}_w(x, y)$ and $\mathrm{Cov}_f(x, y)$ are, respectively, the covariance functions of the deflection and the bending flexibility. Equation (10) then defines the differential equation for the deflection's covariance function.

The procedure described above is developed for statically determinate beams. As such, it takes advantage of the ability to define the bending moment distribution in a closed form. This allows making a mathematical distinction between the input uncertainty and the output uncertainty, see Eq. (4). By doing so, the expectation operator can be later used to produce the stochastic equations (Eqs. (6) and (10)) following two results: Firstly, there is no compound term containing correlated (by definition) input and output uncertainties. Secondly, $m(\mathrm{x})$ and $f(\mathrm{x})$ are (or, if the load is random, can be considered to be) uncorrelated. This enables the formulation of an explicit ordinary differential equation, to be solved later by an analytical integration. In the case of a statically indeterminate beam, the deflections are defined by the solution of the following fourth-order differential equation $\frac{\partial^2}{\partial x^2}\left(\mathrm{EI}(x)\frac{\partial^2 w(x)}{\partial x^2}\right) = q(x)$, in which the input uncertainty and the output uncertainty cannot be separated and isolated on the two sides of the equation. This feature limits the straightforward generalization of the procedure developed here to the case of statically indeterminate beams. Nevertheless, it is noted that facing the challenge of deriving an exact solution for stochastic statically indeterminate beams is considered for future research.

Boundary conditions

Consider the case of a simply supported beam. The boundary conditions for the deflection are:

$$w(0) = w(L) = 0. \tag{11}$$

Applying the expectation operator, we get:

$$\bar{w}(0) = \bar{w}(L) = 0 \tag{12}$$

which are the boundary conditions for Eq. (6).

For Eq. (10), four boundary conditions are needed. The deflection's covariance function is, by definition:

$$\mathrm{cov}_w(x, y) = E\{[w(x) - \bar{w}(x)][w(y) - \bar{w}(y)]\}. \tag{13}$$

Substituting Eqs. (11) and (12) into Eq. (13) yields:

$$\begin{aligned} &\text{a)} \quad \mathrm{cov}_w(0, y) = 0,\\ &\text{b)} \quad \mathrm{cov}_w(x, 0) = 0,\\ &\text{c)} \quad \mathrm{cov}_w(x, L) = 0,\\ &\text{d)} \quad \mathrm{cov}_w(L, y) = 0. \end{aligned} \tag{14}$$

Solving the differential equation for the deflection's covariance

The solution of Eq. (10) is composed of a complementary solution and a particular solution. The particular solution can be obtained by quadruple integration of Eq. (10). Consider now the special case where the flexibility correlation function depends upon $|x - y|$. The integration domain should then be split into two parts—one in which $x \geq y$ and the other with $x < y$.

Denoting the particular solution by $\chi(x, y)$, the procedure starts with integrating Eq. (10) twice with respect to x and y. For a point (u,v) such that $u \geq v$, we get:

$$\frac{\partial^2}{\partial u \partial v}\chi(u, v) = \int_{y=0}^{y=v}\int_{x=0}^{x=y} m(x)m(y)\mathrm{Cov}_f(x, y)\mathrm{d}x\mathrm{d}y + \int_{y=0}^{y=v}\int_{x=y}^{x=u} m(x)m(y)\mathrm{Cov}_f(x, y)\mathrm{d}x\mathrm{d}y \quad \text{for } u \geq v \tag{15}$$

where the first integral covers the domain $x < y$ in which $|x - y| = y - x$, while the second integral covers the domain $x \geq y$ in which $|x - y| = x - y$. These domains are described in Fig. 1.

Then, by integrating twice again, now with respect to u and v, we get:

$$\chi(x, y) = \int_{v=0}^{v=y}\int_{u=0}^{u=v}\left[\frac{\partial^2}{\partial u \partial v}\chi(v, u)\right]\mathrm{d}u\mathrm{d}v + \int_{v=0}^{v=y}\int_{u=v}^{u=x}\left[\frac{\partial^2}{\partial u \partial v}\chi(u, v)\right]\mathrm{d}u\mathrm{d}v \quad \text{for } x \geq y \tag{16}$$

which is the particular solution for any point (x,y) such that $x \geq y$. For the domain $x < y$, the particular solution can be obtained by formally replacing x by y and y by x, due to the symmetry of x and y.

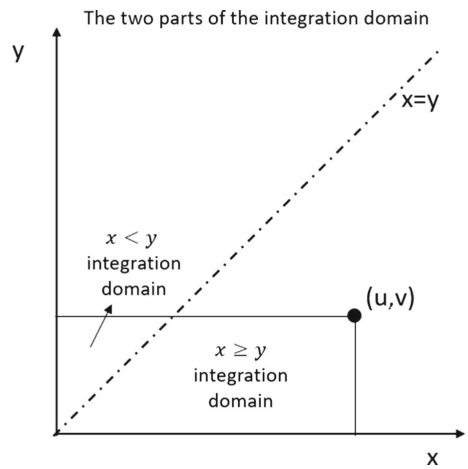

Fig. 1 A description of the integration domain and the way it is separated into 2 different parts, for a point (u, v) such that $u \geq v$

The complementary solution $\omega(x, y)$ can be written as follows:

$$\omega(x, y) = f_1(x) + yf_2(x) + g_1(y) + xg_2(y) \tag{17}$$

where $f_1(x)$, $f_2(x)$, $g_1(y)$and $g_2(y)$ are four arbitrary functions of their respective arguments. Hence,

$$\mathrm{cov}_w(x, y) = f_1(x) + yf_2(x) + g_1(y) + xg_2(y) + \chi(x, y). \tag{18}$$

We would like to get an expression for the complementary solution in terms of the particular solution. We first write the boundary conditions, as shown in Eq. (14), in an explicit form:

$$\begin{aligned}
&\text{a)}\quad \mathrm{cov}_w(0, y) = f_1(0) + yf_2(0) + g_1(y) + \chi(y, 0) = 0,\\
&\text{b)}\quad \mathrm{cov}_w(L, y) = f_1(L) + yf_2(L) + g_1(y) + Lg_2(y) + \chi(L, y) = 0,\\
&\text{c)}\quad \mathrm{cov}_w(x, 0) = f_1(x) + g_1(0) + xg_2(0) + \chi(x, 0) = 0,\\
&\text{d)}\quad \mathrm{cov}_w(x, L) = f_1(x) + Lf_2(x) + g_1(L) + xg_2(L) + \chi(L, x) = 0.
\end{aligned} \tag{19}$$

Extracting $f_1(x)$, $f_2(x)$, $g_1(y)$, and $g_2(y)$, we get:

$$\begin{aligned}
&\text{a)}\quad g_1(y) = -[f_1(0) + yf_2(0) + \chi(y, 0)],\\
&\text{b)}\quad g_2(y) = -\frac{1}{L}[f_1(L) + yf_2(L) + g_1(y) + \chi(L, y)],\\
&\text{c)}\quad f_1(x) = -[g_1(0) + xg_2(0) + \chi(x, 0)],\\
&\text{d)}\quad f_2(x) = -\frac{1}{L}[f_1(x) + g_1(L) + xg_2(L) + \chi(L, x)].
\end{aligned} \tag{20}$$

Substituting these results in Eq. (18), we get:

$$\begin{aligned}
\mathrm{cov}_w(x, y) &= \chi(x, y) - \frac{1}{L}[y\chi(L, x) + x\chi(L, y)] + \frac{xy}{L^2}\chi(L, L); \quad \text{for } x \geq y,\\
\mathrm{cov}_w(x, y) &= \chi(y, x) - \frac{1}{L}[y\chi(L, x) + x\chi(L, y)] + \frac{xy}{L^2}\chi(L, L); \quad \text{for } x < y.
\end{aligned} \tag{21}$$

3 Case 1: Load is trapezoidal

Consider the case of a trapezoidally distributed load:

$$q(x) = q_0 + \beta q_0 \frac{x}{L}; \quad q_0, \beta = \text{const.} \tag{22}$$

Two representative forms for the flexibility's covariance function cov_f will be examined, an exponential form and a Matérn form. For the latter, the interested reader can consult with the works by Shen et al. [13], or Matérn [9].

For the case where cov_f has an exponential form, we write:

$$\bar{f}(x) = f_0; \quad \text{cov}_f(x, y) = a^2 \exp\left(-\alpha \frac{|x - y|}{L}\right) \tag{23}$$

where f_0, a and α are constants. For simplicity, we introduce the following non-dimensional axial coordinates: $\xi = x/L; \eta = y/L$. By solving Eqs. (15) and (16), we get:

$$\begin{aligned}
\chi(\xi, \eta) = \frac{a^2 q_0^2 L^8}{36\alpha^{10}} \cdot \{ & \left[\eta(3 + (\eta + 1)\beta)(\eta - 1)\alpha^3 + \left(-6\beta\eta^2 + 2\beta - 12\eta + 6\right)\alpha^2 + (18\beta\eta + 18)\alpha - 24\beta\right] \\
& \cdot \left[(\xi - 1)\xi(3 + (\xi + 1)\beta)\alpha^3 + \left(6\beta\xi^2 - 2\beta + 12\xi - 6\right)\alpha^2 + (18\beta\xi + 18)\alpha + 24\beta\right] e^{\alpha(\eta - \xi)} \\
& - \left[\eta(3 + (\eta + 1)\beta)(\eta - 1)\alpha^3 + \left(6\beta\eta^2 - 2\beta + 12\eta - 6\right)\alpha^2 + (18\beta\eta + 18)\alpha + 24\beta\right] \\
& \cdot \left[\xi(3 + \beta)\alpha^3 + (6\xi + 2\beta + 6)\alpha^2 + (-6\beta\xi + 18)\alpha - 24\beta\right] e^{-\alpha\eta} \\
& - \left[\eta(3 + \beta)\alpha^3 + (6\eta + 2\beta + 6)\alpha^2 + (-6\beta\eta + 18)\alpha - 24\beta\right] \\
& \cdot \left[(\xi - 1)\xi(3 + (\xi + 1)\beta)\alpha^3 + \left(6\beta\xi^2 - 2\beta + 12\xi - 6\right)\alpha^2 + (18\beta\xi + 18)\alpha + 24\beta\right] e^{-\alpha\xi} \\
& + (\eta^4\alpha^9/28)\Big[\left(\eta^4\xi - (7/9)\eta^5 - (56/15)\eta^2\xi + (8/3)\eta^3 + (14/3)\xi - (14/5)\eta\right)\beta^2 \\
& + \left(-6\eta^4 + 8(\xi + 1)\eta^3 + (56/5)(-\xi + 1)\eta^2 + (84/5)(-\xi - 1)\eta + 28\xi\right)\beta - 12\eta^3 \\
& + (84/5)(\xi + 2)\eta^2 + (126/5)(-2\xi - 1)\eta + 42\xi \Big] \\
& + (2\eta^3\alpha^7/5)\Big[\left(\eta^3\xi - (10/7)\eta^4 - (5/2)\xi\eta + (7/2)\eta^2 - (5/3)\right)\beta^2 \\
& + \left(-9\eta^3 + (6\xi + (21/2))\eta^2 + (-(15/2)\xi + 10)\eta - 5\xi - 10\right)\beta - (27/2)\eta^2 \quad + 15(\xi/2 + 2)\eta - 15\xi - 15\Big] \\
& + \xi\eta(3 + \beta)^2\alpha^6 + \Big(\left(2\eta - 2\xi - (12/5)\eta^5 + 8\eta^3\right)\beta^2 \\
& + \left(-12\eta^4 + 24\eta^3 + 12\eta - 12\xi\right)\beta + 18\eta - 18\xi\Big)\alpha^5 \\
& + \Big((-12\xi\eta - 4)\beta^2 + ((-36\xi + 6)\eta + 6\xi - 24)\beta \\
& + (-36\xi + 18)\eta + 18\xi - 36\Big)\alpha^4 + 36\left(\beta^2 + 3\beta + 3\right)(\xi - \eta)\alpha^3 \\
& + \left(324 + (36\xi\eta + 96)\beta^2 + (36\xi + 36\eta + 288)\beta\right)\alpha^2 - 144\beta^2(\xi - \eta)\alpha - 576\beta^2\} \quad for\ \xi > \eta .
\end{aligned} \tag{24}$$

The deflection's covariance function is obtained by substituting this result in Eq. (21) and is given in "Appendix 1" (Eq. 44).

