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ORIGINAL PAPER

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A domain of influence theorem for a natural stress–heat-flux problem in the Moore–Gibson–Thompson thermoelasticity theory

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Abstract The present paper is focused on the Moore–Gibson–Thompson (MGT) thermoelasticity theory. The MGT thermoelasticity theory is a generalized form of the Lord–Shulman (LS) thermoelasticity theory as well as the Green-Naghdi thermoelasticity theory with energy dissipation (GN-III). The present work is aimed at establishing the domain of influence results in the context of this new thermoelasticity theory. We consider a mixed boundary-initial value problem representing natural stress–heat-flux disturbance inside an isotropic and homogeneous medium. We establish an identity regarding this present problem. Further, we derive the domain of influence theorem based on this identity under the MGT thermoelasticity theory. From this theorem, we conclude that for prescribed bounded support of thermomechanical loading and for a finite time, the disturbance generated by the pair of stress and heat flux vanishes outside a bounded domain. It is also analyzed that the domain of influence relies on the thermoelastic material parameters. We further compare the present domain of influence results with the corresponding results of LS thermoelasticity theory.

1 Introduction

It is well understood that the uncoupled thermoelasticity theory suffers from a deficiency that the temperature and elastic changes are not affected by each other. In order to address this shortcoming, an elegant model of coupled thermoelasticity has been formulated by Biot [1]. This theory has eliminated the apparent deficiency of the uncoupled theory and thereby admitted the coupling effects of elasticity and thermal field. However, Biot's theory is developed depending on the classical Fourier's law which leads to the parabolic nature of heat conduction equation. According to this, the resulting heat waves propagate at an infinite speed which violates the physical phenomena. Overcoming this apparent paradox in the classical coupled thermoelasticity theory has been a challenging and interesting area of research since the last few decades. Several efforts have therefore been carried out to develop various non-Fourier heat conduction models. Consequently, various thermoelasticity theories by the use of these non-Fourier heat conduction models have been introduced. We must recall some achievements in this direction as described in review articles and some books [2–11].

It is worth to be mentioned that Lord and Shulman [12] established the first generalized thermoelasticity theory, and this extended theory is known as the LS thermoelasticity theory. The LS theory is based on the modified heat conduction equation reported by Cattaneo and Vernotte [13–15] in which a new parameter defining thermal relaxation time has been incorporated in the classical heat conduction equation. The modified heat conduction law is defined in the Catteneo-Vernotte model as follows:

$$\overrightarrow{q} + \tau \frac{\partial}{\partial t} \overrightarrow{q} = -K \overrightarrow{\nabla} \theta \tag{1}$$

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where τ acts as a nonnegative time relaxation parameter, K represents the thermal conductivity, $\overrightarrow{q}$ represents the heat flux vector, and θ represents the temperature.

Further, the other generalized thermoelasticity theory without affecting the classical Fourier's law has been proposed by Green and Lindsay [16]. This model has been formulated with the idea of introducing the temperature-rate terms among the constitutive relations. Later on, another generalized thermoelasticity theory was established by Green and Naghdi [17–19]. This theory has given rise to three types of thermoelastic models which are now referred to as GN-I, GN-II and GN-III models, respectively. In view of the GN-III model, the modified constitutive law of heat conduction is considered in the following way:

$$\overrightarrow{q} = -K\overrightarrow{\nabla}\theta - K^*\overrightarrow{\nabla}\alpha \tag{2}$$

where K^* is introduced as a new material parameter and is termed as the conductivity rate of the material. Here, α represents a new constitutive variable called thermal displacement with the property that $\dot{\alpha} = \theta$ (see Refs. [17–19]). In a condition by taking $K^* = 0$ in Eq. (2), we obtain the GN-I model and recover the classical Fourier law of heat conduction. Moreover, if we consider $K = 0$, then we obtain the GN-II model with no thermal energy dissipation. Consequently, the GN-I and GN-II models can be easily retrieved from the GN-III model as special cases. However, out of these three models, the GN-II model overcomes the apparent drawback of the infinite propagation of heat waves in thermoelasticity.

The GN-III model asserts that this model leads to a similar deficiency as the classical Fourier's theory. To overcome this deficiency, a modification of Eq. (2) by incorporating a relaxation parameter has been proposed. Hence, the modification of Eq. (2) follows as

$$\left(1 + \tau_q \frac{\partial}{\partial t}\right)\overrightarrow{q} = -K\overrightarrow{\nabla}\theta - K^*\overrightarrow{\nabla}\alpha \tag{3}$$

where τ_q is the parameter of time relaxation. Therefore, Eq. (3) is the generalization of heat conduction equations defined in the LS theory and GN-III theory. Subsequently, by adjoining the energy equation and Eq. (3), we obtain a completely new equation which is referred to as the Moore–Gibson–Thompson (MGT) equation [20]. The MGT equation has dragged the interest of the researchers and prompted them to work in this direction. Recently, another type of generalized thermoelasticity theory has been proposed by Quintanilla [20] where the MGT equation describes the heat conduction equation. Therefore, this generalized thermoelasticity theory is introduced as the MGT thermoelasticity theory. The MGT thermoelasticity theory is a generalized form of the LS theory as well as the GN-III theory. Consequently, the MGT theory is a new generalized thermoelasticity theory proposed in the context of thermoelasticity. The stability and the well posedness of the solutions in this theory are also analyzed by Quintanilla [20]. Subsequently, the thermoelasticity theory of MGT type with history dependence in the temperature has been established by Conti et al. [21] in which an integro-differential form of the MGT equation has been considered. Further, uniqueness and instability for some thermomechanical problems in the theory of MGT thermoelasticity have been discussed by Pellicer and Quintanilla [22] in detail.

