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ORIGINAL PAPER

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Generalized Noether's theorem in classical field theory with variable mass

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Abstract In this paper, the generalized Noether's theorem for mechanical systems is extended to the classical fields with variable mass, i.e., to the corresponding continuous systems. Noether's theorem is based on the modified Lagrangian, which, besides time derivatives of the field function, contains its partial derivatives with respect to the space coordinates. The generalized Noether's theorem for the classical fields systems with variable mass enables us to find transformations of field functions and independent variables for which there are some integrals of motion. In the paper, Noether's theorem is adopted for non-conservative fields, and energy integrals in a broader sense are determined. In the case of non-conservative fields, a complementary approach to the problem is introduced by applying so-called pseudo-conservative fields. It has been demonstrated that the pseudo-conservative systems have the same energy laws as the non-conservative fields where the laws are obtained by means of this generalized Noether's theorem. As the special case, the natural classical fields with standard Lagrangian are considered.

1 Introduction

As is well known, Noether [1] gave a general algorithm for finding a complete set of invariants for any physical system formulated in terms of the Lagrangian and the Hamiltonian formalism. Noether's theorem has been adjusted to the field theory [2] and to analytical mechanics [3]. Later, a large number of studies have been dedicated to the generalization of Noether's theorem (see [4]). In addition, numerous applications and a formulation in modern mathematical language [5] concerning Noether's theorem have been developed.

A special generalization of Noether's theorem was done by Vujanovic and Djukic [6–8] in an indirect way by applying the transformed d'Alembert–Lagrange principle. The aim of the so-formulated Noether's theorem is to find such transformations of generalized coordinates and time for the system with Lagrangian and possibly non-potential forces, for which some integrals (or constants) of motion, including the energy integrals, exist.

Noether's theorem is applicable to non-conservative fields, and the integrals of motion and energy integrals in a broader sense are obtained, representing in certain cases the generalization of the usual energy conservation laws. The generalized Noether's theorem was also extended to systems with variable mass [9,10].

In this paper, the generalized Noether's theorem, developed for mechanical continuous systems [11–13], is further extended to classical fields with variable mass. It is proved that the suggested procedure is appropriate for consideration of non-conservative fields. Using the suggested procedure, constants and energy integrals of

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the non-conservative fields are obtained. In the paper, the complementary approach, given in [14], is adopted for obtaining energy functions in non-conservative fields. Two types of pseudo-conservative systems are analyzed, and for both fields, the energy impact integrals are determined. It is shown that there is an analogy between the pseudo-conservative fields and the non-conservative systems considered in analytical mechanics. As a special case, using the procedure which is convenient for pseudo-conservative fields, the natural classical fields are studied. The paper concludes with an example.

2 Lagrangian equations for mechanical continuous systems with variable mass

2.1 Formulation of problem

Let us consider a physical classical field determined by a set of field functions $\eta^i(t, x, y, z)$ $(i = 1, 2, \ldots, n)$. Suppose that this field can be described by a Lagrangian in the form

$$L = \int_V \mathcal{L}\left(\eta^i, \eta^i_\alpha, x^\alpha\right) \mathrm{d}V, \tag{2.1}$$

where $\mathcal{L}$ is the Lagrangian density and

$$x^\alpha = \left\{x^0 = t, x^1 = x, x^2 = y, x^3 = z\right\}, \quad \eta^i_\alpha = \frac{\partial \eta^i}{\partial x^\alpha}. \tag{2.2}$$

The field defined with (2.1) is usually named 'classical field with Lagrangian.' This field is a continuous system which represents a generalization of the usual, natural continuous system.

For further consideration, let us introduce the relationship between the mechanics of particles and the classical field theory, i.e., the corresponding continuous system. The position of the particle in the field is defined with space coordinates (x, y, z), which become the independent variables additional to time t. Therefore, instead of time t, which is the only independent variable in the mechanics of particles, in the field theory we have four independent variables $x^\alpha = x^\alpha(t, x, y, z)$. As a consequence of this, the Lagrangian obtains the form $\int \mathcal{L}\mathrm{d}V$, and instead of any quantity f concerning this system (e.g., Q^* and P_i), in the field theory we have integral of the similar form $\int \tilde{f}\mathrm{d}V$. In addition, instead of the mass of the νth particle m_ν in the mechanics of particle, in the field theory we concern the mass dm of the elementary volume $\mathrm{d}V$, i.e., $\mathrm{d}m = \rho \mathrm{d}V$. Instead of the generalized coordinates $q^i(t)$ in mechanics of particles, in the field theory we have the field functions $\eta^i(t, x, y, z)$, since their set $\left\{\eta^i\right\}$ similar to the set $\left\{q^i\right\}$ determines the considered system. Thus, instead of the time derivatives $\mathrm{d}q^i/\mathrm{d}t$ in the mechanics of particles, in the field theory we have the partial derivatives $\frac{\partial \eta^i}{\partial x^\alpha} = \eta^i_\alpha$. Therefore, between the mechanics of particles and the classical field theory there is the following relation:

$$t \to x^\alpha = x^\alpha(t, x, y, z), \quad L \to \int_V \mathcal{L}\mathrm{d}V, \quad f \to \int_V \tilde{f}\mathrm{d}V,$$
$$m_\nu \to \mathrm{d}m = \varrho \mathrm{d}V, \quad q^i(t) \to \eta^i(t, x, y, z), \quad \frac{\mathrm{d}q^i}{\mathrm{d}t} \to \frac{\partial \eta^i}{\partial x^\alpha} = \dot{\eta}^i_\alpha. \tag{2.3}$$

In the system with mass variation, besides the influence of any force which acts on the considered field, there is a simultaneous process of separation and annexation of certain mass. For that reason, elements of the considered system are varying in time but they are dependent on space coordinates, too. This process of separation and annexation is described by so-called Meshchersky's force [15] whose formulation is extended to classical fields.

2.2 General Lagrangian equation

In order to formulate equations of motion in the Lagrangian form, let us consider the mechanics of particles with variable mass (see [15])

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = Q^*_i + P_i \qquad (i = 1, 2, \ldots, n), \tag{2.4}$$

where Q_i^* is the non-potential part of the generalized force, while P_i specifies the influence of mass variation

$$P_i = \frac{\mathrm{d}m_{\nu\sigma}}{\mathrm{d}t} u_{\nu\sigma} \frac{\partial r}{\partial q^i}, \tag{2.5}$$

in which $\mathrm{d}m_{\nu 1}/\mathrm{d}t$ and $\mathrm{d}m_{\nu 2}/\mathrm{d}t$ represent change in the mass of the νth particle per time due to mass separation (for $\sigma = 1$) and annexation (for $\sigma = 2$) with corresponding velocities, $\mathbf{u}_{\nu 1}$ and $\mathbf{u}_{\nu 2}$. One can demonstrate that these Lagrangian equations can be obtained from Hamilton's principle in form [16]

$$\int_{t_0}^{t_1} \left\{ \delta L + \left(Q_i^* + P_i \right) \delta q^i \right\} \mathrm{d}t = 0. \tag{2.6}$$

Remark 1. In Eq. (2.6) instead of Q_i^*, which is the only non-potential generalized force, the sum of forces $(Q_i^* + P_i)$ acts. Namely, the term P_i takes into consideration the mass variation.
2. Expression (2.6) can be obtained using Hamilton's principle for classical fields.