This solution is graphically illustrated in Fig. 2, where the deflection's covariance and variance functions are plotted for $\beta = 2, \alpha = 1$. Each function is normalized by the appropriate constant.

For the case where cov_f has a Matérn form, we write:

$$\bar{f}(x) = f_0; \quad \text{cov}_f(x, y) = a^2 \left(1 + \frac{|x - y|}{L}\right) \exp\left(-\frac{|x - y|}{L}\right) \tag{25}$$

By solving Eqs. (15) and (16) we get:

$$\begin{aligned}
\chi(\xi, \eta) = \frac{a^2 q_0^2 L^8}{36} \cdot \{ & \\
& [-\left(\left(\xi^3 + 6\xi^2 + 17\xi + 22\right)\beta + 3\xi^2 + 9\xi + 12\right)\beta\eta^4 \\
& + \left(\left(\xi^4 + 17\xi^3 + 89\xi^2 + 245\xi + 312\right)\beta^2\right.
\end{aligned}$$

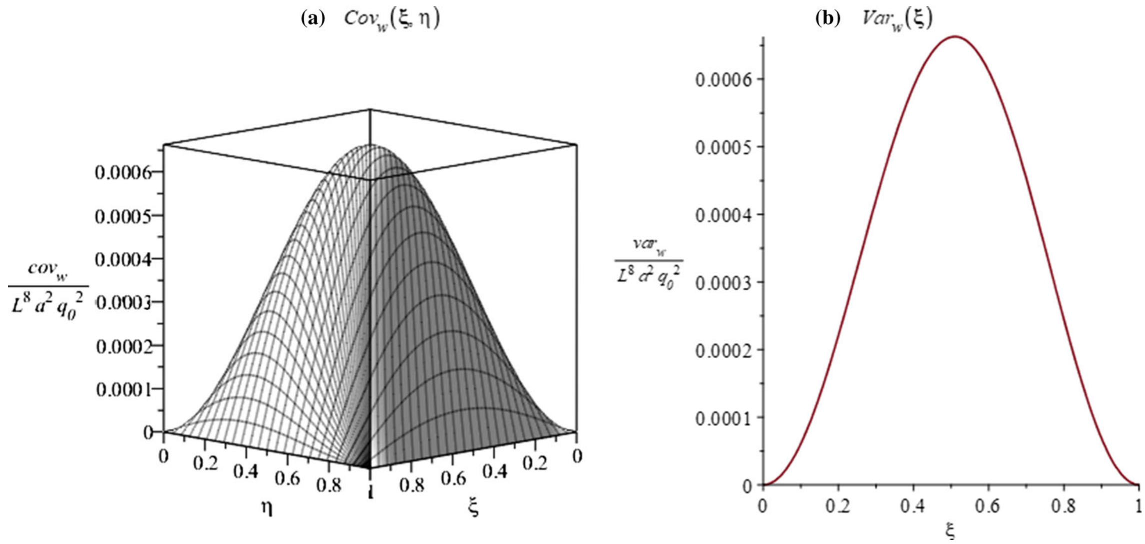

Fig. 2 Covariance function for the deflection of a beam with an exponential covariance function of the bending flexibility, subjected to a deterministic trapezoid load: **a** Covariance function, **b** variance function, both normalized by $L^8a^2q_0^2$, and for $\beta = 2, \alpha = 1$

$$
\begin{aligned}
&+\left(24\xi^2+72\xi+96\right)\beta-9\xi^2-27\xi-36\big)\eta^3\\
&+\ \big(\left(-6\xi^4-89\xi^3-456\xi^2-1249\xi-1586\right)\beta^2\\
&+\left(3\xi^4+30\xi^3+36\xi^2+63\xi+54\right)\beta+9\xi^3\\
&+117\xi^2+342\xi+450\big)\eta^2\\
&+\ \big(\left(17\xi^4+245\xi^3+1249\xi^2+3417\xi+4336\right)\beta^2\\
&+\left(-15\xi^4-150\xi^3-429\xi^2-1062\xi-1266\right)\\
&\beta-45\xi^3-468\xi^2-1359\xi-1782\big)\eta\\
&+\ \left(-22\xi^4-312\xi^3-1586\xi^2-4336\xi-5500\right)\beta^2\\
&+\left(24\xi^4+240\xi^3+786\xi^2+1998\xi+2424\right)\\
&\beta+72\xi^3+702\xi^2+2034\xi+2664\big]e^{\eta-\xi}\\
&+\ \big[\big((5\xi+22)\beta^2+(-9\xi-24)\beta\big)\eta^4\\
&+\big((67\xi+312)\beta^2+(-90\xi-240)\beta-27\xi-72\big)\eta^3\\
&+\ \big((337\xi+1586)\beta^2+(-357\xi-786)\beta-234\xi-702\big)\eta^2\\
&+\big((919\xi+4336)\beta^2+(-936\xi-1998)\,\beta\\
&-675\xi-2034\big)\eta\\
&+\ (1164\xi+5500)\beta^2+(-1158\xi-2424)\beta-882\xi-2664\big]e^{-\eta}\\
&+\ \big[\big(\left(5\xi^4+67\xi^3+337\xi^2+919\xi+1164\right)\beta^2\\
&+\left(-9\xi^4-90\xi^3-357\xi^2-936\xi-1158\right)\beta\\
&-27\xi^3-234\xi^2-675\xi-882\big)\,\eta\\
&+\ \left(22\xi^4+312\xi^3+1586\xi^2+4336\xi+5500\right)\beta^2+\\
&\left(-24\xi^4-240\xi^3-786\xi^2-1998\xi-2424\right)\beta\\
&-72\xi^3-702\xi^2-2034\xi-2664\big]e^{-\xi}\\
&-\ \eta^9\beta^2/18+\left(\beta^2\xi-6\beta\right)\eta^8/14+\left(-6-44\beta^2/3+(4\xi+4)\beta\right)\eta^7/7
\end{aligned}
$$

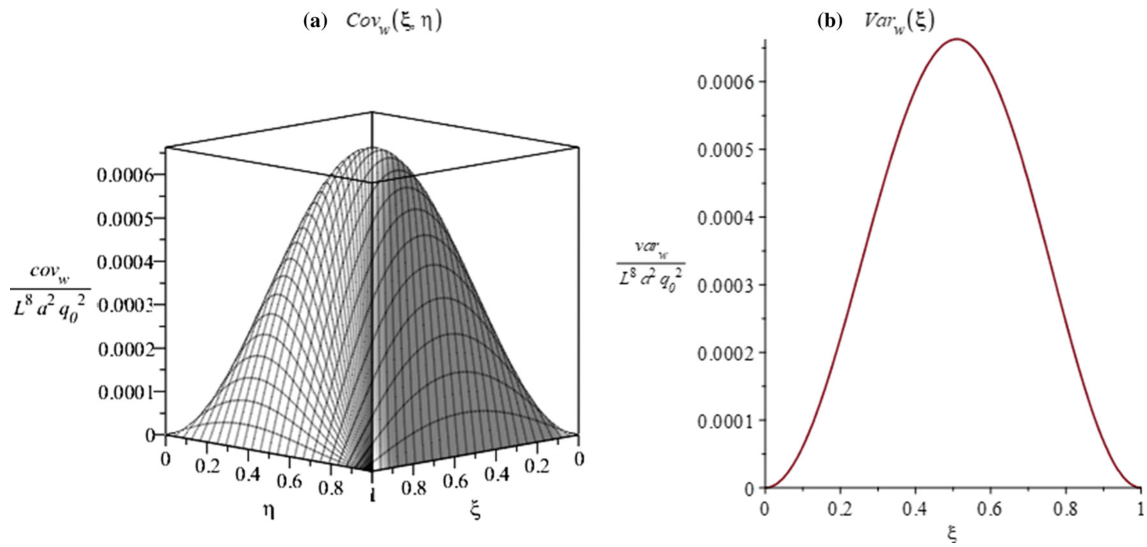

Fig. 3 Covariance function for the deflection of a beam with a Matérn covariance function of the bending flexibility, subjected to a deterministic trapezoidal load: **a** Covariance function, **b** variance function, both normalized by $L^8 a^2 q_0^2$, and for $\beta = 2$

$$
\begin{aligned}
&+ \left(20\beta^2\xi/3 + (-4\xi - 68)\beta + 12 + 6\xi\right)\eta^6/5 \\
&+ \left(-45\beta^2 + (42\xi + 78)\beta - 117 - 18\xi\right)\eta^5/5 + \\
&\left(-11\beta^2\xi/3 + (-56 - 10\xi)\beta + 15\xi + 48\right)\eta^4 \\
&- 8(3+\beta)(-17\beta/3 + \xi + 1)\eta^3 \\
&+ \left((245\xi + 1164)\beta^2 + (-222\xi - 426)\beta - 207\xi - 630\right)\eta + (-1164\xi - 5500)\beta^2 \\
&+ (1158\xi + 2424)\beta + 882\xi + 2664\} \quad for\ \xi > \eta .
\end{aligned} \tag{26}
$$

The deflection's covariance function is obtained by substituting this result in Eq. (21) and is given in "Appendix 1" (Eq. 45).

This solution is graphically illustrated in Fig. 3, where the deflection's covariance and variance functions are plotted for $\beta = 2$, $\alpha = 1$. Each function is normalized by the appropriate constant.

A comparison of the variance functions of the deflections obtained for the two different cases, i.e., ascribing an exponential and a Matérn covariance function to the beam's flexibility, respectively, is given in Fig. 4. It is shown in Fig. 4 that higher mean-square deviations are obtained when ascribing a Matérn covariance function to the beam's flexibility. In this case, the peak deflection's variance is about 18% higher than the peak value obtained when an exponential covariance function is used.

4 Case 2: Sinusoidal load and Fourier series

For a deterministic load of any shape, the solution can be obtained by representing the load as a Fourier sine and cosine series. Consider the case when the load is represented by a sine series, and Eq. (1) then takes the form:

$$
\frac{\partial^2 m(x)}{\partial x^2} = \sum_{n=1}^{\infty} q_n \sin\left(\frac{n\pi x}{L}\right). \tag{27}
$$

Since this is a linear differential equation, the solution *m(x)* can be represented by the sum:

$$
m(x) = \sum_{n=1}^{\infty} q_n m_n(x) \tag{28}
$$

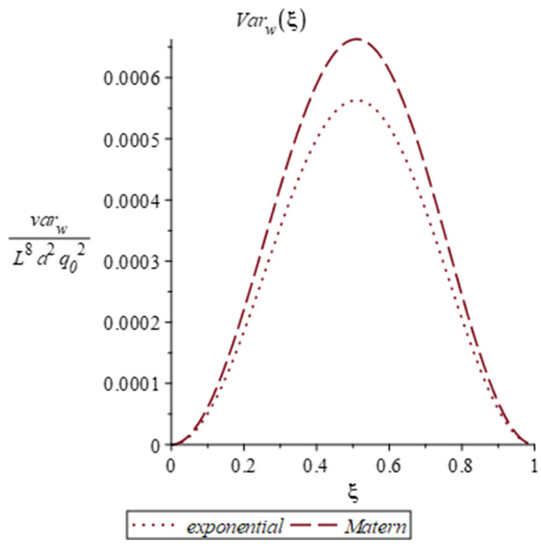

Fig. 4 Variance functions for the deflection of a beam with an exponential (dotted line) and a Matérn (dashed line) covariance function for flexibility, both subjected to a deterministic trapezoidal load, normalized by $L^8a^2q_0^2 and$ for $\beta = 2, \alpha = 1$

where $m_n(x)$ is the solution of:

$$\frac{\partial^2 m_n(x)}{\partial x^2} = \sin\left(\frac{n\pi x}{L}\right). \tag{29}$$

Substituting this expression into the covariance differential equation, as shown in Eq. (10), we get:

$$\frac{\partial^4 Cov_w(x,y)}{\partial x^2 \partial y^2} = \sum_{n=1}^{\infty} q_n m_n(x) \sum_{k=1}^{\infty} q_k m_k(y) \mathrm{Cov}_f(x,y) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} q_n q_k m_n(x) m_k(y) \mathrm{Cov}_f(x,y). \tag{30}$$

Equation (30) is also a linear differential equation, and the solution for the deflection's covariance can be represented by the sum:

$$\mathrm{Cov}_w(x,y) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} q_n q_k \mathrm{Cov}_{w_{nk}}(x,y) \tag{31}$$

where $Cov_{w_{nk}}(x, y)$ is the solution of the following equation:

$$\frac{\partial^4 \mathrm{Cov}_{w_{nk}}(x,y)}{\partial x^2 \partial y^2} = m_n(x) m_k(y) \mathrm{Cov}_f(x,y). \tag{32}$$

This equation is solved by the procedure described above, as shown in Eqs. (15)–(16). It should be noticed that the mixed terms, i.e., $\mathrm{Cov}_{w_{nk}}(x, y)$ with $n \neq k$, cannot be integrated using symmetry of x and y—they should be integrated explicitly over the whole domain.