One of the useful results to understand the deformation of a medium in terms of any thermomechanical disturbance is known as the domain of influence theorem. This theorem implies that, outside a bounded domain, a solution of a given system vanishes for a finite time and in accordance with a data specified in bounded support. Accordingly, this theorem proves the hyperbolicity of the present model. Ignaczak [23] introduced the domain of influence theorem in view of linear thermoelasticity. Further, Ignaczak [24] also discussed the domain of influence theorem under asymmetric elastodynamics. In linear elastodynamics, the theorem of domain of influence with energy inequalities was reported by Carbonaro and Russo [25]. Ignaczak [26] and Hetnarski and Ignaczak [27] presented the domain of influence theorems under the generalized thermoelasticity theories of LS type and also of GL type. Later on, the concepts of domain of influence were described by Ignaczak and Ostoja-Starzewski [28] in detail. Flavin et al. [29] investigated the energy bounds for the transient solutions of the equations in linear and nonlinear elastodynamics. Based on the idea proposed by Flavin et al. [29] in view of linear elastodynamics, the principle of Saint–Venant was obtained by Chirita and Quintanilla [30]. Under the generalized thermoelasticity theory, the spatial decay estimates in terms of Saint–Venant's principle have further been analyzed by Chirita and Quintanilla [31]. The domain of influence theorem had been established by Mukhopadhyay et al. [32] in the context of the DPL model. Further, Kumar and Kumar [33] investigated the domain of influence theorem under the TPL model. Kumari and Mukhopadhyay [34] obtained the domain of influence theorem under the GN-II model in regard to a problem based on stress-heat-flux. Later on, Kumari and Mukhopadhyay [35] also discussed the domain of influence theorem in the context of the GN-II model for the case of a pair of potential to the displacement and temperature.

The present work is motivated to extend the concept of domain of influence for the theory of MGT thermoelasticity. We describe a thermoelastic process corresponding to the natural stress-heat-flux problem in the present context and aim to prove the domain of influence theorem. We start with summarizing the fundamental equations in terms of stress and heat flux pair concerning an isotropic and homogeneous medium under the MGT model. Further, we present the mixed boundary-initial value problem involving stress and heat flux and derive an identity in the present context. Lastly, the domain of influence theorem regarding this identity has been established. According to this theorem, we conclude that the pair of stress with heat flux generates the stress–heat-flux disturbance vanishing outside the bounded domain for a finite time and for a prescribed thermomechanical load support which is bounded. We also find that the domain of influence depends on the thermoelastic coupling constant and other material parameters.

2 Basic equations and problem formulation

We consider that $\tilde{B}$ represents the closure of a connected, open, and bounded set. Let B denote the interior of $\tilde{B}$ and ∂B denote the boundary of $\tilde{B}$ enclosing an isotropic and homogeneous material. In three-dimensional Euclidean space, we consider a rectangular coordinate system, and let n_i denote the components of unit outward normal to ∂B. The basic governing equations and the constitutive relations in view of the MGT theory of thermoelasticity are considered in the following way:

Stress equation of motion:

$$\sigma_{ij,j} + l_i = \rho \ddot{u}_i; \tag{4}$$

Energy equation:

$$-q_{i,i} + r = C_S \dot{\theta} + \theta_0 \alpha \dot{\sigma}_{kk}; \tag{5}$$

Strain-stress-temperature relation:

$$e_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\sigma_{kk}\delta_{ij}\right) + \alpha\theta\delta_{ij}; \tag{6}$$

Heat conduction equation:

$$\left(1 + \tau_q \frac{\partial}{\partial t}\right)\dot{q}_i = -\left(K^* + K\frac{\partial}{\partial t}\right)\theta_{,i}; \tag{7}$$

Strain-displacement relation:

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = u_{(i,j)}. \tag{8}$$

In the above equations, σ_{ij} are the components of the stress tensor, u_i represents the components of the displacement vector, r denotes the external heat source, ρ is the mass density, l_i represents the components of the body force vector, q_i represents the heat flux vector components, C_S denotes the specific heat at constant stress, α represents the linear thermal expansion coefficient, θ represents the temperature variation from θ_0, where θ_0 is the uniform reference temperature, e_{ij} represents the components of the strain tensor, λ and μ are the constants of Lamé elastic, τ_q is the phase-lag parameter, K represents the thermal conductivity, and K^* represents the conductivity rate of the material as termed in the Green-Naghdi theory. The comma notation represents partial derivatives with respect to space variables, and for representing the differentiation with respect to time, we take overdots. The subscripts i, j, k take the values 1, 2, 3, and index repetition implies the summation.