Introducing the variables of the field and the relations [17]

$$L \to \int_V \mathcal{L} \mathrm{d}V, \quad Q_i^* \to \int_V \tilde{Q}_i^* \mathrm{d}V, \quad P_i \to \int_V \tilde{P}_i \mathrm{d}V \;\; \delta q^i \to \delta \eta^i \tag{2.7}$$

into (2.6) yields

$$\int_{t_0}^{t_1} \left\{ \delta \int_V \mathcal{L} \mathrm{d}V + \left(\int_V \tilde{Q}_i^* \mathrm{d}V + \int_V \tilde{P}_i \mathrm{d}V \right) \delta \eta^i \right\} \mathrm{d}t = 0,$$

i.e.,

$$\int_{t_0}^{t_1} \int_V \left\{ \delta \mathcal{L} + \left(Q_i^* + \tilde{P}_i \right) \delta \eta^i \right\} \mathrm{d}V \mathrm{d}t = 0. \tag{2.8}$$

Let us specify the mass variation term in the field theory. According to (2.5) and (2.3), the force $\mathrm{d}P_i$ is substituted with $\tilde{P}_i \, \mathrm{d}V$, while the mass $\frac{\mathrm{d}m_{\nu\sigma}}{\mathrm{d}t}$ and position $\partial r_\nu / \partial q^i$ with $\frac{\mathrm{d}\rho_\sigma \mathrm{d}V}{\mathrm{d}x^\alpha}$ and $\partial r_\nu / \partial \eta^i$, respectively. r is the position vector of the center of masses of the element of continuous system. Using the aforementioned transformation (2.5) gives

$$\tilde{P}_i = \frac{\mathrm{d}\varrho_i}{\mathrm{d}x^\alpha} u_\sigma \frac{\partial r}{\partial \eta^i}. \tag{2.9}$$

It has to be mentioned that this relation depends on time and space coordinates.

Using the variation of $\mathcal{L}$ (2.1), expression (2.8) is

$$\int_{t_1}^{t_0} \int_V \left\{ \frac{\partial \mathcal{L}}{\partial \eta^i} \delta \eta^i + \frac{\partial \mathcal{L}}{\partial \eta^i_\alpha} \delta \eta^i_\alpha + \left(Q_i^* + \tilde{P}_i \right) \delta \eta^i \right\} \mathrm{d}V \mathrm{d}t = 0.$$

For the mathematical transformation

$$\frac{\partial \mathcal{L}}{\partial \eta^i_\alpha} \delta \eta^i_\alpha = \frac{\partial \mathcal{L}}{\partial \eta^i_\alpha} \frac{\mathrm{d}}{\mathrm{d}x^\alpha} \left(\delta \eta^i \right) = \frac{\mathrm{d}}{\mathrm{d}x^\alpha} \left(\frac{\partial \mathcal{L}}{\partial \eta^i_\alpha} \delta \eta^i \right) - \delta \eta^i \frac{\mathrm{d}}{\mathrm{d}x^\alpha} \frac{\partial \mathcal{L}}{\partial \eta^i_\alpha},$$

we have

$$\int_{t_1}^{t_0} \int_V \left(\frac{\partial \mathcal{L}}{\partial \eta^i} + Q_i^* + \tilde{P}_i - \frac{\mathrm{d}}{\mathrm{d}x^\alpha} \frac{\partial \mathcal{L}}{\partial \eta^i_\alpha} \right) \delta \eta^i \mathrm{d}V \mathrm{d}t + \frac{\partial \mathcal{L}}{\partial \eta^i_\alpha} \delta \eta^i \bigg|_{\eta^i_0}^{\eta^i_1} = 0, \tag{2.10}$$

i.e.,

$$\int_{t_1}^{t_0} \int_V \left(\frac{\partial \mathcal{L}}{\partial \eta^i} + Q_i^* + \tilde{P}_i - \frac{\mathrm{d}}{\mathrm{d}x^\alpha} \frac{\partial \mathcal{L}}{\partial \eta^i_\alpha} \right) \delta \eta^i \mathrm{d}V \mathrm{d}t = 0. \tag{2.11}$$

For arbitrary values $\delta \eta^i$, the Lagrangian equations for classical fields with variable mass follow as

$$\frac{\mathrm{d}}{\mathrm{d}x^\alpha} \frac{\partial \mathcal{L}}{\partial \eta^i_\alpha} - \frac{\partial \mathcal{L}}{\partial \eta^i} = Q_i^* + \tilde{P}_i \quad i = 1, 2, \ldots, n. \tag{2.12}$$

2.3 Natural classical fields

The fields which represent the real ones in nature (e.g., the fluid field, the elastic body) are named the 'natural classical fields.' They are determined by a Lagrangian density function $\mathcal{L}$ which is the difference between the kinetic T and potential energy $\mathcal{U}$ of the system, i.e., $\mathcal{L} = T - \mathcal{U}$. In these fields, the partial derivatives of field functions with respect to the space coordinates do not appear and the Lagrangian has the form

$$\mathcal{L}\left(\eta^i, \eta^i_\alpha, x^\alpha\right) = \frac{1}{2} a_{ik}\dot{\eta}^i\dot{\eta}^k - \mathcal{U}\left(\eta^i, \eta^i_k, x^\alpha\right) \quad \left(\eta^i_k = \frac{\partial \eta^i}{\partial x^k}\right), \tag{2.13}$$

where a_{ik} are independent and constant values. However, the space coordinates may appear in the potential energy (e.g., in the oscillating elastic wire). In this case, the first term in the Lagrangian equations (2.12) is reduced only to the first part for $x^\alpha = t$ and the equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\eta}^i} - \frac{\partial \mathcal{L}}{\partial \eta^i} = Q^*_i + \tilde{P}_i \quad (i = 1, 2, \ldots, n), \tag{2.14}$$

which have the similar form as the corresponding equation in the mechanics of particles.

3 Total variation of action

The total variation of Hamilton action on which Noether's theorem is based is

$$\Delta W = \int_{t_1}^{t_0}\int_V \mathcal{L}\left(\tilde{\eta}^i, \tilde{\eta}^i_\alpha, \tilde{x}^\alpha\right)\mathrm{d}\tilde{V}\mathrm{d}\tilde{t} - \int_{t_1}^{t_0}\int_V \mathcal{L}\left(\eta^i, \eta^i_\alpha, x^\alpha\right)\mathrm{d}V\mathrm{d}t, \tag{3.1}$$

where

$$\tilde{\eta}^i = \eta^i + \Delta\eta^i, \quad \tilde{\eta}^i_\alpha = \eta^i_\alpha + \Delta\eta^i_\alpha, \quad \tilde{t} = t + \Delta t. \tag{3.2}$$

Let us denote the set of initial values at the instant t_0 and domain V_0 as $R_0 = \{t_0, V_0\}$ and the set of their final values by $R_1 = \{t_1, V_1\}$ and $d^4x = \mathrm{d}V\mathrm{d}t$. Introducing the notation into (3.1) and after some simplification, the approximate value of the total variation of Hamiltonian action is obtained as

$$\int_{R_0+\Delta R_0}^{R_1+\Delta R_1} \mathcal{L}d^4\tilde{x} \quad \approx \int_{R_0}^{R_1} \mathcal{L}d^4\tilde{x},$$

i.e., according to (3.1) and (3.2) it is

$$\Delta W \approx \int_{R_0}^{R_1} \mathcal{L}\left(\eta^i + \Delta\eta^i, \eta^i_\alpha + \Delta\eta^i_\alpha, x^\alpha + \Delta x^\alpha\right)d^4\tilde{x} - \int_{R_0}^{R_1} \mathcal{L}\left(\eta^i, \eta^i_\alpha, x^\alpha\right)d^4x. \tag{3.3}$$