For this example, as for the previous one, an exponential and a Matérn form for the flexibility's covariance function, cov_f, will be examined hereinafter.

Consider the case where cov_f is exponential:

$$\bar{f}(x) = f_0; \quad \mathrm{cov}_f(x,y) = a^2 \exp\left(-\alpha\frac{|x-y|}{L}\right) \tag{33}$$

where f_0, a, and α are constants. For simplicity, we introduce the following non-dimensional axial coordinates: $\xi = x/L; \eta = y/L$ here as well. By solving Eq. (32) for a general n and k, the general expression for the particular solution is given by means of the solution to Eqs. (15) and (16). This expression is given in "Appendix 1" (Eq. 46). For the sake of simplicity, explicit expressions are only given hereafter for the first term in the series:

$$q(x) = q_0 \sin\left(\frac{\pi x}{L}\right). \tag{34}$$

The particular solution for this case reads:

$$\begin{aligned}
\chi(\xi,\eta) = {} & \frac{L^8 q_0^2 a^2}{2\left(\pi^2+\alpha^2\right)^4 \pi^7}\cdot\{ \\
& \left(\sin(\pi\xi)\left(\pi^2-\alpha^2\right)-2\pi\alpha\cos(\pi\xi)\right)\left(\sin(\pi\eta)\left(\pi^2-\alpha^2\right)+2\pi\alpha\cos(\pi\eta)\right)2\pi^3 e^{\alpha(\eta-\xi)} \\
& + \left(2\cos(\pi\eta)\pi\alpha+\sin(\pi\eta)\left(\alpha^2-\pi^2\right)\right)\left(\pi^2\xi+\alpha^2\xi+2\alpha\right)2\pi^4 e^{-\alpha\eta} \\
& + \left(2\cos(\pi\xi)\pi\alpha+\sin(\pi\xi)\left(\alpha^2-\pi^2\right)\right)\left(\pi^2\eta+\alpha^2\eta+2\alpha\right)2\pi^4 e^{-\alpha\xi} \\
& + \pi\alpha\left(\pi^2+\alpha^2\right)^3(\xi-\eta)\cos^2(\pi\eta) \\
& + \alpha\left(3\pi^2+\alpha^2\right)\left(\pi^2+\alpha^2\right)^2\sin(\pi\eta)\cos(\pi\eta) \\
& + \pi\left[(\eta(\xi-\eta/3)\alpha+2\xi)\eta\pi^8+\left(\eta^2(3\xi-\eta)\alpha^2+4\alpha\eta\xi-5\xi+2\eta\right)\alpha\pi^6\right. \\
& + \left(\left(3\eta^2\xi-\eta^3\right)\alpha^5+2\alpha^4\eta\xi-7\alpha^3\xi-8\alpha^2\right)\pi^4 \\
& \left. + \left(\eta^2(\xi-\eta/3)\alpha^2-3\xi-2\eta\right)\alpha^5\pi^2-\alpha^7\xi\right]\} \quad for\ \xi\ge\eta.
\end{aligned} \tag{35}$$

By substituting this result in Eq. (21), we get:

$$\begin{aligned}
\mathrm{cov}_w(x,y) = {} & \frac{2L^8 q_0^2 a^2}{\left(\pi^2+\alpha^2\right)^4\pi^7}\cdot\{ \\
& \left(\sin(\pi\xi)\left(\pi^2-\alpha^2\right)-2\pi\alpha\cos(\pi\xi)\right)\cdot \\
& \quad \left(\sin(\pi\eta)\left(\pi^2-\alpha^2\right)/2+\pi\alpha\cos(\pi\eta)\right)\pi^3 e^{\alpha(\eta-\xi)} \\
& - \xi\alpha\left(2\cos(\pi\eta)\pi\alpha+\sin(\pi\eta)\left(\pi^2-\alpha^2\right)\right)\pi^4 e^{\alpha(\eta-1)} \\
& - \eta\alpha(2\cos(\pi\xi)\pi\alpha+\sin(\pi\xi)\left(\pi^2-\alpha^2\right))\pi^4 e^{\alpha(\xi-1)} \\
& - \alpha\left(2\cos(\pi\eta)\pi\alpha+\sin(\pi\eta)\left(\alpha^2-\pi^2\right)\right)(\xi-1)\pi^4 e^{-\alpha\eta} \\
& + \alpha\left(-2\cos(\pi\xi)\pi\alpha+\sin(\pi\xi)\left(\pi^2-\alpha^2\right)\right)(\eta-1)\pi^4 e^{-\alpha\xi} \\
& + (\alpha\pi\eta/4)\left(\pi^2+\alpha^2\right)^3(\xi-1)\left[\cos^2(\pi\eta)+\cos^2(\pi\xi)\right] \\
& - (\alpha/4)\left(3\pi^2+\alpha^2\right)\left(\pi^2+\alpha^2\right)^2\cdot \\
& \quad [(\xi-1)\sin(\pi\eta)\cos(\pi\eta)+\eta\sin(\pi\xi)\cos(\pi\xi)] \\
& + (\pi\alpha/12)\left[-24((2\xi-1)\eta-\xi)\alpha\pi^4 e^{-\alpha}\right. \\
& \quad + \eta(\xi-1)\left(-3+\left(\eta^2+\xi^2-2\xi\right)\pi^2\right)\alpha^6 \\
& \quad + 3\eta(\xi-1)\left(-1+\left(\eta^2+\xi^2-2\xi\right)\pi^2\right)\pi^2\alpha^4 \\
& \quad + 3\eta(\xi-1)\left(-7+\left(\eta^2+\xi^2-2\xi\right)\pi^2\right)\pi^4\alpha^2 \\
& \quad - 24((2\xi-1)\eta-\xi+1)\pi^4\alpha \\
& \quad \left. + \left(-21+\left(\eta^2+\xi^2-2\xi\right)\pi^2\right)\eta(\xi-1)\pi^6\right]\} \quad for \quad \xi\ge\eta.
\end{aligned} \tag{36}$$

In the more general case of arbitrary loading expanded by a sine series, a more accurate approximation for the covariance function can take as many components as needed for the load sine series expansion. For example, $\mathrm{Cov}_{w_{nk}}(x, y)$ is given in the figures below for $n = k = 1$, $n = 1$ $k = 2$, $n = 2$ $k = 1$, and $n = k = 2$, which are the solutions needed for a load represented by the two-term sine series:

$$q(x) = q_1 \sin\left(\frac{\pi x}{L}\right) + q_2 \sin\left(\frac{2\pi x}{L}\right). \tag{37}$$

The first four components of the deflection's covariance and variance functions given in Eq. (32), with $\alpha = 1$ and normalized by the appropriate constant, are shown in Figs. 5, 6, 7, and 8.

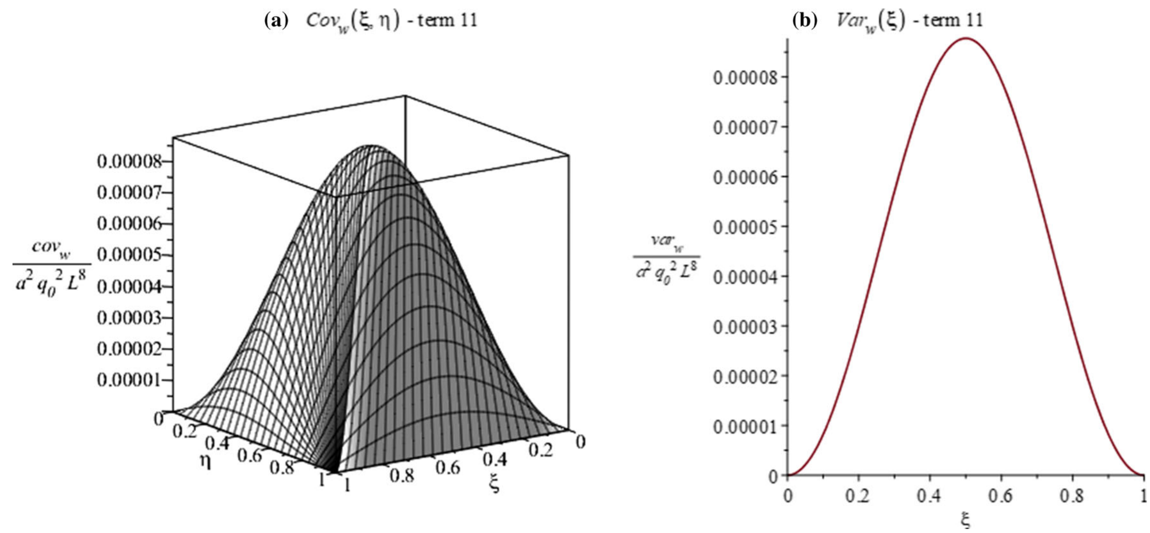

Fig. 5 The first component of the **a** covariance and **b** variance function of the deflections of a beam with an exponential covariance function of the bending flexibility, for a load represented by a two-element sine series, normalized by $q_1^2 a^2 L^8$ and for $\alpha = 1$.

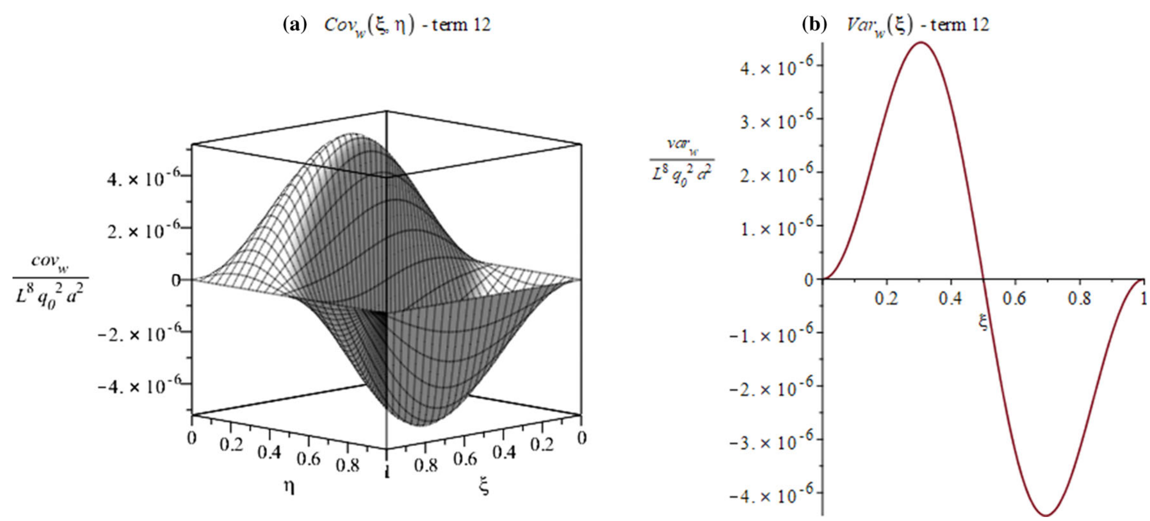

Fig. 6 The mixed term 1-2 of the **a** covariance and **b** variance function of the deflections of a beam with an exponential covariance function of the bending flexibility, for a load represented by a two-element sine series, normalized by $q_1 q_2 a^2 L^8$ for $\alpha = 1$

If cov_f is Matérn:

$$\bar{f}(x) = f_0; \quad \text{cov}_f(x, y) = a^2\left(1 + \frac{|x-y|}{L}\right)\exp\left(-\frac{|x-y|}{L}\right). \tag{38}$$

By solving Eq. (32) for a general n and k, the general expression for the particular solution is given by means of the solution to Eqs. (15) and (16). This expression is given in "Appendix 1" (Eq. 47).