Now, with the help of above Eqs. (4)–(8), we consider a problem on natural stress-heat-flux for an isotropic and homogeneous material under the MGT thermoelasticity theory involving a pair (σ_{ij}, q_i) that satisfies the field equations as follows:

$$\rho^{-1}\sigma_{(ik,kj)} - \left[\frac{1}{2\mu}\left(\ddot{\sigma}_{ij} - \frac{\lambda}{3\lambda + 2\mu}\ddot{\sigma}_{kk}\delta_{ij}\right) - \frac{\theta_0\alpha^2}{C_S}\ddot{\sigma}_{kk}\delta_{ij}\right] + \frac{\alpha}{C_S}\dot{q}_{k,k}\delta_{ij} = \frac{1}{C_S}\alpha\dot{r}\delta_{ij} - \rho^{-1}l_{(i,j)}; \tag{9}$$

$$\left(K^* + K\frac{\partial}{\partial t}\right)\frac{1}{C_S}\left(q_{k,k} + \alpha\theta_0\dot{\sigma}_{kk}\right)_{,i} - \left(1 + \tau_q\frac{\partial}{\partial t}\right)\ddot{q}_i = \left(K^* + K\frac{\partial}{\partial t}\right)\frac{1}{C_S}r_{,i} \tag{10}$$

where we use the notation $l_{(i,j)} = \frac{1}{2}\left(l_{i,j} + l_{j,i}\right)$.

Here, we must recall that Eq. (9) is a generalization of the Ignaczak equation (see Refs. [28,36]).
The material parameters satisfying the following conditions are assumed:

$$\rho > 0,\ \ \mu > 0,\ \ \lambda > 0,\ \ C_S > 0,\ \ \alpha > 0,\ \ \theta_0 > 0,$$
$$\tau_q > 0,\ \ K^* > 0,\ \ 3\lambda + 2\mu > 0. \tag{11}$$

Further, we consider the relation

$$K > K^*\tau_q. \tag{12}$$

We must recall here the fact reported by Quintanilla [20] that the solution under the MGT thermoelasticity theory is exponentially stable if the material parameters τ_q, K and K^* satisfy the relation (12) (see Quintanilla [20]).

Now, we set the following notations in Eqs. (9) and (10):

$$L_{(ij)} = \rho^{-1} l_{(i,j)} - \frac{1}{C_S}\alpha\dot{r}\delta_{ij}, \tag{13}$$

$$m_i = -\left(K^* + K\frac{\partial}{\partial t}\right)\frac{1}{C_S} r_{,i}. \tag{14}$$

Then, from Eqs. (9) and (10), we find

$$\rho^{-1}\sigma_{(ik,kj)} - \left[\frac{1}{2\mu}\left(\ddot{\sigma}_{ij} - \frac{\lambda}{3\lambda+2\mu}\ddot{\sigma}_{kk}\delta_{ij}\right) - \frac{\theta_0\alpha^2}{C_S}\ddot{\sigma}_{kk}\delta_{ij}\right] + \frac{\alpha}{C_S}\dot{q}_{k,k}\delta_{ij} = -L_{(ij)}, \tag{15}$$

$$\left(K^* + K\frac{\partial}{\partial t}\right)\frac{1}{C_S}\left(q_{k,k} + \alpha\theta_0\dot{\sigma}_{kk}\right)_{,i} - \left(1 + \tau_q\frac{\partial}{\partial t}\right)\ddot{q}_i = -m_i. \tag{16}$$

We assume the initial conditions on $x \in B$ to above Eqs. (15) and (16) in the following way:

$$\begin{aligned} \sigma_{ij}(x,0) &= \sigma^0_{ij}, \\ \dot{\sigma}_{ij}(x,0) &= \dot{\sigma}^0_{ij}, \\ q_i(x,0) &= q^0_i, \\ \dot{q}_i(x,0) &= \dot{q}^0_i, \\ \ddot{q}_i(x,0) &= \ddot{q}^0_i, \end{aligned} \tag{17}$$

and boundary conditions on $\partial B \times [0, \infty[$ are taken as:

$$\begin{aligned} \sigma_{ij}n_j &= \sigma'_i, \\ q_i n_i &= q'. \end{aligned} \tag{18}$$

The present problem which is based on the pair of stress and heat flux is known as a natural stress–heat-flux problem [28]. Therefore, a solution to the natural stress–heat-flux problem is called a stress–heat-flux disturbance.

3 Some definitions

Now, we discuss the concepts of the support of thermomechanical load and the set of the domain of influence before moving on to the main results.

Definition 1 We consider that $t \in (0, \infty)$ is a fixed time. Then, the set

$$D_0(t) = \left\{ x \in \tilde{B} : \begin{array}{c} (1)\ \text{for} x \in B,\ \ \sigma^0_{ij} \neq 0\ \text{or}\ \dot{\sigma}^0_{ij} \neq 0\ \text{or}\ q^0_i \neq 0\ \text{or}\ \dot{q}^0_i \neq 0\ \text{or}\ \ddot{q}^0_i \neq 0 \\ (2)\ \text{for}\ (x,\tau) \in \partial B \times [0,t],\ \ \sigma'_i \neq 0\ \text{or}\ q' \neq 0 \\ (3)\ \text{for}\ (x,\tau) \in \partial B \times [0,t],\ \ L_{(ij)} \neq 0\ \text{or}\ m_i \neq 0 \end{array} \right. \tag{19}$$

is said to be the support of the thermomechanical load of the present system (15)–(18) at time t.

Definition 2 We take an open ball $\Omega(x, \nu t)$ with center x and radius νt, where ν is any real parameter. For the above thermomechanical load $D_0(t)$, the set

$$D(t) = \left\{ x \in \tilde{B} : \overline{\Omega(x, \nu t)} \cap D_0(t) \neq \emptyset \right\} \tag{20}$$

which defines the domain of influence. Thus, the set $D(t)$ defines a set of all the points of $\tilde{B}$ which can be accessed by the thermomechanical disturbances propagating from $D_0(t)$ with a finite speed not exceeding ν (see [28]).