For computational reasons, the simplification of relation (3.3) is introduced. Using the Taylor series expansion of the Lagrangian, substituting $d^4\tilde{x} \approx [1 + \frac{\mathrm{d}}{\mathrm{d}x^\alpha}(\Delta x^\alpha)]d^4x$ (see [12]) into (3.3) and neglecting the terms of the higher order than the first, we obtain

$$\Delta W \approx \int_{R_0}^{R_1}\left[\frac{\partial \mathcal{L}}{\partial \eta^i}\Delta\eta^i + \frac{\partial \mathcal{L}}{\partial \eta^i_\alpha}\Delta\eta^i_\alpha + \frac{\partial \mathcal{L}}{\partial x^\alpha}\Delta x^\alpha + \mathcal{L}\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(\Delta x^\alpha\right)\right]d^4x. \tag{3.4}$$

Using the relations $\Delta\eta^i = \delta\eta^i + \eta^i_\alpha\Delta x^\alpha$ and $\Delta\eta^i_\alpha = \delta\eta^i_\alpha + \eta^i_{\alpha\beta}\Delta x^\beta$ suggested by Musicki [13], we have

$$\Delta W = \int_{R_0}^{R_1}\left[\mathcal{L}\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(\Delta x^\alpha\right) + \left(\frac{\partial \mathcal{L}}{\partial \eta^i}\eta^i_\alpha + \frac{\partial \mathcal{L}}{\partial \eta^i_\beta}\eta^i_{\alpha\beta} + \frac{\partial \mathcal{L}}{\partial x^\alpha}\right)\Delta x^\alpha + \frac{\partial \mathcal{L}}{\partial \eta^i}\delta\eta^i + \frac{\partial \mathcal{L}}{\partial \eta^i_\alpha}\delta\eta^i_\alpha\right]d^4x.$$

If the system depends on the mass variable function, the derivative of Lagrangian has an additional term connected with mass variation [18]. Assuming that the mass variation depends on time and space coordinates, the general form of the Lagrangian is

$$\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}x^\alpha} = \frac{\partial \mathcal{L}}{\partial \eta^i}\eta^i_\alpha + \frac{\partial \mathcal{L}}{\partial \eta^i_\beta}\eta^i_{\alpha\beta} + \frac{\partial \mathcal{L}}{\partial x^\alpha} + \frac{\partial \mathcal{L}}{\partial m}\frac{\mathrm{d}m}{\mathrm{d}x^\alpha}. \tag{3.5}$$

It has been said that expression (3.5) is the key point of our investigation. Namely, using relation (3.5) which corresponds to the field with variable mass, expression (3.4) transforms into

$$\Delta W=\int_{R_0}^{R_1}\left[\mathcal{L}\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(\Delta x^\alpha\right)+\left(\frac{\partial\mathcal{L}}{\partial x^\alpha}-\frac{\partial\mathcal{L}}{\partial m}\frac{\mathrm{d}m}{\mathrm{d}x^\alpha}\right)\Delta x^\alpha+\frac{\partial\mathcal{L}}{\partial\eta^i}\delta\eta^i+\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\delta\eta^i_\alpha\right]d^4x.$$

Transforming the last term in the equation

$$\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\delta\eta^i_\alpha=\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(\delta\eta^i\right)=\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\delta\eta^i\right)-\delta\eta^i\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\right),$$

the previous relation becomes

$$\Delta W=\int_{R_0}^{R_1}\left\{\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(\mathcal{L}\Delta x^\alpha+\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\delta\eta^i\right)+\delta\eta^i\left(\frac{\partial\mathcal{L}}{\partial\eta^i}-\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\right)-\frac{\partial\mathcal{L}}{\partial m}\frac{\mathrm{d}m}{\mathrm{d}x^\alpha}\Delta x^\alpha\right\}d^4x. \tag{3.6}$$

Transformation of the expression in the first parenthesis

$$\mathcal{L}\Delta x^\alpha+\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\delta\eta^i=\mathcal{L}\Delta x^\alpha+\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\left(\Delta\eta^i-\eta^i_\beta\Delta x^\beta\right)=\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\Delta\eta^i+\left(\mathcal{L}\delta^\alpha_\beta-\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\eta^i_\beta\right)\Delta x^\beta$$

modifies relation (3.6) into

$$\begin{aligned}\Delta W=\int_{R_0}^{R_1}&\left\{\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left[\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\Delta\eta^i+\left(\mathcal{L}\delta^\alpha_\beta-\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\eta^i_\beta\right)\Delta x^\beta\right]+\left(\frac{\partial\mathcal{L}}{\partial\eta^i}-\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\right)\left(\Delta\eta^i-\eta^i_\alpha\Delta x^\alpha\right)\right.\\&\left.-\frac{\partial\mathcal{L}}{\partial m}\frac{\mathrm{d}m}{\mathrm{d}x^\alpha}\Delta x^\alpha\right\}d^4x,\end{aligned} \tag{3.7}$$

where δ^α_β is Kronecker's symbol.

Relation (3.7) represents the total variation of the considered continuous system with variable mass, expressed consequently by means of total variations $\Delta\eta^i$ and Δx^α. The relation is valid quite generally, independently from the nature of considered problem.

4 Generalized Emmy Noether's theorem

Theorem *The proper transformation of field functions $\Delta\eta^i$ and independent variables Δx^α maintains the Hamiltonian's action of change (3.7) up to so-called calibration term*

$$\Delta W=\int_{R_0}^{R_1}\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\Lambda^\alpha\left(\eta^i,\eta^i_\alpha,x^\alpha\right)d^4x, \tag{4.1}$$

where Λ^α ($\alpha=1,2,3$) is the calibration or gauge function. The gauge function can be an arbitrary function of variables η^i, η^i_α and x^α.

4.1 Proof of the generalized Noether's theorem

Let us suppose that for the considered field the Lagrangian density, non-conservative forces (if this field is non-conservative) and mass change law are given in advance. Let us now start from condition (4.1) where ΔW from expression (3.7) is substituted,

$$\begin{aligned}&\int_{R_0}^{R_1}\left\{\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left[\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\Delta\eta^i+\left(\mathcal{L}\delta^\alpha_\beta-\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\eta^i_\beta\right)\Delta x^\beta\right]+\left(\frac{\partial\mathcal{L}}{\partial\eta^i}-\frac{\partial}{\partial x^\alpha}\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\right)\left(\Delta\eta^i-\eta^i_\alpha\Delta x^\alpha\right)-\frac{\partial\mathcal{L}}{\partial m}\frac{\mathrm{d}m}{\mathrm{d}x^\alpha}\Delta x^\alpha\right\}d^4x\\&=\int_{R_0}^{R_1}\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\Lambda^\alpha\left(\eta^i,\eta^i_\alpha,x^\alpha\right)d^4x.\end{aligned} \tag{4.2}$$

If we substitute the expression from the Lagrangian equations (2.12) and write the relation in the form of one integral, we obtain

$$\int_{R_0}^{R_1}\left\{\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left[\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\Delta\eta^i+\left(\mathcal{L}\delta^\alpha_\beta-\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\eta^i_\beta\right)\Delta x^\beta-\Lambda^\alpha\right]-\left(\tilde{Q}^*_i+\tilde{P}_i\right)\left(\Delta\eta^i-\eta^i_\alpha\Delta x^\alpha\right)\right.$$
$$\left.-\frac{\partial\mathcal{L}}{\partial m}\frac{\mathrm{d}m}{\mathrm{d}x^\alpha}\Delta x^\alpha\right\}d^4x=0. \tag{4.3}$$