As in the previous example, explicit expressions are only given hereafter for the first term in the series:

$$q(x) = q_0 \sin\left(\frac{\pi x}{L}\right). \tag{39}$$

The particular solution for this case reads:

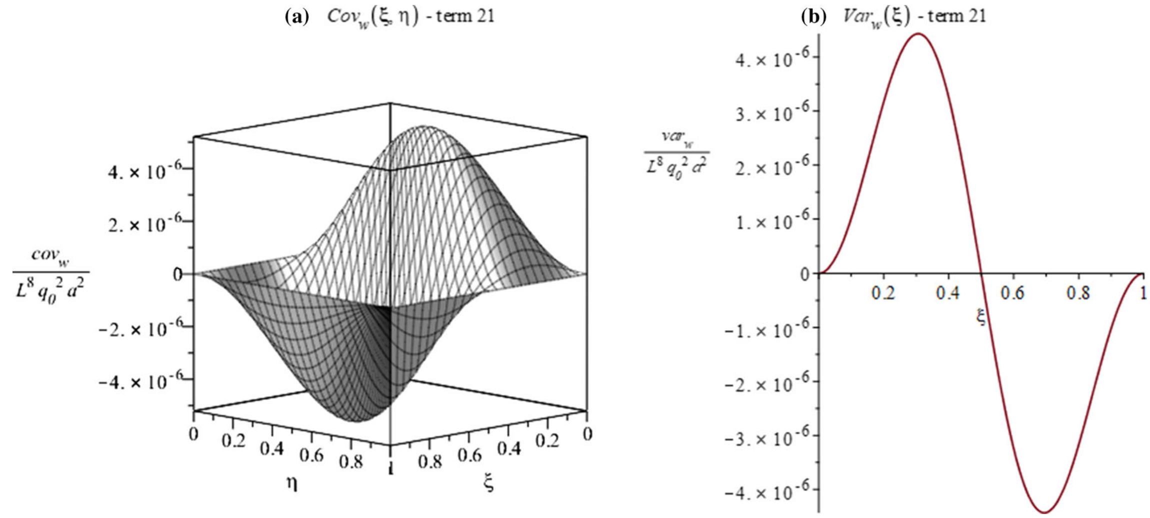

Fig. 7 The mixed term 2-1 of the **a** covariance and **b** variance function of the deflections of a beam with an exponential covariance function of the bending flexibility, for a load represented by a two-element sine series, normalized by $q_1 q_2 a^2 L^8$ for $\alpha = 1$

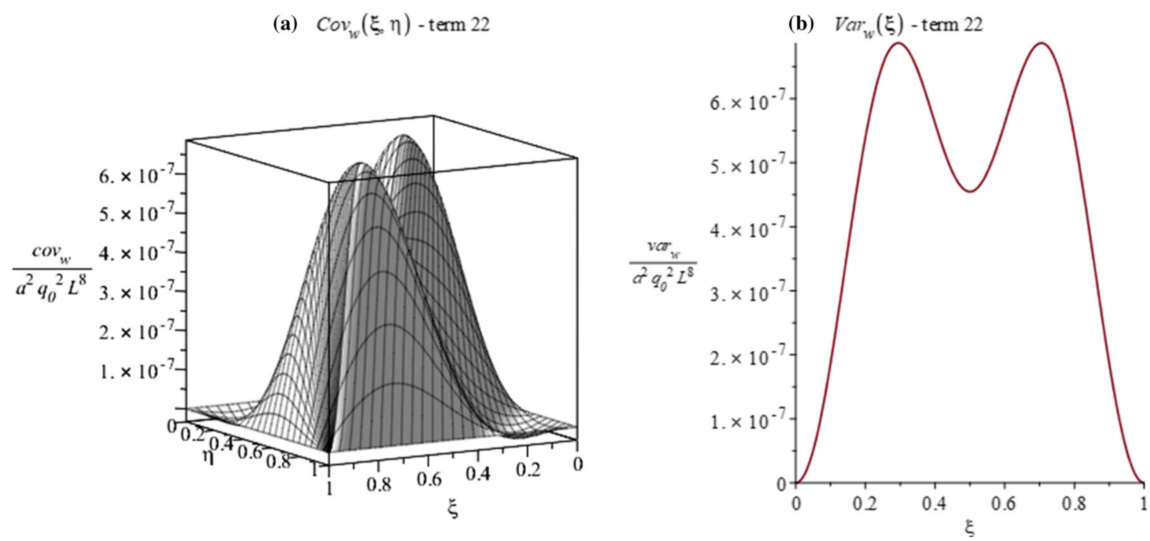

Fig. 8 The fourth component of the **a** covariance and **b** variance function of the deflections of a beam with an exponential covariance function of the bending flexibility, for a load represented by a two-element sine series, normalized by $q_2^2 a^2 L^8$ for $\alpha = 1$

$$\begin{aligned}
\chi(\xi, \eta) = -4 \frac{a^2 L^8 q_0^2}{(\pi^2+1)^5 \pi^7} e^{-\xi-\eta} \cdot \{ \\
& \big\{-(5/4) e^{\eta+\xi} (\pi^2+1)^2 (\pi^2+1/5) \cos(\pi\eta) \\
& + (\pi^3/2) \big\{ -\big[\big((1+\xi-\eta)\pi^4 + 12\pi^2 - \xi + \eta - 5 \big) \cdot \\
& \qquad (\pi+1)(\pi-1) \sin(\pi\xi) e^{2\eta} \big]/2 \\
& \quad + \pi \big[\big((\xi-\eta)\pi^4 + 10\pi^2 - \xi + \eta - 6 \big) \cos(\pi\xi) e^{2\eta} \\
& \quad + \big(\xi(\eta+1)\pi^6 + ((\xi+2)\eta + 9\xi)\pi^4 \\
& \quad + (-\xi\eta + 3\xi + 20)\pi^2 + (-\xi-2)\eta - 5\xi - 12 \big) e^{\xi}/2 \big] \big\} \big\} \sin(\pi\eta) \\
& + \pi \big\{ -(e^{\eta+\xi}/4)(\pi^2+1)^3 (\xi-\eta) \cos^2(\pi\eta)
\end{aligned}$$

$$
\begin{aligned}
&+\pi^3\Big\{-(e^{2\eta}/2)\big((\xi-\eta)\pi^4+10\pi^2-\xi+\eta-6\big)\sin(\pi\xi)\\
&\quad+\ \pi\Big[\cos(\pi\xi)\big((\xi-\eta-1)\pi^2+\xi-\eta+7\big)e^{2\eta}\\
&\qquad-\ (1/2)\big(\pi^4\xi\eta+((2\xi+2)\eta+6\xi-2)\pi^2\\
&\qquad+(\xi+2)\eta+6\xi+14)e^{\xi}\Big]\Big\}\cos(\pi\eta)\\
&+\Big\{-\pi^{10}\eta\xi/4-\eta\big(\xi\eta-(1/3)\eta^2+7\xi\big)\pi^8/4\\
&\quad+\big(\eta^3/4-3\xi\eta^2/4+(-11\xi/4-2)\eta+13\xi/4-1\big)\pi^6\\
&\quad+\big(\eta^3/4-3\xi\eta^2/4+(-5\xi/4-1)\eta+15\xi/4+7\big)\pi^4\\
&\quad+\ \big(-\xi\eta^2/4+\eta+3\xi/4+\eta^3/12\big)\pi^2+\xi/4\Big\}e^{\eta+\xi}\\
&-(e^{\eta}\pi^3/2)\Big\{\Big[-(\eta/2)(\xi+1)\pi^6+((-\xi-9)\eta/2-\xi)\pi^4\\
&\qquad+(-10+(\xi-3)\eta/2)\pi^2+(\xi+5)\eta/2+\xi+6\Big]\sin(\pi\xi)\\
&\qquad+\big(\pi^4\xi\eta+((2\xi+6)\eta+2\xi-2)\pi^2\\
&\qquad+(\xi+6)\eta+2\xi+14)\pi\cos(\pi\xi)\big)\Big\}\Big\}\quad\text{for }\ \xi\geq\eta.
\end{aligned}
\tag{40}
$$

By substituting this result in (21), we get:

$$
\begin{aligned}
\operatorname{cov}_w(\xi,\eta)=&\frac{q_0^2a^2L^8}{(\pi^2+1)^5\pi^7}e^{-\eta-\xi}\cdot\{\\
&4\pi^4\xi\Big(\pi\Big(\pi^2\eta+\eta-8\Big)\cos(\pi\eta)\\
&\qquad+\sin(\pi\eta)\Big((\eta-1)\pi^4-10\pi^2-\eta+7\Big)/2\Big)e^{-1+2\eta+\xi}\\
&+4\eta\pi^4\Big(\pi\Big(\pi^2\xi+\xi-8\Big)\cos(\pi\xi)\\
&\qquad+\sin(\pi\xi)\Big((\xi-1)\pi^4-10\pi^2-\xi+7\Big)/2\Big)e^{-1+2\xi+\eta}\\
&-\ 64\pi^5((\xi-1/2)\eta-\xi/2)e^{\eta-1+\xi}\\
&+\ \{\pi\eta\Big(\pi^2+1\Big)^3(\xi-1)\cos^2(\pi\eta)\\
&\quad-5\sin(\pi\eta)\Big(\pi^2+1/5\Big)(\xi-1)\Big(\pi^2+1\Big)^2\cos(\pi\eta)\\
&\quad+\pi\eta\Big(\pi^2+1\Big)^3(\xi-1)\cos^2(\pi\xi)\\
&\quad-\ 5\sin(\pi\xi)\eta\Big(\pi^2+1/5\Big)\Big(\pi^2+1\Big)^2\cos(\pi\xi)\\
&\quad+\frac{\pi}{3}\Big[\eta(\xi-1)\Big(\eta^2+\xi^2-2\xi\Big)\pi^8\\
&\qquad+\Big((3\xi-3)\eta^3+\Big(3\xi^3-9\xi^2-33\xi+51\Big)\eta-12\xi+12\Big)\pi^6\\
&\qquad+\Big((3\xi-3)\eta^3+\Big(3\xi^3-9\xi^2-219\xi+141\Big)\eta+84\xi-84\Big)\pi^4\\
&\qquad+\eta(\xi-1)\Big(\eta^2+\xi^2-2\xi+3\Big)\pi^2-3\eta(\xi-1)\Big]\Big\}e^{\eta+\xi}\\
&-\ 4\Big\{\Big[\pi\Big((\xi-\eta-1)\pi^2+\xi-\eta+7\Big)e^{2\eta}\cos(\pi\xi)\\
&\qquad-\Big((\xi-\eta)\pi^4+10\pi^2-\xi+\eta-6\Big)e^{2\eta}\sin(\pi\xi)/2\\
&\qquad+\pi\Big((\eta-1)\pi^2+\eta+7\Big)(\xi-1)e^{\xi}\Big]\pi\cos(\pi\eta)\\
&\quad+\Big[\pi\Big((\xi-\eta)\pi^4+10\pi^2-\xi+\eta-6\Big)e^{2\eta}\cos(\pi\xi)/2\\
&\qquad-(\pi+1)(\pi-1)\cdot\\
&\qquad\quad\Big((1+\xi-\eta)\pi^4+12\pi^2-\xi+\eta-5\Big)e^{2\eta}\sin(\pi\xi)/4\\
&\qquad-\pi e^{\xi}\Big(\pi^4\eta+10\pi^2-\eta-6\Big)(\xi-1)/2\Big]\sin(\pi\eta)
\end{aligned}
$$