4 Main results

Now, we formulate an identity regarding the present context, and it is analogous to the identity formulated by Ignaczak and Ostoja-Starzewski [28]. Further, we will derive the domain of influence theorem based on this identity under the MGT thermoelasticity theory. The domain of influence theorem is valid if the inequality is satisfied by ν in the following way:

$$\nu \geq \max(\nu_1, \nu_2, \nu_3, \nu_4), \tag{21}$$

where

$$\nu_1 = \left(\frac{2\mu}{\rho}\right)^{\frac{1}{2}}, \tag{22}$$

$$\nu_2 = \left\{ \frac{K}{\tau_q C_S}\left[1 + \frac{C_S}{C_E}\left(1 - \frac{C_E}{C_S}\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}}, \tag{23}$$

$$\nu_3 = \left\{ \frac{K^*}{C_S}\left[1 + \frac{C_S}{C_E}\left(1 - \frac{C_E}{C_S}\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}}, \tag{24}$$

$$\nu_4 = \left\{ \frac{(3\lambda + 2\mu)}{\rho}\frac{C_S}{C_E}\left[1 - \left(1 - \frac{C_E}{C_S}\right)^{\frac{1}{2}}\right]^{-1}\right\}^{\frac{1}{2}}, \tag{25}$$

where C_E represents the specific heat at constant strain satisfying the following relation with C_S :

$$C_S = C_E + 3\alpha^2 (3\lambda + 2\mu)\,\theta_0. \tag{26}$$

Now, we derive the following energy identity in the context of the present problem in view of the above definitions.

Theorem 1 *Let* (σ_{ij}, q_i) *satisfy the mixed problem* (15)–(18) *with smoothness property, and* $e(x) \in C^1(\tilde{B})$ *is considered to be a scalar field in such a way that the set*

$$J_0 = \left\{ x \in \tilde{B} : e(x) > 0 \right\} \tag{27}$$

is bounded. Then,

$$\begin{aligned} &\frac{1}{2}\int_B \left\{ E_0(x, e(x)) - \left[e(x)\,\dot{E}_0(x, 0) + E_0(x, 0)\right]\right\} \mathrm{d}B + \frac{1}{2}\int_B \left\{ \int_0^{e(x)} E_1(x, t)\,\mathrm{d}t - e(x)\,E_1(x, 0)\right\} \mathrm{d}B \\ &\quad + \int_B \left\{ \int_0^{e(x)} [e(x) - t]\,F(x, t)\,\mathrm{d}t \right\} \mathrm{d}B + \int_B \left\{ \int_0^{e(x)} G_i(x, t)\,e_{,i}(x)\,\mathrm{d}t \right\} \mathrm{d}B \\ &\quad = \int_{\partial B} \left\{ \int_0^{e(x)} [e(x) - t]\,G_i(x, t)\,n_i(x)\,\mathrm{d}t \right\} \mathrm{d}A + \int_B \left\{ \int_0^{e(x)} [e(x) - t]\,H(x, t)\,\mathrm{d}t \right\} \mathrm{d}B, \end{aligned} \tag{28}$$

where

$$E_0(x,t) = \frac{K^*\tau_q}{\theta_0}(\dot{q}_i)^2, \tag{29}$$

$$E_1(x,t) = \rho^{-1}\hat{\sigma}_{ik,k}\hat{\sigma}_{ij,j} + \frac{1}{2\mu}\left(\dot{\hat{\sigma}}_{ij}\dot{\hat{\sigma}}_{ij} - \frac{\lambda}{3\lambda+2\mu}\left(\dot{\hat{\sigma}}_{kk}\right)^2\right) - \frac{\alpha^2\theta_0}{C_S}\left(\dot{\hat{\sigma}}_{kk}\right)^2 + \frac{1}{C_S\theta_0}\left(\hat{q}_{k,k}\right)^2 + \frac{K^*}{\theta_0}(\dot{q}_i)^2 + \frac{K\tau_q}{\theta_0}(\ddot{q}_i)^2, \tag{30}$$

$$F(x,t) = \frac{\left(K - K^*\tau_q\right)}{\theta_0}(\ddot{q}_i)^2, \tag{31}$$

$$G_i(x,t) = \rho^{-1}\dot{\hat{\sigma}}_{ij}\hat{\sigma}_{jk,k} + \frac{1}{C_S}\left(\alpha\dot{\hat{\sigma}}_{kk} + \frac{1}{\theta_0}\hat{q}_{k,k}\right)\left(K^* + K\frac{\partial}{\partial t}\right)\dot{q}_i, \tag{32}$$

$$H(x,t) = L_{(ij)}\dot{\hat{\sigma}}_{ij} + \frac{1}{\theta_0}m_i\dot{\hat{q}}_i, \tag{33}$$

where for any function $g = g(x,t)$ defined on $x \in \tilde{B} \times [0, \infty[$, $\hat{g}()$ is denoted in the following way:

$$\hat{g} = \left(K^* + K\frac{\partial}{\partial t}\right)g. \tag{34}$$

Proof First, we apply the cap operator as represented by Eq. (34) on Eq. (15) and multiply by $\dot{\hat{\sigma}}_{ij}$ both sides of Eq. (15), then we obtain

$$\rho^{-1}\hat{\sigma}_{(ik,kj)}\dot{\hat{\sigma}}_{ij} - \left[\frac{1}{2\mu}\left(\ddot{\hat{\sigma}}_{ij}\dot{\hat{\sigma}}_{ij} - \frac{\lambda}{3\lambda+2\mu}\ddot{\hat{\sigma}}_{kk}\dot{\hat{\sigma}}_{kk}\right) - \frac{\theta_0\alpha^2}{C_S}\ddot{\hat{\sigma}}_{kk}\dot{\hat{\sigma}}_{kk}\right] + \frac{\alpha}{C_S}\dot{\hat{q}}_{k,k}\dot{\hat{\sigma}}_{kk} = -L_{(ij)}\dot{\hat{\sigma}}_{ij}. \tag{35}$$