On the other hand, let us transform the total variation of motion in the following way:

$$\Delta W=\int_{R_0}^{R_1}\Delta\left(\mathcal{L}d^4x\right)=\int_{R_0}^{R_1}\left(\Delta\mathcal{L}d^4x+\mathcal{L}\Delta d^4x\right).$$

The expression Δd^4x actually represents a relationship between the initial and the final value of the d^4x in the process of calculation of the total variation of action (see Hill [2] and Musicki [12]):

$$\Delta d^4x=d^4\bar{x}-d^4x=\left[1+\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(\Delta x^\alpha\right)\right]d^4x-d^4x=\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(\Delta x^\alpha\right)d^4x.$$

The previous relation becomes

$$\Delta W=\int_{R_0}^{R_1}\left[\Delta\mathcal{L}+\mathcal{L}\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(\Delta x^\alpha\right)\right]d^4x. \tag{4.4}$$

If we insert the expression in the second condition (4.2), it can be written in the form

$$\int_{R_0}^{R_1}\left[\Delta\mathcal{L}+\mathcal{L}\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(\Delta x^\alpha\right)-\frac{\mathrm{d}\Lambda^\alpha}{\mathrm{d}x^\alpha}\right]d^4x=0. \tag{4.5}$$

Subtracting (4.5) from relation (4.3) and substituting the term with $\left(\tilde{Q}^*_i+\tilde{P}_i\right)$ and $\frac{\partial\mathcal{L}}{\partial m}$ into the second part, we have

$$\int_{R_0}^{R_1}\left\{\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left[\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\Delta\eta^i+\left(\mathcal{L}\delta^\alpha_\beta-\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\eta^i_\beta\right)\Delta x^\beta-\Lambda^\alpha\right]-\left[\Delta\mathcal{L}+\mathcal{L}\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(\Delta x^\alpha\right)+\left(\tilde{Q}^*_i+\tilde{P}_i\right)\left(\Delta\eta^i\right.\right.\right.$$
$$\left.\left.\left.-\eta^i_\alpha\Delta x^\alpha\right)-\frac{\partial\mathcal{L}}{\partial m}\frac{\mathrm{d}m}{\mathrm{d}x^\alpha}\Delta x^\alpha-\frac{\mathrm{d}\Lambda^\alpha}{\mathrm{d}x^\alpha}\right]\right\}d^4x=0. \tag{4.6}$$

Substituting $\Delta\eta^i_\alpha=\frac{\mathrm{d}(\Delta\eta^i)}{\mathrm{d}x^\alpha}-\eta^i_\beta\frac{\mathrm{d}(\Delta x^\beta)}{\mathrm{d}x^\alpha}$ (see [12]) into the term $\Delta\mathcal{L}(\eta^i,\eta^i_\alpha,x^\alpha)$ and bearing in mind that the mass of the continuous system is not varied, we obtain

$$\Delta\mathcal{L}=\frac{\partial\mathcal{L}}{\partial\eta^i}\Delta\eta^i+\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\left[\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(\Delta\eta^i\right)-\eta^i_\beta\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(\Delta x^\beta\right)\right]+\frac{\partial\mathcal{L}}{\partial x^\alpha}\Delta x^\alpha.$$

The previous relation by certain regrouping can be written in the form

$$\int_{R_0}^{R_1}\left\{\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left[\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\Delta\eta^i+\left(\mathcal{L}\delta^\alpha_\beta-\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\eta^i_\beta\right)\Delta x^\beta-\Lambda^\alpha\right]-\left[\frac{\partial\mathcal{L}}{\partial\eta^i}\Delta\eta^i+\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(\Delta\eta^i\right)+\frac{\partial\mathcal{L}}{\partial x^\alpha}\Delta x^\alpha\right.\right.$$
$$\left.\left.+\left(\mathcal{L}\delta^\alpha_\beta-\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\eta^i_\beta\right)\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(\Delta x^\beta\right)+\left(\tilde{Q}^*_i+\tilde{P}_i\right)\left(\Delta\eta^i-\eta^i_\alpha\Delta x^\alpha\right)-\frac{\partial\mathcal{L}}{\partial m}\frac{\mathrm{d}m}{\mathrm{d}x^\alpha}\Delta x^\alpha-\frac{\mathrm{d}\Lambda^\alpha}{\mathrm{d}x^\alpha}\right]\right\}d^4x=0. \tag{4.7}$$

The relation is the so-called energy–impulse tensor [19] described as

$$T^\alpha_\beta=\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\eta^i_\beta-\mathcal{L}\delta^\alpha_\beta, \tag{4.8}$$

where δ^{α}_{β} is the Kronecker symbol. This quantity represents an extension of the generalized energy in the analytical mechanics to the field theory, but it applies to generalized impulse as well. Namely, for $\alpha = \beta$ it is equivalent to the Hamiltonian density $T^0_{\nu} = \mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{\eta}^i}\dot{\eta}^i - \mathcal{L}$, i.e., to the energy density in a border sense, and for $\alpha \neq \beta$, it is $T^{\alpha}_{\beta} = \frac{\partial \mathcal{L}}{\partial \eta^i_{\alpha}}\eta^i_{\beta}$, i.e., a quantity proportional to the impulse density $p^{\alpha}_i = \frac{\partial \mathcal{L}}{\partial \eta^i_{\alpha}}$ in a broader sense, and only for $x^{\alpha} = I$, the real dynamic quantities are obtained.

Let us introduce the quantities $\Delta\eta^i$ and Δx^{α} in the form of r infinitesimal parameters ε^m (m =1,2,…,r),

$$\Delta\eta^i = \bar{\eta}^i\left(x^{\alpha} + \Delta x^{\alpha}\right) - \eta^i\left(x^{\alpha}\right) = \varepsilon^m \xi^i_m\left(\eta^i, \eta^i_{\alpha}, x^{\alpha}\right),$$
$$\Delta x^{\alpha} = \bar{x}^{\alpha} - x^{\alpha} = \varepsilon^m \xi^i_m\left(\eta^i, \eta^i_{\alpha}, x^{\alpha}\right). \tag{4.9}$$

Substituting $\Lambda^{\alpha} = \varepsilon^m \Lambda^{\alpha}_m$ into (4.7), the relation obtains the form

$$\int_{R_0}^{R_1} \varepsilon^m \left\{ \frac{\mathrm{d}}{\mathrm{d}x^{\alpha}}\left[\frac{\partial \mathcal{L}}{\partial \eta^i_{\alpha}}\xi^i_m - T^{\alpha}_{\xi}\xi^{\alpha\beta}_m - \Lambda^{\alpha}_m\right] - \left[\frac{\partial \mathcal{L}}{\partial \eta^i}\xi^i_m + \frac{\partial \mathcal{L}}{\partial \eta^i_{\alpha}}\frac{\mathrm{d}\xi^i_m}{\mathrm{d}x^{\alpha}} + \frac{\partial \mathcal{L}}{\partial x^{\alpha}}\xi^{0\alpha}_m - T^{\alpha}_{\beta}\frac{\mathrm{d}\xi^{\alpha\beta}_m}{\mathrm{d}x^{\alpha}} \right.\right.$$
$$\left.\left. + \left(\tilde{Q}^*_i + \tilde{P}_i\right)\left(\xi^i_m - \eta^i_{\alpha}\xi^{\alpha\alpha}_m\right) + \frac{\partial \mathcal{L}}{\partial m}\frac{\mathrm{d}m}{\mathrm{d}x^{\alpha}}\xi^{0\alpha}_m - \frac{\mathrm{d}\Lambda^{\alpha}_m}{\mathrm{d}x^{\alpha}}\right]\right\} d^4x = 0. \tag{4.10}$$