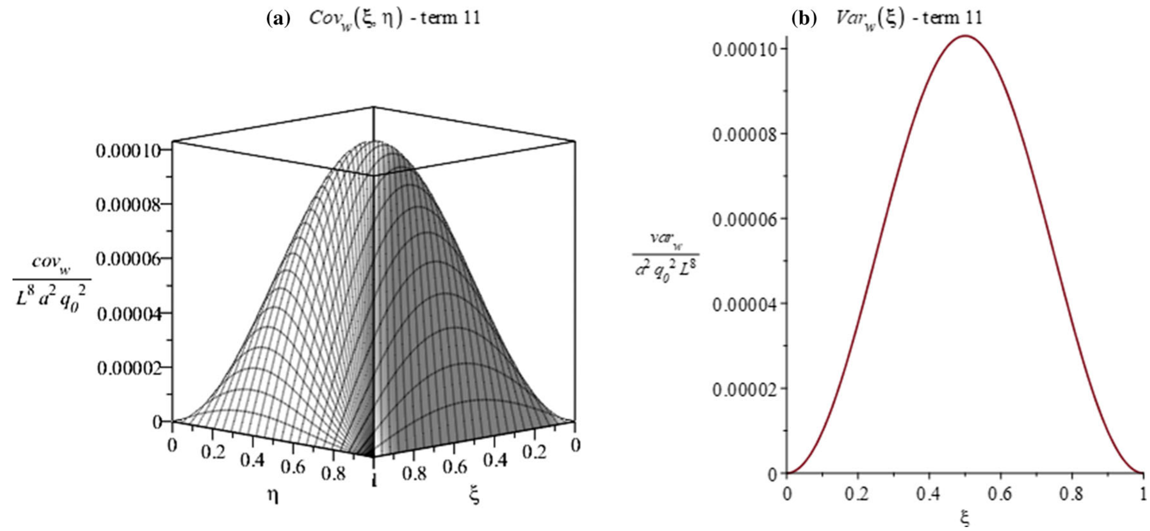

Fig. 9 The first component of the **a** covariance and **b** variance function of the deflections of a beam with an exponential covariance function of the bending flexibility, for a load represented by a two-element sine series, normalized by $q_1^2 a^2 L^8$

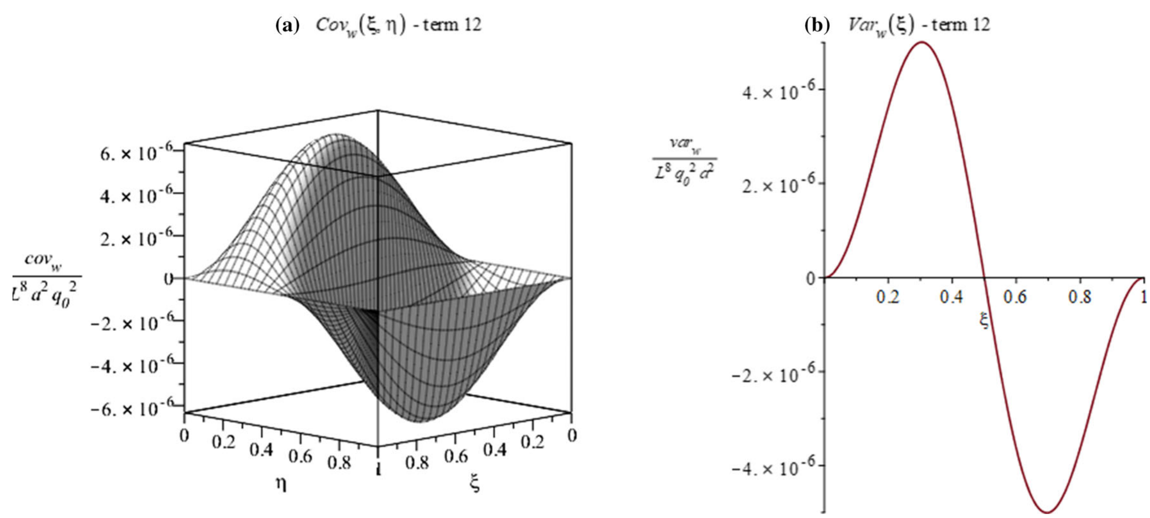

Fig. 10 The mixed term 1-2 of the **a** covariance and **b** variance function of the deflections of a beam with an exponential covariance function of the bending flexibility, for a load represented by a two-element sine series, normalized by $q_1 q_2 a^2 L^8$

$$
\begin{aligned}
&+ e^{\eta}(\eta - 1)\cdot \\
&\Big[\Big((\xi - 1)\pi^2 + \xi + 7\Big)\pi \cos(\pi\xi) \\
&\quad - \sin(\pi\xi)\Big(\pi^4\xi + 10\pi^2 - \xi - 6\Big)/2\Big]\pi\Big\}\pi^3\Big\} \quad for\ \xi \geq \eta.
\end{aligned} \tag{41}
$$

$\mathrm{Cov}_{w_{nk}}(x, y)$ is given graphically in the following Figures for $n = k = 1$, $n = 1\ k = 2$, $n = 2\ k = 1$ and $n = k = 2$, which are the solutions needed for a load represented by the two-term sine series given in Eq. (37).

The first four components of the deflection's covariance and variance functions given in Eq. (32), normalized by the appropriate constant, are shown in Figs. 9, 10, 11, and 12.

A comparison of the variance functions for the deflection obtained for the two different cases, i.e., ascribing an exponential and a Matérn covariance function for the beam's flexibility, respectively, is given in Fig. 13, where $Var_{w_{nk}}(x)$ is given for $n = k = 1$, $n = 1\ k = 2$, $n = 2\ k = 1$, and $n = k = 2$.

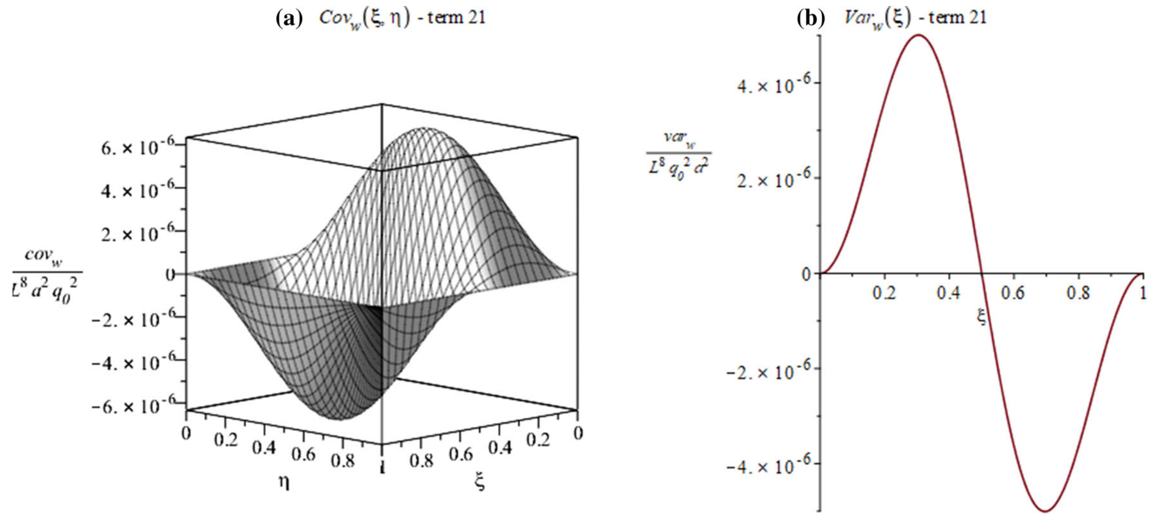

Fig. 11 The mixed term 2-1 of the **a** covariance and **b** variance function of the deflections of a beam with an exponential covariance function of the bending flexibility, for a load represented by a two-element sine series, normalized by $q_1 q_2 a^2 L^8$

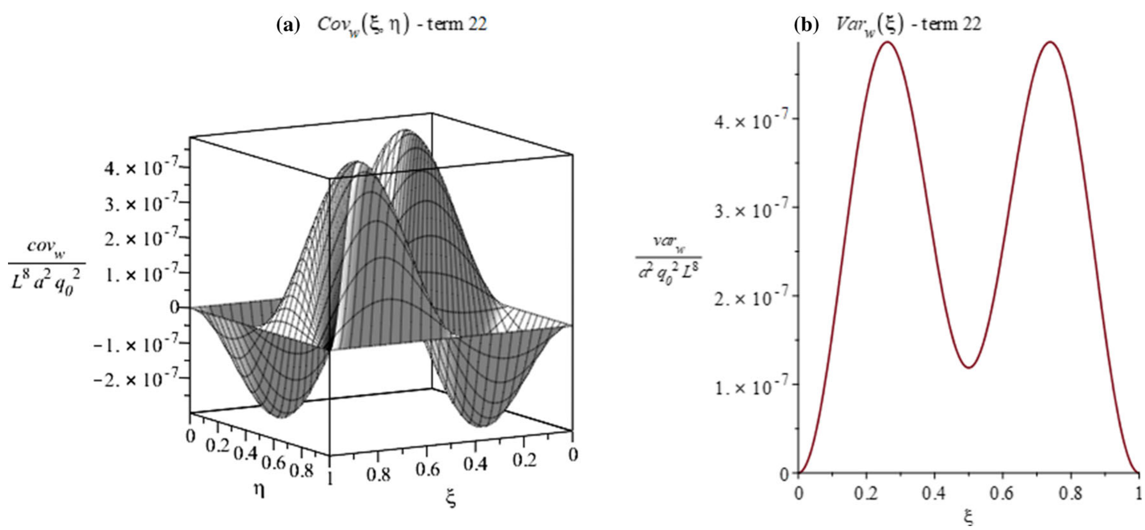

Fig. 12 The fourth component of the **a** covariance and **b** variance function of the deflections of a beam with an exponential covariance function of the bending flexibility, for a load represented by a two-element sine series, normalized by $q_2^2 a^2 L^8$

To demonstrate the use of the sine series approximation, a uniformly distributed load and its representation as such trigonometric series are looked at next. The sine series approximation for a uniformly distributed load is given by:

$$q_0 \approx \sum_{n=1}^{\infty} q_n \sin\left(\frac{n\pi x}{L}\right) = \frac{4q_0}{\pi} \sin\left(\frac{\pi x}{L}\right) + \frac{4q_0}{3\pi} \sin\left(\frac{3\pi x}{L}\right) + \frac{4q_0}{5\pi} \sin\left(\frac{5\pi x}{L}\right) + \cdots \tag{42}$$

Limiting the approximation to only the first two terms in Eq. (42), the deflection's covariance function is given by Eq. (31) as:

$$\mathrm{Cov}_w(x, y) \approx \sum_{n=1,3} \sum_{k=1,3} q_n q_k \mathrm{Cov}_{w_{nk}}(x, y)$$