Now, we multiply both sides of Eq. (16) by $\theta_0^{-1}\dot{\hat{q}}_i$ and get

$$\left\{\left(K^* + K\frac{\partial}{\partial t}\right)\frac{1}{C_S}\left(q_{k,k} + \alpha\theta_0\dot{\sigma}_{kk}\right)_{,i}\right\}\theta_0^{-1}\dot{\hat{q}}_i - \left\{\left(1 + \tau_q\frac{\partial}{\partial t}\right)\ddot{q}_i\right\}\theta_0^{-1}\dot{\hat{q}}_i = -m_i\theta_0^{-1}\dot{\hat{q}}_i. \tag{36}$$

Adding Eqs. (35) and (36) and after some straight-forward manipulations, the following equation is obtained:

$$\frac{1}{2}\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial t}E_0(x,t) + E_1(x,t)\right\} + F(x,t) = G_{i,i}(x,t) + H(x,t), \tag{37}$$

where E_0, E_1, F, G_i, and H are given by Eqs. (29)–(33), respectively.

The following relation holds:

$$\int_0^{e(x)} G_{i,i}[e(x) - t]\,\mathrm{d}t = \int_0^{e(x)}\left\{[G_i(x,t)[e(x) - t]]_{,i} - G_i(x,t)\,e_{,i}(x)\right\}\mathrm{d}t = \left[\int_0^{e(x)} G_i(x,t)[e(x) - t]\,\mathrm{d}t\right]_{,i} - \int_0^{e(x)} G_i(x,t)\,e_{,i}(x)\,\mathrm{d}t. \tag{38}$$

Therefore, taking double integration of Eq. (37) from $t = 0$ to $t = e(x)$ over t and making use of Eq. (38), we get

$$\frac{1}{2}\left[E_0(x, e(x)) - E_0(x,0) - e(x)\,\dot{E}_0(x,0)\right] + \frac{1}{2}\left[\int_0^{e(x)} E_1(x,t)\,dt - e(x)\,E_1(x,0)\right] + \int_0^{e(x)}[e(x) - t]\,F(x,t)\,dt + \int_0^{e(x)} G_i(x,t)e_{,i}(x)\,\mathrm{d}t = \left[\int_0^{e(x)} G_i(x,t)[e(x) - t]\,dt\right]_{,i} + \int_0^{e(x)}[e(x) - t]\,H(x,t)\,\mathrm{d}t. \tag{39}$$

The set J_0 is bounded from Eq. (27). So, each term in Eq. (39) is considered to be bounded. Therefore, taking the integration of Eq. (39) over B and employing the divergence theorem in the RHS, equation (28) is obtained. This proves Theorem 1.

Now, we will derive the following domain of influence theorem for a natural stress–heat-flux problem under the MGT thermoelasticity theory.

Theorem 2 *Let ν denote a real parameter satisfying the inequality defined by Eq. (21). Then, if the domain of influence is defined by the set $D(t)$ at time t and for the thermomechanical load $D_0(t)$ and if the pair (σ_{ij}, q_i) satisfies the problem (15)–(18) with smoothness property, then*

$$\sigma_{ij} = 0, \quad q_i = 0 \quad on \ \left\{\tilde{B} - D(t)\right\} \times [0, t]. \tag{40}$$

Proof We consider a fixed point $(w, \tau) \in \{B - D(t)\} \times (0, t)$. Let

$$\Lambda = \tilde{B} \cap \overline{\Omega(w, \nu\tau)}, \tag{41}$$

and we consider the following:

$$e_\tau(x) = \begin{cases} \tau - \frac{1}{\nu}|x - w| & \text{for } x \in \Lambda \\ 0 & \text{for } x \notin \Lambda \end{cases}, \tag{42}$$

where ν is a parameter given by Eq. (21).

Now, from the definitions of domain $D(t)$ and Λ defined by Eqs. (20) and (41), respectively, and from the inequality $\tau < t$, we find

$$D_0(t) \cap \Lambda = \emptyset. \tag{43}$$

Therefore, we obtain

$$\sigma_{ij}n_j = 0, \quad q_i n_i = 0 \quad \text{on} \ (\Lambda \cap \partial B) \times [0, t] \tag{44}$$

and

$$\dot{\hat{\sigma}}_{ij}n_j = 0, \quad \dot{q}_i n_i = 0, \quad \ddot{q}_i n_i = 0 \quad \text{on} \ (\Lambda \cap \partial B) \times [0, t], \tag{45}$$

$$L_{(ij)} = 0, \quad m_i = 0 \quad \text{on} \ \Lambda \times (0, t). \tag{46}$$

Further, we get

$$\sigma_{ij}(x, 0) = \dot{\sigma}_{ij}(x, 0) = q_i(x, 0) = \dot{q}_i(x, 0) = \ddot{q}_i(x, 0) \quad \text{on} \ \Lambda. \tag{47}$$

Now, in view of Eqs. (32), (42), and (45), we obtain

$$\int_{\partial B} \left\{ \int_0^{e_\tau(x)} [e_\tau(x) - t]\, G_i(x, t)\, n_i(x)\, \mathrm{d}t \right\} \mathrm{d}A = 0. \tag{48}$$