4.2 Conclusion and analysis of obtained results

If the following condition is satisfied:

$$\frac{\partial \mathcal{L}}{\partial \eta^i}\xi^i_m + \frac{\partial \mathcal{L}}{\partial \eta^i_{\alpha}}\frac{\mathrm{d}\xi^i_m}{\mathrm{d}x^{\alpha}} + \frac{\partial \mathcal{L}}{\partial x^{\alpha}}\xi^{0\alpha}_m - T^{\alpha}_{\beta}\frac{\mathrm{d}\xi^{0\beta}_m}{\mathrm{d}x^{\alpha}} + \left(\tilde{Q}^*_i + \tilde{P}_i\right)\left(\xi^i_m - \eta^i_{\alpha}\xi^{0\alpha}_m\right) + \frac{\partial \mathcal{L}}{\partial m}\frac{\mathrm{d}m}{\mathrm{d}x^{\alpha}}\xi^{0\alpha}_m - \frac{\mathrm{d}\Lambda^{\alpha}_m}{\mathrm{d}x^{\alpha}} = 0, \tag{4.11}$$

relation (4.10) reduces to

$$\int_{R_0}^{R_1} \varepsilon^m \left\{ \frac{\mathrm{d}}{\mathrm{d}x^{\alpha}}\left[\frac{\partial \mathcal{L}}{\partial \eta^i_{\alpha}}\xi^i_m - T^{\alpha}_{\xi}\xi^{0\beta}_m - \Lambda^{\alpha}_m\right]\right\} d^4x = 0.$$

Using the arbitrariness of the parameter ε^m and integrating the previous relation, the relations for the existence of the integrals of motion follow:

$$I = \frac{\mathrm{d}}{\mathrm{d}x^{\alpha}}\left[\frac{\partial \mathcal{L}}{\partial \eta^i_{\alpha}}\xi^i_m - T^{\alpha}_{\xi}\xi^{0\beta}_m - \Lambda^{\alpha}_m\right] = 0 \quad m = 1, 2, \ldots, r. \tag{4.12}$$

Accordingly, for each transformation of field functions η^i and independent variables x^{α} given in form (4.9), for which there exists at least one set of particular solutions $(\xi^i_m, \xi^{0\alpha}_m, \Lambda^{\alpha}_m)$ which satisfies condition (4.10), there are r independent integrals (or constants) of motion in form (4.12).

Introducing the function

$$\Theta^{\alpha}_m = \frac{\partial \mathcal{L}}{\partial \eta^i_{\alpha}}\xi^i_m - T^{\alpha}_{\xi}\xi^{0\beta}_m - \Lambda^{\alpha}_m, \tag{4.13}$$

the integrals of motion (4.11) have the following explicit form

$$\frac{\mathrm{d}\Theta^{\alpha}_m}{\mathrm{d}x^{\alpha}} = \frac{\mathrm{d}\Theta^0_m}{\mathrm{d}t} + \frac{\mathrm{d}\Theta^1_m}{\mathrm{d}x} + \frac{\mathrm{d}\Theta^2_m}{\mathrm{d}y} + \frac{\mathrm{d}\Theta^3_m}{\mathrm{d}z} = 0. \tag{4.14}$$

It has been noted that if (4.14) occurs, then the essential meaning of a conservation law disappears. In this case, one would have that all terms in Eq. (4.14) are zero. An expression which is not associated with the fundamental concept that a constant of motion is intimately related to a function which equals a constant (its derivative vanishes) particularly along the solution of the considered problem.

The integral of motion (4.14) can be satisfied if each of its members is equal to zero, i.e.,

$$\frac{\mathrm{d}\Theta_m^0}{\mathrm{d}t} = 0, \frac{\mathrm{d}\Theta_m^k}{\mathrm{d}x^k} = 0, \quad k = 1, 2, 3. \tag{4.15}$$

Relation (4.15) represents a special but very important case of the general solution (4.14). The first of these particular solutions represents an integral of motion of temporal type, and there are integrals of motion of space type. Namely, calculation of the integrals of motion is reduced to finding at least one set of particular solutions $(\xi_m^i, \xi_m^{0\alpha}, \Lambda_m^\alpha)$ for the condition of existence of the integrals of motion (4.10).

Therefore, the fundamental properties of these integrals of motion are similar to those of classical fields, i.e., corresponding continuous systems with permanent mass.

1. In the general case, when the Lagrangian has form (2.1), the integrals of motion are $\frac{\mathrm{d}\Theta_m^\alpha}{\mathrm{d}x^\alpha} = 0$, i.e., the sum of four partial derivatives of function Θ_m^α with respect to time and space coordinates is equal to zero. The difference between (4.14) and the relation for the field with permanent mass is that in (4.14) additional specific terms appear in the condition for existence of the integrals of motion, but the form of the integrals of motion remains unchanged.
2. The energy integral, as the special case of the integral of motion, represents the energy–impulse conservation law, as is the case for fields with permanent mass. The integral can be of temporal and space type. The necessary condition for existence of the energy integral of space type is the existence of the Lagrangian in form (2.1).
3. In the case of natural classical fields, the general form of the integrals of motion is $\frac{\mathrm{d}\Theta_m^0}{\mathrm{d}t} = 0$, i.e., it is reduced to the first term of the total partial derivative of some quantity with respect to time. The energy conservation laws are similar to those obtained for particles.

4.3 The case of natural classical fields

Using expression (4.8), the condition for existence of the integrals of motion (4.10) is simplified to

$$\frac{\partial \mathcal{L}}{\partial \eta^i}\xi_m^i + \frac{\partial \mathcal{L}}{\partial \dot{\eta}_\alpha^i}\dot{\xi}_m^i + \frac{\partial \mathcal{L}}{\partial t}\xi_m^{00} - \mathcal{H}\dot{\xi}_m^{00} + \left(\tilde{Q}_i^* + \tilde{P}_i\right)\left(\xi_m^i - \dot{\eta}^i\xi^{00}\right) + \frac{\partial \mathcal{L}}{\partial m}\frac{\mathrm{d}m}{\mathrm{d}x^\alpha}\xi_m^{0\alpha} - \dot{\Lambda}_m^\alpha = 0 \tag{4.16}$$

and the corresponding integral of motion is reduced only to the first term

$$I_m = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\eta}_\alpha^i}\xi_m^i - \mathcal{H}\xi_m^{00} - \Lambda_m^0\right) = 0 \quad m = 1, 2, \ldots, r. \tag{4.17}$$

Finally, in the case of the natural classical fields with variable mass, for each transformation of the field functions and independent variables which have form (4.1), there exists at least one set of particular solutions $(\xi_m^i, \xi_m^{0\alpha}, \Lambda_m^\alpha)$ which satisfy condition (4.16), and r independent integrals of motion (4.16).

5 Pseudo-conservative systems

5.1 Definitions and condition of pseudo-conservatism

Let us introduce a complementary approach to non-conservative fields by introducing pseudo-conservative fields. In the mechanics of particles and of continuous systems, pseudo-conservative systems are defined as non-conservative systems whose Lagrangian equations can be transformed into Euler–Lagrange equations by introduction of a new suitable Lagrangian [14].