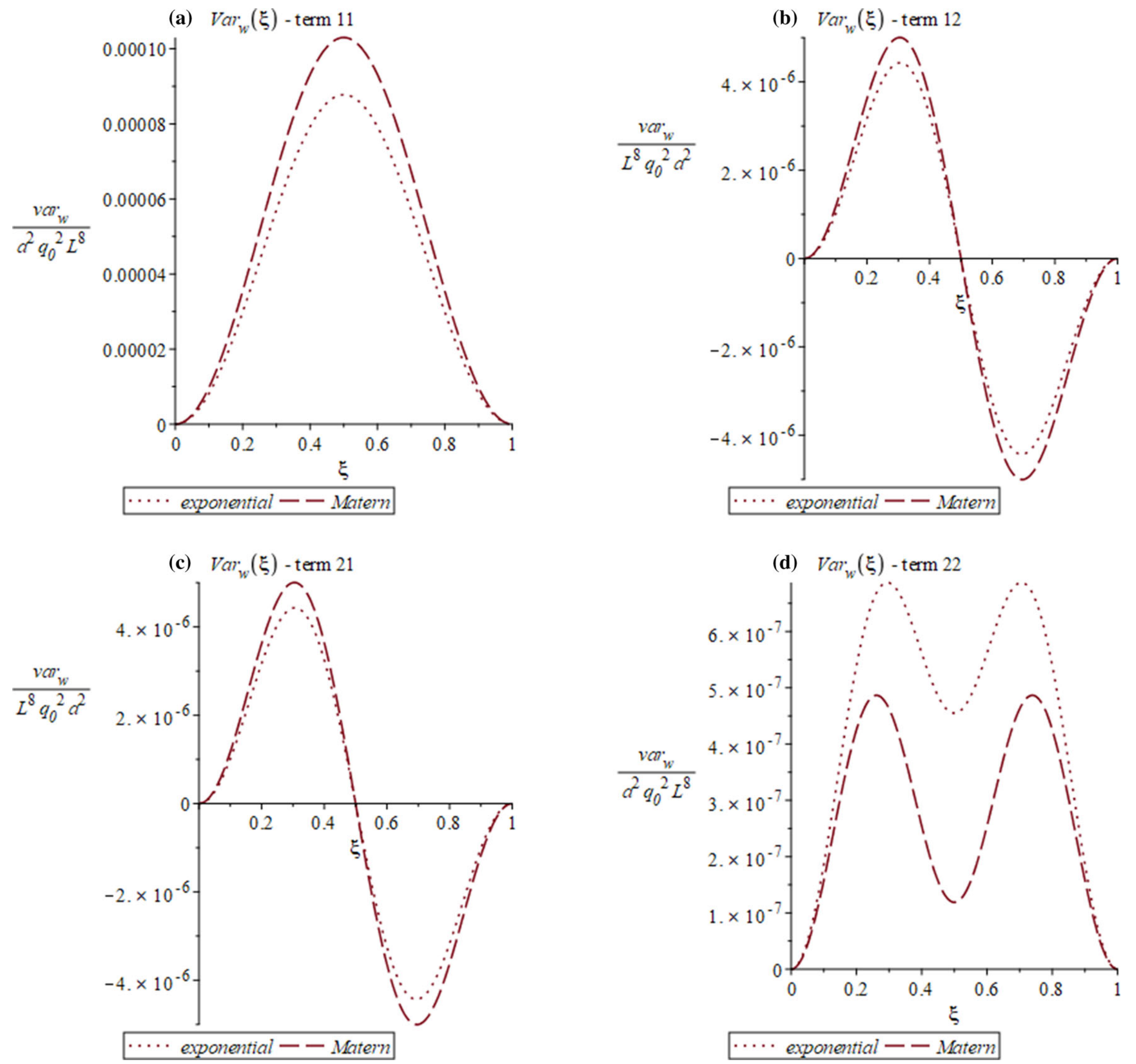

Fig. 13 Comparison of the variance deflection's functions for Cov_f being exponential and Matérn, and for four terms of a load represented by a two-element sine series, normalized by $q_2^2a^2L^8$

$$= \left(\frac{4q_0}{\pi}\right)^2 \mathrm{Cov}_{w_{11}}(x, y) + \left(\frac{4q_0}{\pi}\right)\left(\frac{4q_0}{3\pi}\right)\mathrm{Cov}_{w_{13}}(x, y) + \left(\frac{4q_0}{3\pi}\right)\left(\frac{4q_0}{\pi}\right)\mathrm{Cov}_{w_{31}}(x, y) + \left(\frac{4q_0}{3\pi}\right)^2 \mathrm{Cov}_{w_{33}}(x, y) \tag{43}$$

where $\mathrm{Cov}_{w_{nk}}(x, y)$ are solutions of Eq. (32).

The example given hereafter uses a Matérn covariance function for the beam's flexibility.

The approximated solution can be compared with the exact solution of a uniformly distributed load, obtained by substituting $\beta = 0$ in the solution for the trapezoidally distributed load (obtained by the particular solution Eqs. (26) and (21)). This comparison is given in Fig. 14 for the variance function of the deflections.

Figure 14a shows an excellent agreement between the approximated and the exact solution of a uniformly distributed load, even when only two terms are used for the sine series. The error of the approximated solution, given in Fig. 14b, reaches a peak value of $8.8 \cdot 10^{-8}$. This peak error is four orders of magnitude smaller than the value of the exact solution at that point. This comparison demonstrates the potential of expanding the closed-form solutions derived in this work for a specific pattern of load to the stochastic analysis of simply supported beams subjected to an arbitrary load distribution.

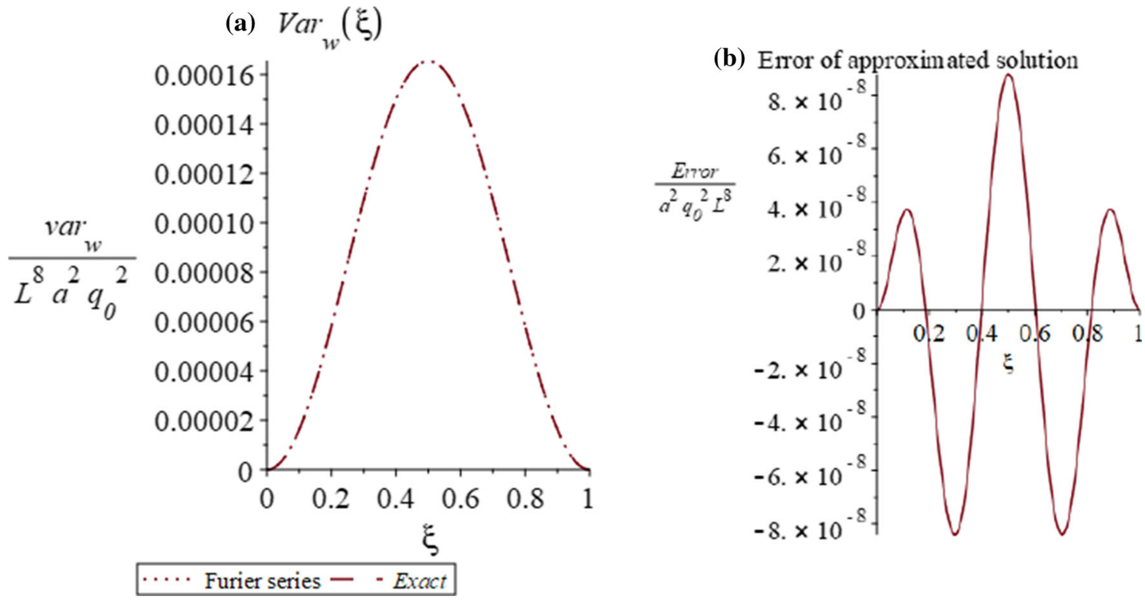

Fig. 14 Variance function of the deflection, for a uniformly distributed load when Cov_f is Matérn: a comparison of exact solution and a two-term sine series approximation of the load. **a** deflection's variance, and **b** error. All functions are normalized by $q_0^2 a^2 L^8$

5 Conclusions

New closed-form solutions for the deflection of a simply supported Bernoulli–Euler beam with stochastic flexibility and subjected to a deterministic loading have been developed. A trapezoidal and a sinusoidal load distribution and two forms for the covariance function of the beam's flexibility have been considered. The sinusoidal load case can be used to represent any desired load distribution, which can be expanded to a sine series. To demonstrate, this the results of the analysis with a sine series expansion of a uniformly distributed load have been looked at. The comparison has revealed an excellent agreement with very small relative differences with only two terms. The obtained solutions can serve as benchmark problems against which the approximate solutions' veracity can be checked.

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Appendix 1 : Expressions for exact solutions

B1 Solution for the deflection's covariance function, where an exponential form is used for the flexibility covariance function cov_f, and the load is a general trapezoidal function:

$$\begin{aligned}
\mathrm{cov}_w(\xi,\eta) = {} & \frac{L^8 a^2 q_0^2}{18\alpha^{10}}\cdot \\
& \big\{\big((3+(\eta+1)\beta)\eta(\eta-1)\alpha^3+(-6\beta\eta^2+2\beta-12\eta+6)\alpha^2+18(\beta\eta+1)\alpha-24\beta\big) \\
& \big((3+(\xi+1)\beta)(\xi-1)\xi\alpha^3+\big(6\beta\xi^2-2\beta+12\xi-6\big)\alpha^2+18(\beta\xi+1)\alpha+24\beta\big)e^{\alpha(\eta-\xi)}/2 \\
& -\ 2\big((3+(\eta+1)\beta)\eta(\eta-1)\alpha^3+(-6\beta\eta^2+2\beta-12\eta+6)\alpha^2 \\
& +18(\beta\eta+1)\alpha-24\beta\big)\big((\beta+3/2)\alpha^2+9/2(\beta+1)\alpha+6\beta\big)\xi e^{\alpha(\eta-1)} \\
& -\ \big((2\beta+3)\alpha^2+9(\beta+1)\alpha+12\beta\big)\eta\big((3+(\xi+1)\beta)(\xi-1)\xi\alpha^3 \\
& +(-6\beta\xi^2+2\beta-12\xi+6)\alpha^2+(18\beta\xi+18)\alpha-24\beta\big)e^{\alpha(\xi-1)} \\
& +\ \big((3+(\eta+1)\beta)\eta(\eta-1)\alpha^3+(6\beta\eta^2-2\beta+12\eta-6)\alpha^2
\end{aligned}$$

$$+18(\beta\eta + 1)\alpha + 24\beta)(\xi - 1)\big((\beta + 3)\alpha^2 + 9\alpha - 12\beta\big)e^{-\alpha\eta}$$
$$+ \ \big((3 + (\xi + 1)\beta)(\xi - 1)\xi\alpha^3 + (6\beta\xi^2 - 2\beta + 12\xi - 6)\alpha^2$$
$$+(18\beta\xi + 18)\alpha + 24\beta)(\eta - 1)\big((\beta + 3)\alpha^2 + 9\alpha - 12\beta\big)e^{-\alpha\xi}$$
$$- \ 2\big((2\beta + 3)\alpha^2 + 9(\beta + 1)\alpha + 12\beta\big)((2\xi - 1)\eta - \xi)\big((\beta + 3)\alpha^2 + 9\alpha - 12\beta\big)e^{-\alpha}$$
$$+ \ \eta(\xi - 1)/72\Big\{\Big(\eta^8 - 24\eta^6/7 + 18\eta^4/5 + \xi\Big(-46/35 - 2\xi^6/7 + 38\xi^4/35 - 26\xi^5/7$$
$$+164\xi^3/35 + \xi^7 - 46\xi^2/35 - 46/35\xi\big)\big)\beta^2$$
$$+ \ \Big(54\eta^7/7 - 72\eta^6/7 - 72\eta^5/5 + 108\eta^4/5 + (54/7)\Big(\xi^6 - 5\xi^5/3 - 5\xi^4/3 + 59\xi^3/15$$
$$-11\xi^2/15 - 11\xi/15 - 11/15\big)\xi\big)\beta$$
$$+ \ 108\eta^6/7 - 216\eta^5/5 + 162\eta^4/5 + (108/35)\Big(5\xi^5 - 16\xi^4 + 31\xi^3/2 - 2\xi^2 - 2\xi - 2\Big)\xi\Big\}\alpha^9$$
$$2/7\Big\{\Big(\eta^6 - 49/20\eta^4 + 7/6\eta^2 + \xi(23/30 + 3\xi^4/10 + \xi^5 - 43\xi^3/20 - 2\xi^2/5 + 23\xi/30)\Big)\beta^2$$
$$+ \ \Big(63\eta^5/10 - 147\eta^4/20 + 7\eta^2 - 21\xi^4/4 + 63\xi^5/10 - 7\eta^3 + 7\xi/2 - 7\xi^3 + 7\xi^2/2\Big)\beta$$
$$+ \ 189\eta^4/20 + 189\xi^4/20 + 21\xi/5 + 21\eta^2/2 - 21\eta^3 - 84\xi^3/5 + 21\xi^2/5\big\}\eta(\xi - 1)\alpha^7$$
$$+ \ (6/5)\eta(\xi - 1)\big(-15 + (\xi^4 + \eta^4 + \xi^3 - 7\xi^2/3 - 10\eta^2/3 - 7\xi/3 - 5/3)\beta^2$$
$$+(5\eta^3 + 5\xi^3 - 10\eta^2 - 5\xi^2 - 5\xi - 10)\beta\big)\alpha^5$$
$$+ \ 2\big(((-5\xi + 1)\eta + \xi - 1)\beta^2 + 6((-3\xi + 1)\eta + \xi - 1)\beta + 9((-2\xi + 1)\eta + \xi - 1)\big)\alpha^4$$
$$+ \ 36\eta(\beta^2 + 3\beta + 3)(\xi - 1)\alpha^3 + \big(((114\xi - 48)\eta - 48\xi + 48)\beta^2 + ((324\xi - 144)\eta$$
$$- \ 144\xi + 144)\beta + (324\xi - 162)\eta - 162\xi + 162)\alpha^2$$
$$- \ 144\beta^2\eta(\xi - 1)\alpha - 576\beta^2((\xi - 1/2)\eta - \xi/2 + 1/2)\big\} \quad \text{for } \xi > \eta\,. \tag{44}$$