Similarly, in view of Eqs. (33), (42), and (46), we obtain

$$\int_B \left\{ \int_0^{e_\tau(x)} [e_\tau(x) - t]\, H(x, t)\, \mathrm{d}t \right\} \mathrm{d}B = 0. \tag{49}$$

Clearly, applying the definitions of $E_0(x, t)$, $E_1(x, t)$, and $e_\tau(x)$, we find the following results after some straight-forward manipulations:

$$E_0(x, e_\tau(x)) - E_0(x, 0) - e_\tau(x)\, \dot{E}_0(x, 0) = \begin{cases} E_0(x, e_\tau(x)) & \text{for } x \in \Lambda \\ 0 & \text{for } x \notin \Lambda \end{cases} \tag{50}$$

and

$$\int_0^{e_\tau(x)} E_1(x, t)\, \mathrm{d}t - e_\tau(x)\, E_1(x, 0) = \begin{cases} \int_0^{e_\tau(x)} E_1(x, t)\, \mathrm{d}t & \text{for } x \in \Lambda \\ 0 & \text{for } x \notin \Lambda \end{cases}. \tag{51}$$

Now, substituting $l_\tau(x)$ into Eq. (28) and making use of Eqs. (48)–(51), we obtain

$$\frac{1}{2}\int_\Lambda E_0(x, e_\tau(x))\,\mathrm{d}B + \frac{1}{2}\int_\Lambda\int_0^{e_\tau(x)} E_1(x,t)\,\mathrm{d}t\mathrm{d}B + \int_\Lambda\left\{\int_0^{e_\tau(x)}[e_\tau(x) - t]\,F(x,t)\,\mathrm{d}t\right\}\mathrm{d}B$$
$$= -\int_\Lambda\left\{\int_0^{e_\tau(x)} G_i(x,t)\,e_{\tau,i}(x)\,\mathrm{d}t\right\}\mathrm{d}B, \tag{52}$$

since Eq. (12) implies that $F \geq 0$ on Λ. Therefore, from Eqs. (42) and (52), we find the inequality as follows:

$$\frac{1}{2}\int_\Lambda E_0(x, e_\tau(x))\,\mathrm{d}B + \frac{1}{2}\int_\Lambda\int_0^{e_\tau(x)} E_1(x,t)\,\mathrm{d}t\mathrm{d}B \leq \frac{1}{\nu}\int_\Lambda\int_0^{e_\tau(x)} |G_i(x,t)\,dt|\,\mathrm{d}B. \tag{53}$$

Now,

$$\frac{1}{\nu}|G_i| \leq \rho^{-1}\left|\frac{\dot{\hat{\sigma}}_{ij}}{\nu}\hat{\sigma}_{jk,k}\right| + \frac{1}{C_S}\left|\alpha\dot{\hat{\sigma}}_{kk} + \frac{1}{\theta_0}\hat{q}_{k,k}\right|\left|\left(K^* + K\frac{\partial}{\partial t}\right)\frac{\dot{q}_i}{\nu}\right|$$
$$\leq \rho^{-1}\left|\frac{\dot{\hat{\sigma}}_{ij}}{\nu}\right|\left|\hat{\sigma}_{jk,k}\right| + \frac{|\alpha|}{C_S}\left|\dot{\hat{\sigma}}_{kk}\right|\left|\frac{K^*\dot{q}_i + K\ddot{q}_i}{\nu}\right| + \frac{1}{C_S\theta_0}\left|\hat{q}_{k,k}\right|\left|\frac{K^*\dot{q}_i + K\ddot{q}_i}{\nu}\right|. \tag{54}$$

To compute each term of the RHS of Eq. (54) and simplify Eq. (54), we use the following relation:

$$\sqrt{mn} \leq \frac{1}{2}\left(\epsilon m + \epsilon^{-1} n\right), \tag{55}$$

where ϵ denotes a positive parameter with no dimension and m and n are nonnegative physical fields with the equal dimension.

In order to compute the first term of Eq. (54), we employ $m = \left(\hat{\sigma}_{jk,k}\right)^2$, $n = \left(\frac{\dot{\hat{\sigma}}_{ij}}{\nu}\right)^2$, and $\epsilon = 1$ in Eq. (55) and then obtain

$$\left|\hat{\sigma}_{jk,k}\right|\left|\frac{\dot{\hat{\sigma}}_{ij}}{\nu}\right| \leq \frac{1}{2}\left(\hat{\sigma}_{ij,j}\hat{\sigma}_{ik,k} + \frac{1}{\nu^2}\dot{\hat{\sigma}}_{ij}\dot{\hat{\sigma}}_{ij}\right). \tag{56}$$

In order to compute the second term of Eq. (54), we use

$$m = \left(\dot{\hat{\sigma}}_{kk}\right)^2, \quad n = \frac{1}{\nu^2(\alpha\theta_0)^2}\left(K^*\dot{q}_i + K\ddot{q}_i\right)^2, \quad \epsilon = \frac{C_E}{C_S}\left(1 - \frac{C_E}{C_S}\right)^{-\frac{1}{2}}. \tag{57}$$