Let us consider a classical field with variable mass, i.e., the corresponding continuous system whose Lagrangian equations according to (2.12) have the form

$$\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\frac{\partial \mathcal{L}}{\partial \dot{\eta}_\alpha^i} - \frac{\partial \mathcal{L}}{\partial \eta^i} = \tilde{Q}_i^* + \tilde{P}_i \quad i = 1, 2, \ldots, n. \tag{5.1}$$

In order to define pseudo-conservative fields, a suitable transformation has to be introduced into formal conservative fields. Two cases can be discerned, depending on whether in the transformed Lagrangian equations the term $\tilde{Q}_i^*$ or both $\tilde{Q}_i^*$ and $\tilde{P}_i$ are taken.

(a) In the first case, let us formulate the following problem: Find a Lagrangian density of the form

$$\tilde{\mathcal{L}}_1\left(\eta^i, \eta^i_\alpha, x^\alpha\right) = f_1(x^\alpha)\mathcal{L}\left(\eta^i, \eta^i_\alpha, x^\alpha\right), \tag{5.2}$$

which transforms the Lagrangian equations (5.1) into an equivalent system of simpler Lagrangian equations without the term $\tilde{Q}^*_i$

$$\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\frac{\partial\tilde{\mathcal{L}}_1}{\partial\dot{\eta}^i_\alpha} - \frac{\partial\tilde{\mathcal{L}}_1}{\partial\eta^i} = \tilde{P}_i \quad i = 1, 2, \ldots, n. \tag{5.3}$$

(b) In the second case, the problem is formulated in a similar way: Find a Lagrangian density of the form

$$\tilde{\mathcal{L}}_2\left(\eta^i, \eta^i_\alpha, x^\alpha\right) = f_2(x^\alpha)\mathcal{L}\left(\eta^i, \eta^i_\alpha, x^\alpha\right), \tag{5.4}$$

for which the system of Lagrangian equations (5.1) would be transformed into an equivalent system of Euler–Lagrange equations (i.e., without both terms $\tilde{Q}^*_i$ and $\tilde{P}_i$)

$$\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\frac{\partial\tilde{\mathcal{L}}_2}{\partial\dot{\eta}^i_\alpha} - \frac{\partial\tilde{\mathcal{L}}_2}{\partial\eta^i} = 0 \quad i = 1, 2, \ldots, n. \tag{5.5}$$

If such new Lagrangian density exists, the non-conservative fields will be named pseudo-conservative (or quasi-conservative) fields of the first or second type, respectively.

Let us find the condition that a non-conservative classical field is pseudo-conservative.

(a) In the first case, if we insert expression (5.2) into the transformed Lagrangian equations (5.3), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(f_1\frac{\partial\mathcal{L}}{\partial\dot{\eta}^i_\alpha}\right) - \frac{\partial(f_1\mathcal{L})}{\partial\eta^i} = \tilde{P}_i.$$

After grouping the similar terms, we obtain

$$\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\frac{\mathrm{d}f_1}{\mathrm{d}x^\alpha} + \left(\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(\frac{\partial\mathcal{L}}{\partial\dot{\eta}^i_\alpha}\right) - \frac{\partial\mathcal{L}}{\partial\eta^i}\right)f_1 = \tilde{P}_i.$$

Substituting the variation derivative by the corresponding expression from the primary Lagrangian equations (5.1), we obtain

$$\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\frac{\mathrm{d}f_1}{\mathrm{d}x^\alpha} + \left(\tilde{Q}^*_i + \tilde{P}_i\right)f_1 = \tilde{P}_i \quad i = 1, 2, \ldots, n. \tag{5.6}$$

(b) In the second case, when the right-hand side of the corresponding Lagrangian equations (5.5) is zero, substituting $\tilde{P}_i = 0$ into the right-hand side of (5.6) we obtain the corresponding condition in the form

$$\frac{\partial\mathcal{L}}{\partial\eta^i_\alpha}\frac{\mathrm{d}f_2}{\mathrm{d}x^\alpha} + \left(\tilde{Q}^*_i + \tilde{P}_i\right)f_2 = 0 \quad i = 1, 2, \ldots, n. \tag{5.7}$$

Let us remark that in Eqs. (5.6) and (5.7) the potential forces are not present, and therefore, the particular solutions of these equations are also independent on potential forces.

Accordingly, in order that a non-conservative field with variable mass would be pseudo-conservative of the first or second type, it is necessary and sufficient condition that there exists at least one particular solution of the system of differential equations (5.6) or (5.7), respectively, which is independent of potential forces.

5.2 Energy relations

In order to examine the energy relations of the pseudo-conservative fields, it is necessary to discern the previously mentioned two types of these fields.

(a) In the case of the first type, we start from the transformed equations (5.3), multiplying them with η^i_β and summarizing over the repeated index

$$\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(\frac{\partial\tilde{\mathcal{L}}_1}{\partial\dot{\eta}^i_\alpha}\right)\eta^i_\beta-\frac{\partial\tilde{\mathcal{L}}_1}{\partial\eta^i}\eta^i_\beta=\tilde{P}_i\eta^i_\beta.$$

Bearing in mind that for

$$\frac{\partial\tilde{\mathcal{L}}_1}{\partial\eta^\beta}=\frac{\partial\tilde{\mathcal{L}}_1}{\partial\eta^i}\eta^i_\beta+\frac{\partial\tilde{\mathcal{L}}_1}{\partial\eta^i}\eta^i_{\alpha\beta}+\frac{\mathrm{d}\tilde{\mathcal{L}}_1}{\mathrm{d}x^\beta}+\frac{\partial\tilde{\mathcal{L}}_1}{\partial m}\frac{\mathrm{d}m}{\mathrm{d}x^\beta}, \tag{5.8}$$

the previous relation can be transformed into

$$\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(\frac{\partial\tilde{\mathcal{L}}_1}{\partial\dot{\eta}^i_\alpha}\right)\eta^i_\beta-\frac{\partial\tilde{\mathcal{L}}_1}{\partial\eta^i}\eta^i_{\alpha\beta}-\frac{\partial\tilde{\mathcal{L}}_1}{\partial\eta^i}\eta^i_\beta=\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(\frac{\partial\tilde{\mathcal{L}}_1}{\partial\dot{\eta}^i_\alpha}\eta^i_\beta\right)-\left(\frac{\partial\tilde{\mathcal{L}}_1}{\partial x^\beta}-\frac{\mathrm{d}\tilde{\mathcal{L}}_1}{\mathrm{d}x^\beta}-\frac{\partial\tilde{\mathcal{L}}_1}{\partial m}\frac{\mathrm{d}m}{\mathrm{d}x^\beta}\right)=\tilde{P}_i\eta^i_\beta.$$

Grouping the two similar terms, the relation obtains the following form:

$$\frac{\mathrm{d}T^\alpha_\beta}{\mathrm{d}x^\alpha}=\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(\frac{\partial\tilde{\mathcal{L}}_1}{\partial\eta^i_\alpha}\eta^i_\beta-\tilde{\mathcal{L}}_1\right)=-\frac{\partial\tilde{\mathcal{L}}_1}{\partial x^\beta}-\frac{\partial\tilde{\mathcal{L}}_1}{\partial m}\frac{\mathrm{d}m}{\mathrm{d}x^\beta}+\tilde{P}_i\eta^i_\beta, \tag{5.9}$$

where the expression in parentheses according to (4.8) represents the energy–impulse tensor. This relation considers the pseudo-conservative fields of the first type.