B2 Solution for the deflection's covariance function, where a Matérn form is used for the flexibility covariance function cov_f, and the load is a general trapezoidal function:

$$\mathrm{cov}_w(\xi,\eta) = \frac{23a^2q_o^2L^8}{18}\cdot\{$$
$$\big[-(\beta/46)\big((\xi^3 + 6\xi^2 + 17\xi + 22)\beta + 3\xi^2 + 9\xi + 12\big)\eta^4$$
$$+\big\{(\xi + 3)(\xi^3 + 14\xi^2 + 47\xi + 104)\beta^2/46$$
$$+(12\xi^2 + 48 + 36\xi)\beta/23 - 18/23 - 9\xi^2/46 - 27\xi/46\big\}\eta^3$$
$$+\big\{\ (-1249\xi/46 - 3\xi^4/23 - 89\xi^3/46 - 228\xi^2/23 - 793/23)\beta^2$$
$$+ (3/46)(\xi + 1)(\xi^3 + 9\xi^2 + 3\xi + 18)\beta$$
$$+ \ 225/23 + 9\xi^3/46 + 117\xi^2/46 + 171\xi/23\big\}\eta^2$$
$$+\big\{(17\xi^4/46 + 245\xi^3/46 + 1249\xi^2/46 + 2168/23 + 3417\xi/46)\beta^2$$
$$+(-15\xi^4/46 - 75\xi^3/23 - 429\xi^2/46 - 633/23 - 531\xi/23)\beta$$
$$-45\xi^3/46 - 234\xi^2/23 - 1359\xi/46 - 891/23\big\}\eta$$
$$+ (-11\xi^4 - 156\xi^3 - 793\xi^2 - 2168\xi - 2750)\beta^2/23$$
$$+ \big(12\xi^4 + 120\xi^3 393\xi^2 + 999\xi + 1212\big)\beta/23$$
$$+ \ 351\xi^2/23 + 36\xi^3/23 + 1017\xi/23 + 1332/23\big]e^{\eta-\xi}$$
$$- ((\xi - 1)/23)\big[(11\beta^2 - 12\beta)\eta^4 + 156(\beta - 1)(\beta + 3/13)\eta^3$$
$$+ (793\beta^2 - 393\beta - 351)\eta^2 + \ (2168\beta^2 - 999\beta - 1017)\eta$$
$$+2750\beta^2 - 1212\beta - 1332\big]e^{-\eta}$$
$$- ((\eta - 1)/23)\big[(11\xi^4 + 156\xi^3 + 793\xi^2 + 2168\xi + 2750)\beta^2$$
$$+ \big(-12\xi^4 - 120\xi^3 - 393\xi^2 - 999\xi - 1212\big)\beta$$

$$
\begin{aligned}
&-36\xi^3 - 351\xi^2 - 1017\xi - 1332\big]e^{-\xi}\\
&+\big[(\beta^2 + 12\beta/23)\eta^4 + (36 - 332\beta^2 - 96\beta)\eta^3/23\\
&\quad + (-459 + 1693\beta^2 - 93\beta)\eta^2/23 + (1827 - 4632\beta^2\\
&\quad + 1461\beta)\eta/23 - 2736/23 + 5878\beta^2/23 - 2736\beta/23\big]\xi e^{\eta-1}\\
&+\eta\big[(5878 + 23\xi^4 - 332\xi^3 + 1693\xi^2 - 4632\xi)\beta^2/23\\
&\quad + (-2736 + 12\xi^4 - 96\xi^3 - 93\xi^2 + 1461\xi)\beta/23\\
&\quad - 2736/23 - 459\xi^2/23 + 1827\xi/23 + 36\xi^3/23\big]e^{\xi-1}\\
&+(11756/23)(-1368/2939 + \beta^2 - 1368\beta/2939)\cdot\\
&\qquad ((\xi - 1/2)\eta - \xi/2)e^{-1}\\
&+(\beta^2/828)(\xi - 1)\eta^9 + 3\beta(\xi - 1)\eta^8/322\\
&+(22/483)(\beta^2 - 3/11\beta + 9/22)(\xi - 1)\eta^7\\
&+(34/115)(\beta - 3/17)(\xi - 1)\eta^6\\
&+(9/46)(\beta^2 - 26\beta/15 + 13/5)(\xi - 1)\eta^5\\
&+(28/23)(\xi - 1)(\beta - 6/7)\eta^4\\
&-(68/69)(3 + \beta)(\beta - 3/17)(\xi - 1)\eta^3\\
&+\Big[\Big(11\xi^4/138 - 68\xi^3/69 + 3914/23 - 822533\xi/2898 + 1\xi^9/828\\
&\quad +9\xi^5/46 - 2\xi^6/69 + 22\xi^7/483 - 1\xi^8/644\Big)\beta^2\\
&\quad + (33\xi^4/23 - 60\xi^3/23 - 2004/23 + 244171\xi/1610\\
&\qquad - 12\xi^5/23 + 36\xi^6/115 - 4\xi^7/161 + 3\xi^8/322)\beta\\
&\quad - 63\xi^4/46 + 24\xi^3/23 + 119538\xi/805\\
&\qquad - 2088/23 + 3\xi^7/161 + 27\xi^5/46 - 9\xi^6/115\Big]\eta\\
&+(2750/23)(\xi - 1)(\beta^2 - 606\beta/1375 - 666/1375)\Big\}\quad for\ \xi \ge \eta.
\end{aligned}
\tag{45}
$$

B3 Particular solution for the deflection's covariance function, where an exponential form is used for the flexibility covariance function cov_f, and the load is a general sinusoidal function:

$$
\begin{aligned}
\chi_{nk}(\xi,\eta) = {} & \frac{a^2 q_0^2 L^8}{(\pi^2 n^2 + \alpha^2)^2 (k^2\pi^2 + \alpha^2)^2 (k-n)^3 (k+n)^3 \alpha^6 n^6 \pi^8 k^6}\cdot\\
&\Big[n^4\Big(\Big(\pi^2\alpha^3 n^2 - \alpha^5\Big)\sin(n\pi\xi) - 2\cos(n\pi\xi)n\pi\alpha^4 + \sin(n\pi)(\pi^2 n^2 + \alpha^2)^2(\alpha\xi + 2)\Big)\cdot\\
&(k-n)^3\Big(\Big(\pi^2\alpha^3 k^2 - \alpha^5\Big)\sin(k\pi\eta) + 2\cos(k\pi\eta)k\pi\alpha^4\\
&\quad + \sin(k\pi)(\pi^2 k^2 + \alpha^2)^2(\alpha\eta - 2)\Big)k^4(k+n)^3\pi^4 e^{\alpha(\eta-\xi)}\\
&\quad + n^4k^4\Big(\Big(\pi^2\alpha^3 k^2 - \alpha^5\Big)\sin(k\pi\eta) - 2\cos(k\pi\eta)k\pi\alpha^4 + \sin(k\pi)(\pi^2 k^2 + \alpha^2)^2(\alpha\eta + 2)\Big)\cdot\\
&\quad\Big(\sin(n\pi)(\pi^2 n^2 + \alpha^2)^2(\alpha\xi + 2) - n\pi\alpha^3(\pi^2 n^2\xi + \alpha^2\xi + 2\alpha)\Big)(k-n)^3(k+n)^3\pi^4 e^{-\alpha\eta}\\
&\quad + n^4k^4\Big(\Big(\pi^2\alpha^3 n^2 - \alpha^5\Big)\sin(n\pi\xi) - 2\cos(n\pi\xi)n\pi\alpha^4 + \sin(n\pi)(\pi^2 n^2 + \alpha^2)^2(\alpha\xi + 2)\Big)\cdot\\
&\quad\Big(\sin(k\pi)(\pi^2 k^2 + \alpha^2)^2(\alpha\eta + 2) - k\pi\alpha^3(\pi^2 k^2\eta + \alpha^2\eta + 2\alpha)\Big)(k-n)^3(k+n)^3\pi^4 e^{-\alpha\xi}\\
&\quad + 2n^4(k^2\pi^2 + \alpha^2)^2\alpha^3\sin(k\eta\pi)\cdot\big\{-k^4\pi^2\alpha^4(\pi^2 n^2 + \alpha^2)(k-n)(k+n)(k^2 + n^2)(\xi - \eta)\sin(\eta n\pi)\\
&\quad + 2n\alpha^4k^4\pi\big(\pi^2k^4 - 3(\pi^2 n^2 + \alpha^2)k^2 - 2\pi^2 n^4 - \alpha^2 n^2\big)\cos(\eta n\pi)\\
&\quad + (\pi^2 n^2 + \alpha^2)^2\big((-2 + \eta(\xi - \eta)\alpha^2)\pi^2 k^2 + 6\alpha^2\big)(k-n)^3\sin(n\pi)(k+n)^3\big\}\\
&\quad + (k/6)\Big\{24n^4(k^2\pi^2 + \alpha^2)^2\alpha^5\pi\cos(k\pi\eta)\\
&\quad\cdot\big[-\alpha^2k^4\big((\pi^2 n^2 - \alpha^2)k^2 - 5\pi^2 n^4 - 3\alpha^2 n^2\big)\sin(\eta n\pi) - (\pi^2 n^2 + \alpha^2)(k-n)\\
&\quad (k+n)k^4 n\pi\alpha^2(\xi - \eta)\cos(\eta n\pi) + (\pi^2 n^2 + \alpha^2)^2(k-n)^3(k+n)^3(\xi - 2\eta)\sin(n\pi)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\ (k-n)(k+n)\cdot\Big\{12\big(k^2\pi^2+\alpha^2\big)^2(k-n)^2(k+n)^2\alpha^7k^3\big(\pi^4\eta(\xi-\eta)n^4\\
&+\ \pi^2\big(10+\eta(\xi-\eta)\alpha^2\big)n^2+6\alpha^2\big)\sin(k\pi)\sin(\eta n\pi)\\
&+\ n\pi\Big\{24\big(k^2\pi^2+\alpha^2\big)^2\big(\pi^2(\xi-3\eta)n^2+\alpha^2(\xi-2\eta)\big)(k-n)^2(k+n)^2\alpha^7k^3\sin(k\pi)\cos(\eta n\pi)\\
&+\big(n^2\pi^2+\alpha^2\big)^2n^3(k-n)^2(k+n)^2\sin(n\pi)\cdot\\
&\Big[\big(k^2\pi^2+\alpha^2\big)^2k^3\pi^3\sin(k\pi)\Big(-24+\eta^4(\xi-3\eta/5)\alpha^5-4\alpha^3\eta^3+6\alpha^2\xi\eta-12(\xi-\eta)\alpha\Big)\\
&-6\alpha^3\big(\pi^6\eta(\alpha\xi-2)k^6+\pi^4\alpha(\alpha\eta+2)(\alpha\xi-2)k^4+8(\xi+\eta/2)\alpha^4k^2\pi^2+4\alpha^6(\xi+\eta)\big)\Big]\\
&-\ 6\alpha^3k^3\Big\{\big(k^2\pi^2+\alpha^2\big)^2(k-n)^2(k+n)^2\sin(k\pi)\cdot\big(\pi^6\xi(\alpha\eta-2)n^6+\pi^4\alpha(\alpha\eta-2)(\alpha\xi+2)n^4\\
&+\ 4(\xi+2\eta)\alpha^4n^2\pi^2+4\alpha^6(\xi+\eta)\big)\\
&-\ n^4\alpha^3k\pi\big[\pi^6\eta\big(n^2\pi^2\xi+\alpha^2\xi-2\alpha\big)k^6-2\big(n^2\pi^2\xi+\alpha^2\xi-2\alpha\big)\pi^4\big(n^2\pi^2\eta-\eta\alpha^2/2-\alpha\big)k^4\\
&+\ \pi^2k^2\Big(n^6\pi^6\xi\eta-\pi^4\alpha(\alpha\xi\eta-2\eta-4\xi)n^4-2\big(-4+\alpha^2\xi\eta-6\alpha(\xi+\eta)\big)\alpha^2n^2\pi^2+8\alpha^5(\xi+\eta/2)\Big)\\
&+\alpha\big(\pi^6n^6\xi(\alpha\eta-2)+\pi^4n^4\alpha(\alpha\eta-2)(\alpha\xi+2)+4\alpha^4\pi^2n^2(\xi+2\eta)+4\alpha^6(\xi+\eta)\big)\big]\}\}\}\}]\quad\text{for }\xi\geq\eta.
\end{aligned}\tag{46}
$$