Therefore, using Eq. (57) in Eq. (55), we obtain

$$\left|\dot{\hat{\sigma}}_{kk}\right|\frac{1}{\nu|\alpha|\theta_0}\left|K^*\dot{q}_i + K\ddot{q}_i\right| \leq \frac{1}{2}\left\{\frac{C_E}{C_S}\left(1 - \frac{C_E}{C_S}\right)^{-\frac{1}{2}}\left(\dot{\hat{\sigma}}_{kk}\right)^2 + \frac{C_S}{C_E}\left(1 - \frac{C_E}{C_S}\right)^{\frac{1}{2}}\frac{1}{\nu^2\alpha^2\theta_0^2}\left(K^*\dot{q}_i + K\ddot{q}_i\right)^2\right\}$$
$$\leq \frac{1}{2}\left\{\frac{C_E}{C_S}\left(1 - \frac{C_E}{C_S}\right)^{-\frac{1}{2}}\left(\dot{\hat{\sigma}}_{kk}\right)^2 + \frac{C_S}{C_E}\left(1 - \frac{C_E}{C_S}\right)^{\frac{1}{2}}\frac{1}{\nu^2\alpha^2\theta_0^2}\left[\left(K^*\dot{q}_i\right)^2 + \left(K\ddot{q}_i\right)^2\right]\right\}$$
$$+\frac{K^*K}{2\nu^2\alpha^2\theta_0^2}\frac{C_S}{C_E}\left(1 - \frac{C_E}{C_S}\right)^{\frac{1}{2}}\frac{d}{dt}\left(\dot{q}_i\right)^2. \tag{58}$$

Now, by fixing $m = \left(\hat{q}_{k,k}\right)^2$, $n = \left(\frac{K^*\dot{q}_i + K\ddot{q}_i}{\nu}\right)^2$, and $\epsilon = 1$ in Eq. (55), we get the last term of Eq. (54) as

$$\left|\hat{q}_{k,k}\right|\left|\frac{K^*\dot{q}_i + K\ddot{q}_i}{\nu}\right| \leq \frac{1}{2}\left\{\left(\hat{q}_{k,k}\right)^2 + \frac{1}{\nu^2}\left(K^*\dot{q}_i + K\ddot{q}_i\right)^2\right\}$$
$$\leq \frac{1}{2}\left\{\left(\hat{q}_{k,k}\right)^2 + \frac{1}{\nu^2}\left[\left(K^*\dot{q}_i\right)^2 + \left(K\ddot{q}_i\right)^2\right]\right\} + \frac{K^*K}{2\nu^2}\frac{d}{dt}\left(\dot{q}_i\right)^2. \tag{59}$$

Thus, Eqs. (54), (56), (58), and (59) yield

$$\begin{aligned}\frac{1}{\nu}\left|G_i\right| \le \frac{\rho^{-1}}{2}\left(\hat{\sigma}_{ij,j}\hat{\sigma}_{ik,k}+\frac{1}{\nu^2}\dot{\hat{\sigma}}_{ij}\dot{\hat{\sigma}}_{ij}\right)\\ +\frac{\alpha^2\theta_0}{2C_S}\left\{\frac{C_E}{C_S}\left(1-\frac{C_E}{C_S}\right)^{-\frac{1}{2}}\left(\dot{\hat{\sigma}}_{kk}\right)^2+\frac{C_S}{C_E}\left(1-\frac{C_E}{C_S}\right)^{\frac{1}{2}}\frac{1}{\nu^2\alpha^2\theta_0^2}\left[\left(K^*\dot{q}_i\right)^2+\left(K\ddot{q}_i\right)^2\right]\right\}\\ +\frac{1}{2C_S\theta_0}\left\{\left(\hat{q}_{k,k}\right)^2+\frac{1}{\nu^2}\left[\left(K^*\dot{q}_i\right)^2+\left(K\ddot{q}_i\right)^2\right]\right\}\\ +\frac{K^*K}{2\nu^2C_S\theta_0}\left\{1+\frac{C_S}{C_E}\left(1-\frac{C_E}{C_S}\right)^{\frac{1}{2}}\right\}\frac{d}{dt}\left(\dot{q}_i\right)^2 .\end{aligned} \tag{60}$$

Further, the relation (26) implies that

$$\frac{\alpha^2\theta_0}{C_S}=\frac{1}{3\left(3\lambda+2\mu\right)}\left(1-\frac{C_E}{C_S}\right). \tag{61}$$

Therefore, in view of the definitions of E_0 and E_1 given by (29),(30), respectively, and using Eqs. (53),(60) and relation (61), we arrive at

$$\begin{aligned}\left(\frac{1}{2\mu}-\frac{1}{\rho\nu^2}\right)\int_\Lambda\int_0^{e_\tau(x)}\left(\dot{\hat{\sigma}}_{ij}-\frac{1}{3}\dot{\hat{\sigma}}_{kk}\delta_{ij}\right)\left(\dot{\hat{\sigma}}_{ij}-\frac{1}{3}\dot{\hat{\sigma}}_{kk}\delta_{ij}\right)\mathrm{d}t\,\mathrm{d}B\\ +\frac{K^*}{\theta_0}\left[1-\frac{K^*}{C_S\nu^2}\left\{1+\frac{C_S}{C_E}\left(1-\frac{C_E}{C_S}\right)^{\frac{1}{2}}\right\}\right]\int_\Lambda\int_0^{e_\tau(x)}(\dot{q}_i)^2\mathrm{d}t\,\mathrm{d}B\\ +\frac{K}{\theta_0}\left[\tau_q-\frac{K}{C_S\nu^2}\left\{1+\frac{C_S}{C_E}\left(1-\frac{C_E}{C_S}\right)^{\frac{1}{2}}\right\}\right]\int_\Lambda\int_0^{e_\tau(x)}(\ddot{q}_i)^2\mathrm{d}t\,\mathrm{d}B\\ +\frac{1}{3}\left[\frac{1}{3\lambda+2\mu}\frac{C_E}{C_S}\left\{1-\left(1-\frac{C_E}{C_S}\right)^{\frac{1}{2}}\right\}-\frac{1}{\rho\nu^2}\right]\int_\Lambda\int_0^{e_\tau(x)}\left(\dot{\hat{\sigma}}_{kk}\right)^2\mathrm{d}t\,\mathrm{d}B\\ +\frac{K^*}{\theta_0}\left[\tau_q-\frac{K}{C_S\nu^2}\left\{1+\frac{C_S}{C_E}\left(1-\frac{C_E}{C_S}\right)^{\frac{1}{2}}\right\}\right]\int_\Lambda(\dot{q}_i)^2\mathrm{d}B\le 0.\end{aligned} \tag{62}$$