(b) For Eq. (5.5), without $\tilde{P}_i$, the law of the energy–impulse change for the pseudo-conservative systems of the second type is

$$\frac{\mathrm{d}T^\alpha_\beta}{\mathrm{d}x^\alpha}=\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\left(\frac{\partial\tilde{\mathcal{L}}_2}{\partial\eta^i_\alpha}\eta^i_\beta-\delta^\alpha_\beta\tilde{\mathcal{L}}_2\right)=-\frac{\partial\tilde{\mathcal{L}}_2}{\partial x^\beta}-\frac{\partial\tilde{\mathcal{L}}_2}{\partial m}\frac{\mathrm{d}m}{\mathrm{d}x^\beta}. \tag{5.10}$$

Under certain conditions using the energy laws, the corresponding integrals (usual or in a broader sense) of motion are obtained. If we establish that the corresponding classical field is pseudo-conservative, the Lagrangian density of the first type is (5.2) and of the second type (5.4).

Finally, we conclude that the pseudo-conservative fields may have two types of energy change laws: (5.9) and (5.10), depending whether condition (5.6) or (5.7) is satisfied.

5.3 The case of natural classical fields

Let us find the condition for the non-conservative field to be pseudo-conservative. Two cases are considered depending on whether the pseudo-conservative fields are of the first or second type.

(a) The non-conservative field is the pseudo-conservative system of the first type if condition (5.6) is satisfied. According to the definition of the natural classical fields (2.13), we have $\frac{\partial\mathcal{L}}{\partial\dot{\eta}^i}=a_{ik}\dot{\eta}^k$ and the condition has the form

$$a_{ik}\dot{\eta}^k\frac{\mathrm{d}f_1}{\mathrm{d}t}+\left(\tilde{Q}^*_i+\tilde{P}_i\right)f_1=\tilde{P}_i. \tag{5.11}$$

Bearing in mind that the solution of equation must be independent of the potential forces, it must be independent from the variables η^i and $\dot{\eta}^i$. Let us introduce the forces in the form

$$\tilde{Q}_i^* = -2\mu a_{ik}\dot{\eta}^k,\ \tilde{P}_i = c_2 a_{ik}\dot{\eta}^k. \tag{5.12}$$

Substituting (5.12) into (5.11) and using the expression

$$2\mu' = 2\mu - c_2, \tag{5.13}$$

it follows that

$$\frac{\mathrm{d}f_1}{\mathrm{d}t} - 2\mu' f_1 = c_2. \tag{5.14}$$

One particular solution of the linear equation (5.14) is

$$f_1(t) = -\frac{c_2}{2\mu'} \equiv K. \tag{5.15}$$

Using (5.15) and expression (5.2), the Lagrangian density follows:

$$\tilde{\mathcal{L}}_1\left(\eta^i, \eta^i_\alpha, x^\alpha\right) = f_1(t)\mathcal{L}\left(\eta^i, \eta^i_\alpha, x^\alpha\right) = K\mathcal{L}\left(\eta^i, \eta^i_\alpha, x^\alpha\right). \tag{5.16}$$

(b) In the case of the second type of pseudo-conservative field, the term $\tilde{P}_i$ in relation (5.7) is zero, i.e., $c_2 = 0$. Expression (5.14) simplifies into

$$\frac{\mathrm{d}f_2}{\mathrm{d}t} - 2\mu f_2 = 0. \tag{5.17}$$

One particular solution of Eq. (5.17) is

$$f_2(t) = \exp(2\mu t). \tag{5.18}$$

Using (5.18), the Lagrangian density according to (5.4) is

$$\tilde{\mathcal{L}}_2\left(\eta^i, \eta^i_\alpha, x^\alpha\right) = \exp(2\mu t)\mathcal{L}\left(\eta^i, \eta^i_\alpha, x^\alpha\right). \tag{5.19}$$

Applying the energy–impulse tensor T^α_β which is reduced to the Hamiltonian density, i.e., density of energy, the energy law (5.9) has the form

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\tilde{\mathcal{L}}_1}{\partial\dot{\eta}^i}\dot{\eta}^i - \tilde{\mathcal{L}}_1\right) = -\frac{\partial\tilde{\mathcal{L}}_1}{\partial t} - \frac{\partial\tilde{\mathcal{L}}_1}{\partial m}\frac{\mathrm{d}m}{\mathrm{d}t} - \tilde{P}_i\dot{\eta}^i. \tag{5.20}$$

In the case of the second type of pseudo-conservative fields, the energy law is according to (5.10)

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\tilde{\mathcal{L}}_2}{\partial\dot{\eta}^i} - \tilde{\mathcal{L}}_2\right) = -\frac{\partial\tilde{\mathcal{L}}_2}{\partial t} - \frac{\partial\tilde{\mathcal{L}}_2}{\partial m}\frac{\mathrm{d}m}{\mathrm{d}t}. \tag{5.21}$$

For the field with variable mass, the term $\frac{\partial\mathcal{L}}{\partial m}\frac{\mathrm{d}m}{\mathrm{d}t}$ is almost never zero, except in the case when the separation and annexation are mutually canceled. Therefore, even in the case when $\frac{\partial\mathcal{L}}{\partial t} = 0$ and $\tilde{P}_i\dot{\eta}^i = 0$ the energy conservation law almost never can be valid. However, from these energy laws one can obtain some information how the energy of the considered field changes in the course of time.

6 One example

Let us consider a natural continuous mechanical system in a viscous medium, whose elements oscillate along a straight line, which can be taken as x-axis. Such motion has only one degree of freedom and can be determined by the elongation of any element of the system from its balance position as field function denoted as $\eta(x,t)$. The oscillating continuous system is determined with the Lagrange density

$$\mathcal{L}\left(\eta^i, \eta^i_\alpha, x^\alpha\right) = \frac{1}{2}\varrho\dot{\eta}^2 - \frac{1}{2}\varrho k\eta^2. \tag{6.1}$$

Suppose that in every time interval $(t, t + \mathrm{d}t)$ some small part of the element of the continuous system separates with velocity proportional to the velocity of the element $\mathrm{u} = \lambda\dot{\eta}$. Diminution of the mass is assumed to be proportional to the mass m and time interval $\mathrm{d}t$, i.e., $\mathrm{d}m = -\beta m\mathrm{d}t$. The mass variation function is

$$m = m_0 \exp(-\beta t). \tag{6.2}$$

The resistance force, due viscous friction is $\tilde{Q}^* = -2\mu\rho\lambda\dot{\eta}$. According to (5.12), the system is pseudo-conservative. Using the unit mass ($m = \rho$) and its velocity of variation, the quantity $\tilde{P}^*_i$ is

$$\tilde{P}_i = -\beta\rho\lambda\dot{\eta}. \tag{6.3}$$

Lagrangian density concerns to volume unit, and it also concerns to corresponding quantities.

We see that the quantities $\tilde{Q}^*_i$ and $\tilde{P}_i$ satisfy the necessary condition (5.12) for non-conservative continuous system to be pseudo-conservative. Thus, the corresponding differential equations for functions f_1 and f_2, (5.6) and (5.7), will be reduced to Eqs. (5.14) and (5.17), respectively, and give the satisfying particular solutions in forms (5.15) and (5.18). This means that the considered system is simultaneously a pseudo-conservative system of the first and second type, but this does not mean that this system necessarily has the both corresponding energy integrals.

Using the generalized Noether's theorem, let us examine whether the energy integral exists.