B4 Particular solution for the deflection's covariance function, where a Matérn form is used for the flexibility covariance function cov_f, and the load is a general sinusoidal function:

$$
\begin{aligned}
\chi_{nk}(\xi,\eta)=&\ \frac{q_0^2a^2L^8e^{-\eta-\xi}}{\big(\pi^2n^2+1\big)^3\big(\pi^2k^2+1\big)^3(k-n)^3(k+n)^3k^6n^6\pi^8}\cdot\\
&\big\{n^4\big\{e^{2\eta}k^4(k+n)^3(k-n)^3\big(\big(\pi^4(1+\xi-\eta)n^4+6\pi^2n^2-\xi+\eta-3\big)\pi^4k^4+\big(6\pi^6n^4-6\pi^2\big)k^2\\
&-\ \pi^4(\xi-\eta+3)n^4-6\pi^2n^2+\xi-\eta+5\big)\pi^4\sin(n\pi\xi)\\
&-2e^{2\eta}k^4(k+n)^3n(k-n)^3\big(\big(\pi^2(\xi-\eta)n^2+\xi-\eta+4\big)\pi^4k^4+\big(6\pi^4n^2+6\pi^2\big)k^2\\
&-\ \pi^2(\xi-\eta+2)n^2-\xi+\eta-6\big)\pi^5\cos(n\pi\xi)\\
&-4e^{\eta+\xi}k^4\pi^2\big(\pi^2n^2+1\big)\big(\pi^2k^2+1\big)^3(k-n)(k+n)\big(k^2+n^2\big)(\xi-\eta)\sin(\eta n\pi)\\
&+16k^4\big(\pi^2k^2+1\big)^3n\big(\pi^2k^4+\big(-\pi^2n^2-1\big)3k^2/2-3\pi^2n^4/2-n^2/2\big)e^{\eta+\xi}\pi\cos(\eta n\pi)\\
&+(k+n)^3\Big\{\big(\pi^2n^2+1\big)^3\big[k^4(((-\xi-2)\eta\\
&+\ \ \xi^2+5\xi+8\big)\pi^4k^4+6\pi^2(\xi+2)k^2+(\xi+2)\eta-\xi^2-7\xi-12\big)\pi^4e^{2\eta}\\
&+4\big(\pi^2k^2+1\big)^3\big(6+\pi^2\big(-\eta^2+\eta\xi-4\big)k^2\big)e^{\eta+\xi}+k^4e^{\xi}\pi^4\big(((\xi+2)\eta+3\xi+8)\pi^4k^4+6\pi^2(\xi+2)k^2\\
&+\ (-\xi-2)\eta-5\xi-12)]\sin(n\pi)\\
&-\big[\big(\pi^4\xi(\eta+1)n^4+2((\xi+1)\eta+2\xi)\pi^2n^2+(\xi+2)\eta+3\xi+8\big)\pi^4k^4+6\pi^2\big(\pi^2n^2+1\big)\big(\pi^2n^2\xi+\xi+2\big)k^2\\
&-\pi^4\xi(\eta+3)n^4-2((\xi+1)\eta+4\xi+2)\pi^2n^2+(-\xi-2)\eta-5\xi-12\big]k^4e^{\xi}n\pi^5\Big\}(k-n)^3\Big\}\sin(k\pi\eta)\\
&+k\big\{2n^4\big\{e^{2\eta}k^4(k+n)^3(k-n)^3\big(\pi^2\big(\pi^4(\xi-\eta)n^4+6\pi^2n^2-\xi+\eta-2\big)k^2+\pi^4(\xi-\eta+4)n^4\\
&+6\pi^2n^2-\xi+\eta-6\big)\pi^4\sin(n\pi\xi)\\
&-2\big(\big(\pi^2(\xi-\eta-1)n^2+\xi-\eta+3\big)\pi^2k^2+\pi^2(\xi-\eta+3)n^2+\xi-\eta+7\big)e^{2\eta}k^4(k+n)^3n(k-n)^3\pi^5\cos(n\pi\xi)\\
&-12k^4\big(\pi^2k^2+1\big)^3\big(\big(\pi^2n^2-1/3\big)k^2-7\pi^2n^4/3-n^2\big)e^{\eta+\xi}\sin(\eta n\pi)+(k+n)(k-n)\\
&\Big\{-4e^{\eta+\xi}k^4n\pi\big(\pi^2n^2+1\big)\big(\pi^2k^2+1\big)^3(\xi-\eta)\cos(\eta n\pi)+(k+n)^2(k-n)^2\big\{(\pi^2n^2+1)^3\cdot\\
&\Big[k^4\big(((-\xi-2)\eta+\xi^2+4\xi+6\big)\pi^2k^2+(-\xi-2)\eta+\xi^2+8\xi+14\big)\pi^4e^{2\eta}+4\big(\pi^2k^2+1\big)^3(\xi-2\eta)e^{\eta+\xi}\\
&-k^4e^{\xi}\big(((\xi+2)\eta+2\xi+6)\pi^2k^2+(\xi+2)\eta+6\xi+14\big)\pi^4\big]\sin(n\pi)\\
&+\ \ \big(\big(n^4\pi^4\xi\eta+2((\xi+1)\eta+\xi-1)\pi^2n^2+(\xi+2)\eta+2\xi+6\big)\pi^2k^2+\pi^4\xi(\eta+4)n^4\\
&+2((\xi+1)\eta+5\xi+3)\pi^2n^2+(\xi+2)\eta+6\xi+14\big)k^4e^{\xi}n\pi^5\Big\}\Big\}\Big\}\pi\cos(k\pi\eta)\\
&+\ (k+n)(k-n)\big\{k^3(k+n)^2n^4(k-n)^2\cdot\Big\{\big(\pi^2k^2+1\big)^3\big[\big(\big(-\eta^2+(\xi+5)\eta-2\xi-8\big)\pi^4n^4
\end{aligned}
$$

$$
\begin{aligned}
&+\ 6\pi^2(\eta-2)n^2+\eta^2+(-\xi-7)\eta+2\xi+12\big)\,e^{2\eta}\\
&+\big(((\xi+3)\eta+2\xi+8)\pi^4n^4+6\pi^2(\eta+2)n^2+(-\xi-5)\eta-2\xi-12\big)e^{\eta}\big]\sin(k\pi)\\
&-ke^{\eta}\big[\big(\pi^4(\xi+1)n^4+6\pi^2n^2-\xi-3\big)\eta\pi^4k^4+2\pi^2\big(((\xi+2)\eta+\xi)\pi^4n^46\pi^2(\eta+1)n^2+(-\xi-4)\eta-\xi-2\big)k^2\\
&+\ ((\xi+3)\eta+2\xi+8)\pi^4n^4+6\pi^2(\eta+2)n^2+(-\xi-5)\eta-2\xi-12\big]\pi\big\}\pi^4\sin(n\pi\xi)\\
&-\ 2\big\{\big(\big(\big(-\eta^2+(\xi+4)\eta-2\xi-6\big)\pi^2n^2-\eta^2+(\xi+8)\eta-2\xi-14\big)e^{2\eta}\\
&+\ \big(((\xi+2)\eta+2\xi+6)\pi^2n^2+(\xi+6)\eta+2\xi+14\big)e^{\eta}\big)\left(\pi^2k^2+1\right)^3\sin(k\pi)\\
&-ke^{\eta}\pi\left(\eta\pi^4\big(\pi^2n^2\xi+\xi+4\big)k^4+2\big(((\xi+1)\eta+\xi-1)\pi^2n^2+(\xi+5)\eta+\xi+3\big)\pi^2k^2\right.\\
&+\ ((\xi+2)\eta+2\xi+6)\pi^2n^2+(\xi+6)\eta+2\xi+14)\big\}k^3(k+n)^2n^5(k-n)^2\pi^5\cos(n\pi\xi)\\
&+4k^3\left(\pi^2k^2+1\right)^3(k+n)^2\big(6+\eta\pi^4(\xi-\eta)n^4+\pi^2\big(-\eta^2+\eta\xi+14\big)n^2\big)(k-n)^2e^{\eta+\xi}\sin(k\pi)\sin(\eta n\pi)\\
&+\big\{8k^3\big(\pi^2(\xi-4\eta)n^2+\xi-2\eta\big)\left(\pi^2k^2+1\right)^3(k+n)^2(k-n)^2e^{\eta+\xi}\sin(k\pi)\cos(\eta n\pi)\\
&+(\pi^2n^2+1)^3(k+n)^2n^3(k-n)^2\big\{k^3\left(\pi^2k^2+1\right)^3\big[\big((-\xi-2)\eta^2+\big(\xi^2+9\xi+16\big)\eta-2\xi^2\\
&-\ 16\xi-28\big)\,e^{2\eta}+\Big(-\eta^5/5+\xi\eta^4/3-8\eta^3/3+(12+5\xi)\eta-12\xi-28\Big)e^{\eta+\xi}\\
&+\big(\big(\xi^2+7\xi+12\big)\eta+2\xi^2+16\xi+28\big)e^{\eta}+\ \big((\xi+2)\eta^2+(7\xi+16)\eta+12\xi+28\big)e^{\xi}\big]\pi^3\sin(k\pi)\\
&+\big(-(3\xi-8)\eta\pi^8k^8-4((2\xi-5)\eta+\xi-3)\pi^6k^6-((5\xi-12)\eta+12\xi-28)\pi^4k^4\\
&-\ 24(\xi+\eta/3)\pi^2k^2-8\xi-8\eta\big)e^{\eta+\xi}\\
&-k^4\left(\eta\pi^4\big(\xi^2+5\xi+8\big)k^4+2\big(\big(\xi^2+6\xi+10\big)\eta+\xi^2+4\xi+6\big)\pi^2k^2\right.\\
&+\ \big(\xi^2+7\xi+12\big)\eta+2\xi^2+16\xi+28\big)e^{\eta}\pi^4\big\}\sin(n\pi)\\
&-3k^3\big\{\left(\pi^2k^2+1\right)^3(k+n)^2\big\{\big(\xi\pi^8(-8+3\eta)n^8+4((2\xi+1)\eta-5\xi-3)\pi^6n^6+((5\xi+12)\eta\\
&-\ \ 12\xi-28)\pi^4n^4+8\pi^2(\xi+3\eta)n^2+8\xi+8\eta\big)\,e^{\eta+\xi}/3\\
&+e^{\xi}n^4\left(\pi^4\xi\big(\eta^2+5\eta+8\big)n^4+2\big((\xi+1)\eta^2+(6\xi+4)\eta+10\xi+6\big)\pi^2n^2+(\xi+2)\eta^2\right.\\
&+\ (7\xi+16)\eta+12\xi+28)\pi^4/3\big\}(k-n)^2\sin(k\pi)\\
&-kn^4\big\{\eta\pi^8\big(\pi^4n^4\xi+4\pi^2n^2\xi+3\xi-8\big)\,k^8-2\big(n^6\pi^6\xi\eta+2n^4\pi^4\xi\eta\\
&-\ \pi^2(3(\xi+2)\eta+2\xi+2)n^2+(-4\xi+10)\eta-2\xi+6\big)\pi^6k^6\\
&+\big(n^8\pi^8\xi\eta-4n^6\pi^6\xi\eta+(-18\eta\xi-8)\pi^4n^4-4((2\xi-9)\eta-3\xi-3)\pi^2n^2+(5\xi-12)\eta+12\xi-28\big)\pi^4k^4\\
&+2\big(2n^8\pi^8\xi\eta+((3\xi+2)\eta+6\xi+2)\pi^6n^6-2((2\xi-3)\eta-9\xi-3)\pi^4n^4\\
&-\ ((5\xi-24)\eta-24\xi-28)\pi^2n^2+12\xi+4\eta\big)\pi^2k^2\\
&\ \ +\xi\pi^8(-8+3\eta)n^8+4((2\xi+1)\eta-5\xi-3)\pi^6n^6+((5\xi+12)\eta-12\xi-28)\pi^4n^4\\
&+\quad 8\pi^2(\xi+3\eta)n^2+8\xi+8\eta\big\}e^{\eta+\xi}\pi/3\big\}\big\}n\pi\big\}\big\}\big\}\quad \text{for } \xi\geq\eta.
\end{aligned}
\tag{47}
$$

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