From Eq. (21), we conclude that the coefficients of all the integrals in Eq. (62) are nonnegative. Thus, the nonnegativity of all integrals with the equality sign in this equation implies the vanishing of each term of Eq. (62) on Λ.

Particularly, we have

$$\dot{\hat{\sigma}}_{ij}\left(x,e_\tau(x)\right)=0,\quad \dot{q}_i\left(x,e_\tau(x)\right)=0\quad \text{on } \Lambda. \tag{63}$$

Since $\left(\sigma_{ij},q_i\right)$ is sufficiently smooth, and from the definition of $e_\tau\left(x\right)$, we get

$$\left.\begin{aligned}\dot{\hat{\sigma}}_{ij}\left(x,e_\tau(x)\right)\to\dot{\hat{\sigma}}_{ij}\left(w,\tau\right)\\ \dot{q}_i\left(x,e_\tau(x)\right)\to\dot{q}_i\left(w,\tau\right)\end{aligned}\right\}\ \text{as}\, x\to w. \tag{64}$$

Consequently, we take the limit $x\to w$ in Eq. (63) and find the following from Eq. (42):

$$\dot{\hat{\sigma}}_{ij}\left(w,\tau\right)=0,\quad \dot{q}_i\left(w,\tau\right)=0\quad \text{on } \left\{\tilde{B}-D(t)\right\}\times[0,t]. \tag{65}$$

In view of an arbitrary point (w,τ) of $\left\{\tilde{B}-D(t)\right\}\times(0,t)$ and from the smoothness property of $\left(\sigma_{ij},q_i\right)$ in $\tilde{B}\times[0,\infty)$, we find that

$$\dot{\hat{\sigma}}_{ij}=0,\quad \dot{q}_i=0\quad \text{on } \left\{\tilde{B}-D(t)\right\}\times[0,t]. \tag{66}$$

Now, with regard to $(x, \tau) \in \left\{\tilde{B} - D(t)\right\} \times [0, t]$, Eq. (66) implies the following results:

$$\sigma_{ij}(x, \tau) = \sigma_{ij}(x, 0) + \left\{1 - e^{-\frac{K^*}{K}\tau}\right\} \frac{K}{K^*} \dot{\sigma}_{ij}(x, 0) \tag{67}$$

and

$$q_i(x, \tau) = q_i(x, 0) \tag{68}$$

Since the definition of $D(t)$ yields

$$\sigma_{ij}(x, 0) = \dot{\sigma}_{ij}(x, 0) = q_i(x, 0) = 0 \quad \text{on} \quad \left\{\tilde{B} - D(t)\right\}, \tag{69}$$

hence, by combining Eqs. (67) and (68) with Eq. (69), we finally obtain

$$\sigma_{ij} = 0, \quad q_i = 0 \quad \text{on} \quad \left\{\tilde{B} - D(t)\right\} \times [0, t]. \tag{70}$$

Thus, this proves the Theorem 2.

This theorem indicates that the pair (σ_{ij}, q_i) satisfying the system (15)–(18) under the MGT model generates the stress–heat-flux disturbance vanishing outside the bounded set $D(t)$ for a prescribed bounded support of thermomechanical load and for a finite time t if the condition (12) holds. Furthermore, this theorem implies that if the relation given by Eq. (12) is considered, then we find that the stress–heat-flux disturbance propagates with finite speed not exceeding ν defined by Eq. (21). Clearly, ν is observed to be dependent on the phase lag τ_q, K, K^* and some other thermoelastic parameters. We also conclude that for a given load the associated domain of influence is specified with a boundary layer of νt thickness. We can also find the upper bound of the speed of stress–heat-flux disturbances from this theorem. We must mention that the condition given by (12) is considered here in view of the fact as analyzed by Quintanilla [20] that if the condition $K^*\tau_q < K$ holds, then the solution under the MGT theory is exponentially stable, otherwise leads to instability solution (see Ref. [20]). Therefore, we can conclude that the inequality $\nu \geq \max\{\nu_1, \nu_2, \nu_3, \nu_4\}$ reduces to $\nu \geq \max\{\nu_1, \nu_2, \nu_4\}$ which is interestingly the same inequality as obtained by Ignaczak and Ostoja-Starzewski [28] for the case of the generalized thermoelasticity theory of Lord and Shulman. Hence, we can conclude that the hyperbolicity of the MGT thermoelastic model is established if condition (12) is satisfied and the maximum speed of propagation of disturbance of stress-heat-flux under the MGT model depends on material parameters in a similar way like in the Lord-Shulman theory.

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