(a) For simplicity, let us start with the second type of pseudo-conservative field. In this case, the system is determined by the Lagrangian density of form (5.19)

$$\tilde{\mathcal{L}}_2\left(\eta^i, \eta^i_\alpha, x^\alpha\right) = \exp(2\mu t)\left(\frac{1}{2}\rho\dot{\eta}^2 - \frac{1}{2}\rho k\eta^2\right). \tag{6.4}$$

Applying the generalized Noether's theorem to the field as pseudo-conservative system of the second type, where $\tilde{Q}^*_i = 0$ and $\tilde{P}_i = 0$, the condition for existence of the integrals of motion for the natural classical fields (4.15) for the mass unit ($m = \rho$) is

$$\frac{\partial\tilde{\mathcal{L}}_2}{\partial\eta}\xi^1 + \frac{\partial\tilde{\mathcal{L}}_2}{\partial\dot{\eta}}\dot{\xi}^1 + \frac{\partial\tilde{\mathcal{L}}_2}{\partial t}\xi^{00} + \left(\tilde{\mathcal{L}}_2 - \frac{\partial\tilde{\mathcal{L}}_2}{\partial\dot{\eta}}\dot{\eta}\right)\dot{\xi}^{00} + \frac{\partial\tilde{\mathcal{L}}_2}{\partial\varrho}\frac{\mathrm{d}\varrho}{\mathrm{d}t}\xi^{00} - \dot{\Lambda}^0 = 0. \tag{6.5}$$

We choose the transformation functions in the form which corresponds to the oscillator with linear viscous damping (see Vujanovic and Jones [7], pp. 98–99)

$$\xi^1 = -A\eta, \quad \xi^{00} = 1, \quad \Lambda^0 = 0. \tag{6.6}$$

Substituting (6.6) into (6.4) and (6.5), it follows that

$$\varrho\left(A - \mu + \frac{1}{2}\beta\right)\left(k\eta^2 - \dot{\eta}^2\right) = 0. \tag{6.7}$$

If $k\eta^2 - \dot{\eta}^2 \neq 0$, relations $(6.6)_1$ and (6.7) are satisfied for

$$A = \mu - \frac{1}{2}\beta, \quad \xi^1 = \left(\frac{1}{2}\beta - \mu\right)\eta. \tag{6.8}$$

Namely, with such choose of A, condition (6.5) is satisfied for any η and $\dot{\eta}$. Finally, for (6.6)–(6.8), condition (6.5) is fulfilled.

Remark Condition (6.8) is obtained due to the fact that the condition for existence of the integral of motion F has the form

$$F = \varphi\left(\eta^i, \eta^i_\alpha\right)\psi\left(\xi^i_m, \xi^{0\alpha}_m, \Lambda^\alpha_m\right) = 0.$$

The transformation functions $\xi^i_m, \xi^{0\alpha}_m, \Lambda^\alpha_m$ have an important role in the problem and must be independent from other factors. In this case, they are determined due to the fact that the condition for existence of the integrals of motion is presented in form (6.7), which enables to obtain functions independent on η^i, η^i_α. In this general case, form (6.8) can be considered as a necessary condition for finding at least of one set of particular solutions $(\xi^i_m, \xi^{0\alpha}_m, \Lambda^\alpha_m)$ which satisfies the condition for existence of the integral of motion.

(b) For the continuous system as a pseudo-conservative one of the first type, where $\tilde{Q}^*_i = 0$ and $\tilde{P}_i \neq 0$, the Lagrangian density (5.16) is assumed in the form

$$\tilde{\mathcal{L}}_1\left(\eta^i, \eta^i_\alpha, x^\alpha\right) = K\left(\frac{1}{2}\rho\dot{\eta}^2 - \frac{1}{2}\rho k\eta^2\right), \tag{6.9}$$

and condition (4.15) transforms into

$$\frac{\partial\tilde{\mathcal{L}}_1}{\partial\eta}\xi^1 + \frac{\partial\tilde{\mathcal{L}}_1}{\partial\dot{\eta}}\dot{\xi}^1 + \frac{\partial\tilde{\mathcal{L}}_1}{\partial t}\xi^{00} + \left(\tilde{\mathcal{L}}_1 - \frac{\partial\tilde{\mathcal{L}}_1}{\partial\dot{\eta}}\dot{\eta}\right)\dot{\xi}^{00} + \tilde{P}_i(\xi^1 - \dot{\eta}\xi^{00}) + \frac{\partial\tilde{\mathcal{L}}_1}{\partial\varrho}\frac{\mathrm{d}\varrho}{\mathrm{d}t}\xi^{00} - \dot{\Lambda}^0 = 0. \tag{6.10}$$

On the basis of (6.9) and bearing in mind that the density of energy of the pseudo-conservative system is $\tilde{\varepsilon} = \frac{\partial\tilde{\mathcal{L}}_1}{\partial\dot{\eta}}\dot{\eta} - \tilde{\mathcal{L}}_1 = K(\frac{1}{2}\rho\dot{\eta}^2 + \frac{1}{2}\rho k\eta^2)$, the relation becomes

$$\begin{aligned}&\eta^2\left(\frac{1}{2}K\varrho\dot{\xi}^{00} + \frac{1}{2}K\beta\rho k\xi^{00}\right) + \dot{\eta}^2\left(-\frac{1}{2}K\varrho\dot{\xi}^{00} + \beta\rho\lambda\xi^{00} - \frac{1}{2}K\beta\rho\xi^{00}\right) + \eta\left(-K\rho k\xi^1\right)\\ &\quad + \dot{\eta}\left(K\rho\dot{\xi}^1 - \beta\rho\lambda\xi^1\right) - \dot{\Lambda}^0 = 0.\end{aligned} \tag{6.11}$$

Since in this relation the variables $(\eta, \dot{\eta})$ and $(\xi^1\xi^{00}, \Lambda^0)$ are mutually mixed up, it is impossible to present it in form (6.8). Therefore, it is impossible to find at least one set of particular solutions $(\xi^1, \xi^{00}, \Lambda^0)$ which satisfy condition (6.10) for existence of the corresponding integrals of motion. Thus, although this continuous system is pseudo-conservative of the first type, there is no integral of motion. The necessary condition for existence of motion integrals is that the system is pseudo-conservative of the second type.

As in the considered continuous system mass is continuously decreasing and is pseudo-conservative of the second type, the integral of motion has form (4.16), i.e.,

$$I = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\partial\tilde{\mathcal{L}}_2}{\partial\dot{\eta}}\xi^1 + \left(\tilde{\mathcal{L}}_2 - \frac{\partial\tilde{\mathcal{L}}_2}{\partial\dot{\eta}}\dot{\eta}\right)\xi^{00} - \Lambda^0\right\} = 0, \tag{6.12}$$

and according to (6.6) and (6.8), it obtains an explicit form

$$I = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\exp\left(2\mu t\right)\left[\left(\frac{1}{2}\rho\dot{\eta}^2 + \frac{1}{2}\rho k\eta^2\right) + \left(\frac{1}{2}\beta - \mu\right)\rho\eta\dot{\eta}\right]\right\}. \tag{6.13}$$

Integral of motion represents an energy integral in a broader sense, which is equivalent to

$$\varepsilon = \exp\left(2\mu t\right)\left\{\left(\frac{1}{2}\rho\dot{\eta}^2 + \frac{1}{2}\rho k\eta^2\right) + \left(\frac{1}{2}\beta - \mu\right)\rho\eta\dot{\eta}\right\} = \text{const.} \tag{6.14}$$

This energy integral for the continuous system is according to its form similar to the corresponding energy integrals in mechanics of particles with variable mass for systems which satisfy the necessary condition for the existence of integrals of motion (see Vujanovic and Jones [7], pp. 97–100).

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